On $\text{SL}_2(\mathbb{Z})$ and $\text{SL}_3(\mathbb{Z})$
Kloosterman sums

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Introduction

Classical Kloosterman sum (or $\text{SL}_2(\mathbb{Z})$ Kloosterman sum) is defined by

$$S(m, n, c) = \sum_{d \equiv 1 (\text{mod } c) \atop \gcd(c, d) = 1} e\left(\frac{m \bar{d} + nd}{c}\right)$$

with $d \bar{d} \equiv 1 (\text{mod } c)$. First, the given sum was discovered by Henri Poincaré in 1911 in the paper [13] on modular forms.

Few years later in 1926 Hendrik Kloosterman also obtained the same sum while he was solving the problem of finding asymptotic expression of the number of representations of a large integer $n$ by a quadratic form in four variables\(^1\), i.e. the number of solutions $(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$ of an equation

$$ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 = n,$$

where $a, b, c, d$ are fixed positive integers.

Already the fact that Kloosterman sum appeared in two problems of different origins emphasizes its importance. Afterwards a number of other applications in number theory have been found. One of them is Conrey’s theorem [3] that at least two-fifths of the zeros of the Riemann zeta function are simple and on the critical line, which uses results of J.-M. Deshouillers and H. Iwaniec on averages of Kloosterman sums.

Another application is the generalization of Ramanujan conjecture to non-holomorphic cusp forms associated to arithmetic discrete sub-

\(^1\)See [11].
groups of $\text{GL}_n(\mathbb{R})$ with $n \geq 2$. Trying to find an approach to the generalized Ramanujan conjecture for $\text{GL}_2(\mathbb{R})$ and $\text{GL}_3(\mathbb{R})$ via Kloosterman sums, Bump, Friedberg and Goldfeld computed Fourier expansion of $\text{SL}_3(\mathbb{Z})$ Poincaré series. As a part of Fourier coefficients they obtained six exponential sums, among which we can distinguish two new types different from classical Kloosterman sums, called $\text{SL}_3(\mathbb{Z})$ Kloosterman sums.

The main goal of this work is to study the connection between Kloosterman sums and automorphic forms associated with groups $\text{SL}_2(\mathbb{Z})$ and $\text{SL}_3(\mathbb{Z})$. We start with classical Kloosterman sums and obtain them as a part of Fourier coefficients of $\text{SL}_2(\mathbb{Z})$ Poincaré series in the first chapter. In the second chapter we compute Fourier expansion of $\text{SL}_3(\mathbb{Z})$ Poincaré series and introduce two new types of exponential sums called $\text{SL}_3(\mathbb{Z})$ Kloosterman sums. We also describe some properties of Kloosterman sums and discuss the problem of distribution of Kloosterman angles.
Chapter 1

Classical Kloosterman sums

In this chapter we construct a particular example of $\text{SL}_2(\mathbb{Z})$ modular forms. Our construction leads us to Poincaré series, whose Fourier expansion turns out to contain classical Kloosterman sums.

1.1 $\text{SL}(2)$ modular forms

The group

$$\text{SL}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \text{ and } a, b, c, d \in \mathbb{R} \right\}$$

acts on the Poincaré upper-half plane

$$\mathbb{H}^2 = \{ z \in \mathbb{C} : \Im z > 0 \}$$

by linear fractional transformations. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{R})$, then for any $z \in \mathbb{H}^2$

$$\gamma(z) = \frac{az + b}{cz + d} \in \mathbb{H}^2$$

since

$$\Im(\gamma(z)) = \frac{\Im(z)}{|cz + d|^2}. \quad (1.2)$$
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The action of $\text{SL}_2(\mathbb{R})$ on the set $\mathbb{H}^2$ has one orbit because we can reach any point in $\mathbb{H}^2$ from the point $i$:

$$\begin{bmatrix} \sqrt{y} & x \\ 0 & \sqrt{y} \end{bmatrix} (i) = x + iy.$$  

And the stabializer $K$ of $i$ is equal to

$$\text{SO}_2(\mathbb{R}) = \{ k(\theta) \} \text{ for } k(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.$$  

Indeed,

$$\frac{ai + b}{ci + d} = i \Rightarrow ai + b = di - c \Rightarrow d = a, c = -b.$$  

So

$$K = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right\} \text{ with } a^2 + b^2 = 1$$  

and we can write $a = \cos \theta$ and $b = \sin \theta$ to obtain the result.

This gives an alternative way to represent the upper-half plane as a quotient space $\text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})$. Each element of the later group has a unique representative of the form $\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}$, where $y > 0$, by Iwasawa decomposition$^1$.

In this section we are mainly interested in a discrete subgroup $\text{SL}_2(\mathbb{Z})$ of $\text{SL}_2(\mathbb{R})$ with $a, b, c, d \in \mathbb{Z}$, called the modular group.

The group $\text{SL}_2(\mathbb{Z})$ is generated$^2$ by two elements

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ such that } S^2 = (ST)^3 = -1.$$  

And the standard fundamental domain$^3$ for the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}^2$ is

$$F = \left\{ z \in \mathbb{H}^2, \ |\Re(z)| \leq \frac{1}{2}, \ |z| \geq 1 \right\}.$$  

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$^1$The proof is given in a general case in 2.1.1

$^2$See [9], theorem 1.1

$^3$See [9], theorem 1.2
1. Classical Kloosterman sums

To define a notion of automorphic form with respect to $\text{SL}_2(\mathbb{Z})$, we use the following operator.

**Definition 1.1.1.** Let $k$ be a positive integer. Define a **weight $k$ slash operator** of

$$
\text{GL}_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; \ a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}
$$
on the set of all functions $f : \mathbb{H}^2 \to \mathbb{C}$ as follows. If $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2^+(\mathbb{R})$ and

$$j(\gamma, z) = cz + d,$$let

$$\left( f \mid_k \gamma \right) (z) = \frac{\det(\gamma)^{k/2}}{j(\gamma, z)^k} \cdot f(\gamma z). \quad (1.3)$$

**Remark 1.1.2.** Formula (1.3) defines a right action of $\text{GL}_2^+(\mathbb{R})$ on the set of all functions $f : \mathbb{H}^2 \to \mathbb{C}$; in particular,

$$f \mid_{\gamma_1 \gamma_2} (z) = (f \mid_{\gamma_1}^k) \mid_{\gamma_2}^k (z). \quad (1.4)$$

The last equation is a consequence of the cocycle property

$$j(\gamma_1 \gamma_2, z) = j(\gamma_1, \gamma_2 z) \cdot j(\gamma_2, z). \quad (1.5)$$

From now on we fix $\Gamma_2 = \text{SL}_2(\mathbb{Z})$. Then the equation (1.3) can be written as

$$\left( f \mid_k \gamma \right) (z) = \frac{f(\gamma z)}{j(\gamma, z)^k}. \quad (1.6)$$

**Definition 1.1.3.** A function $f : \mathbb{H}^2 \to \mathbb{C}$ is called **modular form** of weight $k$ with respect to the group $\Gamma_2$ if

- $f$ is holomorphic on $\mathbb{H}^2$
- $f \mid_{\gamma}^k = f$ for every $\gamma \in \Gamma_2$
- $f$ is holomorphic at infinity.

**Remark 1.1.4.** The last condition can be explained as follows.
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Since \[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\in \text{SL}_2(\mathbb{Z}),
\] \(f\) is a periodic function:

\[f(z + 1) = f(z), \ z \in \mathbb{H}^2.\]

Thus, \(f\) has a \textit{Fourier expansion at infinity}

\[f(z) = \sum_{n=-\infty}^{+\infty} a_n(f)q^n, \ q = e^{2\pi iz}.\]

And we call \(f\) \textit{holomorphic at infinity} if \(a_n(f) = 0\) for every \(n < 0\). If in addition, \(a_0(f) = 0\), function \(f\) is called \textit{cuspidal}.

1.2 Construction of SL(2) Poincaré series

To find functions satisfying definition 1.1.3, we start with the automorphy condition. Let \(h : \mathbb{H}^2 \to \mathbb{C}\) be a holomorphic function. We can write formally

\[f(z) = \sum_{\gamma \in \Gamma_2} \frac{h(\gamma z)}{j(\gamma, z)^k}, \ z \in \mathbb{H}^2. \tag{1.7}\]

The cocycle property (1.5) yields that for every \(\gamma' \in \Gamma_2\)

\[f(\gamma' z) = \sum_{\gamma \in \Gamma_2} \frac{h(\gamma \gamma' z)}{j(\gamma, \gamma' z)^k} = j(\gamma', z)^k \sum_{\gamma \in \Gamma_2} \frac{h(\gamma \gamma' z)}{j(\gamma \gamma', z)^k} = j(\gamma', z)^k f(z).\]

If (1.7) converges absolutely uniformly on compact subsets of \(\mathbb{H}^2\), then \(f(z)\) is a holomorphic function and all formal computations are valid. However, the sum (1.7) does not converge in general. In particular, the sum may diverge if we have infinitely many elements

\[\gamma \in \Gamma_\infty = \left\{ \pm \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, \ n \in \mathbb{Z} \right\} = \langle \pm T \rangle \ 	ext{with} \ j(\gamma, z) \equiv 1.\]

In order to avoid this problem, assume that \(h\) is invariant under \(\Gamma_\infty\) and note that the sum (1.7) depends only on cosets modulo \(\Gamma_\infty\). Indeed, if \(\gamma = \beta \gamma'\) for
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\( \beta \in \Gamma_{\infty}, \gamma,\gamma' \in \Gamma_{2}, \) then

\[ h(\gamma z) = h(\beta \gamma' z) = h(\gamma' z), \]

\[ j(\gamma, z) = j(\beta \gamma', z) = j(\beta, \gamma' z) j(\gamma', z) = j(\gamma', z). \]

So the formula

\[ f(z) = \sum_{\gamma \in \Gamma_{\infty}\backslash \Gamma_{2}} \frac{h(\gamma z)}{j(\gamma, z)^k} \tag{1.8} \]

is the one we are looking for. Now we can choose a particular \( \Gamma_{\infty} \)-invariant function, namely

\[ h(z) = e(mz) = e^{2\pi imz}, \quad m \in \mathbb{Z}. \]

**Definition 1.2.1.** The series

\[ P_m^k(z) = \sum_{\gamma \in \Gamma_{\infty}\backslash \Gamma_{2}} h(\gamma z) = \sum_{\gamma \in \Gamma_{\infty}\backslash \Gamma_{2}} e(m\gamma z) \tag{1.9} \]

is called \( m^th \) Poincaré series of weight \( k. \)

**Proposition 1.2.2.** The Bruhat decomposition of \( \Gamma_{2} \) is given by

\[ \Gamma_{2} = \Gamma_{\infty} \amalg \Pi_{c \in \mathbb{Z}_{>0}} \Pi_{d \text{ } (\text{mod } c)} \left( \Gamma_{\infty} \text{w } \Gamma_{\infty} \right), \quad \Pi \text{ is a disjoint union,} \tag{1.10} \]

\[ \Gamma_{\infty} = < \pm T > \quad \text{and } w \in W_{c,d} = \left\{ \begin{bmatrix} a^* & b^* \\ c & d \end{bmatrix} \right\}, \]

where for given \( c, d \in \mathbb{Z} \) with \( (c, d) = 1 \), integral variables \( a^*, b^* \) satisfy

\[ a^* d - b^* c = 1, \quad \text{i.e. } b^* = \frac{a^* d - 1}{c}. \]

**Proof.** We would like to partition \( \Gamma \) into double cosets with respect to \( \Gamma_{\infty}. \)

First, consider the set of upper triangular matrices

\[ \Delta_1 = \left\{ \begin{bmatrix} a^* & b^* \\ 0 & d^* \end{bmatrix} \in \Gamma_{2} \right\}. \]
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Conditions $a^* d^* = 1$ and $a^*, b^* \in \mathbb{Z}$ imply that $\Delta_1 = \Gamma_\infty$.

Second, any element of $\Delta_1 \setminus \Gamma_2$ can be represented by a matrix

$$\omega = \begin{bmatrix} a & b^* \\ c & d \end{bmatrix}$$

with $c > 0$.

The relation

$$\begin{bmatrix} 1 & n_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b^* \\ c & d \end{bmatrix} \begin{bmatrix} 1 & n_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a + cn_1 & b^*_1 \\ c & d + cn_2 \end{bmatrix}$$

shows that the double coset $\Gamma_\infty \begin{bmatrix} a & b^* \\ c & d \end{bmatrix} \Gamma_\infty$ determines $c$ uniquely, while $a$ and $d$ can be found modulo integral multiples of $c$.

Actually, the given coset does not depend on $a$, because for any two matrices

$$\omega_1 = \begin{bmatrix} a_1 & b_1^* \\ c & d \end{bmatrix} \text{ and } \omega_2 = \begin{bmatrix} a_2 & b_2^* \\ c & d \end{bmatrix} \in \Gamma_2$$

$$\omega_1 \omega_2^{-1} = \begin{bmatrix} 1 & b_3^* \\ 0 & 1 \end{bmatrix},$$

i.e. $a_1$ is congruent to $a_2$ modulo $c$. So

$$\Delta_2 = \Gamma_\infty \omega \Gamma_\infty \in \Gamma_2, \text{ with } \omega = \begin{bmatrix} a^* & b^* \\ c & d \end{bmatrix} \in \Gamma_2.$$

To sum up, $\Gamma_2$ is a disjoint union of $\Delta_1$ and $(\bigcup_{c \in \mathbb{Z} > 0} \bigcup_{(d \mod c) \Delta_2})_{(c,d)=1}$.

Proposition 1.2.3. For $k > 2$, $m \geq 0$ the series $P^k_m(z)$ converges absolutely and uniformly on compact sets of $\mathbb{H}^2$ and defines a holomorphic function on $\mathbb{H}^2$.

Proof. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_2$, then

$$\left| \frac{e(m\gamma z)}{j(\gamma, z)^k} \right| = \left| \frac{e^{2\pi i zm(z)}}{(cz + d)^k} \right|.$$
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By formula 1.2,

\[
\left| e^{2\pi i m \gamma(z)} \right| \leq \frac{1}{|cz + d|^k} e^{-2\pi m \Im(z)} \leq \frac{1}{|cz + d|^k}
\]

for \( m \geq 0 \). According to the Bruhat decomposition of \( \Gamma_2 \), we pick each pair \( (c, d) \) as the second raw of matrices in \( \Gamma_\infty \setminus \Gamma_2 \) at most once. Therefore, \( P_m^k(z) \) is majorated by the series

\[
\sum_{c, d \in \mathbb{Z} \setminus \{(0,0)\}} \frac{1}{|cz + d|^k}.
\]

The last series is known\(^1\) to be convergent uniformly on compact sets of \( \mathbb{H}^2 \) for any \( k > 2 \).

\[\square\]

1.3 Fourier expansion of Poincaré series

It only remains to verify the last condition of 1.1.3 to complete our construction of modular form. With this goal, we find Fourier expansion of the series (1.9). Ultimately, we obtain Kloosterman sums as a part of Fourier coefficients.

Definition 1.3.1. The sum

\[
S(m, n, c) = \sum_{\substack{d \pmod{c} \mid \gcd(c, d) = 1}} e\left( \frac{md + nd}{c} \right)
\]

with \( d \equiv 1 \pmod{c} \) is called classical Kloosterman sum.

Remark 1.3.2. If \( m = 0 \), then Kloosterman sum reduces to Ramanujan sum

\[
R_c(n) = S(0, n, c) = \sum_{\substack{d \pmod{c} \mid \gcd(c, d) = 1}} e\left( \frac{nd}{c} \right).
\]

Theorem 1.3.3. Let \( k > 2 \). The Fourier expansion of Poincaré series \( P_m^k(z) \) is given by

\(^1\)See [8], theorem 1
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- if $m = 0$,
  \[
  P_{m}^{k}(z) = 1 + \frac{(-2\pi i)^{k}}{\Gamma(k)} \sum_{n > 0} \left( \sum_{c > 0} R_{c}(n) \frac{n^{k-1}}{c^{k}} \right) e(nz);
  \]

- if $m > 0$,
  \[
  P_{m}^{k}(z) = e(mz) + \frac{(-2\pi i)^{k}}{m^{\frac{k-1}{2}}} \sum_{n > 0} \left( \sum_{c > 0} S(m, n, c) \frac{n^{k-1}}{c} J_{k-1}(\frac{4\pi \sqrt{mn}}{c}) \right) e(nz),
  \]

where

\[
J_{n}(x) = \sum_{i=0}^{\infty} \frac{(-1)^{i}}{i! \Gamma(i + 1 + n)} \left( \frac{x}{2} \right)^{n+2i}
\]
is the Bessel function of order $n$.

