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INTEGRAL TRANSFORMS OF CONSTRUCTIBLE FUNCTIONS

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Introduction

The main objects of study in this thesis are constructible functions. A constructible function on a real analytic manifold X is an integer valued function which is locally constant on every connected component of a subanalytic stratification of X. On the other hand, an $\mathbb{R}$-constructible sheaf on X is an object of the derived category of sheaves of k-vector spaces (for k a field) having locally constant cohomologies on every component of a subanalytic stratification. It can be shown that the Grothendieck group of the category of $\mathbb{R}$-constructible sheaves is generated by linear combinations of classes of constant sheaves on a locally closed subanalytic sets. Hence one finds an isomorphism of this group with the category of constructible functions, given by:

$$k_Z^m \mapsto m.\mathbf{1}_Z, \quad k_Z[1] \mapsto (-1).\mathbf{1}_Z.$$

Here $Z \subset X$ is a compact contractible subanalytic set, $m > 0$ an integer and $\mathbf{1}_Z$ the characteristic function of Z. This isomorphism allows one to translate the operations and theorems from sheaf theory to the framework of constructible functions.

Among various operations on constructible sheaves, we will highlight the Radon transform. It is defined in the following way:
Assume ϕ is a constructible function on X. Consider the double morphism of real analytic manifolds:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y, \end{tikzcd}$$

the Radon transform of ϕ with the incidence relation S is defined by:

$$\mathcal{R}_S(\phi) = \int_g f^*\phi.$$

Where the operation $\int_g$ plays the same role on consructible functions, as the proper direct image, $Rg_!$, does on derived category of sheaves.
Given a Radon transform, the question of existence of an inverse naturally arises. P. Schapira in [8] considered this problem in general. He proved the existence of an inverse under certain hypotheses. These hypotheses however, are not fulfilled if one looks at the Radon transforms on Grassmannians.
In [7], Y. Matsui, under some assumptions, obtained an inverse for the Radon transform on Grassmannians with the inclusion incidence relation. In order to do it, he considered transforms

associated to different incidence relations and combined them using Cramer's rule to obtain an inverse.
In [6], C. Marastoni considered the transversality incidence relation for dual complex Grassmannians, in the framework of constructible sheaves. Using microlocal techniques he proved that the associated Radon transfom is invertible.
In this thesis we consider the transversality incidence relation for general Grassmannians, in the framework of constructible functions. We have two main results. On one hand, we show that for dual Grassmannians the transversality incidence relation is self adjoint. On the other hand, under certain assumptions, we prove that the associated Radon transform has an inverse. Here, as in Matsui's, the idea is to consider transforms associated to different incidence relations. However, in our proof we avoid the use of Schubert calculus. In addition, our techniques could be adapted to obtain a simplified proof of Matsui's main result.

The first chapter of this thesis is a brief introduction to the category of $\mathbb{R}$-constructible sheaves, in which we give the grounds for defining constructible functions, as well as the operations on them.

In the second chapter, we will introduce a type of integral functor over sheaves. Then, we will consider the analogue of this integral transform in the framework of constructible functions. Which is in particular the Radon transform.

The third chapter is concentrated on Radon transforms on Grassmannians. In this chapter we will prove our results.

The last parts of this thesis are appendix A and appendix B. Appendix A is a collection of duality formulas and theorems related to Euler-Poincaré index with compact support. Theorems and formulas which are employed in the earlier chapters. Appendix B, contains a definition of micro-support which is used in the second chapter. Few basic theorems on micro-supports are also mentioned.

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Chapter 1

Constructible sheaves and constructible functions

This chapter contains basic definitions and theorems for constructible functions. We will state definitions of subanalytic stratification, constructible sheaves and finally constructible functions. Next, we will explain how Euler-Poincaré index provides an isomorphism between Grothendieck group of category of $\mathbb{R}$-constructible sheaves and constructible functions. This isomorphism plays a key role in defining the operations on constructible functions motivated by the operations on constructible sheaves. To justify definitions of operations on constructible functions we will recall series of theorems from theory of sheaves. The main references for this chapter are [5] and [3].

1.1 Constructible sheaves

In this section we first introduce subanalytic stratifications. Next, we will define constructible sheaves which have locally constant cohomology groups and are stable under the Grothendieck operations.

Notation 1.1.1. *Throughout the thesis for an algebraically closed field k and a topological manifold X, we denote by $\mathbf{D}^b(k_X)$, the derived category of bounded complexes of sheaves of k-vector spaces. Moreover, k_X is the locally constant sheaf on X with stalk k. If $F \in Ob(\mathbf{D}^b(k_X))$ to lighten the notation we will write $F \in \mathbf{D}^b(k_X)$.*

1.1.1 Subanalytic stratification

Subanalytic sets, are more general objects than semi-analytic sets (those which are locally defined by inequalities of analytic functions). Family of subanalytic sets is closed under closure, complement, inverse images and proper direct images. More precisely:

Definition 1.1.2. *Let Y be a real analytic manifold and let Z be a subset of Y. One says Z is subanalytic at $y \in Y$ if there exist an open neighborhood U of y, compact manifolds*

$X_j^i (i=1,2, 1 \geq j \geq N)$ and morphisms $f_j^i : X_j^i \to Y$ such that:

$$Z \cap U = U \cap \bigcup_{j=1}^{N} (f_j^1(X_j^1) \backslash f_j^2(X_j^2)).$$

If Z is subanalytic at each $y \in Y$, one says Z is subanalytic in Y.

Subanalytic sets carry the following properties (for a proof, see [5], p.327).

Proposition 1.1.3. *(i) Assume Z is subanalytic in Y. Then $\overline{Z}$ and $Int(Z)$ are subanalytic in Y. Moreover the connected components of Z are locally finite and subanalytic.*

(ii) Assume Z_1 and Z_2 are subanalytic in Y. Then $Z_1 \cup Z_2, Z_1 \backslash Z_2$ and $Z_1 \cap Z_2$ are subanalytic.

(iii) Let $f : X \to Y$ be a morphism of manifolds. If $Z \subset Y$ is subanalytic in Y then $f^{-1}(Z)$ is subanalytic in X. If $W \subset X$ is subanalytic in X and f is proper on $\overline{W}$, then $f(W)$ is subanalytic in Y.

(iv) Let Z be a closed subanalytic subset of Y. Then there exists a manifold X and a proper morphism $f : X \to Y$ such that $f(X) = Z$.

Definition 1.1.4. *For a closed subanalytic subset Y of X, a partition $Y = \bigsqcup_{\alpha \in A} X_\alpha$ is called a subanalytic stratification of Y, if it is locally finite, the X_α's are subanalytic submanifolds and for all pairs $(\alpha, \beta) \in A \times A$ such that $\overline{X}_\alpha \cap X_\beta \neq \emptyset$ one has $X_\beta \subset \overline{X}_\alpha$. Each X_α is called a stratum.*

1.1.2 Definition of constructible sheaves

Definition 1.1.5. *Let F be an object of $\mathbf{D}^b(k_X)$, the derived category of bounded complexes k-vector spaces. Assume that there exists a locally finite covering $X = \bigcup_{i \in I} X_i$ by subanalytic subsets such that for all $j \in \mathbb{Z}$, all $i \in I$, the sheaves $H^j(F)|_{X_i}$ are locally constant of finite rank. Then one says that F is $\mathbb{R}$-constructible. We denote by $\mathbf{D}^b_{\mathbb{R}-c}(k_X)$ the full triangulated subcategory of $\mathbf{D}^b(k_X)$ consisting of $\mathbb{R}$-constructible objects.*

Example 1.1.6. *Assume Z is a locally closed subanalytic subset of an analytic manifold X, and denote by $j : Z \to X$ the embedding. Note if $X = \bigsqcup_\alpha X_\alpha$ is a subanalytic stratification, then so is $X = (\bigsqcup_\alpha X_\alpha \cap Z) \sqcup (\bigsqcup_\alpha X_\alpha \cap (X \backslash Z))$. Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$. For $F_Z = Rj_! j^{-1} F$ one has:*

$$\begin{cases} F_Z|_Z = F|_Z; \\ F_Z|_{X \backslash Z} = 0. \end{cases}$$

As a result F_Z locally constant on both types of strata $X_\alpha \cap Z$ and $X_\alpha \cap (X \backslash Z)$. Therefore, $F_Z \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$.

Unlike the category locally constant sheaves, the category of constructible sheaves is preserved under many natural sheaf theoretic operations.

Theorem 1.1.7. *(i) Let $f : X \to Y$ be an analytic morphism of real analytic spaces. Then the following holds:*

(a) *If* $G \in \mathbf{D}^b_{\mathbb{R}-c}(k_Y)$ *then* $f^{-1}G \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$ *and* $f^!G \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$.

(b) *If* $F \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$ *and restriction of* f *to* $supp(F)$ *is proper, then* $Rf_*(F)$ *and* $Rf_!(F)$ *are* $\mathbb{R}$*-constructible.*

(ii) *If* $F, G \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$ *then* $F \overset{L}{\otimes} G$ *and* $RHom(F, G) \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$.

(iii) *Let* $F \in \mathbf{D}^b(k_X)$. *Then* F *is* $\mathbb{R}$*-constructible if and only if the dual*[1] $D_X F$ *is* $\mathbb{R}$*-constuctible. In particular, the dualizing complex* $\omega_X = D_X k_X$ *is* $\mathbb{R}$*-constructible.*

(iv) *If* $F \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$ *then we have the following:*

(a) *The natural morphism* $F \to D_X(D_X(F))$ *is an isomorphism;*

(b) *For any* $x \in X$ *one has* $(D_X F)_x \simeq RHom(R\Gamma_x(X, F), k)$.

Example 1.1.8. *Let* $X = \mathbb{C}$ *and* $S = \mathbb{C} - \{0\}$ *and* F *a sheaf on* S *with* $supp(F) = \{x_n : n \in \mathbb{N}\}$, *where* $x_n = \frac{1}{n+1}$ *and* $F_{x_n} = \mathbb{C}^n$. *We see that* F *is an* $\mathbb{R}$*-constructible sheaf on* S. *Yet, the sheaf* $i_!F$ *is not. Indeed, if* $i_!F$ *was an* $\mathbb{R}$*-constructible sheaf we would have a subanalytic stratification of* X *and the origin would be in a stratum different from each of* $\{x_n\}$. *Such a stratification though, could not be locally finite.*

Recall that for a continuous map $f : X \to Y$ and sheaf $F \in \mathbf{D}^b(A_X)$ and a sheaf $G \in \mathbf{D}^b(A_Y)$ we have (i) $f^!(DG) \simeq D(f^{-1}G)$ and (ii) $Rf_*(DF) \simeq D(Rf_!F)$. Therefore, by the last theorem we get:

Corollary 1.1.9. *Let* $f : X \to Y$ *be an analytic morphism real analytic spaces. Then,*

(i) *if* $F \in \mathbf{D}^b_{\mathbb{R}-c}(A_X)$, *then* $Rf_!(DF) \simeq D(Rf_*(F))$,

(ii) *if* $G \in \mathbf{D}^b_{\mathbb{R}-c}(A_Y)$, *then* $f^{-1}(DG) \simeq D(f^!(G))$.

Another lemma will be useful later.

Lemma 1.1.10. *Let* $F \in \mathbf{D}^b_{w-\mathbb{R}-c}(A_X)$ *and let* $\phi : X \to \mathbb{R}^n$ *be a real analytic function. Assume* $\phi|_{supp(F)}$ *is proper. Then we have the natural isomorphism:*

(i) $R\Gamma(\phi^{-1}(\overline{B}_\epsilon); F) \simeq R\Gamma(\phi^{-1}(B_\epsilon); F) \simeq R\Gamma(\phi^{-1}(0); F)$, *for* $0 < \epsilon \ll 1$,

(ii) $R\Gamma_{\phi^{-1}(0)}(X; F) \simeq R\Gamma_{\phi^{-1}(\overline{B}_{\epsilon'})}(X; F) \simeq R\Gamma_c(\phi^{-1}(B_\epsilon); F)$, *for* $0 < \epsilon \ll 1$.

[1] For a definition of dual see A.6.

