On $\mathbf{SL}_2(\mathbb{Z})$ and $\mathbf{SL}_3(\mathbb{Z})$ Kloosterman sums

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Classical Kloosterman sum (or $\mathbf{SL}_2(\mathbb{Z})$ Kloosterman sum) is defined by

$$S(m,n,c) = \sum_{\substack{d(\text{mod c})\\ gcd(c,d)=1}} e\left(\frac{m\bar{d}+nd}{c}\right)$$

with $d\bar{d} \equiv 1(\text{mod c})$. First, the given sum was discovered by Henri Poincaré in 1911 in the paper [13] on modular forms.

Few years later in 1926 Hendrik Kloosterman also obtained the same sum while he was solving the problem of finding asymptotic expression of the number of representations of a large integer n by a quadratic form in four variables[1], i.e. the number of solutions $(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$ of an equation

$$ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 = n,$$

where a, b, c, d are fixed positive integers.

Already the fact that Kloosterman sum appeared in two problems of different origins emphasizes its importance. Afterwards a number of other applications in number theory have been found. One of them is Conrey's theorem [3] that at least two-fifths of the zeros of the Riemann zeta function are simple and on the critical line, which uses results of J.-M. Deshouillers and H. Iwaniec on averages of Kloosterman sums.

Another application is the generalization of Ramanujan conjecture to non-holomorphic cusp forms associated to arithmetic discrete sub-

[1]See [11].

groups of $\mathbf{GL}_n(\mathbb{R})$ with $n \geq 2$. Trying to find an approach to the generalized Ramanujan conjecture for $\mathbf{GL}_2(\mathbb{R})$ and $\mathbf{GL}_3(\mathbb{R})$ via Kloosterman sums, Bump, Friedberg and Goldfeld computed Fourier expansion of $\mathbf{SL}_3(\mathbb{Z})$ Poincaré series. As a part of Fourier coefficients they obtained six exponential sums, among which we can distinguish two new types different from classical Kloosterman sums, called $\mathbf{SL}_3(\mathbb{Z})$ Kloosterman sums.

The main goal of this work is to study the connection between Kloosterman sums and automorphic forms associated with groups $\mathbf{SL}_2(\mathbb{Z})$ and $\mathbf{SL}_3(\mathbb{Z})$. We start with classical Kloosterman sums and obtain them as a part of Fourier coefficients of $\mathbf{SL}_2(\mathbb{Z})$ Poincaré series in the first chapter. In the second chapter we compute Fourier expansion of $\mathbf{SL}_3(\mathbb{Z})$ Poincaré series and introduce two new types of exponential sums called $\mathbf{SL}_3(\mathbb{Z})$ Kloosterman sums. We also describe some properties of Kloosterman sums and discuss the problem of distribution of Kloosterman angles.

Chapter 1

Classical Kloosterman sums

In this chapter we construct a particular example of $\mathbf{SL}_2(\mathbb{Z})$ modular forms. Our construction leads us to Poincaré series, whose Fourier expansion turns out to contain classical Kloosterman sums.

1.1 SL(2) modular forms

The group

$$\mathbf{SL}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \text{ and } a, b, c, d \in \mathbb{R} \right\}$$

acts on the *Poincaré upper-half plane*

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \Im z > 0\}$$

by linear fractional transformations. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{SL}_2(\mathbb{R})$, then for any $z \in \mathbb{H}^2$

$$\gamma(z) = \frac{az + b}{cz + d} \in \mathbb{H}^2 \tag{1.1}$$

since

$$\Im(\gamma(z)) = \frac{\Im(z)}{|cz + d|^2}. \tag{1.2}$$

The action of $\mathbf{SL}_2(\mathbb{R})$ on the set $\mathbb{H}^2$ has one orbit because we can reach any point in $\mathbb{H}^2$ from the point i:

$$\begin{bmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} (i) = x + iy.$$

And the stabilizer K of i is equal to

$$\mathbf{SO}_2(\mathbb{R}) = \{k(\theta)\} \text{ for } k(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Indeed,

$$\frac{ai+b}{ci+d}` = i \Rightarrow ai + b = di - c \Rightarrow d = a,\ c = -b.$$

So

$$K = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right\} \text{ with } a^2 + b^2 = 1$$

and we can write $a = \cos\theta$ and $b = \sin\theta$ to obtain the result.

This gives an alternative way to represent the upper-half plane as a quotient space $\mathbf{SL}_2(\mathbb{R})/\mathbf{SO}_2(\mathbb{R})$. Each element of the later group has a unique representative of the form $\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}$, where $y > 0$, by *Iwasawa decomposition*[1].

In this section we are mainly interested in a discrete subgroup $\mathbf{SL}_2(\mathbb{Z})$ of $\mathbf{SL}_2(\mathbb{R})$ with $a, b, c, d \in \mathbb{Z}$, called the *modular group*.

The group $\mathbf{SL}_2(\mathbb{Z})$ is generated [2] by two elements

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ such that } S^2 = (ST)^3 = -1.$$

And the *standard fundamental domain* [3] for the action of $\mathbf{SL}_2(\mathbb{Z})$ on $\mathbb{H}^2$ is

$$F = \left\{ z \in \mathbb{H}^2,\ |\Re(z)| \leq \frac{1}{2},\ |z| \geq 1 \right\}.$$

[1] The proof is given in a general case in 2.1.1
[2] See [9], theorem 1.1
[3] See [9], theorem 1.2

To define a notion of automorphic form with respect to $\mathbf{SL}_2(\mathbb{Z})$, we use the following operator.

Definition 1.1.1. Let k be a positive integer. Define a *weight k slash operator* of

$$\mathbf{GL}_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

on the set of all functions $f : \mathbb{H}^2 \to \mathbb{C}$ as follows. If $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{GL}_2^+(\mathbb{R})$ and $j(\gamma, z) = cz + d$, let

$$\left(f \mid_\gamma^k\right)(z) = \frac{det(\gamma)^{k/2}}{j(\gamma, z)^k} \cdot f(\gamma z). \tag{1.3}$$

Remark 1.1.2. Formula (1.3) defines a right action of $\mathbf{GL}_2^+(\mathbb{R})$ on the set of all functions $f : \mathbb{H}^2 \to \mathbb{C}$; in particular,

$$f \mid_{\gamma_1\gamma_2}^k (z) = \left(f \mid_{\gamma_1}^k\right) \mid_{\gamma_2}^k (z). \tag{1.4}$$

The last equation is a consequence of the cocycle property

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z) \cdot j(\gamma_2, z). \tag{1.5}$$

From now on we fix $\Gamma_2 = \mathbf{SL}_2(\mathbb{Z})$. Then the equation (1.3) can be written as

$$\left(f \mid_\gamma^k\right)(z) = \frac{f(\gamma z)}{j(\gamma, z)^k}. \tag{1.6}$$

Definition 1.1.3. A function $f : \mathbb{H}^2 \to \mathbb{C}$ is called *modular form* of weight k with respect to the group Γ_2 if

- f is holomorphic on $\mathbb{H}^2$
- $f \mid_\gamma^k = f$ for every $\gamma \in \Gamma_2$
- f is holomorphic at infinity.

Remark 1.1.4. The last condition can be explained as follows.

Since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathbf{SL}_2(\mathbb{Z})$, f is a periodic function:

$$f(z+1) = f(z),\ z \in \mathbb{H}^2.$$

Thus, f has a *Fourier expansion at infinity*

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n(f) q^n,\ q = e^{2\pi i z}.$$

And we call f *holomorphic at infinity* if $a_n(f) = 0$ for every $n < 0$. If in addition, $a_0(f) = 0$, function f is called *cuspidal*.

1.2 Construction of SL(2) Poincaré series

To find functions satisfying definition 1.1.3, we start with the automorphy condition. Let $h : \mathbb{H}^2 \to \mathbb{C}$ be a holomorphic function. We can write formally

$$f(z) = \sum_{\gamma \in \Gamma_2} \frac{h(\gamma z)}{j(\gamma, z)^k},\ z \in \mathbb{H}^2. \tag{1.7}$$

The cocycle property (1.5) yields that for every $\gamma' \in \Gamma_2$

$$f(\gamma' z) = \sum_{\gamma \in \Gamma_2} \frac{h(\gamma\gamma' z)}{j(\gamma, \gamma' z)^k} = j(\gamma', z)^k \sum_{\gamma \in \Gamma_2} \frac{h(\gamma\gamma' z)}{j(\gamma\gamma', z)^k} = j(\gamma', z)^k f(z).$$

If (1.7) converges absolutely uniformly on compact subsets of $\mathbb{H}^2$, then $f(z)$ is a holomorphic function and all formal computations are valid. However, the sum (1.7) does not converge in general. In particular, the sum may diverge if we have infinitely many elements

$$\gamma \in \Gamma_\infty = \left\{ \pm \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix},\ n \in \mathbb{Z} \right\} = < \pm T > \ \text{ with } j(\gamma, z) \equiv 1.$$

In order to avoid this problem, assume that h is invariant under Γ_∞ and note that the sum (1.7) depends only on cosets modulo Γ_∞. Indeed, if $\gamma = \beta\gamma'$ for

$\beta \in \Gamma_\infty$, $\gamma, \gamma^{'} \in \Gamma_2$, then

$$h(\gamma z) = h(\beta\gamma^{'} z) = h(\gamma^{'} z),$$

$$j(\gamma, z) = j(\beta\gamma^{'}, z) = j(\beta, \gamma^{'} z) j(\gamma^{'}, z) = j(\gamma^{'}, z).$$

So the formula

$$f(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_2} \frac{h(\gamma z)}{j(\gamma, z)^k} \tag{1.8}$$

is the one we are looking for. Now we can choose a particular Γ_∞-invariant function, namely

$$h(z) = e(mz) = e^{2\pi imz},\ m \in \mathbb{Z}.$$

Definition 1.2.1. The series

$$P_m^k(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_2} h \mid_\gamma^k (z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_2} \frac{e(m\gamma z)}{j(\gamma, z)^k} \tag{1.9}$$

is called m^{th} *Poincaré series of weight* k.

Proposition 1.2.2. *The Bruhat decomposition of* Γ_2 *is given by*

$$\Gamma_2 = \Gamma_\infty \amalg (\amalg_{c \in \mathbb{Z}_{>0}} \amalg_{\substack{d (mod\ c) \\ (c,d)=1}} (\Gamma_\infty w \Gamma_\infty)),\ \amalg \textit{ is a disjoint union,} \tag{1.10}$$

$$\Gamma_\infty =< \pm T > \quad \textit{and } w \in W_{c,d} = \left\{ \begin{bmatrix} a^* & b^* \\ c & d \end{bmatrix} \right\},$$

where for given c, $d \in \mathbb{Z}$ *with* $(c, d) = 1$, *integral variables* a^*, b^* *satisfy*

$$a^*d - b^*c = 1,\ i.e.\ b^* = \frac{a^*d - 1}{c}.$$

Proof. We would like to partition Γ into double cosets with respect to Γ_∞.

First, consider the set of upper triangular matrices

$$\Delta_1 = \left\{ \begin{bmatrix} a^* & b^* \\ 0 & d^* \end{bmatrix} \in \Gamma_2 \right\}.$$

Conditions $a^*d^* = 1$ and $a^*, b^*, d^* \in \mathbb{Z}$ imply that $\Delta_1 = \Gamma_\infty$.

Second, any element of $\Delta_1 \setminus \Gamma_2$ can be represented by a matrix

$$\omega = \begin{bmatrix} a & b^* \\ c & d \end{bmatrix} \text{ with } c > 0.$$

The relation

$$\begin{bmatrix} 1 & n_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b^* \\ c & d \end{bmatrix} \begin{bmatrix} 1 & n_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a + cn_1 & b_1^* \\ c & d + cn_2 \end{bmatrix}$$

shows that the double coset $\Gamma_\infty \begin{bmatrix} a & b^* \\ c & d \end{bmatrix} \Gamma_\infty$ determines c uniquely, while a and d can be found modulo integral multiples of c.

Actually, the given coset does not depend on a, because for any two matrices $\omega_1 = \begin{bmatrix} a_1 & b_1^* \\ c & d \end{bmatrix}$ and $\omega_2 = \begin{bmatrix} a_2 & b_2^* \\ c & d \end{bmatrix}$ in Γ_2

$$\omega_1 {\omega_2}^{-1} = \begin{bmatrix} 1 & b_3^* \\ 0 & 1 \end{bmatrix},$$

i.e. a_1 is congruent to a_2 modulo c. So

$$\Delta_2 = \Gamma_\infty \omega \Gamma_\infty \in \Gamma_2, \text{ with } \omega = \begin{bmatrix} a^* & b^* \\ c & d \end{bmatrix} \in \Gamma_2.$$

To sum up, Γ_2 is a disjoint union of Δ_1 and $(\amalg_{c \in \mathbb{Z}_{>0}} \amalg_{\substack{d (\text{mod c}) \\ (c,d)=1}} \Delta_2)$.

$\square$

Proposition 1.2.3. *For $k > 2$, $m \geq 0$ the series $P_m^k(z)$ converges absolutely and uniformly on compact sets of $\mathbb{H}^2$ and defines a holomorphic function on $\mathbb{H}^2$.*

Proof. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_2$, then

$$\left| \frac{e(m\gamma z)}{j(\gamma, z)^k} \right| = \left| \frac{e^{2\pi i m \gamma(z)}}{(cz+d)^k} \right|.$$

By formula 1.2,

$$\left|\frac{e^{2\pi i m\gamma(z)}}{(cz+d)^k}\right| = \frac{1}{|cz+d|^k} e^{-\frac{2\pi m \Im(z)}{(cz+d)^2}} \leq \frac{1}{|cz+d|^k}$$

for $m \geq 0$. According to the Bruhat decomposition of Γ_2, we pick each pair (c,d) as the second raw of matrices in $\Gamma_\infty \setminus \Gamma_2$ at most once. Therefore, $P_m^k(z)$ is majorated by the series

$$\sum_{\substack{c,d\in Z \\ (c,d)\neq(0,0)}} \frac{1}{|cz+d|^k}.$$

The last series is known[1] to be convergent uniformly on compact sets of $\mathbb{H}^2$ for any $k > 2$. □

1.3 Fourier expansion of Poincaré series

It only remains to verify the last condition of 1.1.3 to complete our construction of modular form. With this goal, we find Fourier expansion of the series (1.9). Ultimately, we obtain Kloosterman sums as a part of Fourier coefficients.

Definition 1.3.1. The sum

$$S(m,n,c) = \sum_{\substack{d(\text{mod c}) \\ gcd(c,d)=1}} e\left(\frac{m\overline{d}+nd}{c}\right)$$

with $d\overline{d} \equiv 1(\text{mod c})$ is called *classical Kloosterman sum.*

Remark 1.3.2. If $m = 0$, then Kloosterman sum reduces to *Ramanujan sum*

$$R_c(n) = S(0,n,c) = \sum_{\substack{d(\text{mod c}) \\ gcd(c,d)=1}} e\left(\frac{nd}{c}\right).$$

Theorem 1.3.3. *Let $k > 2$. The Fourier expansion of Poincaré series $P_m^k(z)$ is given by*

[1] See [8], theorem 1

- *if* $m = 0$,

$$P_0^k(z) = 1 + \frac{(-2\pi i)^k}{\Gamma(k)} \sum_{n>0} \left(\sum_{c>0} R_c(n) \frac{n^{k-1}}{c^k} \right) e(nz);$$

- *if* $m > 0$,

$$P_m^k(z) = e(mz) + \frac{(-2\pi i)^k}{m^{\frac{k-1}{2}}} \sum_{n>0} \left(\sum_{c>0} S(m,n,c) \frac{n^{\frac{k-1}{2}} J_{k-1}(\frac{4\pi\sqrt{mn}}{c})}{c} \right) e(nz),$$

where

$$J_n(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\Gamma(i+1+n)} \left(\frac{x}{2}\right)^{n+2i}$$

is the Bessel function of order n.