**Proof.** Consider the series (1.9). Applying the Bruhat decomposition,

\[
\sum_{c > 0} \sum_{\substack{d \equiv 0 \mod c \ \gcd(c, d) = 1 \ \omega \in W_{c,d} \ \beta \in \Gamma_{\infty}}} h |_{\omega\beta}^{k} (z).
\]

Take

\[
\omega = \left[ \frac{a^{*}}{c} \frac{a^{*}d-1}{d} \right] \in W_{c,d} \text{ and } \beta = \left[ \begin{array}{c} 1 \\ n \\ 0 \\ 1 \end{array} \right] \in \Gamma_{\infty},
\]

then for every $z \in \mathbb{H}^{2}$

\[
(\omega \beta)z = \frac{a^{*}z + a^{*}n + a^{*}d-1}{cz + cz + d} = \frac{a^{*}}{c} - \frac{1}{c(c(z + n) + d)}
\]

and

\[
h |_{\omega\beta}^{k} (z) = (c(z + n) + d)^{-k} e \left( m \left( \frac{a^{*}}{c} - \frac{1}{c(c(z + n) + d)} \right) \right).
\]

Thus,

\[
P_{m}(z) = e(mz) + \sum_{c > 0} \sum_{\substack{d \equiv 0 \mod c \ \gcd(c, d) = 1 \ d \equiv (0, 1, 2, \ldots, c-1) \ \omega \in W_{c,d}}} I(c, d, z)
\] (1.11)
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with

\[ I(c, d, z) = \sum_{n \in \mathbb{Z}} g(n) \text{ and } g(n) = h_{\omega_{\beta}}^k(z). \]

By Poisson summation formula,

\[ I(c, d, z) = \sum_{n \in \mathbb{Z}} \hat{g}(n) = \sum_{n \in \mathbb{Z}} \left( \int_{\mathbb{R}} g(t)e(-nt)dt \right) \]

\[ = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{1}{(c(z + t) + d)^k} e \left( -nt + m \left( \frac{a^*}{c} - \frac{1}{c(c(z + n) + d)} \right) \right) dt. \]

Let us write \( z = x + iy \) and make change of variables \( t \to t' = z + t + \frac{d}{c} \) in the integral. Then we obtain

\[ I(c, d, z) = \sum_{n \in \mathbb{Z}} e(n(z + \frac{d}{c}) + \frac{ma^*}{c}) \int_{t'=-\infty+iy}^{+\infty+iy} \frac{1}{(ct')^k} e(-nt' - \frac{m}{c^2t'})dt'. \quad (1.12) \]

Denote the inner integral \(^1\) by

\[ L_c(m, n) = \int_{t'=-\infty+iy}^{+\infty+iy} \frac{1}{(ct')^k} e(-nt' - \frac{m}{c^2t'})dt'. \]

and distinguish the following cases:

1. If \( n \leq 0 \), then we can move the line of integration upwards, i.e. let \( y \to \infty \), and estimate the absolute value of \( L_c(m, n) \) to see that

\[ L_c(m, n) = 0. \quad (1.13) \]

Therefore, all terms with \( n \leq 0 \) in the sum (1.12) vanish.

2. If \( n > 0 \) and \( m = 0 \), then \(^2\)

\[ L_c(0, n) = \left( \frac{2\pi}{ic} \right)^k \frac{n^{k-1}}{\Gamma(k)}. \quad (1.14) \]

\(^1\)Note that \( L_c(m, n) \) does not depend on \( y \) by Cauchy’s theorem

\(^2\)see [7], 8.315.1

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3. If $n, m > 0$, then

$$L_c(m, n) = \frac{2\pi}{i^k c} \left( \frac{n}{m} \right)^{\frac{k-1}{2}} J_{k-1} \left( \frac{4\sqrt{mn}}{c} \right).$$

(1.15)

Finally, substitute

$$I(c, d, z) = \sum_{n>0} e(nz) e\left( \frac{nd + ma^*}{c} \right) L_c(m, n)$$

in the formula (1.11) and change the order of summation to obtain

$$P_m(z) = e(mz) + \sum_{n>0} \left( \sum_{c>0} \left( \sum_{d \equiv (\mod c)} \sum_{\gcd(c, d) = 1} e\left( \frac{nd + ma^*}{c} \right) L_c(m, n) \right) e(nz) \right)$$

$$= e(mz) + \sum_{n>0} \left( \sum_{c>0} S(m, n, c) L_c(m, n) \right) e(nz).$$

Now just replace $L_c(m, n)$ by its value and the assertion follows.

\[\square\]

**Corollary 1.3.4.** The series $P_m^k(z)$ is holomorphic at infinity for $m \geq 0$ and cuspidal for $m \geq 1$.

**Remark 1.3.5.** In 1965 Selberg [14] introduced non-holomorphic Poincaré series

$$P_m(z, s) = \sum_{\gamma \in \Gamma \setminus \Gamma_2} (\Im(\gamma z))^s e(m\gamma z), \Re(s) > 1.$$  
(1.16)

The Fourier expansion of the series 1.16 also contains classical Kloosterman sums.

Let $z = x + iy$, then

$$P_m(z, s) = \sum_{h \in \mathbb{Z}} p_m(h; y, s) e(hz)$$

\[\text{see [7], 8.412.2}\]
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with
\[ p_m(h; y, s) = \delta(m, h) + \sum_{c \geq 1} c^{-2s} S(m, n, c) B(m, h, c, y, s) \]
and
\[ B(m, h, c, y, s) = y^s \int_{-\infty}^{+\infty} (x^2 + y^2)^{-s} e\left(-hx - \frac{m}{c^2(x + iy)}\right) dx. \]

1.4 Some properties of Kloosterman sums

The sum
\[ S(m, n, c) = \sum_{\substack{d \pmod{c} \\gcd(c, d) = 1}} e\left(\frac{md + nd}{c}\right) \]
with \( d \bar{d} \equiv 1 \pmod{c} \) has some interesting properties.

**Proposition 1.4.1.** The Kloosterman sum depends only on the residue class of \( m, n \) modulo \( c \).

**Proof.** This is clear since \( e^{2\pi ik} = 1 \) for every \( k \in \mathbb{Z} \). \qed

**Proposition 1.4.2.** The value of \( S(m, n; c) \) is always a real number.

**Proof.** Consider complex conjugate of Kloosterman sum
\[ \overline{S}(m, n; c) = \sum_{\substack{d \pmod{c} \\gcd(c, d) = 1}} e\left(\frac{-md - nd}{c}\right). \]

Let \( d = -d' \), then
\[ \overline{S}(m, n; c) = S(m, n; c) \]
since \(-d\) runs again over all residue classes modulo \( c \). \qed

**Proposition 1.4.3.**
\[ S(m, n; c) = S(n, m; c) \]

**Proof.**
\[ S(m, n; c) = \sum_{\substack{d \pmod{c} \\gcd(c, d) = 1}} e\left(\frac{md + nd}{c}\right). \]
Then the substitution $d = m\bar{a}d'$ leads to the required result. \hfill \Box

**Proposition 1.4.4.**

$$S(ma, n; c) = S(m, na, c) \text{ if } (a, c) = 1$$

**Proof.**

$$S(ma, n, c) = \sum_{\substack{d \pmod{c} \\
\gcd(c, d) = 1}} e\left(\frac{ma\bar{d} + nd}{c}\right).$$

Then the substitution $d = ad'$ leads to the required result. \hfill \Box

**Proposition 1.4.5.** (twisted multiplicativity) If $(c_1, c_2) = 1$, then

$$S(m, n; c_1c_2) = S(m\bar{c}_2, n\bar{c}_2, c_1)S(m\bar{c}_1, n\bar{c}_1, c_2).$$

**Proof.** Let $c = c_1c_2$. The proof is based on the Chinese Remainder theorem, i.e. if

$$d \equiv d_i \pmod{c_i}, \ i = 1, 2, \ (c_1, c_2) = 1,$$

then

$$d \equiv d_1b_1c_2 + d_2b_2c_1 \pmod{c},$$

with

$$b_1c_2 \equiv 1 \pmod{c_1},$$

$$b_2c_1 \equiv 1 \pmod{c_2}.$$ 

Then

$$S(m, n, c) = \sum_{\substack{d \pmod{c} \\
\gcd(c, d) = 1}} e\left(\frac{ma\bar{d} + nd}{c}\right)$$

$$= \sum e\left(\frac{m(d_1b_1c_2 + d_2b_2c_1) + n(d_1b_1c_2 + d_2b_2c_1)}{c_1c_2}\right),$$

summation is over all $d_1 \pmod{c_1}$, $d_2 \pmod{c_2}$, $(c_1, d_1) = 1$, $(c_2, d_2) = 1$. So that

$$S(m, n, c_1c_2) = S(m\bar{c}_2, n\bar{c}_2, c_1)S(m\bar{c}_1, n\bar{c}_1, c_2).$$
1. Classical Kloosterman sums

1.5 Distribution of Kloosterman angles

As a consequence of the Riemann Hypothesis for curves over functional fields, A. Weil obtained the following bound\(^1\)

\[
|S(m, 1, p)| \leq 2\sqrt{p},
\]

where \(p\) is a prime number and \(m\) is an integer coprime with \(p\). Therefore, there is a unique Kloosterman angle \(\theta(p, m) \in [0, \pi]\) such that

\[
S(m, 1, p) = 2\sqrt{p} \cos \theta(p, m).
\]

There are two kinds of distribution of Kloosterman angles:

- **vertical**

\[
\{\theta(p, m)\}_{1 \leq m \leq p , (m, p)=1} \quad p \to \infty;
\]

- **horizontal**

\[
\{\theta(p, m)\}_{1 \leq p \leq P , (m, p)=1 , m \text{ fixed}} \quad P \to \infty.
\]

In the vertical case, we have the following theorem by Katz\(^2\).

**Theorem 1.5.1.** Let \(p \to \infty\), then the angles

\[
\{\theta(p, m)\}_{1 \leq m \leq p , (m, p)=1}
\]

are equidistributed with respect to the Sato-Tate measure on \([0, \pi]\)

\[
d\mu_{ST}(\theta) = \frac{2}{\pi} \sin^2(\theta) d\theta.
\]

\(^1\)See [16]  
\(^2\)See [10]
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Equivalently, for any interval $I = [a, b] \in [0, \pi]$, 

$$
\lim_{p \to \infty} \frac{\# \{1 \leq m \leq p - 1, \theta(m, p) \in I\}}{p - 1} = \mu_{ST}(I) = \frac{2}{\pi} \int_{a}^{b} \sin^2(\theta) d\theta.
$$

The horizontal case is still a conjecture.

**Conjecture 1.5.2.** Let $P \to \infty$, $m$ is a fixed non-zero integer, then the angles 

$$
\{\theta(p, m)\}_{1 \leq p \leq P}^{(m, p) = 1}
$$

are equidistributed with respect to the Sato-Tate measure on $[0, \pi]$.

Equivalently, for any interval $I = [a, b] \in [0, \pi]$, 

$$
\lim_{P \to \infty} \frac{\# \{p \leq P, (m, p) = 1, \theta(m, p) \in I\}}{\# \{p \leq P\}} = \mu_{ST}(I) = \frac{2}{\pi} \int_{a}^{b} \sin^2(\theta) d\theta.
$$
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Figure 1.1: Difference between vertical and Sato-Tate measures on the interval [1, 2]
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Figure 1.2: Vertical (red) and Sato-Tate (green) distribution functions
Figure 1.3: Difference between horizontal and Sato-Tate measures on the interval [1, 2], \( m = 1 \)
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Figure 1.4: Horizontal (red) and Sato-Tate (green) distribution functions
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1.6 Numerical computation of Poincaré series

1.6.1 Poincaré series and fundamental domain reduction algorithm

Current section provides a PARI/GP code for the calculation of Poincaré series

\[ P_m(z) = e(mz) + \sum_{n>0} (F(m,n)) e(nz), \]  

(1.17)

with Fourier coefficients given by

\[ F(m,n) = \sum_{c>0} S(m,n,c) L_c(m,n). \]  

(1.18)

First, we compute the Kloosterman sum

\[ S(m,n,c) = \sum_{d \text{ (mod } c)} \sum_{\gcd(d,c)=1} e \left( \frac{md + nd}{c} \right). \]

```
gp>{klsum (m, n , c , sum , dinv , t)=
  sum=0;
  for (d=0,c-1,
    if (gcd (d , c)==1, 
      dinv= lift (1/Mod(d , c ));
      t=(m*dinv+n*d)/c ;
      sum=sum+exp (2*Pi*I*t );
    )
  )}
```

According to the formulas 1.13, 1.14, 1.15, the value of \( L_c(m,n) \) can be found as follows.

```
gp>{coeffL (m,n,c,k,L)=
  L=0;
  if ((n>0)&&(m>=0),
    if (m==0,
```
1. Classical Kloosterman sums

\[ L = \left( \frac{2 \pi}{\Gamma} \right)^{\frac{1}{2}} \frac{n^{\frac{1}{2} (k-1)}}{\Gamma(k)} \cdot \text{besselj}(k-1,4 \pi \sqrt{m \cdot n}/c) \]

Note that for the values of \( z \in \mathbb{H}^2 \) with a small imaginary part \( P_m(z) \) may converge very slow. However, the second property of definition 1.1.3 allows us to compute \( P_m(\gamma z) \) for some \( \gamma \in \Gamma \) and then recover the original series by the formula \( P_m(z) = P_m(\gamma z)/j(\gamma,z)^k \). Furthermore, if a point \( z_1 = \gamma z \) is in the fundamental domain \( F \), then we have the estimate

\[ |e(nz)| \leq e^{(-\pi \sqrt{3})^n} < \left( \frac{1}{230} \right)^n, \]

which provides a very good convergence.

Now we describe a fundamental domain reduction algorithm, which on input \( z \in \mathbb{H}^2 \) returns a matrix \( \gamma \in \Gamma_2 \) such that \( \gamma z \) lies in the fundamental domain. In order to find such \( \gamma \) with \( z_1 = \gamma z \in F \), we first apply \( \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix} \) with \( n = \lfloor \Re(z) \rfloor \) to translate \( z \) into the strip \( |\Re(z)| \leq 1/2 \).

Now if \( z \not\in F \), then \( |z| < 1 \) and

\[ \Im(-1/z) = \Im(z/|z|^2) > \Im(z). \]

Replace \( z \) by \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (z) \) and repeat the process. Note that there are only finitely many integer pairs \((c, d)\) such that \( |cz + d| < 1 \), and so, by the formula 1.2, there are only finitely many transforms of \( z \) with a larger imaginary part. Thus, the algorithm below terminates after a finite number of steps.
1. Classical Kloosterman sums

Finally, we compute Fourier coefficients as in the formula 1.18

\[
\{ \text{coeffF}(c_{\text{max}}, m, n, k, F) = \\
F = 0; \\
\text{for } (c = 1, c_{\text{max}}) \\
F = F + k\text{sum}(m, n, c) \times \text{coeffL}(m, n, c, k)
\}
\]

and Poincaré series as in the formula 1.17.

\[
\{ \text{poincareS}(z, k, m, n_{\text{max}}, c_{\text{max}}, A, z_1, P) = \\
A = \text{transM}(z); \\
z_1 = (A[1, 1] \times z + A[1, 2]) / (A[2, 1] \times z + A[2, 2]); \\
P = \exp(2 \times \Pi \times I \times m \times z_1); \\
\text{for } (n = 1, n_{\text{max}}), \\
P = P + \text{coeffF}(c_{\text{max}}, m, n, k) \times \exp(2 \times \Pi \times I \times n \times z_1) \\
P = P / (A[2, 1] \times z + A[2, 2]) ^ k
\}
\]

1.6.2 Absolute error estimate

Notice that both sums on \(n\) in 1.17 and on \(c\) in 1.18 are infinite. But for the purpose of computing, we truncate these sums to a finite number of terms \(n_{\text{max}}\) and \(c_{\text{max}}\), respectively. This leads to some incorrectness in our computations, which can be measured in terms of the absolute error.

**Definition 1.6.1.** Let \(X\) be a true value of the quantity and \(X_1\) its approximate value, then the absolute error is defined to be a numerical difference \(X - X_1\). An upper limit on the magnitude of the absolute error \(\Delta X\), such that

\[
E_X = |X_1 - X| \leq \Delta X,
\]

is said to measure absolute accuracy.
1. Classical Kloosterman sums

**Remark 1.6.2.** This type of accuracy is convenient when we are dealing with sums, because the magnitude of the absolute error in the result is the sum of the magnitudes of the absolute errors in the summands.

In our case,

\[ X = P_m(z) \text{ and } X_1 = e(mz) + \sum_{0 < n \leq n_{\text{max}}} \left( \tilde{F}(m, n) \right) e(nz) \]

with \( \tilde{F}(m, n) = \sum_{0 < c \leq c_{\text{max}}} S(m, n, c) L_c(m, n) \).

