Research project **DR**

Distribution of reduction types of abelian varieties

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Research project short description:

Let k be a field. An *abelian variety* over k is a smooth proper connected group scheme over $\operatorname{Spec} k$ - a higher-dimensional generalisation of an elliptic curve. Given C/k a smooth proper curve, we can construct the *jacobian* $J(C)$ of C - this is an abelian variety over k, of dimension equal to the genus of C.

As with elliptic curves, it is useful to study the reduction of an abelian variety $A/\mathbb{Q}$ at a prime p. This is defined to be the special fibre at p of the Néron model of A over $\operatorname{Spec} \mathbb{Z}$ [1]. For most choices of p, this reduction A_p will again be an abelian variety (over the finite field $\mathbb{F}_p$); in general it is a smooth commutative group scheme over $\mathbb{F}_p$. We define $\mathcal{L}$ to be the following list of properties of group schemes over finite fields:

- semi-abelian;
- semi-abelian with toric part of rank n for some fixed n;
- isomorphic to the jacobian of a smooth proper curve;
- abelian and ordinary;
- abelian and superspecial.

Fix an integer $d \geq 0$. Let $\mathcal{P} \in \mathcal{L}$ be a property from the list above. **Question 1:** *Does every abelian variety $A/\mathbb{Q}$ of dimension d have a prime p such that the reduction A_p over $\mathbb{F}_p$ satisfies property $\mathcal{P}$? Does it have infinitely many such primes?*

Sometimes this question is easy (for example, if $d = 0$), but sometimes it is not. Recall that an abelian variety $A/\mathbb{F}_p$ of dimension g is *ordinary* if $A[p](\overline{\mathbb{F}}_p) \cong (\mathbb{Z}/p\mathbb{Z})^g$, i.e. the p-torsion is as large as possible. An elliptic curve over a finite field is *superspecial* (equivalently supersingular) if and only if it is not ordinary. If $d = 3$ and $\mathcal{P}$ is 'abelian and ordinary', then question 1 is open. If $d = 1$ and $\mathcal{P}$ is 'abelian and superspecial', then a celebrated theorem of Elkies [2] shows there are in fact infinitely many primes p such that A_p has property $\mathcal{P}$.

We can also ask the same question but with our attention restricted to certain classes of abelian varieties, and we can also look at 'asymptotic' versions of the question. There are various ways to formalise this idea; we give a simple one in the following section.

Asymptotic statements for jacobians

Given $n \in \mathbb{N}$, let $F(n)$ denote the set of separable polynomials in one variable x over $\mathbb{Q}$ of degree 8 all of whose coefficients have height less than n (this is a finite set). Given such a polynomial f, write C_f for the hyperelliptic genus 3 curve defined by the equation $y^2 = f(x)$. Define $G(n) \subset F(n)$ to be the subset of polynomials f such that there exists a prime p with the property that the jacobian of C_f has ordinary reduction at p.

Question 2: *Is it true that*

$$\lim_{n\to\infty} \frac{\#G(n)}{\#F(n)} = 1?$$

If we fix a prime p, and look at the set of degree-8 polynomials in one variable over $\overline{\mathbb{F}}_p$, we will find that a Zariski dense subset are separable, and moreover that a Zariski-dense subset have the property that the jacobian of the curve $y^2 = f(x)$ is ordinary [3]. In other words, at a given (sufficiently large) prime p, 'most' genus 3 hyperelliptic jacobians varieties over $\mathbb{F}_p$ are ordinary. As a result, the answer to the above question should be 'yes'.

One aim of this project will be to provide a proof of a positive answer to question 2. A reasonable strategy to do so would be to estimate the proportion of abelian varieties which have good ordinary reduction at a fixed prime p (probably making use of the Lang-Weil estimates [4]), and then combine this information at all primes.

Generalisations

There are many related questions which can also be studied:

- investigate the (presumably easier) cases of jacobians of hyperelliptic curves of genus 1 and 2;
- investigate the (presumably harder) cases of jacobians of hyperelliptic curves of genus > 3;
- instead of studying ordinary reduction, study other properties in the list $\mathcal{L}$;
- generalise to number fields;
- instead of looking at the jacobians of hyperelliptic curves (which have an obvious parametrisation), study other unirational families of abelian varieties.

References:

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[4] Lang, Weil *Number of points of varieties in finite fields*, Amer. J. Math. 76, (1954). 819-827.
[5] Noot, *Abelian varieties-Galois representation and properties of ordinary reduction*, Compositio Mathematica, 97 no. 1-2 (1995), p. 161-171.
[6] Ogus, *Hodge cycles and crystalline cohomology*, In: (P. Deligne, J. S. Milne, A. Ogus and K.-y. Shih) Hodge Cycles, Motives, and Shimura Varieties, Chapter VI. Lecture Notes in Math. 900. Springer-Verlag (1982) pp. 357-414.