1.2 Constructible functions

Definition 1.2.1. *A function $\phi : X \to \mathbb{Z}$ is constructible if:*

(i) for all $m \in \mathbb{Z}, \phi^{-1}(m)$ is subanalytic,

(ii) the family $\{\phi^{-1}(m)\}_{m\in\mathbb{Z}}$ is locally finite.

We denote by $CF(X)$ the set of constructible functions on X. Which is a ring under usual operations of addition and multiplication. The presheaf $U \to CF(U)$ (U open in X) is a sheaf. We denote it by CF_X.

In the above definition $\{\phi^{-1}(m)\}_{m\in\mathbb{Z}}$ gives a subanalytic, locally finite covering of X. Therefore, by Hardt triangulation theorem([5], Proposition 8.2.5), there exists a locally finite, subanalytic stratification $\bigsqcup_{\alpha\in I} K_\alpha$, such that all K_α's are compact and contractible. Hence we simplify:

Lemma 1.2.2. *A function $\phi : X \to \mathbb{Z}$ is constructible if and only if there exists a locally finite family of compact subanalytic contractible subsets $\{K_i\}_{i\in I}$, such that:*

$$\phi = \sum_{i\in I} c_i \mathbf{1}_{K_i},$$

where $c_i \in \mathbb{Z}$ and $\mathbf{1}_A$ is characteristic function of the subset A, i.e. $\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A. \end{cases}$

1.2.1 Euler-Poincaré index

Theory of constructible functions is based on an observation on the local Euler-Poincaré Index of a constructible sheaf. Recall that for F an object of $\mathbf{D}^b_f(k_X)$, the derived category of finitely generated k-vector spaces, the local Euler-Poincarè index of F is defined to be:

$$\chi(F) = \sum_j (-1)^j dim H^j(F).$$

Definition 1.2.3. *Let X be an analytic manifold and $F \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$. For any $x \in X$ one sets:*

$$\chi(F)(x) = \chi(F_x),$$

$$\chi_c(F)(x) = \chi(R\Gamma_x(X; F)) = \chi(D(F))(x).$$

Moreover, if $R\Gamma(X; F)$ (resp. $R\Gamma_c(X; F)$) belongs to $\mathbf{D}^b_f(k_X)$, one sets:

$$\chi(X; F) = \chi(R\Gamma(X; F))$$

$$\text{(resp. } \chi_c(X; F) = \chi(R\Gamma_c(X; F))\text{)}.$$

The integer $\chi(X; F)$ is called the Euler-Poincaré index of F and the function $\chi(F)(x)$ is called the local Euler-Poincaré index of F.

Remark 1.2.4. *The equality $\chi_c(F)(x) = \chi(D(F))(x)$ in the preceding definition is implied by Theorem 1.1.7*

Example 1.2.5. *(i) If $F \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$, then by definition there exists a subanalytic stratification $\bigsqcup_\alpha X_\alpha$ such that $F|_{X_\alpha}$ are locally constant, and so for a connected component of each X_α, say Z, $H^j(F)|_Z$ is a constant integer. Therefore $\chi(F)$ is a constructible function.*

(ii) It is easy to see that $\chi(\mathbb{R}, k^5_{[0,1]} + k_{[2,3]}[+1]) = 5\mathbf{1}_{[0,1]} - \mathbf{1}_{[2,3]}$.

(iii) Assume Y is a locally closed subset of X, then:

$$\chi(R\Gamma_c(X;k_Y)) = \chi(R\Gamma_c(X;i_!k_Y)) = \chi(R\Gamma_c(Y;i_!i^{-1}k_X)) = \chi_c(Y).$$

Recall that χ satisfies:

$$\chi(F \oplus G) = \chi(F) + \chi(G)$$
$$\chi(F \otimes G) = \chi(F).\chi(G)$$

for $F, G \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$. And for a distinguished triangle $F' \to F \to F'' \xrightarrow{+1}$, we have:

$$\chi(F) = \chi(F') + \chi(F'').$$

As these properties suggest, it is suitable to look at Grothendieck group of $\mathbf{D}^b_{\mathbb{R}-c}(k_X)$.

Definition 1.2.6. *Let $\mathscr{C}$ be an abelian (resp. triangulated) category. One denotes by $\mathbf{K}(\mathscr{C})$ the abelian group obtained as the quotient of the free abelian group generated by the objects of $\mathscr{C}$ by the relation $X = X' + X''$ if there is an exact sequence $0 \to X' \to X \to X'' \to 0$ (resp. a distinguished triangle $X' \to X \to X'' \xrightarrow{+1}$ in $\mathscr{C}$). One calls $\mathbf{K}(\mathscr{C})$ the* Grothendieck Group *of $\mathscr{C}$.*

Remark 1.2.7. *Let $\mathscr{C}$ be an abelian category and, $\mathbf{D}^b(\mathscr{C})$ the derived category of bounded complexes of $\mathscr{C}$. Consider the distinguished triangles, (see [5], p.47):*

$$\tau^{\leq n}(X) \to X \to \tau^{\geq n+1}(X) \xrightarrow{+1},$$

$$\tau^{\leq n-1}(X) \to \tau^{\leq n}(X) \to H^n(X)[-n] \xrightarrow{+1},$$
$$H^n(X)[-n] \to \tau^{\geq n}(X) \to \tau^{\geq n+1}(X) \xrightarrow{+1}.$$

By induction one can easily see that for an $X \in \mathbf{D}^b(\mathscr{C})$, class of X, $[X] \in \mathbf{K}(\mathbf{D}^b(\mathscr{C}))$, can be represented by:

$$\sum_j [H^j(X)[-j]].$$

Therefore, $i : \mathscr{C} \to \mathbf{D}^b(\mathscr{C})$, $X \mapsto X$, induces a group isomorphism $\mathbf{K}(\mathscr{C}) \simeq \mathbf{K}(\mathbf{D}^b(\mathscr{C}))$, which its inverse is given by $\sum_j (-1)^j [H^j(X)]$.

We will denote the Grothendieck group of $\mathbf{D}^b_{\mathbb{R}-c}(k_X)$ by $\mathbf{K}_{\mathbb{R}-c}(k_X)$.

Theorem 1.2.8. *The map χ induces a group isomorphism from $\mathbf{K}_{\mathbb{R}-c}(k_X)$ to $CF(X)$.*

Proof. We explain the proof of this theorem from [5]. The machinery of applying induction on a subanalytic stratification is often useful.

I. Surjectivity. This is done exactly as in the Example 1.2.5. For a $\phi \in CF_X$, choose a subanalytic stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ such that $\phi = \sum_{\alpha \in A} m_\alpha \mathbf{1}_{X_\alpha}, m \in \mathbb{Z}$. Let $\epsilon_\alpha = sgn(m_\alpha)$ for $m_\alpha \neq 0$ and define:

$$F = \bigoplus_{\alpha \in A'} k_{X_\alpha}^{|m_\alpha|} \left[\frac{1-\epsilon_\alpha}{2} \right]$$

where $A' = \alpha \in A; m_\alpha \neq 0$. Clearly $F \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$, and $\chi(F) = \phi$. So the surjectivity follows.

II. Injectivity. To prove the injectivity, first observe that every element u in $\mathbf{K}_{\mathbb{R}-c}(k_X)$, is represented by a finite sum $u = \sum_j a_j[F_j]$, with $a_j \in \mathbb{Z}$. For any $F \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$, we will set $F^n = F \oplus ... \oplus F$ (n times), for $n \in \mathbb{N}$ and $F^n = F^{-n}[+1]$ for $n \in \mathbb{Z}$ and $n < 0$. Therefore, we may rewrite $u = [\bigoplus_j F_j^{a_j}]$. Hence any $u \in \mathbf{K}_{\mathbb{R}-c}(k_X)$ is represented by a single object $F = \oplus_j F_j^{a_j} \in \mathbf{D}^b_{\mathbb{R}-c}$.
Let $X = \bigsqcup_\alpha Z_\alpha$ be a subanalytic stratification such that $H^j(F)|_{Z_\alpha}$ is constant for all j, all α. Let X_n denote the union of the n-codimensional strata. Using the distinguished triangle $F_{X_0} \to F \to F_{X \setminus X_0} \xrightarrow{+1}$, gives:

$$[F] = [F_{X_0}] + [F_{X \setminus X_0}]$$

and by induction we get $F = \sum_n [F_{X_n}]$. Remark 1.2.7 implies:

$$[F] = \sum_{j,n} (-1)^j [H^j(F)_{X_n}] = \sum_{j,n} (-1)^j [k_{X_n}^{dim(H^j(F)_{X_n})}]. \tag{1.1}$$

Hence $\chi(F) = 0$ implies that for any α:

$$dim \bigoplus_{j\,even} H^j(F)_{Z_\alpha} = dim \bigoplus_{j\,odd} H^j(F)_{Z_\alpha}.$$

Since k is a field, this implies:

$$\bigoplus_{j\,even} H^j(F)_{Z_\alpha} = \bigoplus_{j\,odd} H^j(F)_{Z_\alpha}$$

and

$$\bigoplus_{j\,even} H^j(F)_{X_\alpha} = \bigoplus_{j\,odd} H^j(F)_{X_\alpha}.$$

This shows $[F] = \sum_{j,n} (-1)^j [H^j(F)_{X_n}] = 0$. $\square$

Remark 1.2.9. *The Equation 1.1 implies that $[k_{X_\alpha}]$'s linearly generate $\mathbf{K}_{\mathbb{R}-c}(k_X)$.*

1.2.2 Operations on constructible functions

Having the isomorphism of the Theorem 1.2.8 at hand, we can define many operations on constructible functions analogous to those of constructible sheaves.

I. **External Product**

$$CF_X \boxtimes CF_Y \to CF_{X\times Y}$$

For $\phi, \psi \in CF_X$, we can find $F, G \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$, such that $\phi = \chi(F)$ and $\psi = \chi(G)$. We have $\phi \boxtimes \psi = \chi(F \boxtimes G)$. On the other hand by Künneth formula A.14, $\chi(F \boxtimes G) = \chi(F)\chi(G)$, which suggests us to define:

$$(\phi \boxtimes \psi)(x, y) = \phi(x).\psi(y).$$

II. **Inverse Image**

Let $f : X \to Y$ be morphism of manifolds. Defining:

$$f^* : f^{-1}CF_Y \to CF_X$$

$$f^*\phi(x) = \phi(f(x)),$$

we readily see that the diagram:

$$\begin{tikzcd}
\mathbf{K}_{\mathbb{R}-c}(A_Y) \arrow[r, "f^{-1}"] \arrow[d, "\chi"'] & \mathbf{K}_{\mathbb{R}-c}(k_X) \arrow[d, "\chi"'] \\
f^{-1}CF_Y \arrow[r, "f^*"'] & CF_X
\end{tikzcd}$$

is commutative.

III. **Integral**

For $\phi \in \Gamma_c(X; CF_X)$ we can choose $F \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$ such that $supp(F) = supp(\phi)$ (following the proof of the Theorem 1.2.8 we see this choice is possible). One sets:

$$\int_X \phi = \chi(X; F).$$

Additivity of the functor $\chi(X, .)$ with respect to distinguished triangles and 1.2.8 show that the number is only depends on ϕ. By 1.2.2 we may refine the expression $\phi = \sum_\alpha m_\alpha \mathbf{1}_{X_\alpha}, m \in \mathbb{Z}$, to the case when $\{X_\alpha\}$ is a family of locally finite, compact, contractible sets. For which:

$$\int_X \phi = \sum_\alpha m_\alpha \chi(X; k_{X_\alpha}) = \sum_\alpha m_\alpha.$$

Note that $(\int_X \phi)(x) = \chi(X; F_x)$, for $x \in X$.