Thus,

\[ E_X = |X_1 - X| = \left| \sum_{0 < n \leq n_{\text{max}}} \left( \tilde{F}(m, n) \right) e(nz) - \sum_{n > 0} (F(m, n)) e(nz) \right| \]

\[ \leq \sum_{0 < n \leq n_{\text{max}}} |\tilde{F}(m, n) - F(m, n)| |e(nz)| + \sum_{n > n_{\text{max}}} |F(m, n)| |e(nz)| . \]

Let us denote

\[ E_1 = \sum_{0 < n \leq n_{\text{max}}} |\tilde{F}(m, n) - F(m, n)| |e(nz)| \]

and

\[ E_2 = \sum_{n > n_{\text{max}}} |F(m, n)| |e(nz)| . \]

The key ingredient of our computation is the bound for the Kloosterman sum \( S(m, n, c) \). The optimal result for the prime values of \( c \) can be obtained using Weil’s bound\(^1\), but for our purpose it is enough to consider the trivial estimate

\[ |S(m, n, c)| \leq c. \quad (1.20) \]

The next step is to bound the value of \( L_c(m, n) \). In case \( m = 0 \),

\[ |L_c(0, n)| = \beta(k) \frac{n^{k-1}}{c^k}, \quad \text{where} \quad \beta(k) = \frac{2^k \pi^k}{(k - 1)!} . \]

\(^1\)See [16]
Now suppose $m > 0$. Using the following estimate $^1$

$$|J_k(x)| \leq \frac{|x/2|^k}{k!},$$

we obtain

$$|L_c(m, n)| \leq \beta(k) \frac{n^{k-1}}{c^{k-1}}.$$

So for any $m \geq 0$,

$$|S(m, n, c) L_c(m, n)| \leq \beta(k) \frac{n^{k-1}}{c^{k-1}}.$$

Then

$$E_F = |\tilde{F}(m, n) - F(m, n)| \leq \beta(k) n^{k-1} \sum_{c > c_{\text{max}}} \frac{1}{c^{k-1}} \leq \beta(k) n^{k-1} \int_{c_{\text{max}}}^{\infty} \frac{1}{x^{k-1}} dx$$

$$= \frac{\beta(k) n^{k-1}}{(k - 2) c_{\text{max}}^{k-2}}$$

and

$$|F(m, n)| \leq \beta(k) n^{k-1} \sum_{c > 0} \frac{1}{c^{k-1}} = \beta(k) n^{k-1} \left(1 + \int_{1}^{\infty} \frac{1}{x^{k-1}} dx\right) = \beta(k) n^{k-1} \frac{k - 1}{k - 2}.$$

Therefore,

$$E_1 \leq \frac{\beta(k)}{(k - 2) c_{\text{max}}^{k-2}} \sum_{0 < n \leq n_{\text{max}}} n^{k-1} \frac{1}{230^n} \leq \frac{\beta(k) n_{\text{max}}^{k-1}}{(k - 2) c_{\text{max}}^{k-2}} \sum_{0 < n \leq n_{\text{max}}} \frac{1}{230^n}$$

$$= \frac{\beta(k) n_{\text{max}}^{k-1}}{(k - 2) c_{\text{max}}^{k-2}} \frac{230^n_{\text{max}} - 1}{229(230)^{n_{\text{max}}}} \leq \frac{\beta(k) n_{\text{max}}^{k-1}}{229(k - 2) c_{\text{max}}^{k-2}}$$

and

$$E_2 \leq \beta(k) \frac{k - 1}{k - 2} \sum_{n > n_{\text{max}}} n^{k-1} e^{-\pi \sqrt{3} n} \leq \beta(k) \frac{k - 1}{k - 2} \int_{n_{\text{max}}}^{\infty} \frac{x^{k-1} dx}{e^{\pi \sqrt{3} x}}$$

$$= \beta(k) \frac{k - 1}{k - 2} \frac{\pi \sqrt{3} n_{\text{max}}}{\pi \sqrt{3}} + \frac{(k - 1)(k - 2) n_{\text{max}}^{k-3}}{(\pi \sqrt{3})^2} + \ldots + \frac{(k - 1)!}{(\pi \sqrt{3})^{k-1}}$$

$^1$See [12], ex.9.6
1. Classical Kloosterman sums

\[
\leq \beta(k) \frac{k - 1}{k - 2} e^{-\pi \sqrt{3n_{\text{max}}}} \frac{k!}{\pi \sqrt{3}} n_{\text{max}}^{k - 1},
\]

Finally,

\[
E_X \leq E_1 + E_2 \leq \alpha(k) \left( \frac{n_{\text{max}}^{k-1}}{c_{k-2}} + \frac{n_{\text{max}}^{k-1}}{e^{\pi \sqrt{3n_{\text{max}}}}} \right),
\]

where

\[
\alpha(k) = \max \left( \frac{\beta(k)}{229(k - 2)}, \frac{\beta(k)(k - 1)k!}{(k - 2)\pi \sqrt{3}} \right) = \frac{\beta(k)(k - 1)k!}{(k - 2)\pi \sqrt{3}}.
\]
Chapter 2

SL(3) Kloosterman sums

Following the work [2], we generalize results of the previous chapter and obtain Kloosterman sums associated to the group \( \text{SL}_3(\mathbb{Z}) \) as a part of Fourier coefficients of \( \text{SL}_3(\mathbb{Z}) \) Poincaré series.

2.1 Generalized upper-half space and Iwasawa decomposition

Let \( G_n = \text{GL}_n(\mathbb{R}) \) and \( \Gamma_n = \text{SL}_n(\mathbb{Z}) \).

In order to define a notion of generalized upper-half space associated to the group \( G_n \) with \( n \geq 2 \), we prove the following theorem.

Theorem 2.1.1. (Iwasawa decomposition) Every \( g \in G_n \) decomposes as

\[
g = nak
\]

with

\[
n = n(x_{i,j}) \in N = \begin{cases} 
1 & \cdots & * \\
0 & x_{1,2} & \cdots & * \\
0 & 0 & \cdots & x_{n-1,n} \\
0 & 0 & \cdots & 1 
\end{cases}, \quad x_{i,j} \in \mathbb{R}, \ i < j 
\]
2. $\text{SL}_3(\mathbb{Z})$ Kloosterman sums

$$a = a(y_k) \in A = \left\{ \begin{bmatrix} y_1 & 0 & \ldots & 0 \\ 0 & y_2 & \ldots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & \ldots & y_n \end{bmatrix}, \ y_i > 0 \right\},$$

$$k \in K = \text{O}_n(\mathbb{R}) - \text{orthogonal group}.$$

The map

$$N \times A \times K \to G_n,$$

$$(n, a, k) \to nak$$

is a homeomorphism of topological spaces. In particular, the decomposition is unique.

Proof. We need to show that the map

$$f : (n, a, k) \to nak$$

is a continuous bijective map such that $f^{-1}$ is also continuous.

1. The map is injective.

The group $NA$ is a group of upper triangular matrices with positive elements on the diagonal and $K$ is a group of orthogonal matrices. If $g \in NA \cap K$, then $g^{-1} \in NA \cap K$ since $NA \cap K$ is a group. Furthermore, since $g$ is orthogonal, $g = (g^{-1})^T$. So that $g$ is upper triangular and lower triangular at the same time, i.e. it is diagonal. And the only orthogonal diagonal matrix with positive elements on the diagonal is the identity matrix, $g = I$.

We conclude that $NA \cap K = I$.

Suppose $nak = n'a'k'$, then $(na)^{-1}n'a' = (a^{-1}n^{-1}n'a)a^{-1}a' = k(k')^{-1}$.

Note that $A$ normalizes $N$, i.e. for all $a \in A$

$$a^{-1}Na = N.$$

Then

$$(a^{-1}n^{-1}n'a)a^{-1}a' \in NA.$$
Since $NA \cap K = I$, we have that $k = k'$ and $na = n'a'$. Finally, $n = n'$ and $a = a'$ because $N \cap A = I$.

2. The map is surjective.

We apply Gramm-Schmidt orthogonalization process to the columns $g_1, g_2, \ldots, g_n \in \mathbb{R}^n$ of matrix $g^{-1}$. Vectors $g_1, g_2, \ldots, g_n$ form a basis in $\mathbb{R}^n$ because $g^{-1}$ is invertible. Define $h_1, h_2, \ldots, h_n \in \mathbb{R}^n$ and $h_1', h_2', \ldots, h_n' \in \mathbb{R}^n$ as follows:

$$h_1 = g_1, \quad h_1' = \frac{h_1}{||h_1||},$$

$$h_2 = -(g_2|h_1')h_1' + g_2, \quad h_2' = \frac{h_2}{||h_2||},$$

$$\vdots$$

$$h_i = -\sum_{j=1}^{i-1} (g_i|h_j)h_j' + g_i, \quad h_i' = \frac{h_i}{||h_i||},$$

$$\vdots$$

$$h_n = -\sum_{j=1}^{n-1} (g_n|h_j)h_j' + g_n, \quad h_n' = \frac{h_n}{||h_n||}.$$

Note that $\{h_i\}$ form an orthogonal and $\{h_i'\}$ orthonormal bases of $\mathbb{R}^n$. Matrix $g^{-1}$ sends canonical basis $\{e_i\}$ to $\{g_i\}$ via composition

$$e_i \rightarrow h_i' \rightarrow h_i \rightarrow g_i, \quad i = 1, \ldots, n.$$

The first map is an application of $k \in K$ (to the canonical basis), the second is an action of $ka^{-1}$ with $a = diag(||h_1||, \ldots, ||h_2||) \in A$ and the third is $(ka)n(ka)^{-1}$. So that

$$g^{-1} = (ka)n(ka)^{-1}ka^{-1}k = kan$$

and

$$g = nak.$$
3. The map is continuous and its inverse is also continuous.

Given map is polynomial whence continuous. To show the continuity of inverse map notice that $K$ is compact and $B = NA$ is closed subgroups of $G_n$. Let $g, g_m \in G_n, k, k_m \in K, a, a_m \in A$ and $n, n_m \in N$. If the sequence

$$g_m = n_m a_m k_m \xrightarrow{m \to \infty} g = nak,$$

then since $K$ is compact

$$k_m \xrightarrow{m \to \infty} k' \in K.$$

So that

$$b_m = n_m a_m \xrightarrow{m \to \infty} b' = n'a' \in B$$

since $B$ is closed. Therefore, $g = n'a'k'$ and $n = n'$, $a = a'$, $k = k'$, i.e

$$n_m \xrightarrow{m \to \infty} n,$$

$$a_m \xrightarrow{m \to \infty} a$$

and

$$k_m \xrightarrow{m \to \infty} k.$$ 

As a corollary, we derive Iwasawa decomposition of the group $\text{SL}_n(\mathbb{R})$.

**Corollary 2.1.2.**

$$\text{SL}_n(\mathbb{R}) = NA\tilde{SO}_n(\mathbb{R}),$$

with $\tilde{A} = \{a \in A, y_n = 1\}$.

**Remark 2.1.3.** For later applications, it is convenient to write elements $a \in \tilde{A}$ as

$$a = \begin{bmatrix}
y_1 y_2 \cdots y_{n-1} & 0 & \cdots & 0 \\
0 & y_1 y_2 \cdots y_{n-2} & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & y_1 & 0 \\
0 & \cdots & \cdots & 1
\end{bmatrix}.$$
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Given change of variables is valid since \( y_i \neq 0 \) for all \( i = 1, \ldots, n - 1 \).

Now in the analogous manner to the case \( n = 2 \), for \( n > 2 \) we define generalized upper-half space

\[
\mathbb{H}^n \cong \frac{\text{SL}_n(\mathbb{R})}{\text{SO}_n(\mathbb{R})} \cong \frac{G_n}{(O_n(\mathbb{R}) \cdot \mathbb{R}^\times)}.
\]

The space \( \mathbb{H}^n \) plays the same role for \( \text{GL}_n(\mathbb{R}) \) that \( \mathbb{H}^2 \) played for \( \text{GL}_2(\mathbb{R}) \).

By the Iwasawa decomposition, every \( z \in \mathbb{H}^n \) can be uniquely written as

\[
\begin{bmatrix}
1 & x_{1,2} & \cdots & * \\
0 & 1 & x_{2,3} & * \\
0 & 0 & \ddots & x_{n-1,n} \\
0 & 0 & 0 & 1
\end{bmatrix}\begin{bmatrix}
y_1 y_2 \ldots y_{n-1} & 0 & \cdots & 0 \\
0 & y_1 y_2 \ldots y_{n-2} & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & y_1 & 0 \\
0 & \cdots & \cdots & 1
\end{bmatrix},
\]

where \( x_{i,j} \in \mathbb{R} \) for \( j > i \), \( y_1, \ldots, y_{n-1} > 0 \). In particular, the generalized upper half plane \( \mathbb{H}^3 \) is the set of all matrices \( z = na \) with

\[
n = \begin{bmatrix}
1 & x_{1,2} & x_{1,3} \\
0 & 1 & x_{2,3} \\
0 & 0 & 1
\end{bmatrix},
a = \begin{bmatrix}
y_1 y_2 & 0 & 0 \\
0 & y_1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

where \( x_{1,2}, x_{1,3}, x_{2,3} \in \mathbb{R}, y_1, y_2 > 0 \).

2.2 Automorphic forms and Fourier expansion

The group \( G_3 \) acts on \( \mathbb{H}^3 \) by matrix multiplication. It is generated by diagonal matrices, upper triangular matrices with 1s on the diagonal and the Weyl group \( W_3 \) consisting of all \( 3 \times 3 \) matrices with exactly one 1 in each row and column.

The approximation of fundamental domain for \( G_3 \) can be given by the Siegel set \( \Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}} \).

Here \( \Sigma_{a,b} \subset \mathbb{H}^3 \) (\( a, b \geq 0 \)) is the set of all matrices

\(^1\)See [6], section 1.3.
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\[
\begin{bmatrix}
1 & x_{1,2} & x_{1,3} \\
0 & 1 & x_{2,3} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_1 y_2 & 0 & 0 \\
0 & y_1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

with $|x_{i,j}| \leq b$ for $1 \leq i < j \leq 3$ and $y_i > a$ for $i = 1, 2$.

The group $G_3$ is a Lie group whose Lie algebra $\mathfrak{gl}(3, \mathbb{R})$ is the additive vector space (over $\mathbb{R}$) of all $n \times n$ matrices with coefficients in $\mathbb{R}$ with Lie brackets given by

\[
[a, b] = a \cdot b - b \cdot a
\]

for all $a, b \in \mathfrak{gl}(3, \mathbb{R})$, where $\cdot$ denotes matrix multiplication. Define the set $S$ be a space of smooth(infinitely differential) functions $F : G_3 \to \mathbb{C}$.

\textbf{Definition 2.2.1.} Let $F \in S$, $g \in G_3$ and $\alpha \in \mathfrak{gl}(3, \mathbb{R})$. Then we define the \textbf{differential operator} $D_\alpha$ acting on $F$ as

\[
D_\alpha F(g) = \frac{\partial}{\partial t} F(g \cdot \exp(t\alpha))|_{t=0} = \frac{\partial}{\partial t} F(g + t(g\alpha))|_{t=0}.
\]

\textbf{Remark 2.2.2.} The differential operators $D_\alpha$ with $\alpha \in \mathfrak{gl}(3, \mathbb{R})$ generate an associative algebra $D^n$ defined over $\mathbb{R}$. And the ring of differential operators $D_\alpha$ is a realization of the universal enveloping algebra of the Lie algebra $\mathfrak{gl}(3, \mathbb{R})$.\footnote{See [6], section 2.2}

Consider the center $\Delta$ of $D^n$. Every $D \in \Delta$ satisfies $D \cdot D' = D' \cdot D$ for all $D' \in D^n$. We would like to construct an eigenfunction of all differential operators $D \in \Delta$.

Let $\nu_1, \nu_2$ be complex parameters and

\[

\begin{bmatrix}
1 & x_{1,2} & x_{1,3} \\
0 & 1 & x_{2,3} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_1 y_2 & 0 & 0 \\
0 & y_1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\in \mathbb{H}^3.
\]

We define a generalization of imaginary part function on the classical upper-half plane to $\mathbb{H}^3$ by

\[
I_{\nu_1, \nu_2} : \mathbb{H}^3 \to \mathbb{C}
\]
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\[ I_{\nu_1,\nu_2}(z) = y_1^{2\nu_1+\nu_2}y_2^{\nu_1+2\nu_2}. \]  

(2.1)

Below we prove that the function \( I_{\nu_1,\nu_2} \) is an eigenfunction of every \( D \in \Delta \). Thus it determines a character \( \lambda_{\nu_1,\nu_2} \) on \( \Delta \), i.e.

\[ DI_{\nu_1,\nu_2} = \lambda_{\nu_1,\nu_2}(D)I_{\nu_1,\nu_2}. \]  

(2.2)

**Theorem 2.2.3.** Let us define \( D_{i,j} = D_{E_{i,j}} \), where \( E_{i,j} \in \mathfrak{gl}(3,\mathbb{R}) \) is the matrix with an 1 at the \( i,j \) component and zeros elsewhere. Then for all \( 1 \leq i < j \leq 3 \) and \( k = 1, 2, \ldots \)

\[ D_{i,j}^k I_{\nu_1,\nu_2}(z) = \begin{cases} 
\nu_{3-i}^{k}I_{\nu_1,\nu_2}(z) & \text{if } i = j; \\
0 & \text{otherwise.}
\end{cases} \]

where \( D_{i,j}^k \) denotes the composition of differential operators \( D_{i,j} \) iterated \( k \) times.

**Proof.** Let

\[ z = na = \begin{bmatrix} 1 & x_{1,2} & x_{1,3} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{H}^3. \]

Note that the function \( I_{\nu_1,\nu_2}(z) \) depends only on variables \( y_1, y_2 \). So that

\[ I_{\nu_1,\nu_2}(na) = I_{\nu_1,\nu_2}(a) \]

and

\[ D_{i,j}I_{\nu_1,\nu_2}(na) = D_{i,j}I_{\nu_1,\nu_2}(a). \]

We distinguish three different cases.

1. If \( i < j \), then by definition 2.2.1

\[ D_{i,j}I_{\nu_1,\nu_2}(a) = \frac{\partial}{\partial t} I_{\nu_1,\nu_2}(a + taE_{i,j})|_{t=0} = y_1 \cdot \ldots \cdot y_{3-i-1} \frac{\partial}{\partial x_{i,j}} I_{\nu_1,\nu_2}(a) = 0. \]

2. If \( i = j \), then

\[ D_{i,i}I_{\nu_1,\nu_2}(a) = \frac{\partial}{\partial t} I_{\nu_1,\nu_2}(a + taE_{i,i})|_{t=0} = \left( y_{3-i} \frac{\partial}{\partial y_3} - \sum_{l=3-i+1}^{2} y_l \frac{\partial}{\partial y_l} \right) I_{\nu_1,\nu_2}(a) = \nu_{3-i}I_{\nu_1,\nu_2}(a). \]
Similarly,

\[ D^k_{i,i} I_{\nu_1,\nu_2}(a) = \left( \frac{\partial}{\partial t} \right)^k I_{\nu_1,\nu_2}(ae^{tE_{i,i}}) = \nu^k_{3-i} I_{\nu_1,\nu_2}(a). \]

3. Now let \( i > j \). As before

\[ D_{i,j} I_{\nu_1,\nu_2}(a) = \frac{\partial}{\partial t} I_{\nu_1,\nu_2}(a(Id + tE_{i,j})) |_{t=0}, \]

where \( Id \) is the identity matrix. First we will show that

\[ (Id + tE_{i,j}) \equiv M (\text{mod } O_3(\mathbb{R}) \cdot \mathbb{R}^\times), \]

where \( M \) a matrix such that \((t^2 + 1)^{-1/2}\) occurs at the position \{\( j, j \}\}, \((t^2 + 1)^{1/2}\) at the position \{\( i, i \}\}, all the other diagonal entries are ones, \( \frac{t}{(t^2 + 1)^{-1/2}} \) occurs at the position \{\( j, i \}\} and all other entries are zeros. Indeed, let \( h = Id + tE_{i,j} \). Then

\[ hh^t = (Id + tE_{i,j})(Id + tE_{j,i}) = Id + tE_{i,j} + tE_{j,i} + t^2E_{i,i}. \]

Define a matrix \( u = Id - \frac{t}{(t^2 + 1)} E_{j,i} \), then \( uhh^t u^t \) must be a diagonal matrix \( d \). Let \( d = a^{-1}(a^t)^{-1} \). Then by direct computations,

\[ uhh^t u^t = Id + t^2E_{i,i} - \frac{t}{t^2 + 1} E_{j,j}, \]

\[ u^{-1} = Id + \frac{t}{(t^2 + 1)} E_{j,i}, \]

\[ a^{-1} = Id + \left( \frac{1}{\sqrt{t^2 + 1} - 1} \right) E_{j,j} + \left( \sqrt{t^2 + 1} - 1 \right) E_{i,i}. \]

Therefore,

\[ M = u^{-1} a^{-1} = Id + \left( \frac{1}{\sqrt{t^2 + 1} - 1} \right) E_{j,j} + \frac{t}{\sqrt{t^2 + 1}} E_{j,i}. \]

Since

\[ auh(h^t u^t a^t) = Id, \]
we have 
\[ auh \in O_3(\mathbb{R}) \]
and
\[ h \equiv M(\text{mod } O_3(\mathbb{R}) \cdot \mathbb{R}^\times) \]
as required.