Proof. Consider the series (1.9). Applying the Bruhat decomposition,

$$P_m(z) := P_m^k(z) = e(mz) + \sum_{\substack{c>0 \\ d(\text{mod c}) \\ gcd(c,d)=1}} \sum_{\substack{\beta\in\Gamma_\infty \\ \omega\in W_{c,d}}} h \mid_{\omega\beta}^k (z).$$

Take

$$\omega = \begin{bmatrix} a^* & \frac{a^*d-1}{c} \\ c & d \end{bmatrix} \in W_{c,d} \text{ and } \beta = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in \Gamma_\infty,$$

then for every $z \in \mathbb{H}^2$

$$(\omega\beta)z = \frac{a^*z + a^*n + \frac{a^*d-1}{c}}{cz + cz + d} = \frac{a^*}{c} - \frac{1}{c(c(z+n)+d)}$$

and

$$h \mid_{\omega\beta}^k (z) = (c(z+n)+d)^{-k} e\left(m \left(\frac{a^*}{c} - \frac{1}{c(c(z+n)+d)} \right) \right).$$

Thus,

$$P_m(z) = e(mz) + \sum_{c>0} \sum_{\substack{d(\text{mod c}) \\ gcd(c,d)=1}} I(c,d,z) \tag{1.11}$$

with
$$I(c,d,z)=\sum_{n\in\mathbb{Z}} g(n) \text{ and } g(n)=h\,|_{\omega\beta}^{k}\,(z).$$
By Poisson summation formula,
$$I(c,d,z)=\sum_{n\in\mathbb{Z}}\widehat{g(n)}=\sum_{n\in\mathbb{Z}}\left(\int_{\mathbb{R}} g(t)e(-nt)dt\right)$$
$$=\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\frac{1}{(c(z+t)+d)^k}e\left(-nt+m\left(\frac{a^*}{c}-\frac{1}{c(c(z+n)+d)}\right)\right)dt.$$
Let us write $z=x+iy$ and make change of variables $t\to t'=z+t+\frac{d}{c}$ in the integral. Then we obtain
$$I(c,d,z)=\sum_{n\in\mathbb{Z}} e(n(z+\frac{d}{c})+\frac{ma^*}{c})\int_{t'=-\infty+iy}^{+\infty+iy}\frac{1}{(ct')^k}e(-nt'-\frac{m}{c^2t'})dt'. \tag{1.12}$$
Denote the inner integral [1] by
$$L_c(m,n)=\int_{t'=-\infty+iy}^{+\infty+iy}\frac{1}{(ct')^k}e(-nt'-\frac{m}{c^2t'})dt'$$
and distinguish the following cases:

1. If $n\le 0$, then we can move the line of integration upwards, i.e. let $y\to\infty$, and estimate the absolute value of $L_c(m,n)$ to see that
$$L_c(m,n)=0. \tag{1.13}$$
Therefore, all terms with $n\le 0$ in the sum (1.12) vanish.

2. If $n>0$ and $m=0$, then [2]
$$L_c(0,n)=\left(\frac{2\pi}{ic}\right)^k\frac{n^{k-1}}{\Gamma(k)}. \tag{1.14}$$

[1] Note that $L_c(m,n)$ does not depend on y by Cauchy's theorem
[2] see [7], 8.315.1

3. If $n, m > 0$, then [1]

$$L_c(m,n) = \frac{2\pi}{i^k c}\left(\frac{n}{m}\right)^{\frac{k-1}{2}} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{1.15}$$

Finally, substitute

$$I(c,d,z) = \sum_{n>0} e(nz)e(\frac{nd+ma^*}{c})L_c(m,n)$$

in the formula (1.11) and change the order of summation to obtain

$$P_m(z) = e(mz) + \sum_{n>0}\left(\sum_{c>0}\left(\sum_{\substack{d (\text{mod c})\\ gcd(c,d)=1}} e(\frac{nd+ma^*}{c})\right) L_c(m,n)\right) e(nz)$$

$$= e(mz) + \sum_{n>0}\left(\sum_{c>0} S(m,n,c)L_c(m,n)\right) e(nz).$$

Now just replace $L_c(m,n)$ by its value and the assertion follows.

□

Corollary 1.3.4. *The series $P_m^k(z)$ is holomorphic at infinity for $m \geq 0$ and cuspidal for $m \geq 1$.*

Remark 1.3.5. In 1965 Selberg [14] introduced non-holomorphic Poincaré series

$$P_m(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_2} (\Im(\gamma z))^s e(m\gamma z), \; \Re(s) > 1. \tag{1.16}$$

The Fourier expansion of the series 1.16 also contains classical Kloosterman sums.

Let $z = x + iy$, then

$$P_m(z,s) = \sum_{h\in\mathbb{Z}} p_m(h;y,s)e(hz)$$

[1]see [7], 8.412.2

with

$$p_m(h; y, s) = \delta(m, h) + \sum_{c \geq 1} c^{-2s} S(m, n, c) B(m, h, c, y, s)$$

and

$$B(m, h, c, y, s) = y^s \int_{-\infty}^{+\infty} (x^2 + y^2)^{-s} e\left(-hx - \frac{m}{c^2(x + iy)}\right) dx.$$

1.4 Some properties of Kloosterman sums

The sum

$$S(m, n, c) = \sum_{\substack{d(\text{mod c}) \\ gcd(c,d)=1}} e\left(\frac{m\overline{d} + nd}{c}\right)$$

with $d\overline{d} \equiv 1(\text{mod c})$ has some interesting properties.

Proposition 1.4.1. *The Kloosterman sum depends only on the residue class of m, n modulo c.*

Proof. This is clear since $e^{2\pi ik} = 1$ for every $k \in \mathbb{Z}$. □

Proposition 1.4.2. *The value of $S(m, n; c)$ is always a real number.*

Proof. Consider complex conjugate of Kloosterman sum

$$\overline{S(m, n; c)} = \sum_{\substack{d(\text{mod c}) \\ gcd(c,d)=1}} e\left(\frac{-m\overline{d} - nd}{c}\right).$$

Let $d = -d'$, then

$$\overline{S(m, n; c)} = S(m, n; c)$$

since $-d$ runs again over all residue classes modulo c.

□

Proposition 1.4.3.

$$S(m, n; c) = S(n, m, c)$$

Proof.

$$S(m, n, c) = \sum_{\substack{d(\text{mod c}) \\ gcd(c,d)=1}} e\left(\frac{m\overline{d} + nd}{c}\right).$$

Then the substitution $d = m\overline{n}d'$ leads to the required result. $\square$

Proposition 1.4.4.

$$S(ma, n; c) = S(m, na, c) \text{ \textit{if} } (a, c) = 1$$

Proof.

$$S(ma, n, c) = \sum_{\substack{d(\text{mod c})\\ gcd(c,d)=1}} e\left(\frac{ma\overline{d} + nd}{c}\right).$$

Then the substitution $d = ad'$ leads to the required result. $\square$

Proposition 1.4.5. *(twisted multiplicativity) If* $(c_1, c_2) = 1$*, then*

$$S(m, n; c_1c_2) = S(m\overline{c_2}, n\overline{c_2}, c_1)S(m\overline{c_1}, n\overline{c_1}, c_2).$$

Proof. Let $c = c_1c_2$. The proof is based on the Chinese Remainder theorem, i.e. if

$$d \equiv d_i(\text{mod } c_i),\ i = 1, 2,\ (c_1, c_2) = 1,$$

then

$$d \equiv d_1b_1c_2 + d_2b_2c_1(\text{mod } c)$$

with

$$b_1c_2 \equiv 1(\text{mod } c_1),$$

$$b_2c_1 \equiv 1(\text{mod } c_2).$$

Then

$$S(m, n, c) = \sum_{\substack{d(\text{mod c})\\ gcd(c,d)=1}} e\left(\frac{m\overline{d} + nd}{c}\right)$$

$$= \sum e\left(\frac{m(\overline{d_1b_1c_2 + d_2b_2c_1}) + n(d_1b_1c_2 + d_2b_2c_1)}{c_1c_2}\right),$$

summation is over all $d_1(\text{mod } c_1)$, $d_2(\text{mod } c_2)$, $(c_1, d_1) = 1$, $(c_2, d_2) = 1$. So that

$$S(m, n, c_1c_2) = S(m\overline{c_2}, n\overline{c_2}, c_1)S(m\overline{c_1}, n\overline{c_1}, c_2).$$

□

1.5 Distribution of Kloosterman angles

As a consequence of the Riemann Hypothesis for curves over functional fields, A. Weil obtained the following bound[1]

$$|S(m,1,p)| \leq 2\sqrt{p},$$

where p is a prime number and m is an integer coprime with p. Therefore, there is a unique *Kloosterman angle* $\theta(p,m) \in [0,\pi]$ such that

$$S(m,1,p) = 2\sqrt{p}\cos\theta(p,m).$$

There are two kinds of distribution of Kloosterman angles:

- *vertical*
$$\{\theta(p,m)\}_{\substack{1\leq m\leq p \\ (m,p)=1}}, \; p \to \infty;$$
- *horizontal*
$$\{\theta(p,m)\}_{\substack{1\leq p\leq P \\ (m,p)=1}}, \; m \text{ fixed}, \; P \to \infty.$$

In the vertical case, we have the following theorem by Katz[2].

Theorem 1.5.1. *Let $p \to \infty$, then the angles*

$$\{\theta(p,m)\}_{\substack{1\leq m\leq p \\ (m,p)=1}}$$

are equidistributed with respect to the Sato-Tate measure on $[0,\pi]$

$$d\mu_{ST}(\theta) = \frac{2}{\pi}\sin^2(\theta)d\theta.$$

[1] See [16]
[2] See [10]

Equivalently, for any interval $I = [a,b] \in [0,\pi]$,

$$lim_{p\to\infty} \frac{\#\{1 \le m \le p-1,\ \theta(m,p) \in I\}}{p-1} = \mu_{ST}(I) = \frac{2}{\pi}\int_a^b \sin^2(\theta) d\theta.$$

The horizontal case is still a conjecture.

Conjecture 1.5.2. *Let* $P \to \infty$, *m is a fixed non-zero integer, then the angles*

$$\{\theta(p,m)\}_{\substack{1\le p\le P \\ (m,p)=1}}$$

are equidistributed with respect to the Sato-Tate measure on $[0,\pi]$.

Equivalently, for any interval $I = [a,b] \in [0,\pi]$,

$$lim_{P\to\infty} \frac{\#\{p \le P,\ (m,p)=1,\ \theta(m,p) \in I\}}{\#\{p \le P\}} = \mu_{ST}(I) = \frac{2}{\pi}\int_a^b \sin^2(\theta) d\theta.$$

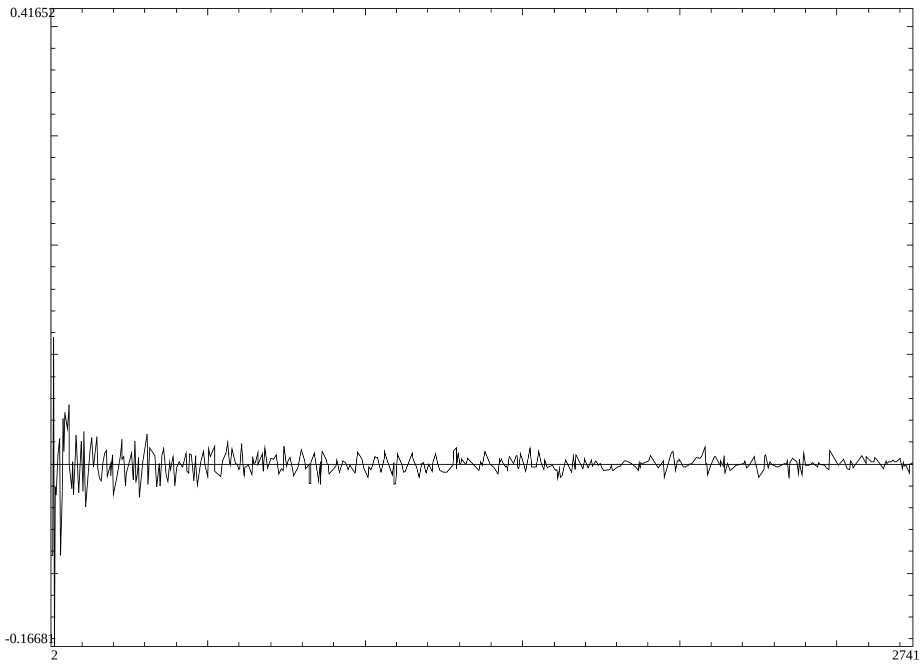


Figure 1.1: Difference between vertical and Sato-Tate measures on the interval $[1, 2]$

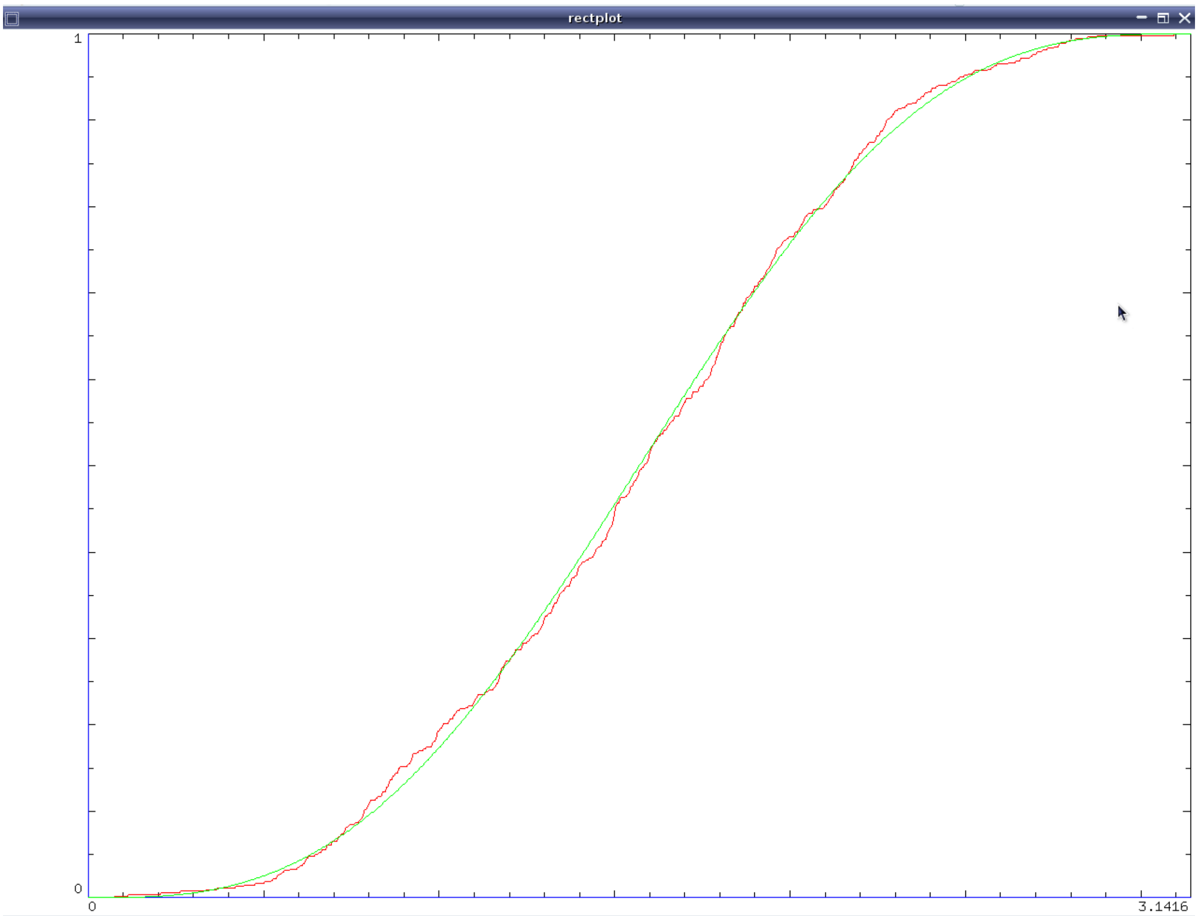


Figure 1.2: Vertical (red) and Sato-Tate (green) distribution functions

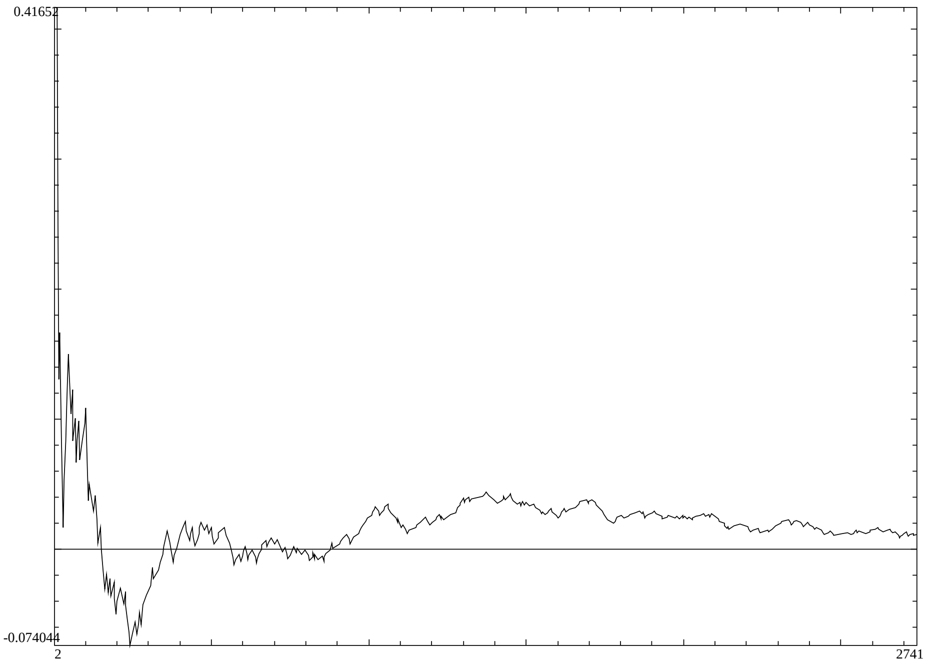


Figure 1.3: Difference between horizontal and Sato-Tate measures on the interval $[1, 2]$, $m = 1$