IV. **Direct Image**

Let $f : X \to Y$ be a morphism of manifolds, and $\psi \in CF_X$. Assume f is proper on $supp(\psi)$. One defines the direct image of ψ by f:

$$(\int_f \psi)(y) = \int_X \psi.\mathbf{1}_{f^{-1}(y)}.$$

Which gives a morphism of sheaves:

$$\int_f : f_! CF_X \to CF_Y.$$

Note that $\chi(Rf_!G)(y) = \chi(Y; G \otimes k_{f^{-1}(y)})$. Since the fibers formula, A.3, gives:

$$\begin{aligned}(Rf_!G)_y \simeq R\Gamma_c(f^{-1}(y); G|_{f^{-1}(y)}) \simeq R\Gamma_c(f^{-1}(y); i^{-1}G) \\ = R\Gamma(X; i_! i^{-1}G) \simeq R\Gamma(Y; G \otimes k_{f^{-1}(y)}).\end{aligned}$$

Where $i : f^{-1}(y) \to X$ is the inclusion. We will therefore have $\chi(f_!G) = \int_f \psi$.

V. **Dual**

There are two types of dual operators which we will introduce.

(a) For a $\phi \in CF_X$ and $F \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$, such that $\chi(F) = \phi$ one defines $D_X : CF_X \to CF_X$ by $D_X(\phi) = \chi(D_X F)$. This definition is well-defined, since D_X is a triangulated functor. On the other hand, by Definition 1.2.3:

$$(D_X\phi)(x) = \chi_c(F)(x) = \chi_c(R\Gamma_x(X; F)).$$

Now from Lemma 1.1.10, we can find a chart (U, ξ) in X, containing x, such that $\xi(U) \subset B(0, \epsilon)$. Therefore:

$$R\Gamma_x(X; F) \simeq R\Gamma_c(\xi^{-1}(B(0, \epsilon)); F) \simeq R\Gamma(X; F \otimes k_{\xi^{-1}(B(0,\epsilon))}).$$

Writing $\xi^{-1}(B(0, \epsilon))$ as $B(x, \epsilon)$ gives an equivalent definition for the Dual:

$$(D_X\phi)(x) = \int_X \mathbf{1}_{B(x,\epsilon)}\phi$$

for $0 < \epsilon \ll 1$.

(b) Second type of dual[2], is a counterpart for D'_X.

If X is an oriented manifold, then by proposition A.7 one has the isomorphism:

$$\omega_X \simeq k_X[+n].$$

Therefore, for a $F \in D^b(k_X)$:

$$D'_X F = RHom(F, k_X) = RHom(F, \omega_X[-n]) = RHom(F[+n], \omega_X) = D_X(F[+n]).$$

Consequently for $\phi \in CF_X$, let $F \in \mathbf{D}^b_{\mathbb{R}-c}(k_X)$ such that $\phi = \chi(F)$. Define:

$$D'_X(\phi) = \chi(D'_X F).$$

Note that $D'_X(\phi) = \chi(D_X(F[+n])) = (-1)^n D_X(\phi)$.

[2]Refer to Definition A.6.

We end this chapter by few relations which are followed easily from the Theorem 1.2.8 and similar formulas for constuctible sheaves.

Proposition 1.2.10. *(i) Let $\phi \in CF_X$. Then*

$$D_X \circ D_X \phi = \phi.$$

(ii) Let $f : X \to Y$ be a morphism of manifolds and let $\psi \in \Gamma(X; f_! CF_X)$. Then

$$\int_f D_X \psi = D_Y \int_f \psi.$$

(iii) Consider a Cartesian square of morphism of real analytic manifolds:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(Recall that this means $X' \simeq \{(y', x) \in Y' \times X : g(y') = f(x)\}$.)
Then for $\psi \in \Gamma(X; f_! CF_X)$ one has:

$$g^* \int_f \psi = \int_{f'} (h^* \psi).$$

Chapter 2

Integral transforms

In the first chapter we defined constructible functions and constructible sheaves and explained how the isomorphism lead to defining operations on constructible functions analogous to those of constructible sheaves. In this chapter we introduce certain integral transforms in each setting. We will introduce a so called *Radon transform* of constructible functions. We will prove general properties of Radon transforms.

2.1 Integral transforms of sheaves

We recall the definition of *integral transforms for sheaves* from [6].

Let X, Y and Z be real analytic manifolds and k be a field. Consider the diagram:

$$\begin{tikzcd}
 & & X \times Y \times Z \arrow[dll, "q_{12}"'] \arrow[d, "q_{13}"] \arrow[drr, "q_{23}"] & & \\
X \times Y \arrow[d, "q_1"'] \arrow[drr, "q_2"] & & X \times Z \arrow[dll, "q'_1"'] \arrow[drr, "q'_2"] & & Y \times Z \arrow[dll, "q''_1"'] \arrow[d, "q''_2"] \\
X & & Y & & Z.
\end{tikzcd}$$

where all the maps are projections.

Definition 2.1.1. *Let X, Y and Z be manifolds, and let $K \in \mathbf{D}^b(k_{X\times Y})$, $K' \in \mathbf{D}^b(k_{Y\times Z})$. We set composition $K \circ K'$ to be:*

$$K \circ K' = Rq_{13!}(q_{12}^{-1}K \otimes q_{23}^{-1}K') \in \mathbf{D}^b(k_{X\times Z}).$$

If in the above definition we put $X = \{pt\}$, $Y = X$ and $Z = Y$ then by identification of $\{pt\} \times X$ with X and $\{pt\} \times Y$ with Y we get the definition of the integral transforms for sheaves.

Definition 2.1.2. *For any $K \in \mathbf{D}^b(k_{X\times Y})$, the integral transform of sheaves from X to Y with kernel K is given by:*

$$. \circ K : \mathbf{D}^b(k_X) \to \mathbf{D}^b(k_Y), \quad F \circ K = Rq_{2!}(q_1^{-1}F \otimes K).$$

Using the following lemma one can compose integrals.

Lemma 2.1.3. *Using the notations in Definitions 2.1.2 and 2.1.1 we have:*

$$(F \circ K) \circ K' \simeq F \circ (K \circ K').$$

Proof. Since the square:

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "q_{12}"'] \arrow[dr, "q_{23}"] & \\ X \times Y \arrow[dr, "q_2"'] & \square & Y \times Z \arrow[dl, "q_1''"] \\ & Y & \end{tikzcd}$$

is Cartesian (see Corollary A.4), therefore:

$$\begin{aligned} (F \circ K) \circ K' &= Rq''_{2!} \left(K' \otimes q_2'^{-1}(Rq_{2!}(K \otimes q_1^{-1} F)) \right) \\ &\simeq Rq''_{2!} \left(K' \otimes Rq_{23!} q_{12}^{-1}(K \otimes q_1^{-1} F)) \right) \end{aligned}$$

and

$$\begin{aligned} &\simeq Rq''_{2!} \left(K' \otimes Rq_{23!}(q_{12}^{-1} K \otimes q_{12}^{-1} q_1^{-1} F)) \right) \\ &= Rq''_{2!} \left(K' \otimes Rq_{23!}(q_{12}^{-1} K \otimes (q_1 q_{12})^{-1} F)) \right) \\ &= Rq''_{2!} \left(K' \otimes Rq_{23!}(q_{12}^{-1} K \otimes (q_1' q_{13})^{-1} F)) \right). \end{aligned}$$

By projection formula (Proposition A.1.15):

$$\begin{aligned} &\simeq Rq''_{2!} Rq_{23!} \left(q_{23}^{-1} K' \otimes (q_{12}^{-1} K \otimes q_{13}^{-1} q_1'^{-1} F)) \right) \\ &\simeq R(q_2'' q_{23})_! \left(q_{23}^{-1} K' \otimes (q_{12}^{-1} K \otimes q_{13}^{-1} q_1'^{-1} F)) \right) \\ &\simeq Rq'_{2!} Rq_{13!} \left(q_{23}^{-1} K' \otimes (q_{12}^{-1} K \otimes q_{13}^{-1} q_1'^{-1} F)) \right). \end{aligned}$$

Again by projection formula:

$$\begin{aligned} &\simeq Rq'_{2!} \left(Rq_{13!}(q_{23}^{-1} K' \otimes q_{12}^{-1} K) \otimes q_1'^{-1} F) \right) \\ &= F \circ (K \circ K'). \end{aligned}$$

$\square$

Corollary 2.1.4. *In the situation of the preceding lemma assume $Z = X$, $K \circ K' \simeq k_{\Delta_X}[l]$ and $K' \circ K \simeq k_{\Delta_Y}[l']$ for some shifts l and l'(Δ_X and Δ_Y denote the diagonal of $X \times X$ and $Y \times Y$, respectively). Then $l = l'$ and $. \circ K$ and $. \circ K'$ are equivalences of categories. One calls $. \circ K'$ the inverse of the functor $. \circ K$.*

Proof. We only need to observe that for an $F \in \mathbf{D}^b(A_X)$,

$$F \circ k_{\Delta_X} = Rq_{2!}((q^{-1}F)_\Delta) = F \tag{2.1}$$

and use Lemma 2.1.3. Note that the isomorphisms $K[l] \simeq K \circ K' \circ K \simeq K[l']$, give $l = l'$. □

Marastoni in [6] puts forward inverses for integral operators of sheaves under certain *geometric conditions*[1].

Theorem 2.1.5 ([6], Lemma 2.2)**.** *Assume X and Y are real analytic compact orientable manifolds of the same dimension n. Ω an open subanalytic subset of $X \times Y$. Let Ω^t be the image of Ω under the map $.^t : X \times Y \to Y \times X, (x,y) \mapsto (y,x)$. Denote by j (resp. j^t) the embedding of Ω into $X \times Y$ (resp. of Ω^t into $Y \times X$). For any $x \in X$ we set:*

$$\Omega_x = \{y \in Y : (x,y) \in \Omega\} \subset Y,$$

and similarly for $y \in Y$. Moreover consider the kernels:

$$K_\Omega = \mathbb{C}_\Omega = j_! j^{-1}\mathbb{C}_{X\times Y} \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_{X\times Y}),$$

$$K^*_\Omega = D'_{X\times Y}\mathbb{C}_\Omega = Rj_* j^{-1}\mathbb{C}_{X\times Y} \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_{X\times Y}).$$

Under the hypotheses

(i) *X is simply connected,*

(ii) $R\Gamma(\Omega_x; \mathbb{C}_{\Omega_{x'}}) = \begin{cases} 0 & \text{for } x \neq x' \\ \mathbb{C} & \text{for } x = x', \end{cases}$

(iii) $SS(\mathbb{C}_\Omega) \cap (T^*_X X \times T^*Y) \subset T^*_{X\times Y}(X \times Y)$,

and similar conditions by interchanging X and Y, one has:

$$K \circ K^*_{\Omega^t} \simeq \mathbb{C}_{\Delta_X}[-n] \quad \text{and} \quad K^*_{\Omega^t} \circ K \simeq \mathbb{C}_{\Delta_Y}[-n].$$

In particular, the categories $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ and $\mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$ are equivalent.

2.2 Integral transforms of constructible functions

Consider the diagram of morphisms of real analytic manifolds:

$$\begin{tikzcd} & X \times Y \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ X & & Y, \end{tikzcd}$$

[1] For the definition of micro-support see the Appendix B

where q_1 and q_2 are projections. Assume $k \in CF_{X\times Y}$. Similar to 2.1.2, an integral transform from CF_X to CF_Y with kernel $k(x,y)$ is defined by the map:

$$\phi \mapsto \int_{q_2} k(x,y)\ q_1^*\phi(x).$$

Note that by definition of proper image, p_2 needs to be proper on $supp(k(x,y))$, for the integral to make sense. Since $k \in CF_{X\times Y}$, by the Lemma 1.2.2, one writes:

$$k(x,y) = \sum_\alpha c_\alpha \mathbf{1}_{S_\alpha}(x,y),$$

such that $\{S_\alpha\}_\alpha$ is a locally finite family of subanalytic, compact and contractible subsets of $X\times Y$. As a consequence, an integral transform of constructible functions can be reduced to the integral transform with kernels $\mathbf{1}_S$, where S is locally closed subanalytic subset of $X\times Y$. Namely, the Radon transform: such that $\{S_\alpha\}_\alpha$ is a locally finite family of subanalytic, compact and contractible subsets of $X\times Y$. As a consequence, an integral transform of constructible functions can be reduced to the integral transform with kernels $\mathbf{1}_S$, where S is locally closed subanalytic subset of $X \times Y$. Namely, the Radon transform:

Definition 2.2.1. *Let S be a locally closed subanalytic subset of $X \times Y$. We denote by p_1 and p_2 the first and second projections defined on $X \times Y$, and by f and g the restrictions of p_1 and p_2 to S.*
We assume

$$p_2 \text{ is proper on } \bar{S}, \text{ the closure of } S \text{ in } X \times Y. \tag{2.2}$$

Then, for a $\phi \in CF_X$, we set:

$$\mathcal{R}_S(\phi) = \int_g f^*\phi = \int_{q_2} \mathbf{1}_S(q_1^*\phi).$$

We call $\mathcal{R}_s(\phi)$ the Radon transform of ϕ.