Finally, taking the derivative of any of diagonal values and setting \( t = 0 \), we obtain zero as an answer. So the only contribution comes from non-diagonal entry \( \frac{t}{(t^2 + 1)^{1/2}} \). Thus,
\[ D_{\nu_1,\nu_2} I_{\nu_1,\nu_2}(a) = a_1 \cdot \ldots \cdot a_{3-i} \frac{\partial}{\partial x_{i,j}} I_{\nu_1,\nu_2}(a) = 0. \]

Now we can define the notion of automorphic form for the group \( \Gamma_3 \) and compute its Fourier expansion.

**Definition 2.2.4.** A function \( f \) on \( \mathbb{H}^3 \) is called an automorphic form (of type \( \nu_1, \nu_2 \)) for \( \Gamma_3 \) if

- \( f(\gamma z) = f(z) \) for \( \gamma \in \Gamma_3, \ z \in \mathbb{H}^3 \)
- \( Df = \lambda_{\nu_1,\nu_2} \cdot f \), where \( D \in \Delta \) and \( \lambda_{\nu_1,\nu_2} \) as in 2.2.
- \( f(z) \) has a polynomial growth in \( y_1, y_2 \) on the region \( \{ z : y_1, y_2 \geq 1 \} \).

**Remark 2.2.5.** If in addition, \( f \) satisfies
\[
\int_0^1 \int_0^1 f \left( \begin{bmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z \right) \, d\xi_1 d\xi_3 = 0
\]
\[
\int_0^1 \int_0^1 f \left( \begin{bmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z \right) \, d\xi_2 d\xi_3 = 0,
\]
for all \( z \in \mathbb{H}^3 \), then \( f \) is called a *cusp form*.
Theorem 2.2.6. (Fourier expansion)

Let $f$ be an automorphic form with respect to $\Gamma_3 = \text{SL}_3(\mathbb{Z})$ and 

\[ \Gamma_{3,\infty} = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \in \text{SL}_3(\mathbb{Z}) \right\} \]

be a minimal parabolic subgroup of $\Gamma_3$.

Then $f$ has a Fourier expansion given by 

\[ f(z) = \sum_{n=-\infty}^{\infty} F_{0,n}(z) + \sum_{\gamma \in \Gamma_{3,\infty}^2 \setminus \Gamma_{3,\infty}^2} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} F_{m,n}(\gamma z), \]

where

\[ F_{m,n}(z) = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} f \left( \begin{bmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z \right) e(-m\xi_1 - n\xi_2)d\xi_1d\xi_2d\xi_3, \quad (2.3) \]

\[ \Gamma_{3,+}^2 = \left\{ \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid A, B, C, D \in \mathbb{Z}, \ AD - BC = 1 \right\} \]

and

\[ \Gamma_{3,\infty}^2 = \Gamma_{3}^2 \cap \Gamma_{3,\infty} = \left\{ \begin{bmatrix} 1 & B & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid B \in \mathbb{Z} \right\}. \]

Proof. Since $f$ is automorphic with respect to $\text{SL}_3(\mathbb{Z})$,

\[ f(z) = f \left( \begin{bmatrix} 1 & 0 & n_3 \\ 0 & 1 & n_1 \\ 0 & 0 & 1 \end{bmatrix} z \right), \quad n_1, n_3 \in \mathbb{Z}. \]
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Then analogously to one-dimensional Fourier expansion we can write

\[ f(z) = \sum_{n_1,n_3 \in \mathbb{Z}} f_{n_1,n_3}(z) \]

(2.4)

with

\[ f_{n_1,n_3}(z) = \int_0^1 \int_0^1 f \left( \begin{bmatrix} 1 & n_2 & 0 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \right) e(-n_1 \xi_1 - n_3 \xi_3) d\xi_1 d\xi_3. \]

(2.5)

The function \( f_{n_1,n_3}(z) \) satisfies the following properties:

- Let \( n_2 \in \mathbb{Z} \), then

\[ f_{n_1,n_3} \left( \begin{bmatrix} 1 & n_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z \right) = f_{n_1+n_2 n_3,n_3}(z). \]

(2.6)

By 2.5 the left-hand side is equal to

\[ \int_0^1 \int_0^1 \int_0^1 f \left( \begin{bmatrix} 1 & n_2 & 0 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \right) e(-n_1 \xi_1 - n_3 \xi_3) d\xi_1 d\xi_3 = \]

\[ \int_0^1 \int_0^1 f \left( \begin{bmatrix} 1 & n_2 & 0 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \xi_3 - n_2 \xi_1 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \right) e(-n_1 \xi_1 - n_3 \xi_3) d\xi_1 d\xi_3. \]

Since \( f \) is automorphic,

\[ f \left( \begin{bmatrix} 1 & n_2 & 0 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \xi_3 - n_2 \xi_1 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \right) = f \left( \begin{bmatrix} 1 & 0 & \xi_3 - n_2 \xi_1 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \right). \]

And the following change of variables

\[ \tilde{\xi}_1 = \xi_1 \]
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$$\tilde{\xi}_3 = \xi_3 - n_2 \xi_1,$$

leads us to the result.

- Let $A, B, C, D, m \in \mathbb{Z}$, $AD - BC = 1$, $m > 0$.

$$f_{mD, mC}(z) = f_{m, 0} \left( \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} z \right). \quad (2.7)$$

Let us consider the left-hand side

$$f_{mD, mC}(z) = \int_0^1 \int_0^1 f \left( \begin{bmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z \right) e^{-mD \xi_1 - mC \xi_3} d\xi_1 d\xi_3 =$$

$$\int_0^1 \int_0^1 f \left( \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z \right) e^{-mD \xi_1 - mC \xi_3} d\xi_1 d\xi_3$$

because $f$ is automorphic. The last expression can be written as

$$\int_0^1 \int_0^1 f \left( \begin{bmatrix} 1 & 0 & B \xi_1 + A \xi_3 \\ 0 & 1 & D \xi_1 + C \xi_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} z \right) e^{-mD \xi_1 - mC \xi_3} d\xi_1 d\xi_3.$$

Changing variables

$$\tilde{\xi}_1 = D \xi_1 + C \xi_3,$$

$$\tilde{\xi}_3 = B \xi_1 + A \xi_3,$$

we obtain the result.

In view of property 2.7, the formula 2.4 takes the form

$$f(z) = f_{0, 0}(z) + \sum_{\gamma \in \Gamma_3^+ \backslash \Gamma_3} \sum_{m=1}^{\infty} f_{m, 0}(\gamma z). \quad (2.8)$$
Note that by 2.6, \( f_{m,0} \) is invariant under the action of the matrices of the form
\[
\begin{bmatrix}
1 & n & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
with \( n \in \mathbb{Z} \).

Therefore,
\[
f_{m,0}(z) = \sum_{n=-\infty}^{\infty} F_{m,n}(z)
\]
with
\[
F_{m,n}(z) = \int_0^1 \int_0^1 \int_0^1 f\left(\begin{bmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z\right) e^{(-m\xi_1 - n\xi_3)d\xi_1 d\xi_2 d\xi_3}.
\]

Finally,
\[
f(z) = \sum_{n=-\infty}^{\infty} F_{0,n}(z) + \sum_{\gamma \in \Gamma_{3,\infty} \setminus \Gamma_{3,+}} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} F_{m,n}(\gamma z).
\]

**Corollary 2.2.7.** If \( f \) is a cusp form, then
\[
F_{0,n} = F_{m,0} = 0 \text{ for every } n, m \in \mathbb{Z}
\]
and Fourier expansion is given by
\[
f(z) = \sum_{\gamma \in \Gamma_{3,\infty} \setminus \Gamma_{3,+}} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} F_{m,n}(\gamma z).
\]

### 2.3 SL(3) Poincaré series

For \( z \in \mathbb{H}^3 \) let
\[
L_{\nu_1,\nu_2}(z) = y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2}.
\]
And for every two integers \( n_1, n_2 \) define \( E \)-function as

\[
E_{n_1, n_2} : \mathbb{H}^3 \rightarrow \mathbb{C},
\]
satisfying

\[
E_{n_1, n_2} \left( \begin{bmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z \right) = e(n_1 \xi_1 + n_2 \xi_2)E_{n_1, n_2}(z) \quad \text{for all } \xi_1, \xi_2, \xi_3 \in \mathbb{R}, \quad (2.12)
\]

\[
E_{n_1, n_2}(z) = O(1) \quad \text{for } z \in \mathbb{H}^3, \quad y_1, y_2 = O(1). \quad (2.13)
\]

**Definition 2.3.1.** Let \( \nu_1, \nu_2 \) be two complex variables such that \( \Re(\nu_i) > \frac{2}{3}, \ i = 1, 2 \). Then the series

\[
P_{n_1, n_2}(z; \nu_1, \nu_2) = \sum_{\gamma \in \Gamma_3, \infty \backslash \Gamma_3} I_{\nu_1, \nu_2}(\gamma z)E_{n_1, n_2}(\gamma z) \quad (2.14)
\]
is called **general Poincaré series** for the minimal parabolic subgroup \( \Gamma_{3,\infty} \).

**Lemma 2.3.2.** The series (2.14) converges absolutely uniformly on compact sub-sets of \( \mathbb{H}^3 \) when \( \Re(\nu_i) > \frac{2}{3}, \ i = 1, 2 \).

**Proof.** For every

\[
z = \begin{bmatrix} y_1 y_2 & x_{1,2} y_1 & x_{1,3} \\ 0 & y_1 & x_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{H}^3,
\]

the left invariant \( GL_3(\mathbb{R}) \) – measure\(^1\) on \( \mathbb{H}^3 \) is given by

\[
d^\ast z = dx_{1,2}dx_{1,3}dx_{2,3} \frac{dy_1 dy_2}{(y_1 y_2)^3}.
\]

Let us also recall the notion of **Siegel set** \( \Sigma_{a,b} \subset \mathbb{H}^3 \ (a, b \geq 0) \) that is the set of all matrices

---

\(^1\)For details see [6], prop. 1.5.3
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\[
\begin{pmatrix}
1 & x_{1,2} & x_{1,3} \\
0 & 1 & x_{2,3} \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_1y_2 & 0 & 0 \\
0 & y_1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

with $|x_{i,j}| \leq b$ for $1 \leq i < j \leq 3$ and $y_i > a$ for $i = 1, 2$.

Since 2.13, $E$–function $E_{n_1,n_2}(\gamma z)$ is bounded and it is enough to prove the theorem with respect to the series

\[\sum_{\gamma \in \Gamma_{3,\infty} \backslash \Gamma_3} I_{\nu_1,\nu_2}(\gamma z),\]

i.e. that for every point $z_0 \in \mathbb{H}^3$ and some non-zero volume compact subset $C_{z_0}$ of $\mathbb{H}^3$ such that $z_0 \in C_{z_0}$ the integral

\[\int_{C_{z_0}} \left| \sum_{\gamma \in \Gamma_{3,\infty} \backslash \Gamma_3} I_{\nu_1,\nu_2}(\gamma z) \right| d^* z\]

converges. Without loss of generality, assume $\nu_1, \nu_2$ to be real. So we can write

\[\int_{C_{z_0}} \sum_{\gamma \in \Gamma_{3,\infty} \backslash \Gamma_3} I_{\nu_1,\nu_2}(\gamma z) d^* z = \int_{(\Gamma_{3,\infty} \backslash \Gamma_3) \cdot \mathbb{C}_{z_0}} I_{\nu_1,\nu_2}(z) d^* z.\]

According to the theorem of Siegel\(^1\), there are only finitely many $\gamma \in \Gamma_{3,\infty} \backslash \Gamma_3$ such that $\gamma z_0 \in \Sigma_{\sqrt{3} \over 2}$. By continuity, for a sufficiently small $C_{z_0}$ there are only finitely many $\gamma \in \Gamma_{3,\infty} \backslash \Gamma_3$ such that $\gamma z \in \Sigma_{\sqrt{3} \over 2}$ for all $z \in C_{z_0}$. Thus, there is some $a \geq \sqrt{3} \over 2$ such that

\[\gamma z \notin \Sigma_{a, \sqrt{3} \over 2}\]

for all $\gamma \in \Gamma_{3,\infty} \backslash \Gamma_3$ and $z \in C_{z_0}$. Consequently,

\[\int_{(\Gamma_3 \backslash \Gamma_3) \cdot C_{z_0}} I_{\nu_1,\nu_2}(z) d^* z \leq \int_0^1 \int_0^1 \int_0^a \int_0^a y_1^{2\nu_1+\nu_2-3} y_2^{\nu_1+2\nu_2-3} dx_{1,2} dx_{1,3} dx_{2,3} dy_1 dy_2.\]

And the last integral converges absolutely if $\nu_1, \nu_2$ are sufficiently large.

\(^1\)See, for example, [6] prop. 1.3.2
2.4 Bruhat decomposition and Plucker coordinates

Similarly to \( \text{SL}_2(\mathbb{Z}) \) case, it is necessary to know the Bruhat decomposition of the group \( \text{GL}_3(\mathbb{R}) \) to compute Fourier expansion of Poincaré series.

**Theorem 2.4.1. (Bruhat decomposition)** The group \( \text{GL}_3(\mathbb{R}) \) can be decomposed as

\[
\text{GL}_3(\mathbb{R}) = B_3W_3B_3,
\]

where

- \( B_3 \) is the standard Borel subgroup of \( \text{GL}_3(\mathbb{R}) \), i.e. the group of invertible upper triangular matrices,

- \( W_3 \) is the Weyl group consisting of all \( 3 \times 3 \) matrices which have exactly one 1 in each row and column and zeros elsewhere.

**Proof.** Consider the element

\[
g = \begin{bmatrix}
g_{1,1} & g_{1,2} & g_{1,3} \\
g_{2,1} & g_{2,2} & g_{2,3} \\
g_{3,1} & g_{3,2} & g_{3,3}
\end{bmatrix} \in \text{GL}_3(\mathbb{R}).
\]

Let \( g_{3,k} \) be the first non-zero element of the third row of matrix \( g \). Without loss of generality, assume \( k = 1 \). Then we can always choose \( b_1 \in B_3 \) such that

\[
gb_1 = \begin{bmatrix}
g'_{1,1} & g'_{1,2} & g'_{1,3} \\
g'_{2,1} & g'_{2,2} & g'_{2,3} \\
1 & 0 & 0
\end{bmatrix}.
\]
Now multiplying on the left by a suitable element

\[ b'_1 = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{bmatrix} \in B_3, \]

we obtain a matrix of the form

\[ b'_1 gb_1 = \begin{bmatrix} 0 & g''_{1,2} & g''_{1,3} \\ 0 & g''_{2,2} & g''_{2,3} \\ 1 & 0 & 0 \end{bmatrix}. \]

Applying the same procedure to the first non-zero element of the second row, we can change the value of this entry to 1 and the rest of the entries in the corresponding row and column to 0 using suitable matrices \( b_2 \) and \( b'_2 \).

Finally, repeating the process with a non-zero element of the first row, we have

\[ b'_1 b'_2 b'_3 gb_1 b_2 b_3 \in W_3 \]

with exactly one 1 in each row and column.

**Corollary 2.4.2.** Let

\[ G_\infty = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \subseteq B_3 \]

and

\[ D = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}, \ det(D) \neq 0 \}

Then Bruhat decomposition can be written as

\[ GL_3(\mathbb{R}) = \bigcup_{w \in W_3} G_w, \] \hspace{1cm} (2.15)

where

\[ G_w = G_\infty D w G_\infty = G_\infty w D G_\infty. \] \hspace{1cm} (2.16)
Proof. The proof consists of several facts. First, \( B_3 = G_\infty D = DG_\infty \). Second, let \( w \in W_3, d = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \in D \), then products \( wdw^{-1} \) and \( w^{-1}dw \) are in \( D \). This can be checked by direct computations for all 6 elements of \( W_3 \). For instance, if \( w = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), then

\[
wdw^{-1} = w^{-1}dw = \begin{bmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{bmatrix}.
\]

So that \( wD = Dw \). Finally, notice that the product of 2 diagonal matrices is again diagonal.

\( \square \)

For \( \gamma \in G_3 \) define the involution

\[
^i\gamma = w^t\gamma w, \quad w = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

If

\[
\gamma = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ A_1 & A_2 & A_3 \end{bmatrix}
\]

then

\[
^i\gamma = \begin{bmatrix} a_{1,1}a_{2,2} - a_{1,2}a_{2,1} & a_{1,3}a_{2,1} - a_{1,1}a_{2,2} & a_{1,2}a_{2,3} - a_{1,3}a_{2,2} \\ a_{1,1}A_1 - a_{1,1}B_1 & a_{1,1}C_1 - a_{1,3}A_1 & a_{1,3}B_1 - a_{1,2}C_1 \\ a_{2,1}B_1 - a_{2,2}A_1 & a_{2,3}A_1 - a_{2,1}C_1 & a_{2,2}C_1 - a_{2,3}B_1 \end{bmatrix}.
\]

(2.17)

Definition 2.4.3. Let us denote elements of the bottom row of \(^i\gamma \) as

\[
A_2 = a_{2,1}B_1 - a_{2,2}A_1,
\]

\[ \]
Then the vectors \( \rho_1 = \{A_1, B_1, C_1\} \) and \( \rho_2 = \{A_2, B_2, C_2\} \)

are called the Plucker coordinates of \( \gamma \).