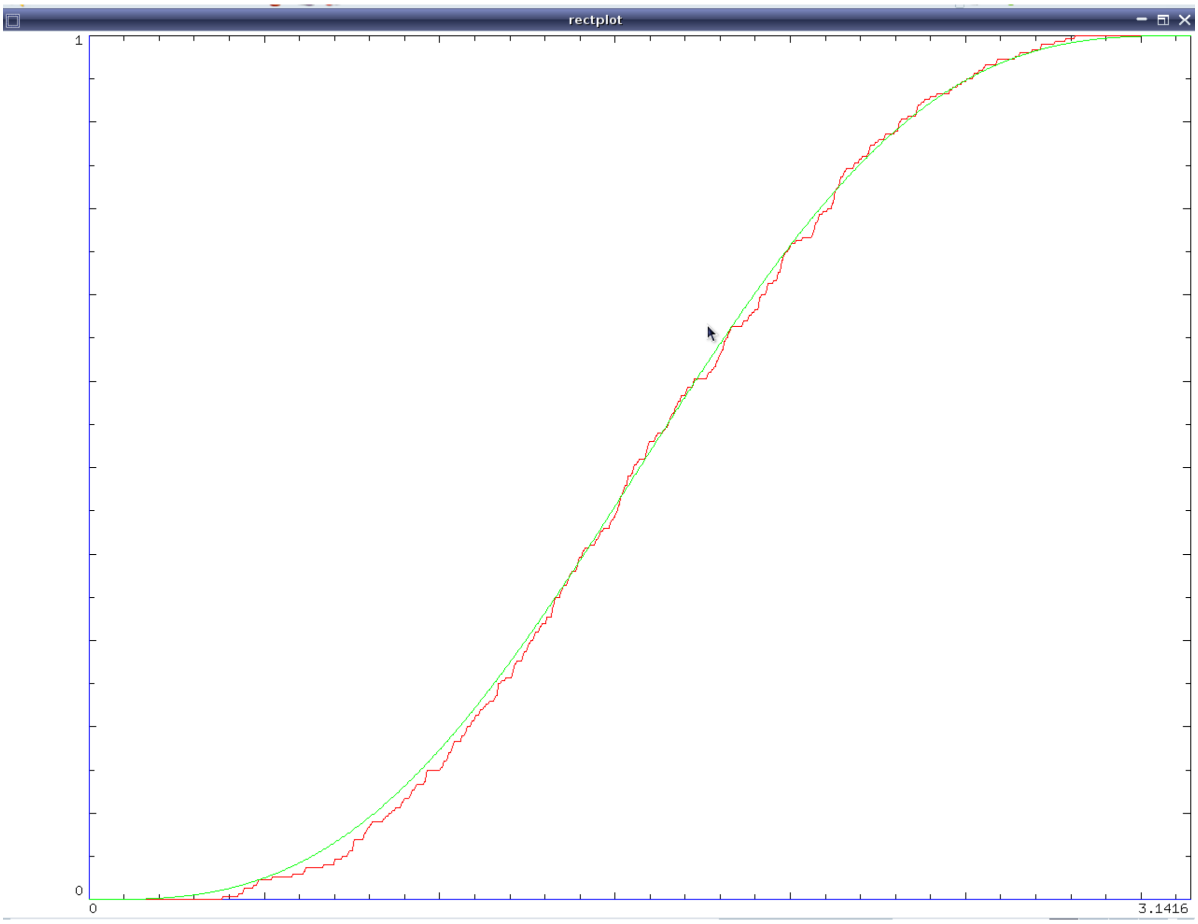


Figure 1.4: Horizontal (red) and Sato-Tate (green) distribution functions

1.6 Numerical computation of Poincaré series

1.6.1 Poincaré series and fundamental domain reduction algorithm

Current section provides a PARI/GP code for the calculation of Poincaré series

$$P_m(z) = e(mz) + \sum_{n>0} (F(m,n))\, e(nz), \tag{1.17}$$

with Fourier coefficients given by

$$F(m,n) = \sum_{c>0} S(m,n,c) L_c(m,n). \tag{1.18}$$

First, we compute the Kloosterman sum

$$S(m,n,c) = \sum_{\substack{d (\text{mod c}) \\ gcd(c,d)=1}} e\left(\frac{m\bar{d} + nd}{c}\right).$$

```
gp>{klsum(m,n,c,sum,dinv,t)=
gp>        sum=0;
gp>        for(d=0,c-1,
gp>            if(gcd(d,c)==1,
gp>                dinv=lift(1/Mod(d,c));
gp>                t=(m*dinv+n*d)/c;
gp>                sum=sum+exp(2*Pi*I*t);
gp>                )
gp>            )}
```

According to the formulas 1.13, 1.14, 1.15, the value of $L_c(m,n)$ can be found as follows.

```
gp>{coeffL(m,n,c,k,L)=
gp>  L=0;
gp>  if((n>0)&(m>=0),
gp>    if(m==0,
```

```
gp>     L=((2*Pi/I/c)^k)*(n^(k-1))/gamma(k),
gp>     L=2*Pi/c/(I^k)*((n/m)^((k-1)/2))*besselj(k-1,4*Pi*sqrt(m*n)/c)
gp>         )
gp>       )}
```

Note that for the values of $z \in \mathbb{H}^2$ with a small imaginary part $P_m(z)$ may converge very slow. However, the second property of definion 1.1.3 allows us to compute $P_m(\gamma z)$ for some $\gamma \in \Gamma$ and then recover the original series by the formula $P_m(z) = P_m(\gamma z)/j(\gamma, z)^k$. Furthermore, if a point $z_1 = \gamma z$ is in the fundamental domain F, then we have the estimate

$$|e(nz)| \leq e^{(-\pi\sqrt(3))n} < (\frac{1}{230})^n, \tag{1.19}$$

which provides a very good convergence.

Now we describe a fundamental domain reduction algorithm, which on input $z \in \mathbb{H}^2$ returns a matrix $\gamma \in \Gamma_2$ such that γz lies in the fundamental domain. In order to find such γ with $z_1 = \gamma z \in F$, we first apply $\begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$ with $n = \lfloor \Re(z) \rceil$ to translate z into the strip $|\Re(z)| \leq 1/2$.

Now if $z \notin F$, then $|z| < 1$ and

$$\Im(-1/z) = \Im(z/\left|z\right|^2) > \Im(z).$$

Replace z by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}(z)$ and repeat the process. Note that there are only finitely many integer pairs (c, d) such that $|cz + d| < 1$, and so, by the formula 1.2, there are only finitely many transforms of z with a larger imaginary part. Thus, the algorithm below terminates after a finite number of steps.

```
gp> {transM(z,A,z1,flag)=
gp>         A=[1,0;0,1];
gp>         flag=1;
gp>         z1=z;
gp>         while(flag==1,
gp>                 n=round(real(z1));
gp>                 z1=z1-n;
```

```
gp>                    A=[1,-n;0,1]*A;
gp>                    m=z1*conj(z1);
gp>                    if(abs(m)>=1,flag=0,A=[0,-1;1,0]*A;z1=-1/z1;);
gp>                    )
gp>              A}
```

Finally, we compute Fourier coefficients as in the formula 1.18

```
gp>{coeffF(cmax,m,n,k,F)=
gp>          F=0;
gp>          for(c=1,cmax,
gp>                F=F+ klsum(m,n,c)*coeffL(m,n,c,k)
gp>                )}
```

and Poincaré series as in the formula 1.17.

```
gp>{poincareS(z,k,m,nmax,cmax,A,z1,P)=
gp>              A=transM(z);
gp>              z1=(A[1,1]*z+A[1,2])/(A[2,1]*z+A[2,2]);
gp>              P=exp(2*Pi*I*m*z1);
gp>              for(n=1,nmax,
gp>                    P=P+coeffF(cmax,m,n,k)*exp(2*Pi*I*n*z1)
gp>                    );
gp>              P=P/(A[2,1]*z+A[2,2])^k}
```

1.6.2 Absolute error estimate

Notice that both sums on n in 1.17 and on c in 1.18 are infinite. But for the purpose of computing, we truncate these sums to a finite number of terms n_{max} and c_{max}, respectively. This leads to some incorrectness in our computations, which can be measured in terms of the absolute error.

Definition 1.6.1. Let X be a true value of the quantity and $X1$ its approximate value, then the *absolute error* is defined to be a numerical difference $X - X1$. An upper limit on the magnitude of the absolute error ΔX, such that

$$E_X = |X1 - X| \leq \Delta X,$$

is said to measure *absolute accuracy*.

Remark 1.6.2. This type of accuracy is convenient when we are dealing with sums, because the magnitude of the absolute error in the result is the sum of the magnitudes of the absolute errors in the summands.

In our case,

$$X = P_m(z) \text{ and } X1 = e(mz) + \sum_{0<n\leq n_{max}} \left(\widetilde{F}(m,n)\right) e(nz)$$

$$\text{with } \widetilde{F}(m,n) = \sum_{0<c\leq c_{max}} S(m,n,c)L_c(m,n).$$

Thus,

$$E_X = |X1 - X| = \left| \sum_{0<n\leq n_{max}} \left(\widetilde{F}(m,n)\right) e(nz) - \sum_{n>0} (F(m,n))\, e(nz) \right|$$

$$\leq \sum_{0<n\leq n_{max}} \left|\widetilde{F}(m,n) - F(m,n)\right| |e(nz)| + \sum_{n>n_{max}} |F(m,n)|\, |e(nz)|\,.$$

Let us denote

$$E_1 = \sum_{0<n\leq n_{max}} \left|\widetilde{F}(m,n) - F(m,n)\right| |e(nz)|$$

and

$$E_2 = \sum_{n>n_{max}} |F(m,n)|\, |e(nz)|\,.$$

The key ingredient of our computation is the bound for the Kloosterman sum $S(m,n,c)$. The optimal result for the prime values of c can be obtained using Weil's bound[1], but for our purpose it is enough to consider the trivial estimate

$$|S(m,n,c)| \leq c. \tag{1.20}$$

The next step is to bound the value of $L_c(m,n)$. In case $m=0$,

$$|L_c(0,n)| \ = \beta(k)\frac{n^{k-1}}{c^k}, \text{ where } \beta(k) = \frac{2^k\pi^k}{(k-1)!}.$$

[1] See [16]

Now suppose $m > 0$. Using the following estimate [1]

$$|J_k(x)| \le \frac{|x/2|^k}{k!},$$

we obtain

$$|L_c(m,n)| \ \le \beta(k)\frac{n^{k-1}}{c^k}.$$

So for any $m \ge 0$,

$$|S(m,n,c)L_c(m,n)| \le \beta(k)\frac{n^{k-1}}{c^{k-1}}.$$

Then

$$E_F = \left|\widetilde{F}(m,n) - F(m,n)\right| \le \beta(k)n^{k-1}\sum_{c>c_{max}}\frac{1}{c^{k-1}} \le \beta(k)n^{k-1}\int_{c_{max}}^{\infty}\frac{1}{x^{k-1}}dx$$

$$= \frac{\beta(k)n^{k-1}}{(k-2)c_{max}^{k-2}}$$

and

$$|F(m,n)| \le \beta(k)n^{k-1}\sum_{c>0}\frac{1}{c^{k-1}} = \beta(k)n^{k-1}\left(1+\int_1^{\infty}\frac{1}{x^{k-1}}dx\right) = \beta(k)n^{k-1}\frac{k-1}{k-2}.$$

Therefore,

$$E1 \le \frac{\beta(k)}{(k-2)c_{max}^{k-2}}\sum_{0<n\le n_{max}} n^{k-1}\frac{1}{230^n} \le \frac{\beta(k)n_{max}^{k-1}}{(k-2)c_{max}^{k-2}}\sum_{0<n\le n_{max}}\frac{1}{230^n}$$

$$= \frac{\beta(k)n_{max}^{k-1}}{(k-2)c_{max}^{k-2}}\frac{230^{n_{max}}-1}{229(230)^{n_{max}}} \le \frac{\beta(k)n_{max}^{k-1}}{229(k-2)c_{max}^{k-2}}$$

and

$$E_2 \le \beta(k)\frac{k-1}{k-2}\sum_{n>n_{max}} n^{k-1}e^{(-\pi\sqrt{3})n} \le \beta(k)\frac{k-1}{k-2}\int_{n_{max}}^{\infty}\frac{x^{k-1}dx}{e^{\pi\sqrt{3}x}}$$

$$= \beta(k)\frac{k-1}{k-2}\frac{e^{-\pi\sqrt{3}n_{max}}}{\pi\sqrt{3}}\left(n_{max}^{k-1} + \frac{(k-1)n_{max}^{k-2}}{\pi\sqrt{3}} + \frac{(k-1)(k-2)n_{max}^{k-3}}{(\pi\sqrt{3})^2} + \ldots + \frac{(k-1)!}{(\pi\sqrt{3})^{k-1}}\right)$$

[1] See [12], ex.9.6

$$\leq \beta(k)\frac{k-1}{k-2}\frac{e^{-\pi\sqrt{3}n_{max}}}{\pi\sqrt{3}}k!n_{max}^{k-1}.$$

Finally,

$$E_X \leq E_1 + E_2 \leq \alpha(k)\left(\frac{n_{max}^{k-1}}{c_{max}^{k-2}} + \frac{n_{max}^{k-1}}{e^{\pi\sqrt{3}n_{max}}}\right),$$

where

$$\alpha(k) = max\left(\frac{\beta(k)}{229(k-2)}, \frac{\beta(k)(k-1)k!}{(k-2)\pi\sqrt{3}}\right) = \frac{\beta(k)(k-1)k!}{(k-2)\pi\sqrt{3}}.$$

Chapter 2

SL(3) Kloosterman sums

Following the work [2], we generalize results of the previous chapter and obtain Kloosterman sums associated to the group $\mathbf{SL}_3(\mathbb{Z})$ as a part of Fourier coefficients of $\mathbf{SL}_3(\mathbb{Z})$ Poincaré series.

2.1 Generalized upper-half space and Iwasawa decomposition

Let $G_n = \mathbf{GL}_n(\mathbb{R})$ and $\Gamma_n = \mathbf{SL}_n(\mathbb{Z})$.