Remark 2.2.2. *In the situation of the preceding theorem, we have $\mathcal{R}_S(\mathbf{1}_{\{x\}})(y) = \mathbf{1}_S(x,y)$. Therefore, $\mathcal{R}_S = 0$ implies $S = \emptyset$. Since Radon transform is additive, we learn that the inverse of a Radon transform, if it exists, it is unique.*

By the same fashion of Lemma 2.1.3 one can prove a similar formula for of Radon transforms. The following lemma is a part of Theorem 3.1 in [8].

Let $S' \subset Y \times X$ be a locally closed subanalytic subset, and again denote by p_2 and p_1 the first and second projections defined on $Y \times X$, and by f' and g' the restrictions of p_1 and p_2 to S' and by r the projection $S \times_Y S' \to X \times X$.
We assume:

$$p_1 \text{ is proper on } \bar{S}', \text{ the closure of } S' \text{ in } Y \times X. \tag{2.3}$$

Lemma 2.2.3. *Under assumptions 2.2 and 2.3, for any $\phi \in CF_X$, we have:*

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) = \int_{q_2} \left(\int_r \mathbf{1}_{S\times_Y S'} \right) q_1^*\phi. \tag{2.4}$$

First proof. Denote by h and h' the projections from $S \times_Y S'$ to S and S' respectively. The square:

$$\begin{tikzcd} S\times_Y S' \arrow[r, "h'"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & S' \arrow[d, "g'"] \\ S \arrow[r, "g"'] & Y \end{tikzcd}$$

is Cartesian. Therefore one writes:

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) = \int_{f'} (g'^* \int_g (f^*\phi)) = \int_{f'\circ h'} ((f \circ h)^*\phi).$$

Considering the diagram:

$$\begin{tikzcd} & & & S\times_Y S' \arrow[dl, "h"'] \arrow[d, "r"] \arrow[dr, "h'"] & & & \\ & & S \arrow[dl, "f"'] \arrow[dr, "g"] & X \times X \arrow[dll, "q_1"] \arrow[drr, "q_2"'] & S' \arrow[dl, "g'"'] \arrow[dr, "f'"] & & \\ & X & & Y & & X, & \end{tikzcd}$$

gives:

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) = \int_{q_2}\int_r (r^* q_1^*\phi) = \int_{q_2} \left(\int_r r^*\mathbf{1}_{X\times X}\right) q_1^*\phi.$$

Note that we have used the fact that for any $\psi \in CF_{X\times X}$

$$\left(\int_r r^*\mathbf{1}_{X\times X}\right)\psi = \int_r r^*\psi = \left(\int_r \mathbf{1}_{S\times_Y S'}\right)\psi.$$

For the last step (one can prove this by checking on characteristic functions of compact subsets of $X \times X$, or by noting that it is followed by projection formula A.13).

Second proof. Let k be a field and in the Lemma 2.1.3, put $Z = X$, $K = k_S \in \mathbf{D}^b_{\mathbb{R}-c}(k_{X\times Y})$, $K' = k_{S'} \in \mathbf{D}^b_{\mathbb{R}-c}(k_{Y\times X})$ and note that:

$$\chi(q_{12}^{-1}k_S \otimes q_{23}^{-1}k_{S'}) = \mathbf{1}_{S\times_Y S'};$$

Moreover, the map r plays the same role as q_{13} in 2.1.2. $\square$

To go further in the preceding Lemma, Schapira in [8] assumes:

$$\text{There exists } \lambda \neq \mu \in \mathbb{Z} \text{ such that: } \int_r \mathbf{1}_{S\times_Y S'} = \mu\mathbf{1}_{\Delta_X} + \lambda\mathbf{1}_{X\times X\setminus\Delta_X}.$$

Where Δ_X is diagonal of $X \times X$. Differently put,

$$\text{There exists } \lambda \neq \mu \in \mathbb{Z} \text{ such that: } \chi(r^{-1}(x,x')) = \begin{cases} \lambda & \text{if } x \neq x'; \\ \mu & \text{if } x = x'. \end{cases} \tag{2.5}$$

We will refer to condition (2.5) as Δ-condition.

The main theorem in [8] is the following:

Theorem 2.2.4 ([8], Theorem 3.1)**.** *Assume (2.2), (2.3) and (2.5). Then for any $\phi \in CF_X$, we have:*

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) = (\mu - \lambda)\phi + \left(\int_X \lambda\phi\right) \mathbf{1}_X. \tag{2.6}$$

Proof. Lemma 2.2.3 and (2.5) together with the equalities

$$\int_{q_2} \mathbf{1}_{\Delta_X} q_1^*\phi = \phi \tag{2.7}$$

and

$$\int_{q_2} \mathbf{1}_{X\times X}\ q_1^*\phi = \int_X \phi,$$

offer the statement. □

The upcoming lemma and proposition are useful to simplify (2.6) and the ideas in them will be employed later on.

Definition 2.2.5. *For a $\psi \in CF_Y$, we set*

$$R_0(\psi) = \int_{p_1} \mathbf{1}_{Y\times X}(p_2^*\psi) = \int_{p_1} (p_2^*\psi) = \left(\int_Y \psi\right) \mathbf{1}_X.$$

Lemma 2.2.6 (Proposition 3.1 in [7])**.** *Assume (2.2), (2.3) and (2.5), then for any $\phi \in CF_X$ we have:*

$$R_0 \circ \mathcal{R}_S(\phi) = \int_X (\mu\phi)\mathbf{1}_X.$$

Proof. Having proved the equality for characteristic functions of compact subanalytic subsets $K \subset X$, by linearity of above operations we deduce the assertion.
Since the square

$$\begin{tikzcd} X\times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "a_Y"] \\ X \arrow[r, "a_X"'] & Y \end{tikzcd}$$

is Cartesian, we have:

$$R_0 \circ \mathcal{R}_S(\mathbf{1}_K) = \int_{p_1} p_2^* \int_{p_2} (\mathbf{1}_S.p_1^*\mathbf{1}_K) = a_X^* \int_{a_Y} \int_{p_2} (\mathbf{1}_S.p_1^*\mathbf{1}_K)$$

$$= a_X^* \int_{a_X} \int_{p_1} (\mathbf{1}_S.p_1^*\mathbf{1}_K).$$

For any $\phi \in CF_X$, we write:

$$\left(a_X^* \int_{a_X} \phi\right)(x) = \left(\int_{a_X} \phi\right)(\{pt\}) = \int_X \phi(x')\mathbf{1}_{a_X^{-1}\{pt\}}(x') = \left(\int_X \phi\right) \mathbf{1}_X(x).$$

Since we have set:

$$\mu = \chi((\{x\} \times Y) \cap S) = \chi(\{y \in Y : (x, y) \in S\}).$$

We deduce:

$$\left(\int_{p_1} \mathbf{1}_S . p_1^* \mathbf{1}_K\right)(x) = \int_{X \times Y} \mathbf{1}_{((\{x\} \cap K) \times Y) \cap S}(x', y') = \mu \mathbf{1}_K .$$

Which gives the desired result. □

Definition 2.2.7. *For a $\psi \in CF_Y$, we set:*

$$R^{-1}(\psi) = \int_{p_1} (\mu \mathbf{1}_{S^t} - \lambda \mathbf{1}_{Y \times X})(p_2^* \psi) = \mu \mathcal{R}_{S^t}(\psi) - \lambda R_0(\psi).$$

So the Proposition 3.2 in [7] follows:

Proposition 2.2.8. *Assume (2.2), (2.3) and (2.5) and let $\phi \in CF_Y$. Then we have:*

$$R^{-1} \circ \mathcal{R}_S(\phi) = \mu(\mu - \lambda)\phi.$$

In particular, if $\mu(\mu - \lambda)$ is not zero, we can reconstruct the original constructible function ϕ from its Radon transform $\mathcal{R}_S(\phi)$ by dividing the last term by this constant $\mu(\mu - \lambda)$.

Proof.

$$R^{-1} \circ \mathcal{R}_S(\phi) = \mu \mathcal{R}_{S^t} \circ \mathcal{R}_S(\phi) - \lambda R_0 \circ \mathcal{R}_S(\phi) = \mu(\mu - \lambda)\phi.$$

□

Chapter 3

Radon transform on Grassmannians

In this chapter we concentrate on the Radon transforms of Grassmannians. In the first section we apply the theorems from previous chapter to draw basic conclusions on these Radon transforms. In the second section we prove the main results of this thesis and we will try to extract inversion formulas for the Radon transform with transversality incidence relation.

3.1 Basics

This section is mainly application of the theorems in the second chapter for the Radon transform on Grassmannians.

We first fix few notations.

Definition 3.1.1. *Let V be a vector space of dimention n over $\mathbb{R}$ or $\mathbb{C}$ (we will discuss the two cases in parallel).*

(i) $G_n(p)$ *= set of p-planes in the vector space k^n, the Grassmannian of p-planes in n-space*

(ii) $S_i^{p,q} = \{(x, y) \in G_n(p) \times G_n(q) : dim(x \cap y) = i\}$*, for fixed integers $1 \leq p \leq q \leq n$ and $i = 0, 1, .., p$.*

(iii) $G_n(p, q) = S_p^{p,q}$*, the inclusion incidence relation.*

(iv) $\Omega^{p,q} = S_0^{p,q}$*, the transversality relation.*

In the above, when p and q are fixed we may drop the superscript p, q.

Remark 3.1.2. *Natural action of the group $G := SL(n)$ on each of $G_n(p)$ and $G_n(q)$, induces an action on $G_n(p) \times G_n(q)$ defined by $g(x, y) := (gx, gy)$ for any $g \in G$. Orbits of this action are clearly $S_i^{p,q}$'s. Each S_i is a submanifold of $G_n(p) \times G_n(q)$. For $0 \leq i \leq p$, the closure of S_i, $\bar{S}_i$, is given by $\bigcup_{j \geq i} S_j$. Thus, its complement $\bigcup_{j=0}^{j=i} S_j$ is open. In particular, $\bigsqcup_{i=0}^{p} S_i$ gives a subanalytic stratification for $G_n(p) \times G_n(q)$. Moreover, note that $G_n(p, q)$ (resp. Ω) is compact (resp. open) submanifold of $G_n(p) \times G_n(q)$.*

Example 3.1.3. *For any integers $\lambda_0, \lambda_1, \ldots, \lambda_p$, the function $\sum_{i=0}^{p} \lambda_i \mathbf{1}_{S_i}$ is a constructible function.*

Example 3.1.4 (Inclusion incidence relation)**.** *Considering the diagram:*

$$\begin{tikzcd} & G_n(p,q) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ G_n(p) & & G_n(q), \end{tikzcd}$$

we verify Δ-condition for $S = G_n(p,q)$ and $S' = S^t$. As before, let $r : S \times_{G_n(q)} S' \to G_n(p) \times G_n(p)$ be projection. For $(x,x') \in G_n(p) \times G_n(p)$ we have:

$$r^{-1}(x,x') = \{y \in G_n(q) : x \subset y, x' \subset y\} = \{y \in G_n(q) : x + x' \subset y\}$$