**Remark 2.4.4.** Plucker coordinates \( \{\rho_1, \rho_2\} \) satisfy the following relation

\[
A_1 C_2 + B_1 B_2 + C_1 A_2 = 0 \quad (2.18)
\]

called Plucker relation.

**Theorem 2.4.5.** Let \( G' = \text{SL}_3(\mathbb{R}) \) and \( G_\infty \) is the group of \( 3 \times 3 \) upper triangular unipotent matrices. Then the involution 2.17 induces the bijection of \( G_\infty \setminus G' \) into the set of all \( (A_1, B_1, C_1, A_2, B_2, C_2) \in \mathbb{R}^6 \) such that 2.18 is satisfied. Furthermore, the given orbit of \( G_\infty \setminus G' \) contains an element of \( \Gamma_3 \) if and only if \( A_1, B_1, C_1 \) are coprime integers and also \( A_2, B_2, C_2 \) are coprime integers.

**Proof.** The map is defined as follows: the element

\[
\gamma = \begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3}
\end{bmatrix} \in G_\infty \setminus G' \quad (2.19)
\]

goes to \( (A_1, B_1, C_1, A_2, B_2, C_2) \), where

\[
A_1 = -a_{3,1}, \quad B_1 = -a_{3,2}, \quad C_1 = -a_{3,3} \quad (2.20)
\]

\[
A_2 = a_{2,1}a_{3,2} - a_{2,2}a_{3,1}, \quad B_2 = a_{2,3}a_{3,1} - a_{2,1}a_{3,3}, \quad C_2 = a_{2,2}a_{3,3} - a_{2,3}a_{3,2} \quad (2.21)
\]

We need to show that the given map is bijective. To prove the injectivity, we
show that if there are two matrices
\[
\begin{bmatrix}
    a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
    b_{1,1} & b_{1,2} & b_{1,3} \\
b_{2,1} & b_{2,2} & b_{2,3} \\
b_{3,1} & b_{3,2} & b_{3,3}
\end{bmatrix}
\]
with the same coordinates \((A_1, B_1, C_1, A_2, B_2, C_2)\), then there exist \(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\) such that
\[
\begin{bmatrix}
    1 & \lambda_2 & \lambda_3 \\
    0 & 1 & \lambda_1 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3}
\end{bmatrix}
= \begin{bmatrix}
    b_{1,1} & b_{1,2} & b_{1,3} \\
b_{2,1} & b_{2,2} & b_{2,3} \\
b_{3,1} & b_{3,2} & b_{3,3}
\end{bmatrix}.
\tag{2.22}
\]
The first step is to show that there exist \(\lambda_1 \in \mathbb{R}\) such that
\[
\begin{bmatrix}
    0 & 1 & \lambda_1 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3}
\end{bmatrix}
= \begin{bmatrix}
    b_{2,1} & b_{2,2} & b_{2,3} \\
b_{3,1} & b_{3,2} & b_{3,3}
\end{bmatrix}.
\tag{2.23}
\]
We have
\[
a_{3,1} = -A_1 = b_{3,1},
\]
\[
a_{3,2} = -B_1 = b_{3,2},
\]
\[
a_{3,3} = -C_1 = b_{3,3}
\]
and
\[
a_{2,2}a_{3,1} - a_{2,1}a_{3,2} = A_2 = b_{2,2}b_{3,1} - b_{2,1}b_{3,2},
\]
\[
a_{2,3}a_{3,1} - a_{2,1}a_{3,3} = B_2 = b_{2,3}b_{3,1} - b_{2,1}b_{3,3},
\]
\[
a_{2,2}a_{3,3} - a_{2,3}a_{3,2} = C_2 = b_{2,2}b_{3,3} - b_{2,3}b_{3,2}.
\]
Therefore,
\[
a_{3,1}(a_{2,2} - b_{2,2}) = a_{3,2}(a_{2,1} - b_{2,1}),
\]
\[
a_{3,2}(a_{2,3} - b_{2,3}) = a_{3,3}(a_{2,2} - b_{2,2}),
\]
\[
a_{3,3}(a_{2,1} - b_{2,1}) = a_{3,1}(a_{2,3} - b_{2,3}).
\]
Note that \(a_{3,1}, a_{3,2}, a_{3,3}\), are not all zeros. Without loss of generality, assume
$a_{3,1} \neq 0$. Let us take
\[ \lambda_1 = \frac{b_{2,1} - a_{2,1}}{a_{3,1}}. \]

Then 2.23 is satisfied, as required. Now we need to find $\lambda_2, \lambda_3$ such that 2.22 is true. The values $A_2, B_2, C_2$ are not all zeros. Suppose, for instance, $A_2 = a_{2,2}a_{3,1} - a_{2,1}a_{3,2} \neq 0$. Then the vectors $(a_{2,1}, a_{2,2})$ and $(a_{3,1}, a_{3,2})$ are linearly independent, so there are $\lambda_2, \lambda_3$ such that
\[ \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} = \begin{bmatrix} b_{1,1} \\ b_{1,2} \end{bmatrix}. \tag{2.24} \]

Let us compute the determinants of matrices in 2.22
\[
-b_{1,3}A_2 - b_{1,2}B_2 - b_{1,1}C_2
= -(a_{1,3} + \lambda_2 a_{2,3} + \lambda_3 a_{2,3})A_2 - (a_{1,2} + \lambda_2 a_{2,2} + \lambda_3 a_{3,2})B_2 - (a_{1,1} + \lambda_2 a_{2,1} + \lambda_3 a_{3,1})C_2.
\]
By, 2.24, we have
\[
-(a_{1,3} + \lambda_2 a_{2,3} + \lambda_3 a_{2,3})A_2 - (a_{1,2} + \lambda_2 a_{2,2} + \lambda_3 a_{3,2})B_2 - (a_{1,1} + \lambda_2 a_{2,1} + \lambda_3 a_{3,1})C_2
= -(a_{1,3} + \lambda_2 a_{2,3} + \lambda_3 a_{2,3})A_2 - b_{1,2}B_2 - b_{1,1}C_2.
\]
So that
\[ b_{1,3} = a_{1,3} + \lambda_2 a_{2,3} + \lambda_3 a_{2,3} \]
and 2.22 follows.

The next step is to show surjectivity. Suppose we are given
\[(A_1, B_1, C_1) \neq (0, 0, 0)\]
and
\[(A_2, B_2, C_2) \neq (0, 0, 0)\]
such that 2.18 is satisfied. We may find $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ such that
\[ A_1X_1 + B_1Y_1 + C_1Z_1 = A_2X_2 + B_2Y_2 + C_2Z_2 = 1. \] (2.25)
Let
\[ a_{1,1} = -Z_2, \quad a_{1,2} = -Y_2, \quad a_{1,3} = -X_2, \]
\[ a_{2,1} = Y_1A_2 - Z_1B_2, \quad a_{2,2} = Z_1C_2 - X_1A_2, \quad a_{2,3} = X_1B_2 - Y_1C_2, \]
\[ a_{3,1} = -A_1, \quad a_{3,2} = -B_1, \quad a_{3,3} = -C_1. \]
Using 2.18, one can verify relations 2.20 and 2.21. Likewise, the determinant of $\gamma$ (given by 2.19) is one. This shows that the map is surjective.

The last thing to prove is the characterization of orbits, which contain integer matrices. If 2.19 is an integer matrix and $(A_1, B_1, C_1, A_2, B_2, C_2)$ are given by 2.20, 2.21, then $A_1, B_1, C_1$ have to be coprime since the determinant of 2.19 is equal to 1. The values $A_2, B_2, C_2$ are also coprime since the determinant
\[-a_{1,3}A_2 - a_{1,2}B_2 - a_{1,1}C_2 = 1.\]
Conversely, let $A_1, B_1, C_1$ be coprime integers and $A_2, B_2, C_2$ are also coprime integers such that 2.18 is satisfied. To show that the coset parametrized by this invariants contains an integer matrix, we may find integer values of $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ satisfying 2.25. Then the matrix 2.19 can be constructed as in the proof of surjectivity. Clearly, all entries of this matrix are integral. 

Remark 2.4.6. The theorem above also gives the characterization of the orbits of $\Gamma_{3,\infty} \setminus \Gamma_3$ in terms of their Plucker coordinates since $\Gamma_{3,\infty} \setminus \Gamma_3$ is included injectively in $G_\infty \setminus G'$.

Consider the element
\[ \gamma = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ A_1 & B_1 & C_1 \end{bmatrix} \in \Gamma_{3,\infty} \setminus \Gamma_3 \]
with Plucker coordinates $\rho_1 = \{A_1, B_1, C_1\}$ and $\rho_2 = \{A_2, B_2, C_2\}$. Below we determine explicitly Bruhat decomposition of $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ depending on its Plucker coordinates.

**Proposition 2.4.7.** If $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ have coordinates $A_1 = A_2 = B_1 = B_2 = 0$, $C_1, C_2 \neq 0$, then

$$
\gamma = \begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
0 & 0 & C_1
\end{bmatrix}
= \begin{bmatrix}
a_{1,1} & 0 & 0 \\
0 & C_2 & 0 \\
0 & 0 & C_1
\end{bmatrix}
\begin{bmatrix}
1 & \frac{a_{1,2}}{a_{1,1}} & \frac{a_{1,3}}{a_{1,1}} \\
0 & \frac{1}{C_2} & 0 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & \frac{a_{1,2}B_1}{B_2} & \frac{a_{1,3}B_1}{B_2} \\
0 & \frac{1}{B_2} & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \frac{a_{2,2}B_1}{A_2} & \frac{a_{2,3}B_1}{A_2} \\
0 & \frac{1}{A_2} & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$

**Proposition 2.4.8.** If $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ have coordinates $A_1 = A_2 = B_1 = 0$, $C_1, B_2 \neq 0$, then

$$
\gamma = \begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
0 & 0 & C_1
\end{bmatrix}
= \begin{bmatrix}
1 & -\frac{a_{1,2}C_1}{B_2} & 0 \\
0 & 1 & \frac{a_{2,3}}{C_1} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a_{1,1} & 0 & 0 \\
0 & B_1 & 0 \\
0 & 0 & \frac{1}{a_{1,1}B_1}
\end{bmatrix}
= \begin{bmatrix}
1 & \frac{a_{1,2}B_1}{B_2} & \frac{a_{1,3}B_1}{B_2} \\
0 & \frac{1}{B_2} & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \frac{a_{2,2}B_1}{A_2} & \frac{a_{2,3}B_1}{A_2} \\
0 & \frac{1}{A_2} & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$

**Proposition 2.4.9.** If $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ have coordinates $A_1 = A_2 = B_2 = 0$, $B_1, C_2 \neq 0$, then

$$
\gamma = \begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
0 & B_1 & C_1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & \frac{a_{2,3}}{B_1} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & B_1 & 0 \\
0 & 0 & \frac{1}{a_{1,1}B_1}
\end{bmatrix}
= \begin{bmatrix}
1 & \frac{a_{1,2}B_1}{B_2} & \frac{a_{1,3}B_1}{B_2} \\
0 & \frac{1}{B_2} & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \frac{a_{2,2}B_1}{A_2} & \frac{a_{2,3}B_1}{A_2} \\
0 & \frac{1}{A_2} & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$

**Proposition 2.4.10.** If $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ have coordinates $A_1 = 0$, $B_1, A_2 \neq 0$, then

$$
\gamma = \begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
0 & B_1 & C_1
\end{bmatrix}
= \begin{bmatrix}
1 & a_{1,1}B_1 & -b_{1,1} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
A_2 & 0 \\
B_1 & 0 \\
0 & \frac{1}{A_2}
\end{bmatrix}
= \begin{bmatrix}
1 & \frac{a_{2,2}B_1}{A_2} & \frac{a_{2,3}B_1}{A_2} \\
0 & \frac{1}{A_2} & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \frac{a_{2,2}B_1}{A_2} & \frac{a_{2,3}B_1}{A_2} \\
0 & \frac{1}{A_2} & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$

**Proposition 2.4.11.** If $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ have coordinates $A_2 = 0$, $A_1, B_2 \neq 0$, then

$$
\gamma = \begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
A_1 & B_1 & C_1
\end{bmatrix}
= \begin{bmatrix}
1 & a_{1,1} & 0 \\
0 & 1 & \frac{a_{2,2}}{A_1} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
A_1 & 0 & 0 \\
0 & B_1 & 0 \\
0 & 0 & \frac{B_2}{A_1}
\end{bmatrix}
= \begin{bmatrix}
1 & B_1 & C_1 \\
0 & \frac{1}{B_2} & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$
where $b_{2,2} = a_{1,1}C_1 - a_{1,3}A_1$.

**Proposition 2.4.12.** If $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ have coordinates $A_1, A_2 \neq 0$, then
\[
\gamma = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ A_1 & B_1 & C_1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{b_1}{A_2} & \frac{a_{1,1}}{A_1} \\ 0 & 1 & \frac{a_{2,1}}{A_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ A_1 & 0 & 0 \\ 0 & 0 & \frac{1}{A_1} \end{bmatrix} \begin{bmatrix} 1 & \frac{B_1}{A_1} & \frac{C_1}{A_1} \\ 0 & 1 & -\frac{b_2}{A_2} \\ 0 & 0 & 1 \end{bmatrix},
\]
where $b_{2,1} = a_{1,2}A_1 - a_{1,1}B_1$.

All the propositions above can be verified by direct computation, i.e. multiplying matrices on right-hand side and taking into account 2.4.3, one obtains the result.

**Definition 2.4.13.** Let us define the following group
\[
\Gamma_w = (w^{-1}\Gamma_{3,\infty}w)^t \cap \Gamma_{3,\infty}.
\]
Explicitly,
\[
\Gamma_{w_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ with } w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},
\]
\[
\Gamma_{w_2} = \left\{ \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, m \in \mathbb{Z} \right\} \text{ with } w_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]
\[
\Gamma_{w_3} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}, n \in \mathbb{Z} \right\} \text{ with } w_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},
\]
\[
\Gamma_{w_4} = \left\{ \begin{bmatrix} 1 & 0 & l \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}, n, l \in \mathbb{Z} \right\} \text{ with } w_4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
\]
\[
\Gamma_{w_5} = \left\{ \begin{bmatrix} 1 & m & l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, m, l \in \mathbb{Z} \right\} \text{ with } w_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.
\]
\[ \Gamma_{w_6} = \left\{ \begin{bmatrix} 1 & m & l \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}, \ m, l, n \in \mathbb{Z} \right\} \quad \text{with } w_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \]

**Remark 2.4.14.** Given results could be checked by direct computation. Let us consider, for example, the case \( w = w_5 \). Take an arbitrary matrix

\[ g = \begin{bmatrix} 1 & n & m \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix} \in \Gamma_{3,\infty}, \]

where \( m, l, n \in \mathbb{Z} \). Then

\[ (w^{-1}\Gamma_{3,\infty}w)^t = \begin{bmatrix} 1 & m & l \\ 0 & 1 & 0 \\ 0 & n & 1 \end{bmatrix}. \]

Intersecting the set of such matrices with \( \Gamma_{3,\infty} \), we obtain the required result.

**Proposition 2.4.15.** The group \( \Gamma_w \) acts properly on the right on \( \Gamma_{3,\infty} \setminus \Gamma_3 \cap G_w / U \), where

\[ U = \left\{ \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}, \ \epsilon_i = \pm 1, \ \epsilon_1\epsilon_2\epsilon_3 = 1 \right\} \]

and \( G_w \) is as in 2.16. Thus, \( \Gamma_{3,\infty} \setminus \Gamma_3 \cap G_w / U \Gamma_w \) is a well-defined double coset space.

**Proof.** Note that

\[ \begin{bmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ A_1 & B_1 & C_1 \end{bmatrix} \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} = \begin{bmatrix} \epsilon_1 a_{2,1} & \epsilon_2 a_{2,2} & \epsilon_3 a_{2,3} \\ \epsilon_1 A_1 & \epsilon_2 B_1 & \epsilon_3 C_1 \end{bmatrix}. \]

So that

\[ A_1 \rightarrow \epsilon_1 A_1, \ B_1 \rightarrow \epsilon_2 B_1, \ C_1 \rightarrow \epsilon_3 C_1 \] (2.26)
2. $\text{SL}_3(\mathbb{Z})$ Kloosterman sums

\[ A_2 \rightarrow \epsilon_1 \epsilon_2 A_2, \quad B_2 \rightarrow \epsilon_1 \epsilon_3 B_2, \quad C_2 \rightarrow \epsilon_2 \epsilon_3 C_2. \quad (2.27) \]

Therefore, the representatives of $\Gamma_{3,\infty} \setminus \Gamma_3 \cap G_w(\text{mod } U)$ can be obtained by fixing two signs of non-zero invariants.

Right multiplication by $\Gamma_w$ maps left cosets to left cosets, so $\Gamma_w$ acts on $\Gamma_{3,\infty} \setminus \Gamma_3 \cap G_w/U$.

We need to show that the action is proper, i.e. if $\gamma \in \Gamma_3 \cap G_w$, $\tau \in \Gamma_w$ and

\[ \Gamma_{3,\infty} \gamma \tau U = \Gamma_{3,\infty} \gamma U, \]

then $\tau = \text{id}$. In order to prove this fact we introduce two new sets:

\[ H_1 = w^{-1}G_{\infty}w \cap G_{\infty} \]

and

\[ H_2 = w^{-1}G^t_{\infty} \cap G_{\infty}, \]

where $w \in W_3$. Explicit matrix computation for elements $w_i \in W_3$, $i = 1, 2, \ldots, 6$ shows that every $g \in G_{\infty}$ has unique expressions

\[ g = h_1 h_2 \quad (2.28) \]

and

\[ g = h'_2 h'_1 \quad (2.29) \]

with $h_1, h'_1 \in H_1$, $h_2, h'_2 \in H_2$. More precisely,

\[
H_1 = \begin{bmatrix}
1 & * & * \\
0 & 1 & * \\
0 & 0 & 1
\end{bmatrix}, \quad H_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \text{ if } w = w_1,
\]

\[
H_1 = \begin{bmatrix}
1 & 0 & * \\
0 & 1 & * \\
0 & 0 & 1
\end{bmatrix}, \quad H_2 = \begin{bmatrix}
1 & * & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \text{ if } w = w_2,
\]
2. \( \text{SL}_3(\mathbb{Z}) \) Kloosterman sums

\[
H_1 = \begin{bmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}
\]
if \( w = w_3 \),

\[
H_1 = \begin{bmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
if \( w = w_4 \),

\[
H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
if \( w = w_5 \),

\[
H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}
\]
if \( w = w_6 \).

