ALGANT Erasmus Mundus Joint Doctorate programme intake 2014

Research project

Picard functors of curves over high-dimensional base schemes

Supervisor 1 David Holmes
E-mail holmesdst@math.leidenuniv.nl
Institution Universiteit Leiden

Supervisor 2 Jilong Tong
E-mail jilong.tong@math.u-bordeaux1.fr
Institution Universisé Bordeaux 1

1. Research project short description

1.1. **Introduction.** - Let S be a scheme and $C \to S$ a curve (a proper flat morphism with connected geometric fibres of pure dimension 1). The moduli space of line bundles on C/S, together with the tensor product operation, form a smooth commutative algebraic group stack, the *Picard stack* $\mathrm{Pic}_{C/S}$ of C/S. Under certain mild hypotheses (such as C/S being *split semi-stable*), the Picard stack is a scheme. See [BLR90], [FGI$^+$05] for very much more information. This project will investigate various phenomena related to this Picard stack, including :
- constructing 'compactifications' of the Picard stack using moduli of torsion free rank 1 sheaves ;
- links to Néron models of abelian varieties.

1.2. **Compactifications of Picard schemes.** Let C/S be a curve. We write $\mathrm{Pic}^0_{C/S}$ for the fibrewise connected component of the identity in $\mathrm{Pic}_{C/S}$ (cf. [GD66, 15.6.5]). If $C \to S$ is smooth, then $\mathrm{Pic}^0_{C/S}$ is a proper scheme over S, and so is an *abelian scheme* (a smooth proper group scheme with connected geometric fibres). However, if C/S is not smooth then $\mathrm{Pic}^0_{C/S}$ is not (in general) proper (or a scheme) over S. We are interested in constructing *compactifications*[1] of $\mathrm{Pic}^0_{C/S}$ with various nice properties. One approach to this is to use moduli of more general sheaves on C/S, in particular *torsion free rank 1 sheaves which on geometric fibres can be realised as the push-forward of a line bundle on a connected partial normalisation.*

This construction is fairly well-understood if the geometric fibres of C/S are integral (see for example [AK80] [KK81]), but there are still many open questions, and the general case is very mysterious.

[1]By 'a compactification' we mean an open immersion into a substack which admits an open cover by stacks proper over the base.

1.3. **Relation to Néron models.**

Definition 1.1. Let S be a connected scheme, and $U \subset S$ a dense open subscheme. Let A/U be an abelian scheme. For example, A/U could be an elliptic curve over U. A *Néron model* for A over S is a smooth, separated, finitely presented scheme $\mathscr{A}/S$ which satisfies the *Néron mapping property* :

For each smooth S-scheme Y and U-morphism $f : Y_U \to A$, there exists a unique S-morphism $F : Y \to \mathscr{A}$ such that $F_U = f$.

Note that a Néron model of an abelian scheme is in a canonical way a group scheme (if it exists). See [BLR90] for more information on Néron models.

Theorem 1.2 ([Nér64])**.** *Let S, U, A be as above, and assume that S is a Dedekind scheme (ie. is integral, normal, Noetherian and of dimension 1). Then a Néron model for A exists.*

It is natural to ask whether the same result holds if we allow S to be of higher dimension. The answer is no ; see the discussion at `http://mathoverflow.net/questions/12923`. A better question is :

Question 1.3. Let S, U, A be as above (S arbitrary connected). Does there exist a proper surjective morphism of connected schemes $\pi : S' \to S$, such that $\pi^* A$ (an abelian scheme over $\pi^{-1}U$) admits a Néron model over S' ?

The answer is again 'no' in general, though this seems rather harder to prove. However, if the closure of the unit section in the Picard scheme of a regular model of C/S is flat over S, then the answer seems to be 'yes'. In the case where C/S is semi-stable, it is fairly well-understood when this closure is flat. However, when C/S is allowed to have more general singularities, the picture again becomes very mysterious. It would also be very interesting to investigate the case of moduli of sheaves on families of more general varieties (rather than just families of curves).

Références

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