Suppose $dim(x \cap x') = j$ for any $0 \leq j \leq p$, then $dim(x+x') = 2p - j$. By taking quotient over $x \oplus x'$ we get:

$$r^{-1}(x,x') \simeq G_{n-2p+j}(q - 2p + j).$$

We recall that the formulas for Euler-Poincaré Index of Grassmannians are given by

$$\chi(G_n(p)) = \begin{cases} 0 & \text{if } p(n-p) \text{ is odd} \\ \binom{E(\frac{n}{2})}{E(\frac{p}{2})} & \text{if } p(n-p) \text{ is even} \end{cases}$$

for the real Grassmannian, where $E(x)$ is the integer part of x; And

$$\chi(G_n(p)) = \binom{n}{p}$$

for the complex Grassmannian. Denote this numbers in both Real and Complex case by $\mu_n(p)$. Therefore, when $p > 1$ there is no hope for Δ-condition to hold. On the other hand when $p = 1$, j is either 0 or 1. Hence we have Δ-condition:

$$\chi(r^{-1}(x,x')) = \begin{cases} \mu_{n-2}(q-2) & \text{if } x \neq x'; \\ \mu_{n-1}(q-1) & \text{if } x = x'. \end{cases}$$

Now by 2.6 we can write:

Proposition 3.1.5 ([8], Proposition 4.1)**.** *For any $\phi \in CF_{G_n(1)}$*

$$\mathcal{R}_{G_n(1,q)} \circ \mathcal{R}_{G_n(q,1)}(\phi) = (\mu_{n-1}(q-1) - \mu_{n-2}(q-2))\phi + [\mu_{n-2}(q-2)\int_{G_n(1)} \phi]\mathbf{1}_{G_n(1)}. \tag{3.1}$$

Remark 3.1.6. *The preceding example shows that for $p > 1$ Δ-condition is not satisfied. Although we could extract some information by considering different cases for intersections of $x \cap x'$, these information is not enough for finding an inverse. These cases are the main subject of [7]. To extract more information, Matsui considers $p+1$ cases for S' and uses that to find an inverse (See section 3.3.2 and Remark 3.2.17).*

Example 3.1.7. *Let S^n be the n-dimensional sphere $\mathbb{R}^n - \{0\}/\mathbb{R}^{>0}$. We denote by $(S^n)^*$ the set $(\mathbb{R}^n)^* - \{0\})/\mathbb{R}^{>0}$, the dual of S^n. One considers these spaces as oriented Grassmannians. Let us take $X = S^n$ and $Y = (S^n)^*$. We set the incidence relation to be:*

$$S = \{(x, y) \in X \times Y : x \subset y\}.$$

Let $S' = S^t$ and r the projection $S \times_Y S' \to X \times X$, as before. To verify Δ-condition we calculate $\chi(r^{-1}(x, x'))$, for $x, x' \in X$. Assume $x \neq x'$, then:

$$\begin{cases} \chi(\{y \in (S^n)^* : x \subset y, x' \subset y\}) \simeq (S^{(n-1)})^*; \\ \chi(\{y \in (S^n)^* : x \subset y\} \simeq (S^{(n-2)})^*. \end{cases}$$

Recalling Cohomology Groups of spheres and using Poincaré Duality (Corollary A.8)

$$\chi(r^{-1}(x, x')) = \begin{cases} 2 & \text{if } x = x' \text{ and } n \text{ is odd ;} \\ 0 & \text{if } x \neq x' \text{ and } n \text{ is odd.} \end{cases}$$

For n even, we get the numbers the other way around. Hence we can use (2.6) to get:

Proposition 3.1.8. *Suppose n is odd. In the situation of Example 3.1.7, for each $\phi \in CF_S$ we have:*

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) = \phi.$$

This means that for a subanalytic subset $K \subset S^n$, by knowledge of $\mathcal{R}_S(\mathbf{1}_K)$ at each point of $(S^n)^*$ we can reconstruct K.

Example 3.1.9 ([8], p.8)**.** *Let V be an n-dimensional real affine space, P_n its projective compactificaion, i.e. $P_n = V \sqcup h_\infty$, where h_∞ is the hyperplane at infinity. Let P_n^* be the dual projective space. Therefore, $P_n^* \backslash \{h_\infty\}$ is the set of hyperplanes of V.*
As in Example 3.1.4, consider $X = G_{n+1}(1) = P_n$ and $Y = G_{n+1}(n) = P_n^*$. Let $S = G_{n+1}(1, n)$, the inclusion incidence relation, and K a subanalytic subset of V. For ξ a hyperplane in V one has:

$$\mathcal{R}_S(\mathbf{1}_K)(\xi) = \left(\int_{q_2} \mathbf{1}_S . q^*(\mathbf{1}_K) \right)(\xi) = \int_{X \times Y} \mathbf{1}_S . \mathbf{1}_{\{(x,\xi): x \in X, x \subset \xi\}} . q_1^* \mathbf{1}_K.$$

Hence:

$$\mathcal{R}_S(\mathbf{1}_K)(\xi) = \int_V \mathbf{1}_K . \mathbf{1}_\xi = \chi(K \cap \xi).$$

By (2.6):

$$\mathcal{R}_{P_n^*} \circ \mathcal{R}_{P_n}(\phi) = \begin{cases} \phi & \text{if } n \text{ is odd ,} \\ -\phi + [\int_{P_n} \phi] \mathbf{1}_{P_n} & \text{if } n \text{ is even and } n > 0 \text{ .} \end{cases}$$

The above formula means, that to reconstruct a subanalytic subset $K \subset V$ in odd dimensions, it suffices to know the Euler-Poincaré Index of every section of the set K with hyperplanes. Letting $n = 3$, one might consider this result as an application of Radon transforms in tomography.

3.2 Inversion formulas for $\mathcal{R}_\Omega$

This section contains the main results of this thesis. We will attempt to find inverses for the Radon transform on Grassmannians with transversality incidence relation. In addition to Definition 3.1.1 we fix few others.

Definition 3.2.1. *Assume $1 \leq p \leq q \leq n$ are integers. We set: $(S_i^{p,q})_x = \{y \in G_n(q) : (x,y) \in S_i^{p,q}\}$, for a fixed $x \in G_n(q)$.*

Remark 3.2.2. *Assume that $l \leq p \leq n$ are integers and $z \in G_n(l)$ and $x \in G_n(p)$, are given. Then the set*

$$(S_0^{p,q})_x \cap (S_0^{l,q})_z = \Omega_x^{p,q} \cap \Omega_z^{l,q} = \big\{y \in G_n(q) : y \cap z = y \cap x = \{0\}\big\},$$

is non-empty if and only if $p+q \leq n$. This follows from the fact that if $l = p$ and $p+q = n$ the set is non-empty, as in preceding section. Having $n > p+q$ or $p \geq l$ means more freedom of choice and therefore the set being non-empty. Therefore, we have the sufficiency. Necessity of the condition is obvious.

In the forthcoming sections we consider the cases when $p+q = n$ and the general case, separately. The following lemma will be frequently used.

Remark 3.2.3. *Using Schubert cell decomposition of Grassmanian (see [4]) one can show that the Grassmanians as well as objects of our study in the following sections are CW-complexes. Moreover, as a result of Example 1.2.5, we will only deal with Euler-Poincaré index with compact support. Therefore, in these sections, by* Euler characteristic of a manifod *we mean* Euler-Poincaré index with compact support of a constant sheaf with stalk k over that manifold.

Lemma 3.2.4. *For an $x \in G_n(p)$, the Euler-Poincaré index with compact support of $\Omega_x^{p,q} = (S_0^{p,q})_x \subset G_n(q)$ is given by:*

$$\begin{cases} (-1)^{qp}\chi(G_{n-p}(q)) & \text{in the real case;} \\ \chi(G_{n-p}(q)) & \text{in the complex case.} \end{cases}$$

More explicitly, recalling the formulas for Euler-Poincaré indeces of Grassmannians (see Example 3.1.4), we have:

$$\chi(\Omega_x^{p,q}) = \begin{cases} 0 & \text{if } q(n-q-p) \text{ is odd} \\ (-1)^{qp}\begin{pmatrix} E(\frac{n-p}{2}) \\ E(\frac{q}{2}) \end{pmatrix} & \text{if } q(n-q-p) \text{ is even} \end{cases}$$

for the real Grassmannian;

$$\chi(\Omega_x^{p,q}) = \begin{pmatrix} n-p \\ q \end{pmatrix}$$

for the complex Grassmannian.

Proof. For simplicity we write Ω_x instead of $\Omega_x^{p,q}$. To calculate $\chi_c(\Omega_x)$, it is convenient to fix an ordered basis for $\{e_1, e_2, ..., e_n\}$ for k^n, such that $x = \{e_1, e_2, ..., e_p\}$. Note that if $y = span\{v_1, ..., v_q\}$ is an element of Ω_x, then no linear combination of v_i's, $1 \leq i \leq q$, is entirely in $span\{e_1, ..., e_p\}$. Therefore, y in matrix form has a non-singular $q \times q$ minor in q of the last $n-p$ columns. Let $\pi : span\{e_1, e_2, ..., e_n\} \to span\{e_{p+1}, ..., e_n\}$ be the projection into the $n-p$ last coordinates. The restriction of π to $\{y \in G_n(q) : y \cap span\{e_1, ..., e_p\} = \{0\}\}$ induces a map:

$$\Omega_x = \{y \in G_n(q) : y \cap span\{e_1, ..., e_p\} = \{0\}\} \longrightarrow B = \{q\text{-planes in } span\{e_{p+1}, ..., e_n\}\} \simeq G_{n-p}(q).$$

Which we will also denote it by π. We claim that the map π is surjective, furthermore, for each b in B, the $\pi^{-1}(b)$ is isomorphic to k^{qp}. To see this, note that a point $b \in B$ is q-dimensional. Hence, by a suitable change of coordinates in $span\{e_{p+1}, \ldots, e_n\}$, it can be expressed by:

$$b = \begin{pmatrix} 1_q & 0 \end{pmatrix}_{q \times (n-p).}$$

where 1_q is $q \times q$ identity matrix, and 0 is a $q \times n - p - q$ zero matrix. Accordingly, the fiber of b, $\pi^{-1}(b)$, is given by:

$$\pi^{-1}(b) = \begin{pmatrix} \bigstar & 1_q & 0 \end{pmatrix}_{q \times n},$$

where $\bigstar$ is a $q \times p$ matrix of free variables in k. Inserting these facts into Corollary A.12, for any $b \in B$ we get:

$$\chi_c(A) = \chi_c(B)\chi_c(\pi^{-1}(b)) = \chi(G_{n-p}(q))\chi_c(k^{qp}).$$

Replacing k with $\mathbb{R}$ and $\mathbb{C}$ and using examples A.10 and Example 3.1.4, give the formulas. $\square$

Corollary 3.2.5. *If $p + q = n$, then*

$$\chi_c(\Omega_x) = \begin{cases} (-1)^{pq} & \textit{in the real case;} \\ 1 & \textit{in the complex case.} \end{cases}$$

3.2.1 Inverses of $\mathcal{R}_\Omega$ when $p + q = n$

In this section we will try to find an inverse for the Radon transform $\mathcal{R}_\Omega$ when $p + q = n$. This section is motivated by [6]. Marastoni in [6] proves that for the complex Grassmannians $X = G_n(p)$ and $Y = G_n(q)$, if $p + q = n$, then $\Omega \subset X \times Y$ fulfills the conditions of Theorem 2.1.5. As a consequence the inverse of the integral functor $. \circ \mathbb{C}_\Omega$, is given by $. \circ D'_{Y \times X}\mathbb{C}_{\Omega^t}$. Which indicates that the Radon transform with kernel $D'_{Y \times X}(\mathbf{1}_{\Omega^t})$ $(\in CF_{Y \times X})$, is an inverse for $\mathcal{R}_\Omega$. Explicit calculation of $D'_{Y \times X}(\mathbf{1}_{\Omega^t})$ is not easy. However, we found that $\mathcal{R}_{\Omega^t}$ is an inverse for $\mathcal{R}_\Omega$. As a result, by uniqueness of inverses for Radon transforms (Remark 2.2.2), we learn that $\mathcal{R}_{\Omega^t}$ is the only inverse and we have $\mathbf{1}_{\Omega^t} = D'_{Y \times X}\mathbf{1}_{\Omega^t}$.