By Bruhat decomposition, \( \gamma = b_1 w d b_2 \) with \( b_1, b_2 \in G_\infty \), \( d \in D \) and \( w \in W_3 \). According to 2.28 and 2.29, without loss of generality, we may assume that \( b_2 \in H_1 \). Since

\[
\Gamma_{3,\infty} \gamma U = \Gamma_{3,\infty} \gamma U,
\]

we conclude that

\[
b_2 \tau b_2^{-1} \in H_1 \cap H_2 = \{I\}
\]

and \( \tau = id \) as required.

Finally, for every \( w \in W_3 \) we determine a canonical set of coset representatives \( R_w \) for the quotient space \( \Gamma_{3,\infty} \backslash \Gamma_3 \cap G_w / U \Gamma_w \). We give a proof in case \( w = w_2 \) as an example.

**Proposition 2.4.16.** If \( w = w_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), then

\[
R_w = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.
\]
Proposition 2.4.17. If $w = w_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then

$$R_w = \begin{cases} \begin{bmatrix} a_{1,1} & a_{1,2} & 0 \\ -B_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{if } (B_2, C_2) = 1, B_2 > 0, C_2 \pmod{B_2} \text{ and values } a_{1,1}, a_{1,2} \text{ are chosen uniquely such that } a_{1,1}C_2 + a_{1,2}B_2 = 1 \text{ for each pair } (B_2, C_2). \end{cases}$$

Proof. By proposition 2.4.8, $\Gamma_{3,\infty} \setminus (\Gamma_3 \cap G_w)$ have coordinates $A_1 = A_2 = B_1 = 0$, $C_1, B_2 \neq 0$. By 2.26, 2.27, we can obtain a representative of $\Gamma_{3,\infty} \setminus (\Gamma_3 \cap G_w) \pmod{U}$ by fixing the signs of $C_1, B_2$. Let $C_1, B_2 > 0$. Furthermore, by theorem 2.4.5, Plucker coordinates $B_2 = -a_{2,1}C_1$ and $C_2 = a_{2,2}C_1$ are coprime integers, so that $C_1 = 1$. Consequently, $a_{2,2} = C_2$ and $a_{2,1} = -B_2$. Consider

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ 0 & 0 & C_1 \end{bmatrix} \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{1,1} & ma_{1,1} + a_{1,2} & a_{1,3} \\ a_{2,1} & ma_{2,1} + a_{2,2} & a_{2,3} \\ 0 & 0 & C_1 \end{bmatrix}.$$ 

Thus, to obtain the coset representative modulo $\Gamma_w$, we need to consider $a_{2,2} \pmod{a_{2,1}}$, equivalently $C_2 \pmod{B_2}$. The determinant of obtained matrix

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ -B_2 & C_2 & a_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$$

must be equal to one. Whence, variables $a_{1,1}, a_{1,2}$ are chosen uniquely such that $a_{1,1}C_2 + a_{1,2}B_2 = 1$ for each pair $(B_2, C_2)$. Nor determinant, nor Plucker coordinates depend on the values of $a_{1,3}, a_{2,3}$, so we can let $a_{1,3} = a_{2,3} = 0$. The proposition follows. \[\Box\]
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Proposition 2.4.18. If \( w = w_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \), then

\[
R_w = \begin{cases} 
1 & 0 & 0 \\
0 & \alpha_{2,2} & \alpha_{2,3} \\
0 & B_1 & C_1 
\end{cases}
\]

where \((B_1, C_1) = 1, B_1 > 0, C_1(\text{mod } B_1)\) and values \(\alpha_{2,2}, \alpha_{2,3}\) are chosen uniquely such that \(\alpha_{2,2}C_1 + \alpha_{2,3}B_1 = 1\) for each pair \((B_1, C_1)\).

Proposition 2.4.19. If \( w = w_4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \), then

\[
R_w = \begin{cases} 
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\
\frac{\alpha_2}{B_1} & \alpha C_2 & \beta C_2 \\
0 & B_1 & C_1 
\end{cases}
\]

where \((B_1, C_1) = 1, B_1 > 0, C_1(\text{mod } B_1), \frac{\alpha_2}{B_1}, C_2 = 1, A_2 > 0, C_2(\text{mod } A_2), B_1B_2 + C_1A_2 = 0\) and values \(\alpha, \beta\) are chosen uniquely such that \(\alpha C_2 - \beta B_2 = 1\) for each pair \((B_1, C_1)\). The values \(\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}\) are chosen uniquely such that the matrix has determinant one for every quintuple \((B_1, C_1, A_2, B_2, C_2)\).

Proposition 2.4.20. If \( w = w_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \), then

\[
R_w = \begin{cases} 
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\
\frac{\alpha_2}{B_1} & \frac{\alpha_2B_1}{A_1} & \alpha_{1,3} \\
A_1 & B_1 & C_1 
\end{cases}
\]

where \((B_2, C_2) = 1, B_2 > 0, C_2(\text{mod } B_2), \frac{\alpha_1}{B_1}, C_1 = 1, A_1 > 0, C_1(\text{mod } A_1), A_1C_2 + B_1B_2 = 0\) and the values \(\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,1}, \alpha_{2,3}\) are chosen uniquely such that the matrix has determinant one for every quintuple \((A_1, B_1, C_1, B_2, C_2)\).
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Proposition 2.4.21. If \( w = w_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \), then

\[
R_w = \begin{cases} \\
\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ A_1 & B_1 & C_1 \end{bmatrix}, \\
\end{cases}
\]

where \( A_1, A_2 > 0, B_1, C_1 \pmod{A_1}, B_2, C_2 \pmod{A_2}, (A_1, B_1, C_1) = (A_2, B_2, C_2) = 1, A_1C_2 + B_1B_2 + A_2C_1 = 0 \) and for every sextuple \((A_1, B_1, C_1, A_2, B_2, C_2)\) the values \(a_{2,1}, a_{2,2}, a_{2,3}\) are uniquely chosen such that

\[
A_2 = a_{2,1}B_1 - a_{2,2}A_1, \quad B_2 = a_{2,3}A_1 - a_{2,1}C_1, \quad C_2 = a_{2,2}C_1 - a_{2,3}B_1
\]

and the rest of values \(a_{1,1}, a_{1,2}, a_{1,3}\) are chosen uniquely such that the matrix has determinant one.

2.5 SL(3) Kloosterman sums

There are six Kloosterman sums that occur in the Fourier expansion of Poincaré series (2.14). Let \( m_1, m_2, n_1, n_2 \in \mathbb{Z}, D_1, D_2 \in \mathbb{Z}_{>0}, S(m, n, c)\) be a classical Kloosterman sum and \(W_3\) denote a Weyl group of permutation matrices of \(GL_3(\mathbb{Z})\). By definition,

\[
\delta_{a,b} = \begin{cases} \\
1 & a = b \\
0 & a \neq b \\
\end{cases}
\]

For each \( w_i \in W_3 \) \((i = 1, \ldots, 6)\) we associate a certain Kloosterman sum \(S_{w_i}\). Namely,

\[
S_{w_1} = \delta_{D_1,1}\delta_{D_2,1} \text{ with } w_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]
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\[
S_{w_2} = \delta_{D_1,1} S(m_2, n_2; D_2) \quad \text{with} \quad w_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

\[
S_{w_3} = \delta_{D_2,1} S(m_1, n_1; D_1) \quad \text{with} \quad w_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
\]

The remaining cases, corresponding to the elements

\[
w_4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad w_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad w_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},
\]

are the most interesting: these are exponential sums different from classical Kloosterman sums. Below we define two new types of Kloosterman sums \( K_1 = K_1(m_1, m_2, n_1, n_2; D_1, D_2) \) and \( K_2 = K_2(m_1, n_1, n_2; D_1, D_2) \) so that

\[
S_{w_4} = K_2(m_1, n_1, n_2; D_1, D_2),
\]

\[
S_{w_5} = K_2(m_2, n_2, n_1; D_2, D_1),
\]

\[
S_{w_6} = K_1(m_2, m_1, n_1, n_2; D_2, D_1).
\]

**Definition 2.5.1.** First type \( K_1 = K_1(m_1, m_2, n_1, n_2; D_1, D_2) \) of \( SL_3(\mathbb{Z}) \) Kloosterman sum is

\[
K_1 = \sum_{B_1 \pmod{D_1} \atop C_1 \pmod{D_1}} \sum_{B_2 \pmod{D_2} \atop C_2 \pmod{D_2}} e \left( \frac{m_1 B_1 + n_1 (Y_1 D_2 - Z_1 B_2)}{D_1} + \frac{m_2 B_2 + n_2 (Y_2 D_1 - Z_2 B_2)}{D_2} \right)
\]

where the inner sum satisfies the following conditions

\[
(D_1, B_1, C_1) = 1, \quad (D_2, B_2, C_2) = 1
\]

and

\[
D_1 C_2 + B_1 B_2 + C_1 D_2 \equiv 0 \pmod{D_1 D_2}.
\]
Variables $Y_1, Y_2, Z_1, Z_2$ are chosen such that

$$Y_1 B_1 + Z_1 C_1 \equiv 1 \mod D_1,$$

$$Y_2 B_2 + Z_2 C_2 \equiv 1 \mod D_2.$$

**Lemma 2.5.2.** The sum (2.30) is well-defined, i.e. it is independent of the choice of $Y_1, Y_2, Z_1, Z_2$ and it does not depend on the choice of representatives $B_1, C_1$ and $B_2, C_2$ of the residue classes modulo $D_1$ and $D_2$.

**Proof.** 1. First, we show the independence of the choice of $Y_1, Z_1$ and $(Y_2, Z_2)$, i.e. if $(D_1, B_1, C_1) = 1, D_1 \neq 0,$

$$D_1 C_2 + B_1 B_2 + C_1 D_2 \equiv 0 \mod (D_1 D_2),$$

and if

$$X_1 D_1 + Y_1 B_1 + Z_1 C_1 = X_1' D_1 + Y_1' B_1 + Z_1' C_1,$$

then

$$\frac{Y_1 D_2 - Z_1 B_2}{D_1} \equiv \frac{Y_1' D_2 - Z_1' B_2}{D_1} \mod 1.$$ 

We may assume that

$$D_1 C_2 + B_1 B_2 + C_1 D_2 = 0$$

by changing the value of $C_2$. Then both vectors $(C_2, B_2, D_2)$ and $(\alpha, \beta, \gamma) = (X_1 - X_1', Y_1 - Y_1', Z_1 - Z_1')$ are orthogonal to $(D_1, B_1, C_1)$. Thus, the vector cross product of $(C_2, B_2, D_2)$ and $(\alpha, \beta, \gamma)$ is parallel to $(D_1, B_1, C_1)$. So there is $\lambda \in \mathbb{Q}$ such that

$$(\beta D_2 - \gamma B_2, \gamma C_2 - \alpha D_2, \alpha B_2 - \beta C_2) = \lambda (D_1, B_1, C_1).$$

Since $(D_1, B_1, C_1) = 1$, we deduce that $\lambda \in \mathbb{Z}$ and

$$\frac{Y_1 D_2 - Z_1 B_2}{D_1} = \lambda + \frac{Y_1' D_2 - Z_1' B_2}{D_1}$$

as required. \qed
2. Let us denote the inner sum in 2.30 by

\[ S_{B_1, B_2}(m_1, m_2, n_1, n_2, D_1, D_2) \]

\[ = \sum_{C_1 \pmod{D_1}, C_2 \pmod{D_2}} e \left( \frac{m_1 B_1 + n_1 (Y_1 D_2 - Z_1 B_2)}{D_1} + \frac{m_2 B_2 + n_2 (Y_2 D_1 - Z_2 B_2)}{D_2} \right), \]

where

\[ (D_1, B_1, C_1) = 1, (D_2, B_2, C_2) = 1 \]

and

\[ D_1 C_2 + B_1 B_2 + C_1 D_2 \equiv 0 \pmod{D_1 D_2}. \]

We claim that the given sum depends only on the residue classes of \( B_1 \pmod{D_1} \) (respectively \( B_2 \pmod{D_2} \)):

if \( B_1' = B_1 + \lambda D_1 \), then

\[ S_{B_1, B_2}(m_1, m_2, n_1, n_2, D_1, D_2) = S_{B_1', B_2}(m_1, m_2, n_1, n_2, D_1, D_2). \]

Let \( C_2' = C_2 - \lambda B_2 \), so that

\[ (D_1, B_1', C_1) = (D_2, B_2, C_2') = 1 \]

and

\[ D_1 C_2' + B_1 B_2' + C_1 D_2 \equiv 0 \pmod{D_1 D_2}. \]

We deduce that

\[ Y_1 B_1' + Z_1 C_1 \equiv 1 \pmod{D_1}, \]

\[ Y_2 B_2' + Z_2 C_2' \equiv 1 \pmod{D_2} \]

with \( Y_2' = Y_2 + \lambda Z_2 \). Then

\[ Y_2' D_1 - Z_2 B_1' = Y_2 D_1 - Z_2 B_1. \]

Finally, summing over all \( C_1 \) and \( C_2 \) for the first sum and over all \( C_1 \) and
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\[ C'_2 \text{ for the second sum, we obtain the result} \]
\[ S_{B'_1,B'_2}(m_1,m_2,n_1,n_2,D_1,D_2) = S_{B'_1,B'_2}(m_1,m_2,n_1,n_2,D_1,D_2). \]

\[ \square \]

\textbf{Definition 2.5.3.} Suppose \( D_1 \mid D_2 \). Then the second type of \( \textbf{SL}_3(\mathbb{Z}) \) Kloosterman sums is defined as follows

\[ K'_2 = K_2(m_1,n_1,n_2;D_1,D_2), \]
\[ K_2 = \sum_{\substack{C_1 \equiv 1 (mod \ D_1) \\ C_2 \equiv 1 (mod \ D_2) \atop (C_1,D_1)=(C_2,D_2/D_1)=1}} e \left( \frac{m_1 C_1 + n_1 C_2 C_1^*}{D_1} + \frac{n_2 C_2^*}{D_2/D_1} \right). \quad (2.31) \]

Variables \( C_1^*, C_2^* \) are chosen so that

\[ C_1 C_1^* \equiv 1 (mod \ D_1) \]

and

\[ C_2 C_2^* \equiv 1 (mod \ D_2/D_1). \]

\textbf{Remark 2.5.4.} The sum (2.31) is well-defined, i.e. it is independent of the choice of \( C_1^*, C_2^* \) and it does not depend on the choice of representatives \( C_1, C_2 \) of the residue classes modulo \( D_1 \) and \( D_2 \).

\section{2.6 Some properties of Kloosterman sums}

New types of Kloosterman sums have properties similar to the classical case. Let us list some of them.

\textbf{Proposition 2.6.1.}

\[ S(m_1,m_2,n_1,n_2;D_1,D_2) = S(n_1,n_2,m_1,m_2;D_1,D_2). \]
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Proof. Given $(D_1, B_1, C_1) = (D_2, B_2, C_2) = 1$ such that
\[ D_1C_2 + B_1B_2 + C_1D_2 \equiv 0 \pmod{D_1D_2}, \]
let
\[ X_1D_1 + Y_1B_1 + Z_1C_1 = 1 \]
and
\[ X_2D_2 + Y_2B_2 + Z_2C_2 = 1. \]
Also let
\[ B'_1 = Y_1D_2 - Z_1B_2, \quad B'_2 = Y_2D_1 - Z_2B_1, \]
\[ C'_1 = Z_2, \quad C'_2 = Z_1, \]
\[ Y'_1 = X_2B_1 - Y_2C_1, \quad Y'_2 = X_1B_2 - Y_1C_2, \]
\[ Z'_1 = C_2, \quad C'_2 = Z_1. \]
Thus,
\[ Y'_1B'_1 + Z'_1C'_1 \equiv Y_1B_1 + Z_1C_1 + D_1C_2(X_1Z_2 + Y_1Y_2 + Z_1X_2) \pmod{D_1D_2}. \]
So that
\[ Y'_1B'_1 + Z'_1C'_1 \equiv 1 \pmod{D_1} \]
and similarly
\[ Y'_2B'_2 + Z'_2C'_2 \equiv 1 \pmod{D_2}. \]
Besides,
\[ D_1C'_2 + B'_1B'_2 + C'_1 \equiv D_1D_2(X_1Z_2 + Y_1Y_2 + Z_1X_2) \equiv 0 \pmod{D_1D_2} \]
and
\[ Y'_1D_2 - Z'_1B'_2 \equiv B_1(X_2D_2 + Y_2B_2 + Z_2C_2) \equiv B_1 \pmod{D_1D_2}. \]
In an analogous manner,
\[ Y'_2D_1 - Z'_2B'_1 \equiv B_2 \pmod{D_1D_2}. \]
Finally,
\[
e \left( \frac{m_1 B_1 + n_1 (Y_1 D_2 - Z_1 B_2)}{D_1} + \frac{m_2 B_2 + n_2 (Y_2 D_2 - Z_2 B_1)}{D_2} \right)
= e \left( \frac{n_1 B_1' + m_1 (Y_1' D_2 - Z_1' B_2)}{D_1} + \frac{n_2 B_2' + m_2 (Y_2' D_1 - Z_2' B_1)}{D_2} \right).
\]

Summing, we obtain the result. □

**Proposition 2.6.2.**

\[
S(m_1, m_2, n_1, n_2; D_1, D_2) = S(m_2, m_1, n_2, n_1; D_2, D_1).
\]