In order to define a notion of *generalized upper-half space* associated to the group G_n with $n \geq 2$, we prove the following theorem.

Theorem 2.1.1. ***(Iwasawa decomposition)*** *Every $g \in G_n$ decomposes as*

$$g = nak$$

with

$$n = n(x_{i,j}) \in N = \left\{ \begin{bmatrix} 1 & x_{1,2} & \dots & * \\ 0 & 1 & x_{2,3} & * \\ 0 & 0 & \ddots & x_{n-1,n} \\ 0 & 0 & 0 & 1 \end{bmatrix},\ x_{i,j} \in \mathbb{R},\ i < j \right\},$$

$$a = a(y_k) \in A = \left\{ \begin{bmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & \vdots \\ \vdots & \dots & \ddots & 0 \\ 0 & \dots & \dots & y_n \end{bmatrix}, \, y_i > 0 \right\},$$

$$k \in K = \boldsymbol{O}_n(\mathbb{R})\text{- } \textit{orthogonal group.}$$

The map

$$N \times A \times K \to G_n,$$

$$(n, a, k) \to nak$$

is a homeomorphism of topological spaces. In particular, the decomposition is unique.

Proof. We need to show that the map

$$f : (n, a, k) \to nak$$

is a continuous bijective map such that f^{-1} is also continuous.

1. The map is *injective.*

   The group NA is a group of upper triangular matrices with positive elements on the diagonal and K is a group of orthogonal matrices. If $g \in NA \cap K$, then $g^{-1} \in NA \cap K$ since $NA \cap K$ is a group. Furthermore, since g is orthogonal, $g = (g^{-1})^T$. So that g is upper triangular and lower triangular at the same time, i.e. it is diagonal. And the only orthogonal diagonal matrix with positive elements on the diagonal is the identity matrix, $g = I$. We conclude that $NA \cap K = I$.

   Suppose $nak = n'a'k'$, then $(na)^{-1}n'a' = (a^{-1}n^{-1}n'a)a^{-1}a' = k(k')^{-1}$.

   Note that A normalizes N, i.e. for all $a \in A$

   $$a^{-1}Na = N.$$

   Then

   $$(a^{-1}n^{-1}n'a)a^{-1}a' \in NA.$$

Since $NA \cap K = I$, we have that $k = k'$ and $na = n'a'$. Finally, $n = n'$ and $a = a'$ because $N \cap A = I$.

2. The map is *surjective*.

   We apply Gramm-Schmidt orthogonalization process to the columns $g_1, g_2, \ldots, g_n \in \mathbb{R}^n$ of matrix g^{-1}. Vectors $g_1, g_2, \ldots, g_n$ form a basis in $\mathbb{R}^n$ because g^{-1} is invertible. Define $h_1, h_2, \ldots, h_n \in \mathbb{R}^n$ and $h'_1, h'_2, \ldots, h'_n \in \mathbb{R}^n$ as follows:

$$h_1 = g_1,\ h'_1 = \frac{h_1}{||h_1||},$$

$$h_2 = -(g_2|h'_1)h'_1 + g_2,\ h'_2 = \frac{h_2}{||h_2||},$$

$$\vdots$$

$$h_i = -\sum_{j=1}^{i-1}(g_i|h_j)h'_j + g_i,\ h'_i = \frac{h_i}{||h_i||},$$

$$\vdots$$

$$h_n = -\sum_{j=1}^{n-1}(g_n|h_j)h'_j + g_n,\ h'_n = \frac{h_n}{||h_n||}.$$

   Note that $\{h_i\}$ form an orthogonal and $\{h'_i\}$ orthonormal bases of $\mathbb{R}^n$. Matrix g^{-1} sends canonical basis $\{e_i\}$ to $\{g_i\}$ via composition

$$e_i \to h'_i \to h_i \to g_i,\ i = 1, \ldots, n.$$

   The first map is an application of $k \in K$ (to the canonical basis), the second is an action of kak^{-1} with $a = diag(||h_1||, \ldots, ||h_2||) \in A$ and the third is $(ka)n(ka)^{-1}$. So that

$$g^{-1} = (ka)n(ka)^{-1}kak^{-1}k = kan$$

   and

$$g = nak.$$

3. The map is *continuous* and its *inverse is also continuous.*

   Given map is polynomial whence continuous. To show the continuity of inverse map notice that K is compact and $B = NA$ is closed subgroups of G_n. Let $g, g_m \in G_n$, $k, k_m \in K$, $a, a_m \in A$ and $n, n_m \in N$. If the sequence

   $$g_m = n_m a_m k_m \overset{m\to\infty}{\longrightarrow} g = nak,$$

   then since K is compact

   $$k_m \overset{m\to\infty}{\longrightarrow} k' \in K.$$

   So that

   $$b_m = n_m a_m \overset{m\to\infty}{\longrightarrow} b' = n'a' \in B$$

   since B is closed. Therefore, $g = n'a'k'$ and $n = n'$, $a = a'$, $k = k'$, i.e

   $$n_m \overset{m\to\infty}{\longrightarrow} n,$$

   $$a_m \overset{m\to\infty}{\longrightarrow} a$$

   and

   $$k_m \overset{m\to\infty}{\longrightarrow} k.$$

□

As a corollary, we derive Iwasawa decomposition of the group $\mathbf{SL}_n(\mathbb{R})$.

Corollary 2.1.2.

$$\boldsymbol{SL}_n(\mathbb{R}) = N\tilde{A}\boldsymbol{SO}_n(\mathbb{R}),$$

with $\tilde{A} = \{a \in A, y_n = 1\}$.

Remark 2.1.3. For later applications, it is convenient to write elements $a \in \tilde{A}$ as

$$a = \begin{bmatrix} y_1y_2\cdots y_{n-1} & 0 & \dots & 0 \\ 0 & y_1y_2\cdots y_{n-2} & \dots & \vdots \\ \vdots & \ddots & \dots & 0 \\ 0 & \dots & y_1 & 0 \\ 0 & \dots & \dots & 1 \end{bmatrix}.$$

Given change of variables is valid since $y_i \neq 0$ for all $i = 1, \ldots, n-1$.

Now in the analogous manner to the case $n = 2$, for $n > 2$ we define *generalized upper-half space*

$$\mathbb{H}^n \cong \frac{\mathbf{SL}_n(\mathbb{R})}{\mathbf{SO}_n(\mathbb{R})} \cong \frac{G_n}{(\mathbf{O}_n(\mathbb{R}) \cdot \mathbb{R}^\times)}.$$

The space $\mathbb{H}^n$ plays the same role for $\mathbf{GL}_n(\mathbb{R})$ that $\mathbb{H}^2$ played for $\mathbf{GL}_2(\mathbb{R})$.

By the *Iwasawa decomposition*, every $z \in \mathbb{H}^n$ can be uniquely written as

$$z = \begin{bmatrix} 1 & x_{1,2} & \ldots & * \\ 0 & 1 & x_{2,3} & * \\ 0 & 0 & \ddots & x_{n-1,n} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 y_2 \ldots y_{n-1} & 0 & \ldots & 0 \\ 0 & y_1 y_2 \ldots y_{n-2} & \ldots & \vdots \\ \vdots & \ddots & \ldots & 0 \\ 0 & \ldots & y_1 & 0 \\ 0 & \ldots & \ldots & 1 \end{bmatrix},$$

where $x_{i,j} \in \mathbb{R}$ for $j > i$, $y_1, \ldots, y_{n-1} > 0$. In particular, the generalized upper half plane $\mathbb{H}^3$ is the set of all matrices $z = na$ with

$$n = \begin{bmatrix} 1 & x_{1,2} & x_{1,3} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{bmatrix}, \; a = \begin{bmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $x_{1,2}, x_{1,3}, x_{2,3} \in \mathbb{R}$, $y_1, y_2 > 0$.

2.2 Automorphic forms and Fourier expansion

The group G_3 acts on $\mathbb{H}^3$ by matrix multiplication. It is generated by diagonal matrices, upper triangular matrices with $1s$ on the diagonal and the Weyl group W_3 consisting of all 3×3 matrices with exactly one 1 in each row and column. The approximation of fundamental domain for G_3 can be given by the Siegel set $\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}}$.[1] Here $\Sigma_{a,b} \subset \mathbb{H}^3$ $(a, b \geq 0)$ is the set of all matrices

[1]See [6], section 1.3.

$$\begin{bmatrix} 1 & x_{1,2} & x_{1,3} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $|x_{i,j}| \leq b$ for $1 \leq i < j \leq 3$ and $y_i > a$ for $i = 1, 2$.

The group G_3 is a Lie group whose Lie algebra $\mathbf{gl}(3, \mathbb{R})$ is the additive vector space (over $\mathbb{R}$) of all $n \times n$ matrices with coefficients in $\mathbb{R}$ with Lie brackets given by

$$[a, b] = a \cdot b - b \cdot a$$

for all $a, b \in \mathbf{gl}(3, \mathbb{R})$, where $\cdot$ denotes matrix multiplication. Define the set S be a space of smooth(infinitely differential) functions $F : G_3 \to \mathbb{C}$.

Definition 2.2.1. Let $F \in S$, $g \in G_3$ and $\alpha \in \mathbf{gl}(3, \mathbb{R})$. Then we define the *differential operator* D_α acting on F as

$$D_\alpha F(g) = \frac{\partial}{\partial t} F(g \cdot exp(t\alpha))|_{t=0} = \frac{\partial}{\partial t} F(g + t(g\alpha))|_{t=0}.$$

Remark 2.2.2. The differential operators D_α with $\alpha \in \mathbf{gl}(3, \mathbb{R})$ generate an associative algebra D^n defined over $\mathbb{R}$. And the ring of differential operators D_α is a realization of the universal enveloping algebra of the Lie algebra $\mathbf{gl}(3, \mathbb{R})$.[1]

Consider the center Δ of D^n. Every $D \in \Delta$ satisfies $D \cdot D' = D' \cdot D$ for all $D' \in D^n$. We would like to construct an eigenfunction of all differential operators $D \in \Delta$.

Let ν_1, ν_2 be complex parameters and

$$z = \begin{bmatrix} 1 & x_{1,2} & x_{1,3} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{H}^3.$$

We define a generalization of imaginary part function on the classical upper-half plane to $\mathbb{H}^3$ by

$$I_{\nu_1,\nu_2} : \mathbb{H}^3 \to \mathbb{C}$$

[1]See [6], section 2.2

$$I_{\nu_1,\nu_2}(z) = y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2}. \tag{2.1}$$

Below we prove that the function I_{ν_1,ν_2} is an eigenfunction of every $D \in \Delta$. Thus it determines a character λ_{ν_1,ν_2} on Δ, i.e.

$$DI_{\nu_1,\nu_2} = \lambda_{\nu_1,\nu_2}(D) I_{\nu_1,\nu_2}. \tag{2.2}$$

Theorem 2.2.3. *Let us define $D_{i,j} = D_{E_{i,j}}$, where $E_{i,j} \in \boldsymbol{gl}(3,\mathbb{R})$ is the matrix with an 1 at the i,j component and zeros elsewhere. Then for all $1 \le i < j \le 3$ and $k = 1, 2, \ldots$*

$$D^k_{i,j} I_{\nu_1,\nu_2}(z) = \begin{cases} \nu^k_{3-i} I_{\nu_1,\nu_2}(z) & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases},$$

where $D^k_{i,j}$ denotes the composition of differential operators $D_{i,j}$ iterated k times.

Proof. Let

$$z = na = \begin{bmatrix} 1 & x_{1,2} & x_{1,3} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{H}^3.$$

Note that the function $I_{\nu_1,\nu_2}(z)$ depends only on variables y_1, y_2. So that

$$I_{\nu_1,\nu_2}(na) = I_{\nu_1,\nu_2}(a)$$

and

$$D_{i,j} I_{\nu_1,\nu_2}(na) = D_{i,j} I_{\nu_1,\nu_2}(a).$$

We distinguish three different cases.

1. If $i < j$, then by definition 2.2.1

$$D_{i,j} I_{\nu_1,\nu_2}(a) = \frac{\partial}{\partial t} I_{\nu_1,\nu_2}(a + taE_{i,j})|_{t=0} = y_1 \cdot \ldots \cdot y_{3-i} \frac{\partial}{\partial x_{i,j}} I_{\nu_1,\nu_2}(a) = 0.$$

2. If $i = j$, then

$$D_{i,i} I_{\nu_1,\nu_2}(a) = \frac{\partial}{\partial t} I_{\nu_1,\nu_2}(a+taE_{i,i})|_{t=0} = \left(y_{3-i} \frac{\partial}{\partial y_{3-i}} - \sum_{l=3-i+1}^{2} y_l \frac{\partial}{\partial y_l} \right) I_{\nu_1,\nu_2}(a) = \nu_{3-i} I_{\nu_1,\nu_2}(a).$$

Similarly,

$$D^k_{i,i} I_{\nu_1,\nu_2}(a) = \left(\frac{\partial}{\partial t}\right)^k I_{\nu_1,\nu_2}(ae^{tE_{i,i}}) = \nu^k_{3-i} I_{\nu_1,\nu_2}(a).$$

3. Now let $i > j$. As before

$$D_{i,j} I_{\nu_1,\nu_2}(a) = \frac{\partial}{\partial t} I_{\nu_1,\nu_2}(a(Id + tE_{i,j}))|_{t=0},$$

where Id is the identity matrix. First we will show that

$$(Id + tE_{i,j}) \equiv M(\mathrm{mod}\ \mathbf{O}_3(\mathbb{R}) \cdot \mathbb{R}^\times),$$

where M a matrix such that $(t^2+1)^{-1/2}$ occurs at the position $\{j,j\}$, $(t^2+1)^{1/2}$ at the position $\{i,i\}$, all the other diagonal entries are ones, $\frac{t}{(t^2+1)^{-1/2}}$ occurs at the position $\{j,i\}$ and all other entries are zeros. Indeed, let $h = Id + tE_{i,j}$. Then

$$hh^t = (Id + tE_{i,j})(Id + tE_{j,i}) = Id + tE_{i,j} + tE_{j,i} + t^2 E_{i,i}.$$

Define a matrix $u = Id - \frac{t}{(t^2+1)} E_{j,i}$, then uhh^tu^t must be a diagonal matrix d. Let $d = a^{-1}(a^t)^{-1}$. Then by direct computations,

$$uhh^tu^t = Id + t^2 E_{i,i} - \frac{t}{t^2+1} E_{j,j},$$

$$u^{-1} = Id + \frac{t}{(t^2+1)} E_{j,i},$$

$$a^{-1} = Id + \left(\frac{1}{\sqrt{t^2+1}-1}\right) E_{j,j} + (\sqrt{t^2+1}-1) E_{i,i}.$$

Therefore,

$$M = u^{-1}a^{-1} = Id + \left(\frac{1}{\sqrt{t^2+1}-1}\right) E_{j,j} + \frac{t}{\sqrt{t^2+1}} E_{j,i}.$$

Since

$$auh(h^tu^ta^t) = Id,$$

we have

$$auh \in \mathbf{O}_3(\mathbb{R})$$

and

$$h \equiv M (\mathrm{mod}\ \mathbf{O}_3(\mathbb{R}) \cdot \mathbb{R}^\times)$$

as required.

Finally, taking the derivative of any of diagonal values and setting $t = 0$, we obtain zero as an answer. So the only contribution comes from non-diagonal entry $\frac{t}{(t^2+1)^{-1/2}}$. Thus,

$$D_{i,i} I_{\nu_1,\nu_2}(a) = a_1 \cdot \ldots \cdot a_{3-i} \frac{\partial}{\partial x_{i,j}} I_{\nu_1,\nu_2}(a) = 0.$$

$\square$

Now we can define the notion of automorphic form for the group Γ_3 and compute its Fourier expansion.

Definition 2.2.4. A function f on $\mathbb{H}^3$ is called an automorphic form (of type ν_1, ν_2) for Γ_3 if

- $f(\gamma z) = f(z)$ for $\gamma \in \Gamma_3$, $z \in \mathbb{H}^3$
- $Df = \lambda_{\nu_1,\nu_2} \cdot f$, where $D \in \Delta$ and λ_{ν_1,ν_2} as in 2.2.
- $f(z)$ has a polynomial growth in y_1, y_2 on the region $\{z : y_1, y_2 \geq 1\}$.