We will show that Ω and Ω^t, satisfy Δ-condition, (2.5), in the complex case. To do this for each $(x, x') \in G_n(p) \times G_n(p)$, we will calculate:

$$\chi_c(r^{-1}(x, x')) = \chi_c(\Omega_x \cap \Omega_{x'}) = \chi_c(\{y \in G_q(n) : y \cap x = y \cap x' = \{0\}\}).$$

Assume $(x, x') \in Z_j$ (recall that this means $dim(x \cap x') = j$). When $x = x'$ or equivalently $j = p$ we have:

$$\chi_c(r^{-1}(x, x')) = \chi_c(\{y \in G_q(n) : y \cap x = \{0\}\}).$$

Which is already calculated in Corollary 3.2.5, therefore we only need to calculate the cases when $p - j \geq 1$. First, we fix (x, x') such that $dim(x \cap x') = j$. Then, we find bases for $x = span\{e_1, ..., e_j, e_{j+1}, ..., e_p\}$ and for $x' = span\{e_1, ...e_j, e_{p+1}, ..., e_{2p-j}\}$. We extend $x + x' = span\{e_1, ...e_j, e_{j+1}, ..., e_p, e_{p+1}, ..., e_{2p-j}\}$ to $span\{e_1, e_2, ..., e_n\}$. Accordingly, in matrix form we can represent:

$$x = \begin{pmatrix} 1_j & 0 & 0 & 0 \\ 0 & 1_{p-j} & 0 & 0 \end{pmatrix}_{p\times n} \quad \text{and} \quad x' = \begin{pmatrix} 1_j & 0 & 0 & 0 \\ 0 & 0 & 1_{p-j} & 0 \end{pmatrix}_{p\times n},$$

where 1_m is m by m identity matrix.
Moreover to write the elements of Ω_x in a matrix form, note that a vector $v = \sum_{i=1}^{n} \lambda_i e_i \in y$ cannot be entirely in $x = span\{e_1, ..., e_p\}$, so at least one of the λ_i's for $i > p$ is non-zero. As we are only interested in the span of such vectors, we can assume it is 1. Since we have $dim(y) = q$ we can choose $q = n-p$ independent vectors in this way. By some elementary operations (again since the span of such vectors only matters) on the matrix we can get another basis for y in the matrix form $(\bigstar \mid 1_q)$ or:

$$y = \begin{pmatrix} [b_1]_{p-j\times j} & [b_3]_{p-j\times p-j} & 1_{p-j} & 0_{p-j\times q-p-j} \\ [b_2]_{q-p-j\times j} & [b_4]_{q-p-j\times p-j} & 0_{q-p-j\times p-j} & 1_{q-p-j} \end{pmatrix}.$$

Knowing that:

$$\Omega_x \cap \Omega_{x'} = \{y \in \Omega_x \colon x' \cap y = \{0\}\},$$

for $(x, x') \in Z_j$. One has:

$$y = \begin{pmatrix} b_1 & b_3 & 1_{p-j} & 0 \\ b_2 & b_4 & 0 & 1_{q-p-j} \end{pmatrix} \in \Omega_{x'}$$

if and only if

$$det \begin{pmatrix} 1_j & 0 & 0 & 0 \\ 0 & 0 & 1_{p-j} & 0 \\ b_1 & b_3 & 1_{p-j} & 0 \\ b_2 & b_4 & 0 & 1_{q-p-j} \end{pmatrix} \neq 0.$$

In addition

$$0 \neq \det \begin{pmatrix} 1_j & 0 & 0 & 0 \\ 0 & 0 & 1_{p-j} & 0 \\ b_1 & b_3 & 1_{p-j} & 0 \\ b_2 & b_4 & 0 & 1_{q-p-j} \end{pmatrix} = \det \begin{pmatrix} 0 & 1_{p-j} & 0 \\ b_3 & 1_{p-j} & 0 \\ b_4 & 0 & 1_{q-p-j} \end{pmatrix}$$

$$= \det \begin{pmatrix} 0 & 1_{p-j} \\ b_3 & 1_{p-j} \end{pmatrix} = \pm \det b_3.$$

Therefore, *in the real case*, considering the projection:

$$\pi : \Omega_x \to \mathbb{R}^{(p-j)^2}, \quad \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} \mapsto b_3$$

and application of Corollary A.12 and Example A.10, we get:

$$\chi_c(\Omega_x \cap \Omega_{x'}) = (-1)^{dim(b_1)+dim(b_2)+dim(b_4)} \chi_c(GL_{p-j}).$$

When $p - j = 1$ (in real dimensions), we readily have:

$$\chi_c(GL(p-j)) = -2.$$

For $p - j > 1$ we define the map $\phi : GL(p-j) \to SL(p-j)$ by $z \mapsto \frac{1}{det(z)} z$. Which is obviously surjective with the fiber $k^\times$ above each point. Therefore $\chi_c(GL(p-j)) = \chi_c(k^\times)\chi_c(SL(p-j))$, by Corollary A.12. Since $SL(n)$ is a compact Lie group, by Lemma 3.2.9, we conclude:

$$\chi_c(GL(p-j)) = 0 \quad \text{for } p - j > 1.$$

Note that the natural real structure to $\mathbb{C}$ and considering it as $\mathbb{R}^2$, provides *the complex case*. We recap:

Proposition 3.2.6. *In the real case,*

$$\chi_c(\Omega_x \cap \Omega_{x'}) = \begin{cases} 0 & \text{if } j < p-1 \\ 2(-1)^{pq} & \text{if } j = p-1 \\ (-1)^{pq} & \text{if } j = p \text{ or } x = x'. \end{cases}$$

In the complex case,

$$\chi_c(\Omega_x \cap \Omega_{x'}) = \begin{cases} 0 & \text{if } j < p \text{ or } x \neq x' \\ 1 & \text{if } j = p \text{ or } x = x'. \end{cases}$$

The complex case in Proposition 3.2.6, suits fine in Δ-condition (2.5). Using (2.6), we have shown:

Theorem 3.2.7. *The inverse of Radon transform $\mathcal{R}_\Omega$, when $p + q = n$, in the complex Grassmannian is $\mathcal{R}_{\Omega^t}$. That is to say, for any $\phi \in CF_{G_n(p)}$ we have:*

$$\mathcal{R}_{\Omega^t} \circ \mathcal{R}_\Omega(\phi) = \phi$$

Remark 3.2.8. *Using Theorem 3.2.11 below, one can find the inverse also for the real case.*

We now give a short proof for the fact about Lie groups which is used in the proof of Proposition 3.2.6.

Lemma 3.2.9. *The Euler characteristic of every smooth compact Lie group of dimension greater than zero is zero.*

Proof. Assume G is a compact Lie group and denote by $l_x : G \to G$, $y \mapsto xy$ the left translation by $x \in G$. Pick a non-zero vector X in the tangent space over the identity element $e \in G$. Noting that for every $x \in G$, l_x is a diffeomorphism implies that $\mathcal{V}(x) := T_e(l_x)X$ is a non-zero vector field on G. Hence by Poincaré-Hopf theorem $\chi(G) = 0$. Recall that the Poincaré-Hopf theorem (cf. [2]) states that the Euler characteristic of a compact, orientable and connected manifold of dimension greater than zero, is sum of the indices of the singularities of a vector field on it. In particular, the Euler Characteristic a manifold is zero when a vector field on it has no singularities. $\square$

3.2.2 The general case

In the preceding section we found an inverse for $\mathcal{R}_\Omega$, on complex Grassmannians when $p+q=n$. In this section we consider this problem in a general case for both real and complex case. Unfortunately, the ideas of last section do not work here. To gain our results, we will use a method introduced by Matsui in [7]. However, the computations in [7] include Schubert calculus, and Matsui dealt with one Schubert variety. Our computations brought up the question of intersection of two Schubert varieties; The problem of finding the intersection of Schubert varieties is a long standing problem (see [4]). Formulas which have been extracted to find intersection of two Schubert varieties, solve the question for Schubert cells in the *general position*. In our problem, on the other hand, non-general positions might occur and contribute in the calculations of Euler-Poincaré index. To perform our calculations, we looked at fibrations[1] of the sets. One can easily apply these fibration ideas in the next section to recover main theorem of [7].

Let $X = G_n(p)$ and $Y = G_n(q)$. We assume $p \le q$, since by taking the dual we can deal with the cases when $p > q$. We will use the notation in 3.1.1 and 3.2.1. Therefore $\Omega^{p,q} \subset X \times Y$ and $S_i^{q,p} \subset Y \times X$. Moreover, we set:

$$Z_j = S_j^{p,p} = \{(x,x') \in X \times X : dim(x \cap x') = j\}.$$

For simplicity we write Ω instead of $\Omega^{p,q}$ and S_i instead of $S_i^{q,p}$. By Remark 3.1.2, $Y \times X = \bigsqcup_{i=0}^{p} S_i$ is a subanalytic stratification. For each $0 \le i \le p$ we consider the diagram:

$$\begin{tikzcd}
& & \Omega\times_Y S_i \arrow[dl, "h"'] \arrow[d, "r"] \arrow[dr, "h'"] & & \\
& \Omega \arrow[dl, "f"'] \arrow[dr, "g"] & X \times X \arrow[dll, "q_1"] \arrow[drr, "q_2"'] & S_i \arrow[dl, "g'"'] \arrow[dr, "f'"] & \\
X & & Y & & X.
\end{tikzcd}$$

In which all the maps are projections. Note that $Z_p = \{(x,x') \in X \times X : x = x'\}$, and as in (2.7) for a $\phi \in CF_X$ we have $\int_{q_2} \mathbf{1}_{Z_p} q_1^* \phi = \phi$. The key idea here is to use Cramer's rule and find a kernel such that $\int_r (kernel) = \mathbf{1}_{Z_p}$.

[1]Thanks to Prof. B. Edixhoven for this suggestion.

One writes:

$$(\int_r \mathbf{1}_{\Omega\times_Y S_i})(x,x') = \sum_{j=0}^{p}(\int_{\Omega\times_Y S_i} \mathbf{1}_{r^{-1}(x,x')\cap r^{-1}(Z_j)})\mathbf{1}_{Z_j}. \tag{3.2}$$

Accordingly, for a fixed pair $(x,x') \in Z_j$,

$$\int_{\Omega\times_Y S_i} \mathbf{1}_{r^{-1}(x_1,x_2)\cap r^{-1}(Z_j)} = \chi_c(\{r^{-1}(x,x')\cap \Omega\times_Y S_i\}).$$

Note that:

$$r^{-1}(x,x')\cap\Omega\times_Y S_i = \{y\in G_n(q) : x\cap y = \{0\}, dim(x'\cap y) = i\}.$$

We define the matrix $T = (t_{i,j})_{(p+1)\times(p+1)}$ by:

$$t_{i,j} = \chi_c(r^{-1}(x,x')\cap\Omega\times_Y S_i) \text{ for any } 0\le i,j\le p. \tag{3.3}$$

These numbers are independent of the choice of a pair $(x,x')\in Z_j$. Moreover, $i+j>p$ means $dim(x'\cap x) + dim(x'\cap y) > dim(x')$ which implies that $dim(x'\cap(x\cap y))\ge 1$ and $dim(x\cap y)\ge 1$. From which we understand that $t_{i,j}=0$ for $i+j>p$, i.e. the matrix is anti-triangular. In the section 2.3.2.1 we will provide recursive formulas to calculate all of the entries of the matrix T in both real and complex cases.