*Proof.* Follows from the definition. □

**Proposition 2.6.3.** If \( p_1q_1 \equiv p_2q_2 \equiv 1 \mod D_1D_2 \), \( p_1, q_1, p_2, q_2 \in \mathbb{Z} \), then

\[
S(p_1 m_1, p_2 m_2, q_1 n_1, q_2 n_2; D_1, D_2) = S(m_1, m_2, n_1, n_2; D_1, D_2).
\]

*Proof.* Suppose we are given \((D_1, B_1, C_1) = (D_2, B_2, C_2) = 1\) such that

\[
D_1 C_2 + B_1 B_2 + C_1 D_2 \equiv 0 \mod D_1D_2.
\]

Let \(Y_1 B_1 + Z_1 C_1 \equiv 1 \mod D_1\), \(Y_2 B_2 + Z_2 C_2 \equiv 1 \mod D_2\) and

\[
B_1' = p_1 B_1, \quad B_2' = p_2 B_2,
\]

\[
C_1' = p_1 p_2 C_1, \quad C_2' = p_1 p_2 C_2,
\]

\[
Y_1' = q_1 Y_1, \quad Y_2' = q_2 Y_2,
\]

\[
Z_1' = q_1 q_2 Z_1, \quad Z_2' = q_1 q_2 Z_2.
\]

Then \((D_1, B_1', C_1') = (D_2, B_2', C_2') = 1\),

\[
D_1 C_2' + B_1' B_2' + C_1' D_2 \equiv 0 \mod D_1D_2,
\]

\[
Y_1' B_1' + Z_1' C_1' \equiv 1 \mod D_1, \quad Y_2' B_2' + Z_2' C_2' \equiv 1 \mod D_2.
\]
So we have
\[
e \left( \frac{p_1 m_1 B_1 + q_1 n_1 (Y_1 D_2 - Z_1 B_2)}{D_1} + \frac{p_2 m_2 B_2 + q_2 n_2 (Y_2 D_1 - Z_2 B_1)}{D_2} \right) \\
= e \left( \frac{m_1 B_1' + n_1 (Y'_1 D_2 - Z'_1 B_2)}{D_1} + \frac{m_2 B_2' + n_2 (Y'_2 D_1 - Z'_2 B_1)}{D_2} \right).
\]
Summing, we obtain the result.

\[\Box\]

**Proposition 2.6.4.** *(twisted multiplicativity)* If \((D_1 D_2, D'_1 D'_2) = 1\) and if
\[
D_1' D_1 \equiv D'_1 D_2 \equiv 1(\text{mod } D'_1 D'_2),
\]
\[
\overline{D_1'} D'_1 \equiv \overline{D_2'} D'_2 \equiv 1(\text{mod } D_1 D_2),
\]
then
\[
S(m_1, m_2, n_1, n_2; D_1, D_2, D'_1, D'_2) \\
= S(\overline{D_1'} D_2 m_1, \overline{D_2'} D'_1 m_2, n_1, n_2; D_1, D_2) S(D_1' \overline{D_2} m_1, D_2' \overline{D_1} m_2, n_1, n_2; D'_1, D'_2).
\]

**Proof.** Let \(p, p'\) be such that
\[
p D_1 D_2 + p' D'_1 D'_2 = 1.
\]
Given \((D_1, B_1, C_1) = (D_2, B_2, C_2) = 1\) such that
\[
D_1 C_2 + B_1 B_2 + C_1 D_2 \equiv 0(\text{mod } D_1 D_2)
\]
and \((D'_1, B'_1, C'_1) = (D'_2, B'_2, C'_2) = 1\) such that
\[
D'_1 C'_2 + B'_1 B'_2 + C'_1 D'_2 \equiv 0(\text{mod } D'_1 D'_2),
\]
let
\[
d_1 = D_1 D'_1, \ d_2 = D_2 D'_2,
\]
\[
b_1 = p' D'_1 D'_2 B_1 + p D_1 D_2 B'_1, \ b_2 = p' D'_1 D'_2 B_2 + p D_1 D_2 B'_2,
\]
\[
c_1 = p' D'_1 \overline{D_2} C_1 + p D_2 \overline{D_1} C'_1, \ c_2 = p' D'_1 \overline{D_2} C_2 + p D_2 \overline{D_1} C'_2.
\]
Then
\[(d_1, b_1, c_1) = (d_2, b_2, c_2) = 1,\]
\[d_1 c_2 + b_1 b_2 + c_1 d_2 \equiv 0 (\mod d_1 d_2).\]

Let
\[y_1 = p D_1' D_2 Y_1 + p D_1 D_2 Y_1', \quad y_2 = p D_1' D_2 Y_2 + p D_1 D_2 Y_2',\]
\[z_1 = p D_1' D_2^2 Z_1 + p D_1 D_2^2 Z_1', \quad z_2 = p D_1'^2 D_2 Z_2 + p D_1^2 D_2 Z_2'.\]

Then
\[y_1 b_1 + z_1 c_1 \equiv 1 (\mod d_1), \quad y_2 b_2 + z_2 c_2 \equiv 1 (\mod d_2),\]
\[
\frac{m_1 b_1 + n_1 (y_1 d_2 - z_1 b_2)}{d_1} = \frac{m_1 p D_1' B_1 + n_1 p D_2^2 (Y_1 D_2 - Z_1 B_2)}{D_1} \\
+ \frac{m_1 p D_2 B_1' + n_1 p D_2^2 (Y_1' D_2') - Z_1' B_2'}{D_1'} (\mod 1).
\]

And the identity for \(d_2\) is similar. Summing, we have
\[S(m_1, m_2, n_1, n_2; d_1, d_2) = S(p D_1' m_1, p D_1' m_2, p D_2^2 n_1, p D_1^2 n_2; D_1, D_2) \times S(p D_2 m_1, p D_1 m_2, p D_2^2 n_1, p D_1^2 n_2; D_1', D_2').\]

But
\[D_1' \equiv p D_2 (\mod D_1' D_2'), \quad D_2' \equiv p D_1 (\mod D_1' D_2'),\]
\[D_1'^2 \equiv p D_1' (\mod D_1 D_2), \quad D_2'^2 \equiv p D_2' (\mod D_1 D_2').\]

Now the result follows from proposition 2.6.3.

\[\square\]

**Proposition 2.6.5.**

\[S(m_1, m_2, n_1, n_2; D_1, 1) = S(m_1, n_1; D_1),\]
\[S(m_1, m_2, n_1, n_2; 1, D_2) = S(m_2, n_2; D_2),\]

where \(S(m, n; D)\) is a classical Kloosterman sum.

**Proof.** Follows from the definition. \(\square\)
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Proposition 2.6.6. If $(D_1, D_2) = 1$, then

$$S(m_1, m_2, n_1, n_2; D_1, D_2) = S(D_2m_1, n_1, D_1)S(D_1m_2, n_2, D_2).$$

Proof. Follows from properties 2.6.4 and 2.6.5. \hfill \Box

Now, let us consider the second type of $\text{SL}_3(\mathbb{Z})$ Kloosterman sum and list some of its properties.

Proposition 2.6.7.

$$S(m_1, n_1, n_2, 1, D_2) = R_{D_2}(n_2),$$

where

$$R_{c}(n) = S(0, n, c) = \sum_{d \equiv n \pmod{c}, \gcd(c,d)=1} e\left(\frac{nd}{c}\right)$$

is a Ramanujan sum.

Proof. Follows from the definition. \hfill \Box

Proposition 2.6.8. (twisted multiplicativity) let $(D_2, D'_2) = 1$, $D_1 | D_2$, $D'_1 | D'_2$. Then

$$S(m_1, n_1, n_2, D_1D'_1, D_2D'_2) = S(m_1D'_1, n_1D'_2, n_2D''_2, D_1, D_2)S(m_1D_1, n_1D_2, n_2D''_2, D'_1, D'_2),$$

where

$$D_1D'_1 \equiv 1(\text{mod } D'_1), \quad D_2D'_2 \equiv 1(\text{mod } D'_2),$$

$$D'_1D'_1 \equiv 1(\text{mod } D_1), \quad D'_2D'_2 \equiv 1(\text{mod } D_2).$$

Proposition 2.6.9. Let $p$ be a prime number. Then for $b > a > 0$

$$S(m_1, n_1, n_2; p^a, p^b) = 0$$

unless $b = 2a$, or $n_2 \equiv 0(\text{mod } p^{b-2a})$ and $b > 2a$, or $n_1 \equiv 0(\text{mod } p^{2a-b})$ and $b < 2a$. 

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Proposition 2.6.10.

\[ S(m_1, n_1, n_2; p^a, p^a) = \begin{cases} p^{2a} - p^{2a-1} & \text{if } p^a | m, p^a | n_1 \\ -p^{2a-1} & \text{if } p^a \not| m, p^a | n_1 \\ 0 & \text{otherwise.} \end{cases} \]

Proposition 2.6.11.

\[ S(m_1, n_1, n_2; D_1, D_2) = 0 \]

unless \( n_1 \frac{D_2}{D_1} \in \mathbb{Z}. \)

**Proof.** Follows from propositions 2.6.8, 2.6.9 and 2.6.10. \qed

Proposition 2.6.12. *(Larsen’s bound)*

\[ |S(m_1, n_1, n_2; D_1, D_2)| \leq \min(\tau(D_1)^\alpha(n_2, D_2/D_1)D_1^2, \tau(D_2)(m_1, n_1, D_1)D_2), \]

where \( \alpha = \frac{\log(3)}{\log(2)} \) and \( \tau(n) = \sum_{d|n} 1. \)

2.7 Fourier expansion of Poincaré series

Let us choose an \( E \)-function as

\[ E_{n_1, n_2}(z) = e(n_1(x_1 + iy_1/M) + n_2(x_2 + iy_2/M)) \text{ with } M \in \mathbb{Z}. \]  \hspace{1cm} (2.32)

Since the function does not depend on \( x_3, \) we write

\[ E_{n_1, n_2}(z) = E_{n_1, n_2}(x_1 + iy_1, x_2 + iy_2). \]  \hspace{1cm} (2.33)

Below we compute the Fourier coefficients of \( \text{SL}_3(\mathbb{Z}) \) Poincaré series for this choice of \( E \)-function.

**Theorem 2.7.1.** Let \( \Re(\nu_1), \Re(\nu_2) > \frac{2}{3}. \) Then

\[ \int_0^1 \int_0^1 \int_0^1 P_{n_1, n_2}\left( \begin{bmatrix} 0 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z \right) e(-m_1 \xi_1 - m_2 \xi_2)d\xi_1 d\xi_2 d\xi_3 = \]

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2. \(SL_3(\mathbb{Z})\) Kloosterman sums

\[
e(m_1 x_1 + m_2 x_2)I_{\nu_1, \nu_2}(z) \sum_{w_i \in W} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \sum_{D_1, D_2 = 1} \infty S_{w_i}(\epsilon_1 m_1, \epsilon_2 m_2, n_1, n_2; D_1, D_2) \\
x D_1^{-3\nu_1} D_2^{-3\nu_2} J_{w_i}(y_1, y_2; \nu_1, \nu_2; \epsilon_1 m_1, \epsilon_2 m_2, n_1, n_2; D_1, D_2),
\]

where \(S_{w_i}\) is a \(SL_3(\mathbb{Z})\) Kloosterman sums and

\[J_{w_i} = J_{w_i}(y_1, y_2; \nu_1, \nu_2; \epsilon_1 m_1, \epsilon_2 m_2, n_1, n_2; D_1, D_2)\]

is an integral below corresponding to the element \(w_i\).

Let us write \(\xi_1 = \xi_1 \xi_2 - \xi_3\), \(Z_3 = \xi_3^2 + \xi_2 y_1^2 + y_1^2 y_2^2\) and \(Z_4 = \xi_4^2 + \xi_1 y_3^2 + y_1^2 y_2^2\), then

- \(J_{w_1} = \delta_{m_1, n_1} \delta_{m_2, n_2} E_{n_1, n_2}(y_1, y_2)\),
- \(J_{w_2} = \delta_{n_1, 0} \delta_{m_2, n_2} \int_{-\infty}^{+\infty} (\xi_1^2 + y_1^2)^{-\frac{3\nu_1}{2}} e(-m_2 \xi_1) d\xi_1\),
- \(J_{w_3} = \delta_{m_1, n_1} \delta_{m_2, 0} \int_{-\infty}^{+\infty} (\xi_1^2 + y_1^2)^{-\frac{3\nu_1}{2}} e(-m_1 \xi_1) d\xi_1\),
- \(J_{w_4} = \delta_{m_2 D_1, n_1 D_2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\xi_1^2 + y_1^2)^{-\frac{3\nu_1}{2}} Z_3^{-\frac{3\nu_1}{2}} e(-m_1 \xi_1 - m_2 \xi_2) d\xi_1 d\xi_2 d\xi_3 d\xi_4\).

For the proof, consider the Poincaré series

\[P_{n_1, n_2}(z, \nu_1, \nu_2) = \sum_{\gamma \in \Gamma_3 \backslash \Gamma_3} I_{\nu_1, \nu_2}(\gamma z) E_{n_1, n_2}(\gamma z).\]

Let

\[U = \left\{ \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} , \epsilon_i = \pm 1, \epsilon_1 \epsilon_2 \epsilon_3 = 1 \right\}.\]
Then we can write

\[ P_{n_1,n_2}(z,\nu_1,\nu_2) = \sum_{\gamma \in \Gamma_3,\infty \setminus \Gamma_3/U} \sum_{u \in U} I_{\nu_1,\nu_2}(\gamma uz) E_{n_1,n_2}(\gamma uz). \]

According to the formula 2.3, Fourier coefficients are given by

\[ F_{m_1,m_2}(z) = \int_0^1 \int_0^1 \int_0^1 P_{n_1,n_2} \left( \begin{bmatrix} \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \end{bmatrix} \right) \frac{1}{z} \left( \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right) e^{-m_1 \xi_1 - m_2 \xi_2} d\xi_1 d\xi_2 d\xi_3. \]

Since

\[ \begin{bmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z = \begin{bmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 y_2 & 0 & 0 \\ x_2 + \xi_2 y_1 & \xi_3 + \xi_2 x_1 + x_3 \\ y_1 & \xi_1 + x_1 \end{bmatrix} \begin{bmatrix} 0 & y_1 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]

we can make a change of variables

\[ \xi_1 \to \xi_1 - x_1, \]

\[ \xi_2 \to \xi_2 - x_2, \]

\[ \xi_3 \to \xi_3 - x_3 - \xi_2 x_1 \]

to obtain

\[ F_{m_1,m_2}(z) = e(x_1 m_1 + x_2 m_2) \int_0^1 \int_0^1 \int_0^1 P_{n_1,n_2} \left( \begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \right) e^{-m_1 \xi_1 - m_2 \xi_2} d\xi_1 d\xi_2 d\xi_3 \]

\[ = e(x_1 m_1 + x_2 m_2) \sum_{\gamma \in \Gamma_3,\infty \setminus \Gamma_3/U} \sum_{u \in U} \int_0^1 \int_0^1 \int_0^1 I_{\nu_1,\nu_2}(\gamma uz) E_{n_1,n_2} \left( \begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \right). \]
\[ \times e(-m_1 \xi_1 - m_2 \xi_2) d\xi_1 d\xi_2 d\xi_3. \]

Note that each element \( u \in U, u \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) changes signs of some variables \( \xi_i, 1 \leq i \leq 3 \). So the following substitution

\[ \xi_i \rightarrow \epsilon_i \xi_i, i = 1, \ldots, 3 \]

leads to

\[ F_{m_1, m_2}(z) = e(x_1 m_1 + x_2 m_2) \sum_{\epsilon_1, \epsilon_2 = \pm 1} \sum_{\gamma \in \Gamma_{3, \infty} \cap \Gamma_3 / U} \int_0^1 \int_0^1 \int_0^1 I_{\nu_1, \nu_2} \times E_{n_1, n_2} \left( \gamma \begin{bmatrix} y_1 y_2 & y_1 \xi_3 \\ 0 & y_1 \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \right) e(-m_1 \epsilon_1 \xi_1 - m_2 \epsilon_2 \xi_2) d\xi_1 d\xi_2 d\xi_3. \]

Let us denote \( k_1 = m_1 \epsilon_1, k_2 = m_2 \epsilon_2 \) and

\[ L = \sum_{\gamma \in \Gamma_{3, \infty} \cap \Gamma_3 / U} \int_0^1 \int_0^1 \int_0^1 I_{\nu_1, \nu_2} E_{n_1, n_2} \left( \gamma \begin{bmatrix} y_1 y_2 & y_1 \xi_3 \\ 0 & y_1 \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \right) e(-k_1 \xi_1 - k_2 \xi_2) d\xi_1 d\xi_2 d\xi_3. \]

Now we apply results of the section 2.4 to modify the given sum. Note that

\[ \Gamma_3 = G_3 \cap \Gamma_3 = (\cup_{w \in W_3} G_w) \cap \Gamma_3 \]

by Bruhat decomposition 2.15. So that

\[ \Gamma_{3, \infty} \cap \Gamma_3 / U = \cup_{w \in W_3} \Gamma_{3, \infty} \cap \Gamma_3 \cap G_w / U. \]

Let

\[ e_{n_1, n_2} \begin{bmatrix} 1 & x_2 & * \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix} = e(n_1 x_1 + n_2 x_2). \]
Then, using proposition 2.4.15 and property 2.12, we have

\[ L = \sum_{w \in W_3} \sum_{\gamma \in R_w} \sum_{b_1,b_2 \in G_\infty, d \in D} e_{n_1,n_2}(b_1) \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} I_{n_1,n_2} \]

\[ \times E_{n_1,n_2} \left( wdb_2 t \begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \right) e(-k_1 \xi_1 - k_1 \xi_2) \xi_1 \xi_2 \xi_3. \]

According to the definition 2.4.13, there are six types of groups \( \Gamma_w \) associated to different elements \( w \in W_3 \). We can treat them case by case in order to apply the action of \( t \in \Gamma_w \) to the matrix \( \begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \) and change the domain of integration. Consider, for instance,

\[ \Gamma_{w_5} = \left\{ \begin{bmatrix} 1 & m & l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, m, l \in \mathbb{Z} \right\} \text{ with } w_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \]