Remark 2.2.5. If in addition, f satisfies

$$\int_0^1 \int_0^1 f\left(\left(\begin{bmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix}\right) z\right) d\xi_1 d\xi_3 = 0$$

$$\int_0^1 \int_0^1 f\left(\left(\begin{bmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) z\right) d\xi_2 d\xi_3 = 0,$$

for all $z \in \mathbb{H}^3$, then f is called a *cusp form.*

Theorem 2.2.6. ***(Fourier expansion)***

Let f be an automorphic form with respect to $\Gamma_3 = \boldsymbol{SL}_3(\mathbb{Z})$ and

$$\Gamma_{3,\infty} = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \in \boldsymbol{SL}_3(\mathbb{Z}) \right\}$$

be a minimal parabolic subgroup of Γ_3.

Then f has a Fourier expansion given by

$$f(z) = \sum_{n=-\infty}^{\infty} F_{0,n}(z) + \sum_{\gamma \in \Gamma^2_{3,\infty} \backslash \Gamma^2_{3,+}} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} F_{m,n}(\gamma z),$$

where

$$F_{m,n}(z) = \int_0^1 \int_0^1 \int_0^1 f\left(\begin{bmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z \right) e(-m\xi_1 - n\xi_2) d\xi_1 d\xi_2 d\xi_3, \tag{2.3}$$

$$\Gamma^2_{3,+} = \left\{ \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} | A, B, C, D \in \mathbb{Z},\ AD - BC = 1 \right\}$$

and

$$\Gamma^2_{3,\infty} = \Gamma^2_3 \cap \Gamma_{3,\infty} = \left\{ \begin{bmatrix} 1 & B & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} | B \in \mathbb{Z} \right\}.$$

Proof. Since f is automorphic with respect to $\mathbf{SL}_3(\mathbb{Z})$,

$$f(z) = f\left(\begin{bmatrix} 1 & 0 & n_3 \\ 0 & 1 & n_1 \\ 0 & 0 & 1 \end{bmatrix} z \right), \ n_1, n_3 \in \mathbb{Z}.$$

Then analogously to one-dimensional Fourier expansion we can write

$$f(z) = \sum_{n_1,n_3 \in Z} f_{n_1,n_3}(z) \tag{2.4}$$

with

$$f_{n_1,n_3}(z) = \int_0^1 \int_0^1 f\left(\begin{bmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z\right) e(-n_1\xi_1 - n_3\xi_3) d\xi_1 d\xi_3. \tag{2.5}$$

The function $f_{n_1,n_3}(z)$ satisfies the following properties:

- Let $n_2 \in \mathbb{Z}$, then

$$f_{n_1,n_3}\left(\begin{bmatrix} 1 & n_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z\right) = f_{n_1+n_2n_3,n_3}(z). \tag{2.6}$$

By 2.5 the left-hand side is equal to

$$\int_0^1 \int_0^1 f\left(\begin{bmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & n_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z\right) e(-n_1\xi_1 - n_3\xi_3) d\xi_1 d\xi_3 =$$

$$\int_0^1 \int_0^1 f\left(\begin{bmatrix} 1 & n_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \xi_3 - n_2\xi_1 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z\right) e(-n_1\xi_1 - n_3\xi_3) d\xi_1 d\xi_3.$$

Since f is automorphic,

$$f\left(\begin{bmatrix} 1 & n_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \xi_3 - n_2\xi_1 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z\right) = f\left(\begin{bmatrix} 1 & 0 & \xi_3 - n_2\xi_1 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z\right).$$

And the following change of variables

$$\tilde{\xi_1} = \xi_1$$

$$\tilde{\xi_3} = \xi_3 - n_2\xi_1,$$

leads us to the result.

- Let $A, B, C, D, m \in \mathbb{Z}$, $AD - BC = 1$, $m > 0$.

$$f_{mD,mC}(z) = f_{m,0}\left(\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} z\right). \tag{2.7}$$

Let us consider the left-hand side

$$f_{mD,mC}(z) = \int_0^1 \int_0^1 f\left(\begin{bmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z\right) e(-mD\xi_1 - mC\xi_3) d\xi_1 d\xi_3 =$$

$$\int_0^1 \int_0^1 f\left(\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z\right) e(-mD\xi_1 - mC\xi_3) d\xi_1 d\xi_3$$

because f is automorphic. The last expression can be written as

$$\int_0^1 \int_0^1 f\left(\begin{bmatrix} 1 & 0 & B\xi_1 + A\xi_3 \\ 0 & 1 & D\xi_1 + C\xi_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} z\right) e(-mD\xi_1 - mC\xi_3) d\xi_1 d\xi_3.$$

Changing variables

$$\tilde{\xi_1} = D\xi_1 + C\xi_3,$$

$$\tilde{\xi_3} = B\xi_1 + A\xi_3,$$

we obtain the result.

In view of property 2.7, the formula 2.4 takes the form

$$f(z) = f_{0,0}(z) + \sum_{\gamma \in \Gamma^2_{3,\infty} \backslash \Gamma^2_{3,+}} \sum_{m=1}^{\infty} f_{m,0}(\gamma z). \tag{2.8}$$

Note that by 2.6, $f_{m,0}$ is invariant under the action of the matrices of the form

$$\begin{bmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ with } n \in \mathbb{Z}.$$

Therefore,

$$f_{m,0}(z) = \sum_{n=-\infty}^{\infty} F_{m,n}(z) \tag{2.9}$$

with

$$F_{m,n}(z) = \int_0^1 \int_0^1 \int_0^1 f\left(\begin{bmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z\right) e(-m\xi_1 - n\xi_3) d\xi_1 d\xi_2 d\xi_3. \tag{2.10}$$

Finally,

$$f(z) = \sum_{n=-\infty}^{\infty} F_{0,n}(z) + \sum_{\gamma \in \Gamma^2_{3,\infty} \backslash \Gamma^2_{3,+}} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} F_{m,n}(\gamma z). \tag{2.11}$$

□

Corollary 2.2.7. *If f is a cusp form, then*

$$F_{0,n} = F_{m,0} = 0 \text{ for every } n, m \in \mathbb{Z}$$

and Fourier expansion is given by

$$f(z) = \sum_{\gamma \in \Gamma^2_{3,\infty} \backslash \Gamma^2_{3,+}} \sum_{m=1}^{\infty} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} F_{m,n}(\gamma z).$$

2.3 SL(3) Poincaré series

For $z \in \mathbb{H}^3$ let

$$I_{\nu_1,\nu_2}(z) = y_1^{2\nu_1+\nu_2} y_2^{\nu_1+2\nu_2}.$$

And for every two integers n_1, n_2 define *E-function* as

$$E_{n_1,n_2} : \mathbb{H}^3 \to \mathbb{C},$$

satisfying

$$E_{n_1,n_2}\left(\begin{bmatrix} 1 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z\right) = e(n_1\xi_1 + n_2\xi_2)E_{n_1,n_2}(z) \text{ for all } \xi_1, \xi_2, \xi_3 \in \mathbb{R}, \tag{2.12}$$

$$E_{n_1,n_2}(z) = O(1) \text{ for } z \in \mathbb{H}^3,\ y_1, y_2 = O(1). \tag{2.13}$$

Definition 2.3.1. Let ν_1, ν_2 be two complex variables such that $\Re(\nu_i) > \frac{2}{3}$, $i = 1, 2$. Then the series

$$P_{n_1,n_2}(z; \nu_1, \nu_2) = \sum_{\gamma \in \Gamma_{3,\infty} \backslash \Gamma_3} I_{\nu_1,\nu_2}(\gamma z) E_{n_1,n_2}(\gamma z) \tag{2.14}$$

is called *general Poincaré series* for the minimal parabolic subgroup $\Gamma_{3,\infty}$.

Lemma 2.3.2. *The series* (2.14) *converges absolutely uniformly on compact subsets of* $\mathbb{H}^3$ *when* $\Re(\nu_i) > \frac{2}{3}$, $i = 1, 2$.

Proof. For every

$$z = \begin{bmatrix} y_1 y_2 & x_{1,2} y_1 & x_{1,3} \\ 0 & y_1 & x_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{H}^3,$$

the *left invariant* $\boldsymbol{GL}_3(\mathbb{R})-$ *measure*[1] on $\mathbb{H}^3$ is given by

$$d^*z = dx_{1,2} dx_{1,3} dx_{2,3} \frac{dy_1 dy_2}{(y_1 y_2)^3}.$$

Let us also recall the notion of *Siegel set* $\Sigma_{a,b} \subset \mathbb{H}^3$ $(a, b \geq 0)$ that is the set of all matrices

[1]For details see [6], prop. 1.5.3

$$\begin{bmatrix} 1 & x_{1,2} & x_{1,3} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $|x_{i,j}| \leq b$ for $1 \leq i < j \leq 3$ and $y_i > a$ for $i = 1, 2$.

Since 2.13, $E-$function $E_{n_1,n_2}(\gamma z)$ is bounded and it is enough to prove the theorem with respect to the series

$$\sum_{\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3} I_{\nu_1,\nu_2}(\gamma z),$$

i.e. that for every point $z_0 \in \mathbb{H}^3$ and some non-zero volume compact subset C_{z_0} of $\mathbb{H}^3$ such that $z_0 \in C_{z_0}$ the integral

$$\int_{C_{z_0}} \left| \sum_{\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3} I_{\nu_1,\nu_2}(\gamma z) \right| d^* z$$

converges. Without loss of generality, assume ν_1, ν_2 to be real. So we can write

$$\int_{C_{z_0}} \sum_{\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3} I_{\nu_1,\nu_2}(\gamma z) d^* z = \int_{(\Gamma_{3,\infty} \setminus \Gamma_3) \cdot C_{z_0}} I_{\nu_1,\nu_2}(z) d^* z.$$

According to the theorem of Siegel[1], there are only finitely many $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ such that $\gamma z_0 \in \Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}$. By continuity, for a sufficiently small C_{z_0} there are only finitely many $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ such that $\gamma z \in \Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}$ for all $z \in C_{z_0}$. Thus, there is some $a \geq \frac{\sqrt{3}}{2}$ such that

$$\gamma z \notin \Sigma_{a,\frac{1}{2}}$$

for all $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ and $z \in C_{z_0}$. Consequently,

$$\int_{(\Gamma_{3,\infty} \setminus \Gamma_3) \cdot C_{z_0}} I_{\nu_1,\nu_2}(z) d^* z \leq \int_0^1 \int_0^1 \int_0^1 \int_0^a \int_0^a y_1^{2\nu_1+\nu_2-3} y_2^{\nu_1+2\nu_2-3} dx_{1,2} dx_{1,3} dx_{2,3} dy_1 dy_2.$$

And the last integral converges absolutely if ν_1, ν_2 are sufficiently large.

[1] See, for example, [6] prop. 1.3.2

$\square$

2.4 Bruhat decomposition and Plucker coordinates

Similarly to $\mathbf{SL}_2(\mathbb{Z})$ case, it is necessary to know the Bruhat decomposition of the group $\mathbf{GL}_3(\mathbb{R})$ to compute Fourier expansion of Poincaré series.

Theorem 2.4.1. ***(Bruhat decomposition)*** *The group $\boldsymbol{GL}_3(\mathbb{R})$ can be decomposed as*

$$\boldsymbol{GL}_3(\mathbb{R}) = B_3 W_3 B_3,$$

where

- *B_3 is the standard Borel subgroup of $\boldsymbol{GL}_3(\mathbb{R})$, i.e. the group of invertible upper triangular matrices,*
- *W_3 is the Weyl group consisting of all 3×3 matrices which have exactly one 1 in each row and column and zeros elsewhere.*

Proof. Consider the element

$$g = \begin{bmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,1} & g_{2,2} & g_{2,3} \\ g_{3,1} & g_{3,2} & g_{3,3} \end{bmatrix} \in \mathbf{GL}_3(\mathbb{R}).$$

Let $g_{3,k}$ be the first non-zero element of the third row of matrix g. Without loss of generality, assume $k = 1$. Then we can always choose $b_1 \in B_3$ such that

$$gb_1 = \begin{bmatrix} g'_{1,1} & g'_{1,2} & g'_{1,3} \\ g'_{2,1} & g'_{2,2} & g'_{2,3} \\ 1 & 0 & 0 \end{bmatrix}.$$

Now multiplying on the left by a suitable element

$$b_1' = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{bmatrix} \in B_3,$$

we obtain a matrix of the form

$$b_1' g b_1 = \begin{bmatrix} 0 & g''_{1,2} & g''_{1,3} \\ 0 & g''_{2,2} & g''_{2,3} \\ 1 & 0 & 0 \end{bmatrix}.$$

Applying the same procedure to the first non-zero element of the second row, we can change the value of this entry to 1 and the rest of the entries in the corresponding row and column to 0 using suitable matrices b_2 and b_2'.

Finally, repeating the process with a non-zero element of the first row, we have

$$b_3' b_2' b_1' g b_1 b_2 b_3 \in W_3$$

with exactly one 1 in each row and column. □

Corollary 2.4.2. *Let*

$$G_\infty = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \right\} \subset B_3$$

and

$$D = \left\{ \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}, \, det(D) \neq 0 \right\}.$$

Then Bruhat decomposition can be written as

$$\boldsymbol{GL}_3(\mathbb{R}) = \bigcup_{w \in W_3} G_w, \tag{2.15}$$

where

$$G_w = G_\infty D w G_\infty = G_\infty w D G_\infty. \tag{2.16}$$

Proof. The proof consists of several facts. First, $B_3 = G_\infty D = DG_\infty$. Second, let $w \in W_3$, $d = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \in D$, then products wdw^{-1} and $w^{-1}dw$ are in D. This can be checked by direct computations for all 6 elements of W_3. For instance, if $w = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then

$$wdw^{-1} = w^{-1}dw = \begin{bmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{bmatrix}.$$

So that $wD = Dw$. Finally, notice that the product of 2 diagonal matrices is again diagonal.

□

For $\gamma \in G_3$ define the *involution*

$$^{i}\gamma = w^t \gamma w, \; w = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

If

$$\gamma = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ A_1 & A_2 & A_3 \end{bmatrix}$$

then

$$^{i}\gamma = \begin{bmatrix} a_{1,1}a_{2,2} - a_{1,2}a_{2,1} & a_{1,3}a_{2,1} - a_{1,1}a_{2,2} & a_{1,2}a_{2,3} - a_{1,3}a_{2,2} \\ a_{1,1}A_1 - a_{1,1}B_1 & a_{1,1}C_1 - a_{1,3}A_1 & a_{1,3}B_1 - a_{1,2}C_1 \\ a_{2,1}B_1 - a_{2,2}A_1 & a_{2,3}A_1 - a_{2,1}C_1 & a_{2,2}C_1 - a_{2,3}B_1 \end{bmatrix}. \tag{2.17}$$

Definition 2.4.3. Let us denote elements of the bottom row of $^{i}\gamma$ as

$$A_2 = a_{2,1}B_1 - a_{2,2}A_1,$$

$$B_2 = a_{2,3}A_1 - a_{2,1}C_1,$$

$$C_2 = a_{2,2}C_1 - a_{2,3}B_1.$$

Then the vectors

$$\rho_1 = \{A_1, B_1, C_1\} \text{ and } \rho_2 = \{A_2, B_2, C_2\}$$

are called the *Plucker coordinates* of γ.

Remark 2.4.4. Plucker coordinates $\{\rho_1, \rho_2\}$ satisfy the following relation

$$A_1C_2 + B_1B_2 + C_1A_2 = 0 \tag{2.18}$$

called *Plucker relation*.