Writing (3.2) in matrix form yields:

$$T\begin{pmatrix}\mathbf{1}_{Z_0}\\ \mathbf{1}_{Z_1}\\ \vdots\\ \mathbf{1}_{Z_p}\end{pmatrix} = \begin{pmatrix}\int_r \mathbf{1}_{\Omega\times_Y S_0}\\ \int_r \mathbf{1}_{\Omega\times_Y S_1}\\ \vdots\\ \int_r \mathbf{1}_{\Omega\times_Y S_p}\end{pmatrix}. \tag{3.4}$$

Since the matrix T is anti-triangular, its determinant is given by multiplication of entries of anti-diagonal up to a sign. We will find the elements of anti-diagonal explicitly in Lemma 3.2.15 and in Theorem 3.2.16 we will show when $det(T)\neq 0$.
By Cramer's rule we can solve the equation (3.4) with respect to Z_p and write:

$$det(T)\mathbf{1}_{Z_p} = \begin{pmatrix} t_{0,0} & t_{0,1} & \dots & t_{0,p-1} & \int_r \mathbf{1}_{\Omega\times_Y S_0}\\ t_{1,0} & t_{1,1} & \dots & t_{1,p-1} & \int_r \mathbf{1}_{\Omega\times_Y S_1}\\ t_{2,0} & t_{2,1} & \dots & 0 & \int_r \mathbf{1}_{\Omega\times_Y S_2}\\ \vdots & \vdots & \iddots & \vdots & \\ t_{p-1,0} & t_{p-1,1} & \dots & 0 & \int_r \mathbf{1}_{\Omega\times_Y S_{p-1}}\\ t_{p,0} & 0 & \dots & 0 & \int_r \mathbf{1}_{\Omega\times_Y S_p}\end{pmatrix}.$$

This formula yields coefficients for $\mathcal{R}_{S_i}$'s, say μ_i's, such that $\mu_0\mathcal{R}_{S_0}+\cdots+\mu_p\mathcal{R}_{S_p}$ is an inverse for $\mathcal{R}_\Omega$.

Definition 3.2.10. *(i) We set*

$$K_{p,q} = det \begin{pmatrix} t_{0,0} & t_{0,1} & \dots & t_{0,p-1} & \mathbf{1}_{S_0} \\ t_{1,0} & t_{1,1} & \dots & t_{1,p-1} & \mathbf{1}_{S_1} \\ t_{2,0} & t_{2,1} & \dots & 0 & \mathbf{1}_{S_2} \\ \vdots & \vdots & \iddots & \vdots & \\ t_{p-1,0} & t_{p-1,1} & \dots & 0 & \mathbf{1}_{S_{p-1}} \\ t_{p,0} & 0 & \dots & 0 & \mathbf{1}_{S_p} \end{pmatrix} \in CF_{X\times Y}.$$

(ii) We define the map $R^{-1} : CF_Y \to CF_X$ by

$$\psi \mapsto \int_{p_1} K_{p,q}(p_2^*\psi).$$

The following proposition is an essential part of our main result.

Proposition 3.2.11. *For a $\phi \in CF_X$,*

$$R^{-1} \circ \mathcal{R}_\Omega(\phi) = det(T)\phi.$$

In particular, if $det(T) \neq 0$, then R_Ω is invertible and its inverse is given by $det(T)^{-1}R^{-1}$.

Proof. By definition

$$\begin{aligned} R^{-1} \circ \mathcal{R}_\Omega(\phi) &= \int_{p_1} \left(K_{p,q} \left(p_2^* \int_{p_2} \mathbf{1}_\Omega . p_1^*\phi \right) \right) \\ &= \int_{q_2} \left(det \begin{pmatrix} t_{0,0} & t_{0,1} & \dots & t_{0,p-1} & \int_r \mathbf{1}_{\Omega\times_Y S_0} \\ t_{1,0} & t_{1,1} & \dots & t_{1,p-1} & \int_r \mathbf{1}_{\Omega\times_Y S_1} \\ t_{2,0} & t_{2,1} & \dots & 0 & \int_r \mathbf{1}_{\Omega\times_Y S_2} \\ \vdots & \vdots & \iddots & \vdots & \\ t_{p-1,0} & t_{p-1,1} & \dots & 0 & \int_r \mathbf{1}_{\Omega\times_Y S_{p-1}} \\ t_{p,0} & 0 & \dots & 0 & \int_r \mathbf{1}_{\Omega\times_Y S_p} \end{pmatrix} \right) q_1^*\phi \\ &= \int_{q_2} det(T)\mathbf{1}_{Z_p} q_1^*\phi \\ &= det(T)\phi. \end{aligned}$$

For the second equality we have used Lemma 2.2.3. For the last equality recall that $\int_{q_2} \mathbf{1}_{Z_p} q_1^*\phi = \phi$. $\square$

3.2.2.1 Calculation of the entries of T

In the preceding section we showed how to find the inverse for R_Ω when $det(T) \neq 0$. Recall that $T = (t_{i,j})$ and $t_{i,j} = \chi_c(\Omega_x^{p,q} \cap (S_i^{q,p})_{x'})$ for a pair $(x, x') \in S_j^{p,p}$. In this section we will give a recursive method for finding entries of T. Next, we will use this method to find the entries of anti-diagonal. From which, we realize when T has a non-zero determinant.

Consider the Grassmannians $X = G_n(p)$, $Y = G_n(q)$ and $Z = G_n(l)$, for integers $1 \leq l \leq p \leq q \leq n$. By definition:

$$\begin{cases} S_0^{p,q} \subset X \times Y, \\ S_j^{p,l} \subset X \times Z, \\ S_\beta^{q,l} \subset Y \times Z. \end{cases}$$

Definition 3.2.12. *(i) For a $x \in G_n(p)$, we set*

$$g(n, q, p) = \chi(\Omega_x^{p,q}).$$

(ii) For given $(x, z) \in S_j^{p,l}$ we set:

$$\begin{aligned} f_{p,j}(n, q, l, \beta) &= \chi_c((S_0^{p,q})_x \cap (S_\beta^{l,q})_z) \\ &= \chi_c(\{y \in G_n(q) : y \cap x = \ \{0\}, dim(y \cap z) = \beta\}). \end{aligned}$$

Remark 3.2.13. *The values of $g(n, q, p) = \chi_c(\Omega_x^{p,q}) = \chi_c((S_0^{p,q})_x)$, is calculated in the Lemma 3.2.4.*

Note that $x \cap y = \{0\}$ implies $dim(x \cap z) \leq dim(z) - dim(y \cap z)$, which means that for the quadruple (n, q, l, β) we must have $j \leq l - \beta$. Moreover $l = j$, means $z \subset x$ and implies $\beta = 0$. Therefore:

$$f_{p,j}(n, q, j, 0) = g(n, q, p). \tag{3.5}$$

For an integer $\beta > 0$, let:

$$B = \{\beta - \text{planes in } z \text{ with zero intersection with } (x \cap z)\} \simeq \{b \in G_l(\beta) : b \cap (x \cap z) = \{0\}\}.$$

Therefore

$$\chi_c(B) = g(l, \beta, j). \tag{3.6}$$

Lemma 3.2.14. *The map:*

$$\begin{aligned} \phi : A = \{y \in G_n(q) : y \cap x = \ \{0\}, dim(y \cap z) = \beta\} &\longrightarrow B \\ y &\mapsto y \cap z, \end{aligned}$$

is surjective. Moreover, for each $b \in B$ one has:

$$\chi_c(\phi^{-1}(b)) = f_{p,j}(n - \beta, q - \beta, l - \beta, 0).$$

Proof. Let $b \in B$ be β-plane. Then, in the quotient space k^n/b let $z' = z/b \in G_{n-\beta}(l - \beta)$ and $x' = x/(x \cap b) \in G_{n-\beta}(p)$. Then the set:

$$A' = \{y' \in G_{n-\beta}(q - \beta) : y' \cap x' = \ \{0\}, y' \cap z' = \{0\}\},$$

is non-empty. This is followed from Remark 3.2.2 since $p + q \leq n$ and $l \leq p$. Moreover, for $y' \in A'$, we have $y' \oplus b \in A$ and $\phi(y' \oplus b) = (y' \oplus b) \cap (z' \oplus b) = b$. Second part is proved by taking quotient and noting that $\phi^{-1}(b) \simeq A'$. $\square$

By Lemma 3.2.14 and (3.6) and application of the Corollary A.12 we derive:

$$f_{p,j}(n,q,l,\beta) = f_{p,j}(n-\beta, q-\beta, l-\beta, 0).g(l,\beta,j) \quad \text{for } \beta > 0. \tag{3.7}$$

Using the equality of sets:

$$\begin{aligned}&\big\{y \in G_n(q) : y \cap x = \{0\}, dim(y \cap z) = 0\big\}\\ &= \big\{y \in G_n(q) : y \cap x = \{0\}\big\} - \bigsqcup_{\alpha=1}^{l} \big\{y \in G_n(q) : y \cap x = \{0\}, dim(y \cap z) = \alpha\big\} \qquad (l = dim(z))\end{aligned}$$

and additivity of χ_c, we deduce:

$$f_{p,j}(n,q,l,0) = g(n,q,p) - \sum_{\alpha=1}^{l} f_{p,j}(n,q,l,\alpha).$$

Obviously we only need to consider $0 < \alpha < l - j$. So we can refine the formula to:

$$f_{p,j}(n,q,l,0) = g(n,q,p) - \sum_{\alpha=1}^{l-j} f_{p,j}(n,q,l,\alpha). \tag{3.8}$$

In the above recursive formulas we have two steps. In the first step (3.7) the third and fourth coordinates of the quadruple (n,q,l,β) are reduced by β. On the other hand, if β is zero at one stage then in the second step (3.8), $\alpha \geq 1$ will replace β. Hence, at each time, running the two steps will make the quadruple closer some $(\clubsuit, \spadesuit, j, 0)$, which its f is known by (3.5). This implies that the procedure will end in finite number of steps and that it can calculate the desired number.

We apply the recursive formulas (3.7) and (3.8) to find the entries of anti-diagonal.

Lemma 3.2.15. *Elements of the anti-diagonal of T are given by:*

$$t_{i,j} = t_{i,p-i} = \begin{cases} 0 & \text{if } (n-i)(n-q-p) \text{ is odd} \\ (-1)^{p^2-i^2} \binom{E(\frac{n-p-i}{2})}{E(\frac{q-i}{2})} & \text{if } (n-i)(n-q-p) \text{ is even} \end{cases}$$

in the real case. For the complex case:

$$t_{i,j} = t_{i,p-i} = \binom{n-p-i}{q-i}.$$

Proof. By our notations $t_{i,j} = f(n,q,p,i)$. By (3.7) for $i = p-j > 0$ we have $f(n,q,p,i) = f(n-i, q-i, j, 0).g(n,p,j)$ by Lemma 3.2.4 and (3.5) both numbers on the right hand side are known. If $i = 0$ then $j = p$ or equivalently $x = x'$, then by definition $f(n,q,p,i) = g(n,p,j)$. Recalling the values g from Lemma 3.2.4 gives the assertion. □

Now we can extract the key to the existence problem, and prove the main result of this thesis.

Theorem 3.2.16. *If*

(i) $p+q \leq n$ *in the complex Grassmannian.*

(ii) $p+q \leq n$ *and* $n-p-q$ *is even in the real Grassmannian.*

Then determinant of T *in (3.4) is not zero. Moreover,* R_Ω *is invertible and its inverse is given by* $det(T)^{-1}R^{-1}$.

Proof. By Lemma 3.2.15 and Proposition 3.2.11. □

Remark 3.2.17. *Matsui in [7] proves that* $\mathcal{R}_{G(p,q)}$*, the Radon transform with inclusion incidence relation, has an inverse in the following cases:*

(i) $p+q \leq n$ *in the complex Grassmannian,*

(ii) $p+q \leq n$ *and* $q-p$ *is even, in the real Grassmannian.*

Appendix A

Duality formulas and Euler-Poincaré index with compact support

Here we state some theorems which were used in the preceding chapters. The references for this chapter are [1], [3], [5] and [9].