Then,

\[ \begin{bmatrix} 1 & m & l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} y_1 y_2 & y_1 (\xi_2 + m) & \xi_3 + m \xi_1 + l \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix}. \]

Let us make the following change of variables

\[ \xi_1 \to \xi_1, \]

\[ \xi_2 \to \xi_2 - m, \]

\[ \xi_3 \to \xi_3 - m \xi_1 - l. \]

Summing over all elements in \( \Gamma_{w_5} \) (equivalently, summing over all \( m, l \in \mathbb{Z} \)), the
domain of integration in the space $\xi_3 \times \xi_2 \times \xi_1$ is

$$[-\infty, +\infty] \times [-\infty, +\infty] \times [0, 1].$$

In general,

$$L = \sum_{w \in W_3} \sum_{\gamma \in \Gamma_{3, \infty} \setminus \Gamma_{3} \cap G_w/UT_w} e_{n_1, n_2}(b_1) \int_{\Omega_w} I_{\nu_1, \nu_2} e_{n_1, n_2}(b_1) \int_{\Omega_w} I_{\nu_1, \nu_2}$$

$$\times E_{n_1, n_2} \left( \begin{array}{ccc} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{array} \right) e(-k_1 \xi_1 - k_1 \xi_2) d\xi_1 d\xi_2 d\xi_3,$$

where in the space $\xi_3 \times \xi_2 \times \xi_1$

$$\Omega_{w_1} = [0, 1] \times [0, 1] \times [0, 1] \text{ with } w_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\Omega_{w_2} = [0, 1] \times [-\infty, +\infty] \times [0, 1] \text{ with } w_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\Omega_{w_3} = [0, 1] \times [0, 1] \times [-\infty, +\infty] \text{ if } w_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\Omega_{w_4} = [-\infty, +\infty] \times [0, 1] \times [-\infty, +\infty] \text{ with } w_4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\Omega_{w_5} = [-\infty, +\infty] \times [-\infty, +\infty] \times [0, 1] \text{ with } w_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\Omega_{w_6} = [-\infty, +\infty] \times [-\infty, +\infty] \times [-\infty, +\infty] \text{ if } w_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$
Next we apply
\[
\begin{pmatrix}
1 & \beta_2 & \beta_3 \\
0 & 1 & \beta_1 \\
0 & 0 & 1
\end{pmatrix}
\]
to
\[
\begin{pmatrix}
y_1 y_2 & y_1 \xi_2 & \xi_3 \\
0 & y_1 & \xi_1 \\
0 & 0 & 1
\end{pmatrix}
\]
and make a change of variables
\[
\begin{align*}
\xi_1 & \rightarrow \xi_1 - \beta_1, \\
\xi_2 & \rightarrow \xi_2 - \beta_2, \\
\xi_3 & \rightarrow \xi_3 - \beta_2 \xi_1 - \beta_3.
\end{align*}
\]

Then,
\[
L = I_{\nu_1, \nu_2}(z) E_{n_1, n_2}(y_1, y_2) \int_0^1 \int_0^1 \int_0^1 e((n_1 - k_1)\xi_1 + (n_2 - k_2)\xi_2)d\xi_1 d\xi_2 d\xi_3
\]
\[
+ \sum_{\substack{w \in W_3 \gamma \in \Gamma_3 \setminus G_{w_1} \cap \Gamma_3 / U \Gamma_2 \\
w \neq w_1}} e_{n_1, n_2}(b_1) e_{k_1, k_2}(b_2)
\]
\[
\times \int_{\Omega_{w, b_2}} I_{\nu_1, \nu_2}(z) E_{n_1, n_2} \left( wd \begin{pmatrix}
y_1 y_2 & y_1 \xi_2 & \xi_3 \\
0 & y_1 & \xi_1 \\
0 & 0 & 1
\end{pmatrix} \right) e(-k_1\xi_1 - k_2\xi_2)d\xi_1 d\xi_2 d\xi_3.
\]

According to Bruhat decomposition and propositions 2.4.7-2.4.21, the domain of integration \(\Omega_{w, b_2}\) is given by
\[
\begin{align*}
\Omega_{w_2, b_2} & = [0, 1] \times [-\infty, +\infty] \times [0, 1], \\
\Omega_{w_3, b_2} & = [0, 1] \times [0, 1] \times [-\infty, +\infty], \\
\Omega_{w_4, b_2} & = [-\infty, +\infty] \times \left[ \frac{\alpha C_2 B_1}{A_2}, 1 + \frac{\alpha C_2 B_1}{A_2} \right] \times [-\infty, +\infty],
\end{align*}
\]
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\[
\Omega_{w_3,b_2} = [-\infty, +\infty] \times [-\infty, +\infty] \times \left[ -\frac{b_2B_2}{A_1}, 1 - \frac{b_2B_2}{A_1} \right],
\]
\[
\Omega_{w_6,b_2} = [-\infty, +\infty] \times [-\infty, +\infty] \times [-\infty, +\infty].
\]

The next step is to modify the integral \( \int_{\Omega_{w,d}} \). Since \( wd \equiv \begin{bmatrix} y_1 & y_2 & \xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \mod (O_3(\mathbb{R}) \cdot \mathbb{R}^\times) \), we consider

\[
\begin{bmatrix} y'_1 & y'_2 & x_2 & x_3 \\ 0 & y'_1 & x_1 \\ 0 & 0 & 1 \end{bmatrix} \equiv wd \begin{bmatrix} y_1 & y_2 & \xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \mod (O_3(\mathbb{R}) \cdot \mathbb{R}^\times).
\]

Let

\[
\xi_4 = \xi_1\xi_2 + \xi_3,
\]
\[
Z_3 = \xi_3^2 + y_1^2\xi_2^2 + y_1^2y_2^2,
\]
\[
Z_4 = \xi_4^2 + y_2^2\xi_1^2 + y_1^2y_2^2.
\]

Then the values of \( x'_1, x'_2, x'_3, y'_1, y'_2 \) are as follows.

- If \( w = w_2 \), then

\[
x'_1 = B_2\xi_3,
\]
\[
x'_2 = \frac{-\xi_2}{B_2(\xi_2^2 + y_2^2)},
\]
\[
x'_3 = \frac{\xi_1}{B_2},
\]
\[
y'_1 = B_2y_1(\xi_2^2 + y_2^2)^{0.5},
\]
\[
y'_2 = \frac{y_2}{B_2(\xi_2^2 + y_2^2)}.
\]

- If \( w = w_3 \), then

\[
x'_1 = \frac{-\xi_1}{B_1^2(\xi_1^2 + y_1^2)} ,
\]
\[
x'_2 = B_1(\xi_1\xi_2 - \xi_3),
\]
\[
x'_3 = \frac{\xi_1\xi_3 + \xi_2y_1^2}{B_1(\xi_1^2 + y_1^2)}.
\]
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\[ y'_1 = \frac{y_1}{B_1^2(\xi_1^2 + y_1^2)}, \]
\[ y'_2 = B_1 y_2(\xi_1^2 + y_1^2)^{0.5}. \]

- If \( w = w_4 \), then

\[ x'_1 = \frac{A_2 (\xi_1 \xi_3 + \xi_2 y_1^2)}{B_1^2} \]
\[ x'_2 = -\frac{B_1 \xi_4}{A_2^2 Z_4}, \]
\[ x'_3 = \frac{\xi_1}{B_1 A_2 (\xi_1^2 + y_1^2)}, \]
\[ y'_1 = \frac{A_2 y_1 Z_4^{0.5}}{B_1^2 (\xi_1^2 + y_1^2)}, \]
\[ y'_2 = \frac{B_1 y_2 (\xi_1^2 + y_1^2)^{0.5}}{A_2^2 Z_4}. \]

- If \( w = w_5 \), then

\[ x'_1 = \frac{B_2 \xi_3}{A_1^2 Z_3}, \]
\[ x'_2 = \frac{A_1 (\xi_1 - \xi_2 \xi_3)}{B_2^2 (\xi_2^2 + y_2^2)}, \]
\[ x'_3 = \frac{\xi_1 \xi_3 + \xi_2 y_1^2}{A_1 B_2 Z_3}, \]
\[ y'_1 = \frac{B_2 y_1 (\xi_2^2 + y_2^2)^{0.5}}{A_2^2 Z_3}, \]
\[ y'_2 = \frac{A_1 y_2 Z_3^{0.5}}{B_2^2 (\xi_2^2 + y_2^2)}. \]

- If \( w = w_6 \), then

\[ x'_1 = \frac{-A_2 (\xi_1 \xi_3 + \xi_2 y_1^2)}{A_1^2 Z_3}, \]
\[ x'_2 = \frac{-A_1 (\xi_2 \xi_3 + \xi_2 y_2^2)}{A_2^2 Z_4}, \]
\[ x'_3 = \frac{\xi_3}{A_1 A_2 Z_3}. \]
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\[ y'_1 = \frac{A_2 y_1 Z_4^{0.5}}{A_1^2 Z_3}, \]
\[ y'_2 = \frac{A_1 y_2 Z_3^{0.5}}{A_2^2 Z_4}. \]

Given results can be verified by direct calculations. Consider, for example, case \( w = w_5 \). Let \( r_1 = \sqrt{\xi_1^2 + y_2^2}, r_2 = \sqrt{\xi_2^2 y_1^2 + y_1^2 y_2^2 + \xi_3^2} \), then there are

\[
O = \begin{bmatrix}
\frac{-\xi_3}{r_1} & \frac{y_2}{r_1} & 0 \\
\frac{-\xi y_2}{r_1 r_2} & \frac{-\xi_3}{r_1 r_2} & \frac{r_1 y_2}{r_1 r_2} \\
\frac{y_1 y_2}{r_2} & \frac{y_1 \xi_3}{r_2} & \frac{\xi_3}{r_2}
\end{bmatrix} \in \mathbb{O}_3(\mathbb{R})
\]

and

\[
R = \begin{bmatrix}
A_1 r_2^2 & 0 & 0 \\
0 & A_1 r_2^2 & 0 \\
0 & 0 & A_1 r_2^2
\end{bmatrix} \in \mathbb{R}^3
\]

such that

\[
\begin{bmatrix}
y'_1 y'_2 \\
y'_2 x_2 \\
x_3 \\
0 \\
y'_1 \\
0 \\
0 \\
1
\end{bmatrix}
\begin{bmatrix}
0 \\
y_1 \\
\xi_1 \\
0 \\
1
\end{bmatrix}
\]

\[
OR = w_5 d
\]

Now summing up all the results we immediately obtain the statement of the theorem in case \( w = w_1, w_2, w_3 \). The remained three cases involve some more computations. Let us consider for instance the case \( w = w_5 \):

\[
I_{\nu_1,\nu_2}(z) = A_1^{-3\nu_1} B_2^{-3\nu_2} Z_3^{-\frac{3\nu_1}{2}} (\xi_2^2 + y_2^2)^{-\frac{3\nu_2}{2}}
\]

and according to 2.33

\[
E_{\nu_1,\nu_2}(z) = E_{\nu_1,\nu_2}(x'_1 + iy'_1, x'_2 + iy'_2).
\]

Then

\[
\int_{\Omega_{w_5, w_2}} A_1^{-3\nu_1} B_2^{-3\nu_2} \int_{\frac{b_2 y_2}{A_1}}^{1-\frac{b_2 y_2}{B_2}} \frac{1}{A_1} e((\frac{n_2 A_1}{B_2} - k_1) \xi_1) d\xi_1
\]
2. SL$_3(Z)$ Kloosterman sums

\[
\times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{-3\nu_1}{Z_3^{3/2}} (\xi_2^2 + y_2^2)^{-3\nu_2} E_{n_1,n_2} \left( \frac{B_2}{A_1^2} \left( \frac{\xi_3 + iy_1(\xi_2^2 + y_2^2)^{1/2}}{Z_3} \right), \frac{A_1}{B_2^2} \left( \frac{-\xi_2 \xi_3 + iy_2 Z_3^{1/2}}{(\xi_2^2 + y_2^2)} \right) \right) \times e(-k_2 \xi_2) d\xi_2 d\xi_3.
\]

If \( \frac{n_2 A_1}{B_2^2} \neq k_1 \),

\[
\int_{-\frac{b_2 \cdot n_2}{A_1}}^{1 - \frac{b_2 \cdot n_2}{A_1}} e\left((\frac{n_2 A_1}{B_2^2} - k_1)\xi_1\right) d\xi_1 = \frac{1}{2\pi i} \left( \frac{n_2 A_1}{B_2^2} - k_1 \right)^{-1} e(b_{2,2}(\frac{k_1 B_2}{A_1} - \frac{n_2}{B_2}))(e(\frac{n_2 A_1}{B_2^2} - k_1) - 1) - 1.
\]

So that

\[
\int_{\Omega_{w_5,k_2}} = \mu \cdot A_1^{-3\nu_1} B_2^{-3\nu_2} e(b_{2,2}(\frac{k_1 B_2}{A_1} - \frac{n_2}{B_2}))
\]

\[
\times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{-3\nu_1}{Z_3^{3/2}} (\xi_2^2 + y_2^2)^{-3\nu_2} E_{n_1,n_2} \left( \frac{B_2}{A_1^2} \left( \frac{\xi_3 + iy_1(\xi_2^2 + y_2^2)^{1/2}}{Z_3} \right), \frac{A_1}{B_2^2} \left( \frac{-\xi_2 \xi_3 + iy_2 Z_3^{1/2}}{(\xi_2^2 + y_2^2)} \right) \right) \times e(-k_2 \xi_2) d\xi_2 d\xi_3,
\]

where

\[
\mu = 1 \text{ if } \frac{n_2 A_1}{B_2^2} = k_1
\]

and

\[
\mu = \frac{1}{2\pi i} \left( \frac{n_2 A_1}{B_2^2} - k_1 \right)^{-1} e(\frac{n_2 A_1}{B_2^2} - k_1) - 1, \text{ otherwise.}
\]

Let $D_1 = A_1$ and $D_2 = B_2$, then

\[
\sum_{\gamma} e_{n_1,n_2}(b_1)e_{k_1,k_2}(b_2)e(b_{2,2}(\frac{k_1 B_2}{A_1} - \frac{n_2}{B_2})) = \sum_{C_1 \bmod {D_1}, C_2 \bmod {D_2}} e\left( \frac{n_1 C_1^*}{D_1 D_2^{-1}} + \frac{k_2 C_2}{D_2} + \frac{n_2 C_1 C_2^*}{D_2} \right),
\]

where $(C_2, D_2) = 1$, $(C_1, D_1 D_2^{-1}) = 1$, $C_2 C_2^* \equiv 1 (\mod D_2)$ and $C_1 C_1^* \equiv 1 (\mod D_1 D_2^{-1})$.

By property 2.6.11, the later sum is zero unless $\frac{n_2 D_1}{D_2} \in \mathbb{Z}$. On the other hand, if $\frac{n_2 D_1}{D_2} \in \mathbb{Z}$, then the integral $\int_{\Omega_{w_5,k_2}}$ vanishes for $k_1 \neq \frac{n_2 A_1}{B_2^2}$. This leads to the result. Applying the same procedure, one can also obtain required expressions for Kloosterman sums and integrals $J_w$ in case $w = w_4, w_6$. 

\[\square\]
2.8 SL(3) Kloosterman angles

Let us recall proposition 2.6.6 in case \( n_1 = n_2 = m_1 = m_2 = 1, D_1 = p_1, D_2 = p_2, \) where \( p_1 \neq p_2 \) are prime numbers.

**Proposition 2.8.1.** If \((p_1, p_2) = 1\), then

\[
S(1, 1, 1, 1; p_1, p_2) = S(p_2, 1, p_1)S(p_1, 1, p_2),
\]

where \( S(m, n, c) \) is a classical Kloosterman sum.

According to Weil’s bound,

\[
|S(p_2, 1, p_1)| \leq 2\sqrt{p_1},
\]

\[
|S(p_1, 1, p_2)| \leq 2\sqrt{p_2}.
\]

Thus, there are unique Kloosterman angles \((\theta_{p_2, p_1}, \theta_{p_1, p_2})\) on \([0, \pi] \times [0, \pi]\) such that

\[
S(p_2, 1, p_1) = 2\sqrt{p_1}\cos(\theta_{p_2, p_1})
\]

and

\[
S(p_1, 1, p_2) = 2\sqrt{p_2}\cos(\theta_{p_1, p_2}).
\]

We associate a couple of angles \((\theta_{p_1, p_2}, \theta_{p_2, p_1})\) with \(\text{SL}_3(\mathbb{Z})\) Kloosterman sum \( S(1, 1, 1; p_1, p_2) \).

**Conjecture 2.8.2.** Let \( P_1, P_2 \to \infty \), then the set of Kloosterman angles

\[
\{(\theta_{p_2, p_1}, \theta_{p_1, p_2}) | p_1 \leq P_1, p_2 \leq P_2, p_1 \neq p_2, \theta(p_2, p_1) \in I_1, \theta(p_1, p_2) \in I_2 \}
\]

becomes equidistributed with respect to Sato-Tate measure on \([0, \pi] \times [0, \pi]\). Equivalently, for any \( I_1 \times I_2 = [a_1, b_1] \times [a_2, b_2] \in [0, \pi] \times [0, \pi] \),

\[
l_{P_1 \to \infty, P_2 \to \infty} \frac{\# \{p_1 \leq P_1, p_2 \leq P_2, p_1 \neq p_2, \theta(p_2, p_1) \in I_1, \theta(p_1, p_2) \in I_2 \}}{\# \{p_1 \leq P_1 \} \times (\# \{p_2 \leq P_2 \} - 1)} = \mu_{ST}(I_1 \times I_2) = \frac{4}{\pi^2} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \sin^2(\theta_1) \sin^2(\theta_2) d\theta_1 d\theta_2.
\]
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Figure 2.1: Cumulative distribution function for Kloosterman angles (red) and Sato-Tate cumulative distribution function (blue)
Figure 2.2: Cumulative distribution function for Kloosterman angles (red) and Sato-Tate cumulative distribution function (blue) in one plot.
References


REFERENCES