Theorem 2.4.5. *Let $G' = \boldsymbol{SL}_3(\mathbb{R})$ and G_∞ is the group of 3×3 upper triangular unipotent matrices. Then the involution 2.17 induces the bijection of $G_\infty \backslash G'$ into the set of all $(A_1, B_1, C_1, A_2, B_2, C_2) \in \mathbb{R}^6$ such that 2.18 is satisfied. Furthermore, the given orbit of $G_\infty \backslash G'$ contains an element of Γ_3 if and only if A_1, B_1, C_1 are coprime integers and also A_2, B_2, C_2 are coprime integers.*

Proof. The map is defined as follows: the element

$$\gamma = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \in G_\infty \backslash G' \tag{2.19}$$

goes to $(A_1, B_1, C_1, A_2, B_2, C_2)$, where

$$A_1 = -a_{3,1},\ B_1 = -a_{3,2},\ C_1 = -a_{3,3}, \tag{2.20}$$

$$A_2 = a_{2,1}a_{3,2} - a_{2,2}a_{3,1},\ B_2 = a_{2,3}a_{3,1} - a_{2,1}a_{3,3},\ C_2 = a_{2,2}a_{3,3} - a_{2,3}a_{3,2}. \tag{2.21}$$

We need to show that the given map is bijective. To prove the injectivity, we show that if there are two matrices

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \text{ and } \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{bmatrix}$$

with the same coordinates $(A_1, B_1, C_1, A_2, B_2, C_2)$, then there exist λ_1, λ_2, $\lambda_3 \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 & \lambda_2 & \lambda_3 \\ 0 & 1 & \lambda_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{bmatrix}. \tag{2.22}$$

The first step is to show that there exist $\lambda_1 \in R$ such that

$$\begin{bmatrix} 0 & 1 & \lambda_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} = \begin{bmatrix} b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{bmatrix}. \tag{2.23}$$

We have

$$a_{3,1} = -A_1 = b_{3,1},$$

$$a_{3,2} = -B_1 = b_{3,2},$$

$$a_{3,3} = -C_1 = b_{3,3}$$

and

$$a_{2,2}a_{3,1} - a_{2,1}a_{3,2} = A_2 = b_{2,2}b_{3,1} - b_{2,1}b_{3,2},$$

$$a_{2,3}a_{3,1} - a_{2,1}a_{3,3} = B_2 = b_{2,3}b_{3,1} - b_{2,1}b_{3,3},$$

$$a_{2,2}a_{3,3} - a_{2,3}a_{3,2} = C_2 = b_{2,2}b_{3,3} - b_{2,3}b_{3,2}.$$

Therefore,

$$a_{3,1}(a_{2,2} - b_{2,2}) = a_{3,2}(a_{2,1} - b_{2,1}),$$

$$a_{3,2}(a_{2,3} - b_{2,3}) = a_{3,3}(a_{2,2} - b_{2,2}),$$

$$a_{3,3}(a_{2,1} - b_{2,1}) = a_{3,1}(a_{2,3} - b_{2,3}).$$

Note that $a_{3,1}$, $a_{3,2}$, $a_{3,3}$, are not all zeros. Without loss of generality, assume

$a_{3,1} \neq 0$. Let us take

$$\lambda_1 = \frac{b_{2,1} - a_{2,1}}{a_{3,1}}.$$

Then 2.23 is satisfied, as required. Now we need to find λ_2, λ_3 such that 2.22 is true. The values A_2, B_2, C_2 are not all zeros. Suppose, for instance, $A_2 = a_{2,2}a_{3,1} - a_{2,1}a_{3,2} \neq 0$. Then the vectors $(a_{2,1}, a_{2,2})$ and $(a_{3,1}, a_{3,2})$ are linearly independent, so there are λ_2, λ_3 such that

$$\begin{bmatrix} 1 & \lambda_2 & \lambda_3 \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} = \begin{bmatrix} b_{1,1} & b_{1,2} \end{bmatrix}. \tag{2.24}$$

Let us compute the determinants of matrices in 2.22

$$-b_{1,3}A_2 - b_{1,2}B_2 - b_{1,1}C_2$$

$$= -(a_{1,3}+\lambda_2 a_{2,3}+\lambda_3 a_{2,3})A_2 - (a_{1,2}+\lambda_2 a_{2,2}+\lambda_3 a_{3,2})B_2 - (a_{1,1}+\lambda_2 a_{2,1}+\lambda_3 a_{3,1})C_2.$$

By, 2.24, we have

$$-(a_{1,3} + \lambda_2 a_{2,3} + \lambda_3 a_{2,3})A_2 - (a_{1,2} + \lambda_2 a_{2,2} + \lambda_3 a_{3,2})B_2 - (a_{1,1} + \lambda_2 a_{2,1} + \lambda_3 a_{3,1})C_2$$

$$= -(a_{1,3} + \lambda_2 a_{2,3} + \lambda_3 a_{2,3})A_2 - b_{1,2}B_2 - b_{1,1}C_2.$$

So that

$$b_{1,3} = a_{1,3} + \lambda_2 a_{2,3} + \lambda_3 a_{2,3}$$

and 2.22 follows.

The next step is to show surjectivity. Suppose we are given

$$(A_1, B_1, C_1) \neq (0, 0, 0)$$

and

$$(A_2, B_2, C_2) \neq (0, 0, 0)$$

such that 2.18 is satisfied. We may find $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ such that

$$A_1X_1 + B_1Y_1 + C_1Z_1 = A_2X_2 + B_2Y_2 + C_2Z_2 = 1. \tag{2.25}$$

Let

$$a_{1,1} = -Z_2,\ a_{1,2} = -Y_2,\ a_{1,3} = -X_2,$$

$$a_{2,1} = Y_1A_2 - Z_1B_2,\ a_{2,2} = Z_1C_2 - X_1A_2,\ a_{2,3} = X_1B_2 - Y_1C_2,$$

$$a_{3,1} = -A_1,\ a_{3,2} = -B_1,\ a_{3,3} = -C_1.$$

Using 2.18, one can verify relations 2.20 and 2.21. Likewise, the determinant of γ (given by 2.19) is one. This shows that the map is surjective.

The last thing to prove is the characterization of orbits, which contain integer matrices. If 2.19 is an integer matrix and $(A_1, B_1, C_1, A_2, B_2, C_2)$ are given by 2.20, 2.21, then A_1, B_1, C_1 have to be coprime since the determinant of 2.19 is equal to 1. The values A_2, B_2, C_2 are also coprime since the determinant

$$-a_{1,3}A_2 - a_{1,2}B_2 - a_{1,1}C_2 = 1.$$

Conversely, let A_1, B_1, C_1 be coprime integers and A_2, B_2, C_1 are also coprime integers such that 2.18 is satisfied. To show that the coset parametrized by this invariants contains an integer matrix, we may find integer values of $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ satisfying 2.25. Then the matrix 2.19 can be constructed as in the proof of surjectivity. Clearly, all entries of this matrix are integral.

$\square$

Remark 2.4.6. The theorem above also gives the characterization of the orbits of $\Gamma_{3,\infty} \backslash \Gamma_3$ in terms of their Plucker coordinates since $\Gamma_{3,\infty} \backslash \Gamma_3$ is included injectively in $G_\infty \setminus G'$.

Consider the element

$$\gamma = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ A_1 & B_1 & C_1 \end{bmatrix} \in \Gamma_{3,\infty} \setminus \Gamma_3$$

with Plucker coordinates $\rho_1 = \{A_1, B_1, C_1\}$ and $\rho_2 = \{A_2, B_2, C_2\}$. Below we determine explicitly Bruhat decomposition of $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ depending on its Plucker coordinates.

Proposition 2.4.7. *If $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ have coordinates $A_1 = A_2 = B_1 = B_2 = 0$, $C_1, C_2 \neq 0$, then*

$$\gamma = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & a_{2,3} \\ 0 & 0 & C_1 \end{bmatrix} = \begin{bmatrix} a_{1,1} & 0 & 0 \\ 0 & \frac{C_2}{C_1} & 0 \\ 0 & 0 & C_1 \end{bmatrix} \begin{bmatrix} 1 & \frac{a_{1,2}}{a_{1,1}} & \frac{a_{1,3}}{a_{1,1}} \\ 0 & 1 & \frac{a_{2,3}C_1}{C_2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Proposition 2.4.8. *If $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ have coordinates $A_1 = A_2 = B_1 = 0$, $C_1, B_2 \neq 0$, then*

$$\gamma = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ 0 & 0 & C_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-a_{1,1}C_1}{B_2} & 0 \\ 0 & 1 & \frac{a_{2,3}}{C_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-B_2}{C_1} & 0 & 0 \\ 0 & \frac{1}{B_2} & 0 \\ 0 & 0 & C_1 \end{bmatrix} \begin{bmatrix} 1 & \frac{-C_2}{B_2} & 0 \\ 0 & 1 & a_{1,3}B_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proposition 2.4.9. *If $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ have coordinates $A_1 = A_2 = B_2 = 0$, $B_1, C_2 \neq 0$, then*

$$\gamma = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & a_{2,3} \\ 0 & B_1 & C_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{a_{2,2}}{B_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{1,1} & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & \frac{-1}{a_{1,1}B_1} \end{bmatrix} \begin{bmatrix} 1 & \frac{a_{1,2}}{a_{1,1}} & \frac{a_{1,3}}{a_{1,1}} \\ 0 & 1 & \frac{C_1}{B_1} \\ 0 & 0 & 1 \end{bmatrix}.$$

Proposition 2.4.10. *If $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ have coordinates $A_1 = 0$, $B_1, A_2 \neq 0$, then*

$$\gamma = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ 0 & B_1 & C_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{a_{1,1}B_1}{A_2} & \frac{-b_{1,1}}{A_2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{A_2}{B_1} & & \\ 0 & B_1 & 0 \\ 0 & 0 & \frac{1}{A_2} \end{bmatrix} \begin{bmatrix} 1 & \frac{a_{2,2}B_1}{A_2} & \frac{a_{2,3}B_1}{A_2} \\ 0 & 1 & \frac{C_1}{B_1} \\ 0 & 0 & 1 \end{bmatrix}.$$

Proposition 2.4.11. *If $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ have coordinates $A_2 = 0$, $A_1, B_2 \neq 0$, then*

$$\gamma = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ A_1 & B_1 & C_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{a_{1,1}}{A_1} \\ 0 & 1 & \frac{a_{2,1}}{A_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \frac{1}{B_2} & 0 \\ 0 & 0 & \frac{B_2}{A_1} \end{bmatrix} \begin{bmatrix} 1 & \frac{B_1}{A_1} & \frac{C_1}{A_1} \\ 0 & 1 & \frac{-b_{2,2}B_2}{A_1} \\ 0 & 0 & 1 \end{bmatrix},$$

where $b_{2,2} = a_{1,1}C_1 - a_{1,3}A_1$.

Proposition 2.4.12. *If* $\gamma \in \Gamma_{3,\infty} \setminus \Gamma_3$ *have coordinates* $A_1, A_2 \neq 0$, *then*

$$\gamma = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ A_1 & B_1 & C_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-b_{2,1}}{A_2} & \frac{a_{1,1}}{A_1} \\ 0 & 1 & \frac{a_{2,1}}{A_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \frac{-A_2}{A_1} & 0 \\ 0 & 0 & \frac{1}{A_2} \end{bmatrix} \begin{bmatrix} 1 & \frac{B_1}{A_1} & \frac{C_1}{A_1} \\ 0 & 1 & \frac{-B_2}{A_2} \\ 0 & 0 & 1 \end{bmatrix},$$

where $b_{2,1} = a_{1,2}A_1 - a_{1,1}B_1$.

All the propositions above can be verified by direct computation, i.e. multiplying matrices on right-hand side and taking into account 2.4.3 , one obtains the result.

Definition 2.4.13. Let us define the following group

$$\Gamma_w = (w^{-1}\Gamma_{3,\infty}w)^t \cap \Gamma_{3,\infty}.$$

Explicitly,

$$\Gamma_{w_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ with } w_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\Gamma_{w_2} = \left\{ \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, m \in \mathbb{Z} \right\} \text{ with } w_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\Gamma_{w_3} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}, n \in \mathbb{Z} \right\} \text{ with } w_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\Gamma_{w_4} = \left\{ \begin{bmatrix} 1 & 0 & l \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}, n, l \in \mathbb{Z} \right\} \text{ with } w_4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\Gamma_{w_5} = \left\{ \begin{bmatrix} 1 & m & l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, m, l \in \mathbb{Z} \right\} \text{ with } w_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\Gamma_{w_6} = \left\{ \begin{bmatrix} 1 & m & l \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}, \, m, l, n \in \mathbb{Z} \right\} \text{ with } w_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Remark 2.4.14. Given results could be checked my direct computation. Let us consider, for example, the case $w = w_5$. Take an arbitrary matrix

$$g = \begin{bmatrix} 1 & n & m \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix} \in \Gamma_{3,\infty},$$

where $m, l, n \in \mathbb{Z}$. Then

$$(w^{-1}\Gamma_{3,\infty} w)^t = \begin{bmatrix} 1 & m & l \\ 0 & 1 & 0 \\ 0 & n & 1 \end{bmatrix}.$$

Intersecting the set of such matrices with $\Gamma_{3,\infty}$, we obtain the required result.

Proposition 2.4.15. *The group Γ_w acts properly on the right on $\Gamma_{3,\infty} \backslash \Gamma_3 \cap G_w / U$, where*

$$U = \left\{ \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}, \, \epsilon_i = \pm 1, \, \epsilon_1\epsilon_2\epsilon_3 = 1 \right\}$$

and G_w is as in 2.16. Thus, $\Gamma_{3,\infty} \setminus \Gamma_3 \cap G_w / U\Gamma_w$ is a well-defined double coset space.

Proof. Note that

$$\begin{bmatrix} * & * & * \\ a_{2,1} & a_{2,2} & a_{2,3} \\ A_1 & B_1 & C_1 \end{bmatrix} \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}$$

$$= \begin{bmatrix} * & * & * \\ \epsilon_1 a_{2,1} & \epsilon_2 a_{2,2} & \epsilon_3 a_{2,3} \\ \epsilon_1 A_1 & \epsilon_2 B_1 & \epsilon_3 C_1 \end{bmatrix}.$$

So that

$$A_1 \to \epsilon_1 A_1, \, B_1 \to \epsilon_2 B_1, \, C_1 \to \epsilon_3 C_1 \tag{2.26}$$

$$A_2 \to \epsilon_1\epsilon_2 A_2,\ B_2 \to \epsilon_1\epsilon_3 B_2,\ C_2 \to \epsilon_2\epsilon_3 C_2. \tag{2.27}$$

Therefore, the representatives of $\Gamma_{3,\infty} \setminus \Gamma_3 \cap G_w (\mathrm{mod}\ U)$ can be obtained by fixing two signs of non-zero invariants.

Right multiplication by Γ_w maps left cosets to left cosets, so Γ_w acts on $\Gamma_{3,\infty} \setminus \Gamma_3 \cap G_w/U$.

We need to show that the action is proper, i.e. if $\gamma \in \Gamma_3 \cap G_w$, $\tau \in \Gamma_w$ and

$$\Gamma_{3,\infty}\gamma\tau U = \Gamma_{3,\infty}\gamma U,$$

then $\tau = id$. In order to prove this fact we introduce two new sets:

$$H_1 = w^{-1} G_\infty w \cap G_\infty$$

and

$$H_2 = w^{-1} G_\infty^t \cap G_\infty,$$

where $w \in W_3$. Explicit matrix computation for elements $w_i \in W_3$, $i = 1, 2, \ldots, 6$ shows that every $g \in G_\infty$ has unique expressions

$$g = h_1 h_2 \tag{2.28}$$

and

$$g = h_2' h_1' \tag{2.29}$$

with $h_1, h_1' \in H_1$, $h_2, h_2' \in H_2$. More precisely,

$$H_1 = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix},\ H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ if } w = w_1,$$

$$H_1 = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix},\ H_2 = \begin{bmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ if } w = w_2,$$

$$H_1 = \begin{bmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \text{ if } w = w_3,$$

$$H_1 = \begin{bmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, H_2 = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \text{ if } w = w_4,$$

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, H_2 = \begin{bmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ if } w = w_5,$$

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, H_2 = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \text{ if } w = w_6.$$

By Bruhat decomposition, $\gamma = b_1 w d b_2$ with $b1, b2 \in G_\infty$, $d \in D$ and $w \in W_3$. According to 2.28 and 2.29, without loss of generality, we may assume that $b_2 \in H_1$. Since

$$\Gamma_{3,\infty}\gamma\tau U = \Gamma_{3,\infty}\gamma U,$$

we conclude that

$$b_2\tau b_2^{-1} \in H_1 \cap H_2 = \{I\}$$

and $\tau = id$ as required.