Let k be a field and $f : X \to Y$ be a continuous map of manifolds. Let F be a sheaf on X. Recall that the *proper direct image* of F by f is defined by:

$$(f_!F)(U) = \{s \in \Gamma(f^{-1}(U); F) : f|_{supp(s)} : supp(s) \to U \text{ is proper}\}.$$

This functor is left exact and injective with respect to the family of c-soft sheaves (those sheaves, for which $\Gamma(X; .) \to \Gamma(K; .)$, for any compact $K \subset X$, is surjective). Therefore $f_!$ has a right derived functor $Rf_! : \mathbf{D}^+(k_X) \to \mathbf{D}^+(k_Y)$. Note that if we consider direct proper image by the map $a_X : X \to \{pt\}$, we get:

$$Ra_{X!}(F) \simeq R\Gamma_c(X; F).$$

Proposition A.1. *Let $g : Y \to Z$ be another continuous map of locally compact spaces. Then, there are canonical isomorphisms.*

$$(g \circ f)_! \simeq g_! \circ f_! \qquad \textit{and} \qquad R(g \circ f)_! \simeq Rg_! \circ Rf_!. \tag{A.1}$$

In particular the diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "a_X"'] & & Y \arrow[dl, "a_Y"] \\ & \{pt\}, & \end{tikzcd}$$

gives

$$R\Gamma_c(Y; Rf_!F) \simeq R\Gamma_c(X; F).$$

Remark A.2. *The formula $R\Gamma(Y; Rf_*(F)) = R\Gamma(X; F)$ might be considered as replacement of Leray-Serre spectral sequence in the framework of derived categories. By the same token, the formula $R\Gamma_c(Y; Rf_!F) \simeq R\Gamma_c(X; F)$ might be seen as Leray-Serre sequence with compact supports.*

Proposition A.3 (Fibers formula, [9] Proposition 1.11.4)**.** *For any $F \in \mathbf{D}^+(k_X)$, we have a canonical isomorphism:*

$$[Rf_!(F)]_y \simeq R\Gamma_c(f^{-1}(y); F|_{f^{-1}(y)}).$$

Corollary A.4 (Cartesian square formula, [9] Proposition 1.11.6)**.** *Assume:*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

is a Cartesian square of locally compact spaces (i.e. $Y' \simeq \{(x', y) \in X' \times Y : g(x') = f(x)\}$). Then, we have canonical isomorphisms:

$$f_! g^{-1} \simeq h^{-1} f'_! \qquad \textit{and} \qquad Rf_! g^{-1} \simeq h^{-1} Rf'_!$$

Theorem A.5 (Poincaré-Verdier duality, [5] Theorem 3.1.5)**.** *Let $f : X \to Y$ be a continuous map of locally compact spaces that $f_!$ has finite cohomological dimension(i.e. there exists a non-negative integer such that $R^j f_! = 0$ for $j > r$). Then there exist a functor of triangulated categories $f^! : \mathbf{D}^b(k_Y) \to \mathbf{D}^b(k_X)$ and an isomorphism of bifunctors on $\mathbf{D}^b(k_X) \times \mathbf{D}^b(k_Y)$:*

$$Hom_{\mathbf{D}^b(k_Y)}(Rf_!(.), .) \simeq Hom_{\mathbf{D}^b(k_X)}(., f^!(.)).$$

Namely, $f^!$ is a right adjoint to $Rf_!$.

Definition A.6. *(i) Let $a_X : X \to \{pt\}$, and $F \in \mathbf{D}^b(k_{\{pt\}})$. The dualizing complex on X, denoted by ω_X is:*

$$\omega_X = a_X^! F \in \mathbf{D}^b(k_X).$$

(ii) Assume X has a finite c-soft dimension and let $F \in \mathbf{D}^b(k_X)$. One sets:

$$D_X F = RHom(F, \omega_X) \ , \ D'_X F = RHom(F, k_X).$$

We call $D_X F$, the dual of F. When there is no risk of confusion we might write D instead of D_X. Note that $\omega_X = D_X k_X$.

(iii) The orientation sheaf over X, Or_X, with field coefficients is defined by:

$$U \to Hom(H_c^n(U; k), k).$$

(iv) An orientation of the manifold M relative to k, the field, is an isomorphism:

$$Or_X \to k_X.$$

If such an orientation exists, we will call the manifold orientable.

Proposition A.7 ([1] Proposition 3.5.1)**.** *Let X be an n-dimensional topological manifold with boundary and Or_X be the orientation sheaf on X. Then there is a canonical isomorphism:*

$$\omega_X \simeq Or_X[+n],$$

in $\mathbf{D}^b(k_X)$.

Corollary A.8 (Classical Poincaré duality, [1] p.70)**.** *Let k be a field and X an n-dimensional orientable manifold without boundary. We have the isomorphism:*

$$H^{n-i}(X;k) \simeq Hom(H_c^i(X;k),k)$$

Proof. I. By definition of orientable manifold and Proposition A.7, we have:

$$H^{n-i}(X;k_X) \simeq H^{n-i}(X;Or_X) = H^{-i}(X;Or_X[+n]) \simeq H^{-i}(X;\omega_X).$$

II. For a manifold M, $A \in \mathbf{D}^b(k_M)$ and $U \subset M$ open, and each $i \in \mathbb{Z}$, there exists a short exact sequence (For more general case see Theorem 3.4.4 in [1]):

$$0 \to Ext(H_c^{i+1}(U;A),k) \to H^{-i}(U;D_XA) \to Hom(H_c^i(U;A),k) \to 0.$$

III. Since a field is an injective object of the category of k-vector spaces, $Hom(.,k)$ is an exact functor. Therefore, for any $A \in \mathbf{D}^b(k_M)$, we have:

$$Ext(H_c^{i+1}(M,A),k) = 0$$

IV. By definition and II, III it follows:

$$H^{-i}(X;\omega_X) \simeq H^{-i}(X;D_Xk_X) \simeq Hom(H_c^i(X;k_X),k).$$

Composing IV and I yields the assertion. □

By Classical Poincaré duality it follows that:

Corollary A.9 ([3] Exercise 3.3.13)**.** *Let k be a field and X be an orientable n-dimensional topological manifold. Then for a locally constant k-vector space L on X we have:*

$$\chi_c(X,L) = (-1)^n\chi(X,L).$$

In particular $\chi_c(X,L)$ is invariant under homotopy equivalence up to a sign.

Example A.10. *(i) Using Corollary A.9 we see that $\chi_c(\mathbb{R}^n,\mathbb{R}) = (-1)^n\chi(\mathbb{R}^n;\mathbb{R})$. Since $\chi(\mathbb{R}^n;\mathbb{R})$ is invariant under homotopy equivalence we have $\chi(\mathbb{R}^n,\mathbb{R}) = \chi(\{pt\};\mathbb{R}) = 1$. Since $\mathbb{C}^n$ endows a real analytic structure of $\mathbb{R}^{2n}$, in particular we have $\chi_c(\mathbb{C}^n,\mathbb{R}) = 1$.*

(ii) One should be cautious that in the preceding corollary L is a locally constant sheaf. For instance $R\Gamma_c(\mathbb{R},\mathbb{R}_{[0,\infty)}) = 0$, and therefore $\chi_c(\mathbb{R},\mathbb{R}_{[0,\infty)}) = 0$. Yet, $\mathbb{R}$ is homotopic to the origin. Hence χ_c in general is not invariant by homotopy, even up to a sign.

We state a proposition which is used in Chapter 3 for manifolds which are homotopy equivalent to compact manifolds.

Proposition A.11 ([3] Corollary 2.5.5)**.** *Let $F \to E \to B$ be a locally trivial fibration such that he base B and the fiber F are homotopy equivalent to finite CW-complexes. Then the three Euler characteristics $\chi(B)$, $\chi(F)$ and $\chi(E)$ are defined and:*

$$\chi(E) = \chi(B)\chi(F).$$

Corollary A.12. *In the situation of the preceding proposition, one has:*

$$\chi_c(E) = \chi_c(B)\chi_c(F).$$

where $\chi_c(X) = \chi_c(X;k)$. We read $\chi_c(X)$ the Euler Characteristic with compact support of X.

Proof. By Proposition A.11 and Corollary A.9. Note that the signs cancel out. $\square$

The following formula is used in the second chapter.

Proposition A.13 (Projection formula, [9] Proposition 1.13.4)**.** *Let $f : X \to Y$ be a continuous map between locally compact spaces. Then, for any $F \in \mathbf{D}^+(k_X)$ and any $G \in \mathbf{D}^+(k_Y)$, there is a canonical isomorphism*

$$G \overset{L}{\otimes} Rf_!F \simeq Rf_!(f^{-1}G \overset{L}{\otimes} F).$$

We end the appendix by mentioning the Künneth formula with compact support.

Corollary A.14 (Künneth formula with compact support,[3] Corollary 2.3.30)**.** *Let X_1 and X_2 be two topological spaces, $p_i : X_1 \times X_2 \to X_i$ for $i = 1, 2$ be the two projections. Let $F_i \in \mathbf{D}^b(k_{X_i})$ for $i = 1, 2$. Then:*

$$R\Gamma_c(X_1 \times X_2; p_1^{-1}F_1 \overset{L}{\otimes} p_2^{-1}F_2) = R\Gamma_c(X_1; F_1) \overset{L}{\otimes} R\Gamma_c(X_2; F_2).$$

Appendix B

Micro-support of a sheaf

In this section we introduce the notion of micro-support of a sheaf. Micro-support measures how far a sheaf is from being locally constant. It is also a tool for realizing constructible sheaves. For equivalent definitions see [5] p.221.

Definition B.1. *Let X be a real analytic manifold, and denote by $\pi : T^*X \to X$. Let $F \in \mathbf{D}^b(k_X)$ be a bounded complex. For a point $p = (x_0, \xi_0) \in T^*X$, the micro-support (or characteristic variety) of complex F, denoted by $SS(F)$ is the subset of cotangent bundle T^*X consisting of all the points $p = (x_0, \xi_0)$ such that the condition (C) is failed.*

(C) *There is an open neighborhood U of the point p such that for any point $x_1 \in X$ and any real smooth function f defined in a neighborhood of x_1 and satisfying $f(x_1) = 0$ and $df(x_1) \in U$ we have:*

$$(R\Gamma_{\{x:f(x)\geq 0\}}F)_{x_1} = 0.$$

Definition B.2. *(i) Let (E, σ) be a symplectic vector space and V a linear subspace of E. We set:*

$$V^\perp = \{x \in E : \sigma(x, V) = 0\}.$$

V is called isotropic (resp. Lagrangian, resp. involutive) if $V \subset V^\perp$ (resp. $V = V^\perp$, resp. $V \supset V^\perp$).

(ii) Assume[1] Λ is a conic subanalytic subset of T^*X. One says that Λ is isotropic (resp. Lagrangian, resp. involutive), if at each regular point $p \in \Lambda$, the tangent space of Λ at p, $T_p\Lambda$, has the corresponding property in T_pT^*X, with respect to the natural symplectic form on T^*X.

Following proposition yields some properties of micro-support.

Proposition B.3. *(i) The micro-support $SS(F)$ is a closed conic subset of T^*X and $SS(F) \cap T^*_XX \simeq supp(F)$, where T^*_XX is the zero section of the cotangent bundle of X;*

(ii) $SS(F) = SS(F[1])$;

[1] If Λ is a subset, see [5] for the definition.

(iii) *Let* $F_1 \to F_2 \to F_3 \overset{+1}{\to}$ *be a distinguished triangle in* $\mathbf{D}^b(k_X)$*. Then* $SS(F_i) \subset SS(F_j) \cup SS(F_k)$ *and* $(SS(F_i) \backslash SS(F_j)) \cup (SS(F_j) \backslash SS(F_i)) \subset SS(F_k)$ *for any permutation (i,j,k) of (1,2,3);*

(iv) $SS(F) \subset \bigcup_j SS(H^j(F))$*;*

(v) *Let* M *be a closed submanifold in* X *amd consider its conormal space in* X *given by:*

$$T^*_M X = \{(x,\xi) \in T^*X; \xi|_{T_xM} = 0\}.$$

Then $SS(i_!L) = T^*_M X$ *where* $i : M \to X$ *is the inclusion and* L *is any non-zero locally constant sheaf on* M*.*

We end this appendix with a theorem which relates constructible sheaves to micro-support.

Theorem B.4. *Let* $F \in \mathbf{D}^b(k_X)$*. Then the following conditions are equivalent:*

(i) *There exists a locally finite covering* $X = \bigcup_{i \in I} X_i$ *by subanalytic subsets such that for all* $j \in \mathbb{Z}$*, all* $i \in I$*, the sheaves* $H^j(F)|_{X_i}$ *are locally constant.*

(ii) $SS(F)$ *is contained in a closed conic subanalytic isotropic subset of* T^*X*.*

(iii) $SS(F)$ *is a closed conic subanalytic Lagrangian subset of* T^*X*.*

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