□

Finally, for every $w \in W_3$ we determine a canonical set of coset representatives R_w for the quotient space $\Gamma_{3,\infty} \setminus \Gamma_3 \cap G_w / U\Gamma_w$. We give a proof in case $w = w_2$ as an example.

Proposition 2.4.16. *If* $w = w_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, *then*

$$R_w = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Proposition 2.4.17. *If* $w = w_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$*, then*

$$R_w = \left\{ \begin{bmatrix} a_{1,1} & a_{1,2} & 0 \\ -B_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\},$$

where $(B_2, C_2) = 1$*,* $B_2 > 0$*,* $C_2 (\text{mod } B_2)$ *and values* $a_{1,1}$*,* $a_{1,2}$ *are chosen uniquely such that* $a_{1,1}C_2 + a_{1,2}B_2 = 1$ *for each pair* (B_2, C_2)*.*

Proof. By proposition 2.4.8, $\Gamma_{3,\infty} \setminus \Gamma_3 \cap G_w$ have coordinates $A_1 = A_2 = B_1 = 0$, $C_1, B_2 \neq 0$. By 2.26, 2.27, we can obtain a representative of $\Gamma_{3,\infty} \setminus \Gamma_3 \cap G_w (\text{mod } U)$ by fixing the signs of C_1, B_2. Let $C_1, B_2 > 0$. Furthermore, by theorem 2.4.5, Plucker coordinates $B_2 = -a_{2,1}C_1$ and $C_2 = a_{2,2}C_1$ are coprime integers, so that $C_1 = 1$. Consequently, $a_{2,2} = C_2$ and $a_{2,1} = -B_2$. Consider

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ 0 & 0 & C_1 \end{bmatrix} \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{1,1} & ma_{1,1} + a_{1,2} & a_{1,3} \\ a_{2,1} & ma_{2,1} + a_{2,2} & a_{2,3} \\ 0 & 0 & C_1 \end{bmatrix}.$$

Thus, to obtain the coset representative modulo Γ_w , we need to consider $a_{2,2} (\text{mod } a_{2,1})$, equivalently $C_2 (\text{mod } B_2)$. The determinant of obtained matrix

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ -B_2 & C_2 & a_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$$

must be equal to one. Whence,variables $a_{1,1}$, $a_{1,2}$ are chosen uniquely such that $a_{1,1}C_2 + a_{1,2}B_2 = 1$ for each pair (B_2, C_2). Nor determinant, nor Plucker coordinates depend on the values of $a_{1,3}$, $a_{2,3}$, so we can let $a_{1,3} = a_{2,3} = 0$. The proposition follows. □

Proposition 2.4.18. *If* $w = w_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, *then*

$$R_w = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{2,2} & a_{2,3} \\ 0 & B_1 & C_1 \end{bmatrix} \right\},$$

where $(B_1, C_1) = 1$, $B_1 > 0$, $C_1 (mod\ B_1)$ *and values* $a_{2,2}$, $a_{2,3}$ *are chosen uniquely such that* $a_{2,2}C_1 + a_{2,3}B_1 = 1$ *for each pair* (B_1, C_1).

Proposition 2.4.19. *If* $w = w_4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, *then*

$$R_w = \left\{ \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ \frac{A_2}{B_1} & \alpha C_2 & \beta C_2 \\ 0 & B_1 & C_1 \end{bmatrix} \right\},$$

where $(B_1, C_1) = 1$, $B_1 > 0$, $C_1 (mod\ B_1)$, $(\frac{A_2}{B_1}, C_2) = 1$, $A_2 > 0$, $C_2 (mod\ A_2)$, $B_1B_2 + C_1A_2 = 0$ *and values* α, β *are chosen uniquely such that* $\alpha C_2 - \beta B_2 = 1$ *for each pair* (B_1, C_1). *The values* $a_{1,1}$, $a_{1,2}$, $a_{1,3}$ *are chosen uniquely such that the matrix has determinant one for every quintuple* $(B_1, C_1, A_2, B_2, C_2)$.

Proposition 2.4.20. *If* $w = w_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, *then*

$$R_w = \left\{ \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & \frac{a_{2,1}B_1}{A_1} & a_{2,3} \\ A_1 & B_1 & C_1 \end{bmatrix} \right\},$$

where $(B_2, C_2) = 1$, $B_2 > 0$, $C_2 (mod\ B_2)$, $(\frac{A_1}{B_1}, C_1) = 1$, $A_1 > 0$, $C_1 (mod\ A_1)$, $A_1C_2 + B_1B_2 = 0$ *and the values* $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{2,1}$, $a_{2,3}$ *are chosen uniquely such that the matrix has determinant one for every quintuple* $(A_1, B_1, C_1, B_2, C_2)$.

Proposition 2.4.21. *If* $w = w_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, *then*

$$R_w = \left\{ \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ A_1 & B_1 & C_1 \end{bmatrix} \right\},$$

where $A_1, A_2 > 0$, $B_1, C_1 (mod\ A_1)$, $B_2, C_2 (mod\ A_2)$, $(A_1, B_1, C_1) = (A_2, B_2, C_2) = 1$, $A_1C_2 + B_1B_2 + A_2C_1 = 0$ *and for every sextuple* $(A_1, B_1, C_1, A_2, B_2, C_2)$ *the values* $a_{2,1}$, $a_{2,2}$, $a_{2,3}$ *are uniquely chosen such that*

$$A_2 = a_{2,1}B_1 - a_{2,2}A_1,\ B_2 = a_{2,3}A_1 - a_{2,1}C_1,\ C_2 = a_{2,2}C_1 - a_{2,3}B_1$$

and the rest of values $a_{1,1}$, $a_{1,2}$, $a_{1,3}$ *are chosen uniquely such that the matrix has determinant one.*

2.5 SL(3) Kloosterman sums

There are six Kloosterman sums that occur in the Fourier expansion of Poincaré series (2.14). Let $m_1, m_2, n_1, n_2 \in \mathbb{Z}$, $D_1, D_2 \in \mathbb{Z}_{>0}$, $S(m, n, c)$ be a classical Kloosterman sum and W_3 denote a Weyl group of permutation matrices of $\mathbf{GL}_3(\mathbb{Z})$. By definition,

$$\delta_{a,b} = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}.$$

For each $w_i \in W_3$ $(i = 1, \ldots, 6)$ we associate a certain Kloosterman sum S_{w_i}. Namely,

$$S_{w_1} = \delta_{D_1,1}\delta_{D_2,1} \text{ with } w_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$S_{w_2} = \delta_{D_1,1} S(m_2, n_2; D_2) \text{ with } w_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$S_{w_3} = \delta_{D_2,1} S(m_1, n_1; D_1) \text{ with } w_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The remaining cases, corresponding to the elements

$$w_4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \, w_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } w_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

are the most interesting: these are exponential sums different from classical Kloosterman sums. Below we define two new types of Kloosterman sums $K_1 = K_1(m_1, m_2, n_1, n_2; D_1, D_2)$ and $K_2 = K_2(m_1, n_1, n_2; D_1, D_2)$ so that

$$S_{w_4} = K_2(m_1, n_1, n_2; D_1, D_2),$$

$$S_{w_5} = K_2(m_2, n_2, n_1; D_2, D_1),$$

$$S_{w_6} = K_1(m_2, m_1, n_1, n_2; D_2, D_1).$$

Definition 2.5.1. First type $K_1 = K_1(m_1, m_2, n_1, n_2; D_1, D_2)$ of $\mathbf{SL}_3(\mathbb{Z})$ Kloosterman sum is

$$K_1 = \sum_{\substack{B_1 (\text{mod } D_1) \\ B_2 (\text{mod } D_2)}} \sum_{\substack{C_1 (\text{mod } D_1) \\ C_2 (\text{mod } D_2)}} e\left(\frac{m_1 B_1 + n_1 (Y_1 D_2 - Z_1 B_2)}{D_1} + \frac{m_2 B_2 + n_2 (Y_2 D_1 - Z_2 B_2)}{D_2} \right) \tag{2.30}$$

where the inner sum satisfies the following conditions

$$(D_1, B_1, C_1) = 1, \, (D_2, B_2, C_2) = 1$$

and

$$D_1 C_2 + B_1 B_2 + C_1 D_2 \equiv 0 (\text{mod } D_1 D_2).$$

Variables Y_1, Y_2, Z_1, Z_2 are chosen such that

$$Y_1 B_1 + Z_1 C_1 \equiv 1 (\mathrm{mod}\ D1),$$

$$Y_2 B_2 + Z_2 C_2 \equiv 1 (\mathrm{mod}\ D2).$$

Lemma 2.5.2. *The sum* (2.30) *is well-defined, i.e. it is independent of the choice of Y_1, Y_2, Z_1, Z_2 and it does not depend on the choice of representatives B_1, C_1 and B_2, C_2 of the residue classes modulo D_1 and D_2.*

Proof. 1. First, we show the independence of the choice of Y_1, Z_1 (Y_2, Z_2), i.e. if $(D_1, B_1, C_1) = 1$, $D_1 \neq 0$,

$$D_1 C_2 + B_1 B_2 + C_1 D_2 \equiv 0 (\mathrm{mod}\ D_1 D_2),$$

and if

$$X_1 D_1 + Y_1 B_1 + Z_1 C_1 = X_1' D_1 + Y_1' B_1 + Z_1' C_1,$$

then

$$\frac{Y_1 D_2 - Z_1 B_2}{D_1} \equiv \frac{Y_1' D_2 - Z_1' B_2}{D_1} (\mathrm{mod}\ 1).$$

We may assume that

$$D_1 C_2 + B_1 B_2 + C_1 D_2 = 0$$

by changing the value of C_2. Then both vectors (C_2, B_2, D_2) and $(\alpha, \beta, \gamma) = (X_1 - X_1', Y_1 - Y_1', Z_1 - Z_1')$ are orthogonal to (D_1, B_1, C_1). Thus, the vector cross product of (C_2, B_2, D_2) and (α, β, γ) is parallel to (D_1, B_1, C_1). So there is $\lambda \in \mathbb{Q}$ such that

$$(\beta D_2 - \gamma B_2, \gamma C_2 - \alpha D_2, \alpha B_2 - \beta C_2) = \lambda (D_1, B_1, C_1).$$

Since $(D_1, B_1, C_1) = 1$, we deduce that $\lambda \in \mathbb{Z}$ and

$$\frac{Y_1 D_2 - Z_1 B_2}{D_1} = \lambda + \frac{Y_1' D_2 - Z_1' B_2}{D_1}$$

as required. $\square$

2. Let us denote the inner sum in 2.30 by

$$S_{B_1,B_2}(m_1,m_2,n_1,n_2,D_1,D_2)$$

$$= \sum_{\substack{C_1(\mathrm{mod}\ D_1)\\ C_2(\mathrm{mod}\ D_2)}} e\left(\frac{m_1B_1+n_1(Y_1D_2-Z_1B_2)}{D_1}+\frac{m_2B_2+n_2(Y_2D_1-Z_2B_2)}{D_2}\right),$$

where

$$(D_1,B_1,C_1)=1,\ (D_2,B_2,C_2)=1$$

and

$$D_1C_2+B_1B_2+C_1D_2\equiv 0(\mathrm{mod}\ D_1D_2).$$

We claim that the given sum depends only on the residue classes of $B_1(\mathrm{mod}\ D_1)$ (respectively $B_2(\mathrm{mod}\ D_2)$) :

if $B_1'=B_1+\lambda D_1$, then

$$S_{B_1,B_2}(m_1,m_2,n_1,n_2,D_1,D_2)=S_{B_1',B_2}(m_1,m_2,n_1,n_2,D_1,D_2).$$

Let $C_2'=C_2-\lambda B_2$, so that

$$(D_1,B_1',C_1)=(D_2,B_2,C_2')=1$$

and

$$D_1C_2'+B_1B_2'+C_1D_2\equiv 0(\mathrm{mod}\ D_1D_2).$$

We deduce that

$$Y_1B_1'+Z_1C_1\equiv 1(\mathrm{mod}\ D_1),$$

$$Y_2'B_2+Z_2C_2'\equiv 1(\mathrm{mod}\ D_2)$$

with $Y_2'=Y_2+\lambda Z_2$. Then

$$Y_2'D_1-Z_2B_1'=Y_2D_1-Z_2B_1.$$

Finally, summing over all C_1 and C_2 for the first sum and over all C_1 and

C_2' for the second sum, we obtain the result

$$S_{B_1,B_2}(m_1,m_2,n_1,n_2,D_1,D_2) = S_{B_1',B_2}(m_1,m_2,n_1,n_2,D_1,D_2).$$

□

Definition 2.5.3. Suppose $D_1|D_2$. Then the second type of $\mathbf{SL}_3(\mathbb{Z})$ Kloosterman sums is defined as follows

$$K_2 = K_2(m_1,n_1,n_2;D_1,D_2),$$

$$K_2 = \sum_{\substack{C_1(\text{mod } D_1)\\ C_2(\text{mod } D_2)\\ (C_1,D_1)=(C_2,\frac{D_2}{D_1})=1}} e\left(\frac{m_1C_1+n_1C_2C_1^*}{D_1}+\frac{n_2C_2^*}{D_2/D_1}\right). \tag{2.31}$$

Variables C_1^*, C_2^* are chosen so that

$$C_1C_1^* \equiv 1(\text{mod } D1)$$

and

$$C_2C_2^* \equiv 1(\text{mod } D_2/D_1).$$

Remark 2.5.4. The sum (2.31) is well-defined, i.e. it is independent of the choice of C_1^*, C_2^* and it does not depend on the choice of representatives C_1, C_2 of the residue classes modulo D_1 and D_2.

2.6 Some properties of Kloosterman sums

New types of Kloosterman sums have properties similar to the classical case. Let us list some of them.

Proposition 2.6.1.

$$S(m_1,m_2,n_1,n_2;D_1,D_2) = S(n_1,n_2,m_1,m_2;D_1,D_2).$$

Proof. Given $(D_1, B_1, C_1) = (D_2, B_2, C_2) = 1$ such that

$$D_1C_2 + B_1B_2 + C_1D_2 \equiv 0(\text{mod } D_1D_2),$$

let

$$X_1D_1 + Y_1B_1 + Z_1C_1 = 1$$

and

$$X_2D_2 + Y_2B_2 + Z_2C_2 = 1.$$

Also let

$$B_1^{'} = Y_1D_2 - Z_1B_2,\ B_2^{'} = Y_2D_1 - Z_2B_1,$$

$$C_1^{'} = Z_2,\ C_2^{'} = Z_1,$$

$$Y_1^{'} = X_2B_1 - Y_2C_1,\ Y_2^{'} = X_1B_2 - Y_1C_2,$$

$$Z_1^{'} = C_2,\ C_2^{'} = Z_1.$$

Thus,

$$Y_1^{'}B_1^{'} + Z_1^{'}C_1^{'} \equiv Y_1B_1 + Z_1C_1 + D_1C_2(X_1Z_2 + Y_1Y_2 + Z_1X_2)(\text{mod } D_1D_2).$$

So that

$$Y_1^{'}B_1^{'} + Z_1^{'}C_1^{'} \equiv 1(\text{mod } D_1)$$

and similarly

$$Y_2^{'}B_2^{'} + Z_2^{'}C_2^{'} \equiv 1(\text{mod } D_2).$$

Besides,

$$D_1C_2^{'} + B_1^{'}B_2^{'} + C_1^{'} \equiv D_1D_2(X_1Z_2 + Y_1Y_2 + Z_1X_2) \equiv 0(\text{mod } D_1D_2)$$

and

$$Y_1^{'}D_2 - Z_1^{'}B_2^{'} \equiv B_1(X_2D_2 + Y_2B_2 + Z_2C_2) \equiv B_1(\text{mod } D_1D_2).$$

In an analogous manner,

$$Y_2^{'}D_1 - Z_2^{'}B_1^{'} \equiv B_2(\text{mod } D_1D_2).$$

Finally,

$$e\left(\frac{m_1B_1+n_1(Y_1D_2-Z_1B_2)}{D_1}+\frac{m_2B_2+n_2(Y_2D_2-Z_2B_1)}{D_2}\right)$$

$$=e\left(\frac{n_1B_1^{'}+m_1(Y_1^{'}D_2-Z_1^{'}B_2^{'})}{D_1}+\frac{n_2B_2^{'}+m_2(Y_2^{'}D_1-Z_2^{'}B_1^{'})}{D_2}\right).$$

Summing, we obtain the result. □

Proposition 2.6.2.

$$S(m_1,m_2,n_1,n_2;D_1,D_2)=S(m_2,m_1,n_2,n_1;D_2,D_1).$$

Proof. Follows from the definition. □

Proposition 2.6.3. *If $p_1q_1\equiv p_2q_2\equiv 1(mod\ D_1D_2)$, $p_1,q_1,p_2,q_2\in\mathbb{Z}$, then*

$$S(p_1m_1,p_2m_2,q_1n_1,q_2n_2;D_1,D_2)=S(m_1,m_2,n_1,n_2;D_1,D_2).$$

Proof. Suppose we are given $(D_1,B_1,C_1)=(D_2,B_2,C_2)=1$ such that

$$D_1C_2+B_1B_2+C_1D_2\equiv 0(\text{mod } D_1D_2).$$

Let $Y_1B_1+Z_1C_1\equiv 1(\text{mod } D_1)$, $Y_2B_2+Z_2C_2\equiv 1(\text{mod } D_2)$ and

$$B_1^{'}=p_1B_1,\ B_2^{'}=p_2B_2,$$

$$C_1^{'}=p_1p_2C_1,\ C_2^{'}=p_1p_2C_2,$$

$$Y_1^{'}=q_1Y_1,\ Y_2^{'}=q_2Y_2,$$

$$Z_1^{'}=q_1q_2Z_1,\ Z_2^{'}=q_1q_2Z_2.$$

Then $(D_1,B_1^{'},C_1^{'})=(D_2,B_2^{'},C_2^{'})=1$,

$$D_1C_2^{'}+B_1^{'}B_2^{'}+C_1^{'}D_2\equiv 0(\text{mod } D_1D_2),$$

$$Y_1^{'}B_1^{'}+Z_1^{'}C_1^{'}\equiv 1(\text{mod } D_1),\ Y_2^{'}B_2^{'}+Z_2^{'}C_2^{'}\equiv 1(\text{mod } D_2).$$

So we have

$$e\left(\frac{p_1 m_1 B_1 + q_1 n_1 (Y_1 D_2 - Z_1 B_2)}{D_1} + \frac{p_2 m_2 B_2 + q_2 n_2 (Y_2 D_1 - Z_2 B_1)}{D_2}\right)$$

$$= e\left(\frac{m_1 B_1' + n_1 (Y_1' D_2 - Z_1' B_2')}{D_1} + \frac{m_2 B_2' + n_2 (Y_2' D_1 - Z_2' B_1')}{D_2}\right).$$

Summing, we obtain the result. $\square$

Proposition 2.6.4. *(twisted multiplicativity) If* $(D_1 D_2, D_1' D_2') = 1$ *and if*

$$D_1' D_1 \equiv D_1' D_2 \equiv 1 (mod\ D_1' D_2'),$$

$$\overline{D_1'} D_1' \equiv \overline{D_2'} D_2' \equiv 1 (mod\ D_1 D_2),$$

then

$$S(m_1, m_2, n_1, n_2; D_1 D_1', D_2 D_2')$$
$$= S(\overline{D_1'}^2 D_2' m_1, \overline{D_2'}^2 D_1' m_2, n_1, n_2; D_1, D_2) S(\overline{D_1}^2 D_2 m_1, \overline{D_2}^2 D_1' m_2, n_1, n_2; D_1', D_2').$$

Proof. Let p, p' be such that

$$p D_1 D_2 + p' D_1' D_2' = 1.$$

Given $(D_1, B_1, C_1) = (D_2, B_2, C_2) = 1$ such that

$$D_1 C_2 + B_1 B_2 + C_1 D_2 \equiv 0 (\text{mod } D_1 D_2)$$

and $(D_1', B_1', C_1') = (D_2', B_2', C_2') = 1$ such that

$$D_1' C_2' + B_1' B_2' + C_1' D_2' \equiv 0 (\text{mod } D_1' D_2'),$$

let

$$d_1 = D_1 D_1',\ d_2 = D_2 D_2',$$

$$b_1 = p' D_1' D_2' B_1 + p D_1 D_2 B_1',\ b_2 = p' D_1' D_2' B_2 + p D_1 D_2 B_2',$$

$$c_1 = p'^2 D_1'^2 D_2' C_1 + p^2 D_1^2 D_2 C_1',\ c_2 = p'^2 D_1' D_2'^2 C_2 + p^2 D_1 D_2^2 C_2'.$$

Then

$$(d_1, b_1, c_1) = (d_2, b_2, c_2) = 1,$$

$$d_1c_2 + b_1b_2 + c_1d_2 \equiv 0(\text{mod } d_1d_2).$$

Let

$$y_1 = p'D_1'D_2'Y_1 + pD_1D_2Y_1',\ y_2 = p'D_1'D_2'Y_2 + pD_1D_2Y_2',$$

$$z_1 = p'D_1'D_2'^2Z_1 + pD_1D_2^2Z_1',\ z_2 = p'D_1'^2D_2'Z_2 + pD_1^2D_2Z_2'.$$

Then

$$y_1b_1 + z_1c_1 \equiv 1(\text{mod } d_1),\ y_2b_2 + z_2c_2 \equiv 1(\text{mod } d_2),$$

$$\frac{m_1b_1 + n_1(y_1d_2 - z_1b_2)}{d_1} \equiv \frac{m_1p'D_2'B_1 + n_1p'D_2'^2(Y_1D_2 - Z_1B_2)}{D_1}$$
$$+\frac{m_1pD_2B_1' + n_1pD_2^2(Y_1'D_2') - Z_1'B_2'}{D_1'}(\text{mod } 1).$$

And the identity for d_2 is similar. Summing, we have

$$S(m_1, m_2, n_1, n_2; d_1, d_2) = S(p'D_2'm_1, p'D_1'm_2, p'D_2'^2n_1, p'D_1'^2n_2; D_1, D_2)$$

$$\times S(pD_2m_1, pD_1m_2, pD_2^2n_1, pD_1^2n_2; D_1', D_2').$$

But

$$\overline{D_1} \equiv pD_2(\text{mod } D_1'D_2'),\ \overline{D_2} \equiv pD_1(\text{mod } D_1'D_2'),$$

$$\overline{D_1'} \equiv p'D_2'(\text{mod } D_1D_2),\ \overline{D_2'} \equiv p'D_1'(\text{mod } D_1D_2).$$

Now the result follows from proposition 2.6.3.

□

Proposition 2.6.5.

$$S(m_1, m_2, n_1, n_2; D_1, 1) = S(m_1, n_1; D_1),$$

$$S(m_1, m_2, n_1, n_2; 1, D_2) = S(m_2, n_2; D_2),$$

where $S(m, n; D)$ is a classical Kloosterman sum.

Proof. Follows from the definition. □

Proposition 2.6.6. *If* $(D_1, D_2) = 1$, *then*

$$S(m_1, m_2, n_1, n_2; D_1, D_2) = S(D_2 m_1, n_1, D_1) S(D_1 m_2, n_2, D_2).$$

Proof. Follows from properties 2.6.4 and 2.6.5. □

Now, let us consider the second type of $\mathbf{SL}_3(\mathbb{Z})$ Kloosterman sum and list some of its properties.

Proposition 2.6.7.

$$S(m_1, n_1, n_2, 1, D_2) = R_{D_2}(n_2),$$

where

$$R_c(n) = S(0, n, c) = \sum_{\substack{d (mod\ c) \\ gcd(c,d)=1}} e\left(\frac{nd}{c}\right)$$

is a Ramanujan sum.

Proof. Follows from the definition. □

Proposition 2.6.8. *(twisted multiplicativity) let* $(D_2, D_2') = 1$, $D_1 | D_2$, $D_1' | D_2'$. *Then*

$$S(m_1, n_1, n_2, D_1 D_1', D_2 D_2') = S(m_1 \overline{D_1'}, n_1 D_2', n_2 \overline{D_2'}^2, D_1, D_2) S(m_1 \overline{D_1}, n_1 D_2, n_2 \overline{D_2}^2, D_1', D_2'),$$

where

$$D_1 \overline{D_1} \equiv 1 (mod\ D_1'),\ D_2 \overline{D_2} \equiv 1 (mod\ D_2'),$$
$$D_1' \overline{D_1'} \equiv 1 (mod\ D_1),\ D_2' \overline{D_2'} \equiv 1 (mod\ D_2).$$

Proposition 2.6.9. *Let* p *be a prime number. Then for* $b > a > 0$

$$S(m_1, n_1, n_2; p^a, p^a) = 0$$

unless $b = 2a$, *or* $n_2 \equiv 0 (mod\ p^{b-2a})$ *and* $b > 2a$, *or* $n_1 \equiv 0 (mod\ p^{2a-b})$ *and* $b < 2a$.

Proposition 2.6.10.

$$S(m_1, n_1, n_2; p^a, p^a) = \begin{cases} p^{2a} - p^{2a-1} & \text{if } p^a | m, p^a | n_1 \\ -p^{2a-1} & \text{if } p^a \not| m, p^a | n_1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.6.11.

$$S(m_1, n_1, n_2; D_1, D_2) = 0$$

unless $n_1 \frac{D_2}{D_1^2} \in \mathbb{Z}$.

Proof. Follows from propositions 2.6.8, 2.6.9 and 2.6.10. □

Proposition 2.6.12. *(Larsen's bound)*

$$|S(m_1, n_1, n_2; D_1, D_2)| \leq min(\tau(D_1)^\alpha (n_2, D_2/D_1) D_1^2, \tau(D_2)(m_1, n_1, D_1) D_2),$$

where $\alpha = \frac{log(3)}{log(2)}$ *and* $\tau(n) = \sum_{\substack{d|n \\ d \geq 1}} 1$.

2.7 Fourier expansion of Poincaré series

Let us choose an $E-$function as

$$E_{n_1,n_2}(z) = e(n_1(x_1 + iy_1/M) + n_2(x_2 + iy_2/M)) \text{ with } M \in \mathbb{Z}. \tag{2.32}$$

Since the function does not depend on x_3, we write

$$E_{n_1,n_2}(z) = E_{n_1,n_2}(x_1 + iy_1, x_2 + iy_2). \tag{2.33}$$

Below we compute the Fourier coefficients of $\mathbf{SL}_3(\mathbb{Z})$ Poincaré series for this choice of $E-$function.

Theorem 2.7.1. *Let* $\Re(\nu_1), \Re(\nu_2) > \frac{2}{3}$. *Then*

$$\int_0^1 \int_0^1 \int_0^1 P_{n_1,n_2}\left(\begin{bmatrix} 0 & \xi_2 & \xi_3 \\ 0 & 1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} z \right) e(-m_1\xi_1 - m_2\xi_2) d\xi_1 d\xi_2 d\xi_3 =$$

$$= e(m_1x_1+m_2x_2)I_{\nu_1,\nu_2}(z)\sum_{w_i\in W}\sum_{\epsilon_1,\epsilon_2=\pm1}\sum_{D_1,D_2=1}^{\infty}S_{w_i}(\epsilon_1m_1,\epsilon_2m_2,n_1,n_2;D_1,D_2)$$
$$\times D_1^{-3\nu_1}D_2^{-3\nu_2}J_{w_i}(y_1,y_2;\nu_1,\nu_2;\epsilon_1m_1,\epsilon_2m_2,n_1,n_2;D_1,D_2),$$

where S_{w_i} is a $\boldsymbol{SL}_3(\mathbb{Z})$ Kloosterman sums and

$$J_{w_i}=J_{w_i}(y_1,y_2;\nu_1,\nu_2;\epsilon_1m_1,\epsilon_2m_2,n_1,n_2;D_1,D_2)$$

is an integral below corresponding to the element w_i.

Let us write $\xi_4=\xi_1\xi_2-\xi_3$, $Z_3=\xi_3^2+\xi_2^2y_1^2+y_1^2y_2^2$ and $Z_4=\xi_4^2+\xi_1^2y_2^2+y_1^2y_2^2$, then

- $J_{w_1}=\delta_{m_1,n_1}\delta_{m_2,n_2}E_{n_1,n_2}(y_1,y_2)$,
- $J_{w_2}=\delta_{n_1,0}\delta_{m_1,0}\int_{-\infty}^{+\infty}(\xi_2^2+y_2^2)^{\frac{-3\nu_2}{2}}E_{n_1,n_2}(0,-(\xi_2+iy_2)^{-1}D_2^{-2})e(-m_2\xi_2)d\xi_2$,
- $J_{w_3}=\delta_{n_2,0}\delta_{m_2,0}\int_{-\infty}^{+\infty}(\xi_1^2+y_1^2)^{\frac{-3\nu_2}{2}}E_{n_1,n_2}(-(\xi_1+iy_1)^{-1}D_1^{-2},0)e(-m_1\xi_1)d\xi_1$,
- $J_{w_4}=\delta_{m_2D_1^2,n_1D_2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\xi_1^2+y_1^2)^{\frac{-3\nu_1}{2}}Z_4^{\frac{-3\nu_2}{2}}e(-m_1\xi_1)$
$\times E_{n_1,n_2}((\xi_1\xi_4+iy_1Z_4^{1/2})(\xi_1^2+y_1^2)^{-1}D_2D_1^{-2},(\xi_4+iy_2(\xi_1^2+y_1^2)^{1/2})Z_4^{-1}D_1D_2^{-2})d\xi_1d\xi_4$,
- $J_{w_5}=\delta_{m_1D_2^2,n_2D_1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\xi_2^2+y_2^2)^{\frac{-3\nu_2}{2}}Z_3^{\frac{-3\nu_1}{2}}e(-m_2\xi_2)$
$\times E_{n_1,n_2}((\xi_3+iy_1(\xi_2^2+y_2^2)^{1/2})Z_3^{-1}D_2D_1^{-2},(\xi_2\xi_3+iy_2Z_3^{1/2})(\xi_2^2+y_2^2)^{-1}D_1D_2^{-2})d\xi_2d\xi_3$,
- $J_{w_6}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}Z_3^{\frac{-3\nu_1}{2}}Z_4^{\frac{-3\nu_2}{2}}e(-m_1\xi_1-m_2\xi_2)$
$\times E_{n_1,n_2}((-\xi_1\xi_3-\xi_2y_1^2+iy_1Z_4^{1/2})Z_3^{-1}D_2D_1^{-2},(-\xi_2\xi_4-\xi_1y_2^2+iy_1Z_3^{1/2})Z_4^{-1}D_1D_2^{-2})$
$\times d\xi_1d\xi_2d\xi_3$.

Proof. Consider the Poincaré series

$$P_{n_1,n_2}(z,\nu_1,\nu_2)=\sum_{\gamma\in\Gamma_{3,\infty}\backslash\Gamma_3}I_{\nu_1,\nu_2}(\gamma z)E_{n_1,n_2}(\gamma z).$$

Let

$$U=\left\{\begin{bmatrix}\epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_3\end{bmatrix},\ \epsilon_i=\pm1,\ \epsilon_1\epsilon_2\epsilon_3=1\right\}.$$

Then we can write

$$P_{n_1,n_2}(z,\nu_1,\nu_2)=\sum_{\gamma\in\Gamma_{3,\infty}\backslash\Gamma_3/U}\sum_{u\in U}I_{\nu_1,\nu_2}(\gamma uz)E_{n_1,n_2}(\gamma uz).$$

According to the formula 2.3, Fourier coefficients are given by

$$F_{m_1,m_2}(z)=\int_0^1\int_0^1\int_0^1 P_{n_1,n_2}\left(\begin{bmatrix}1&\xi_2&\xi_3\\0&1&\xi_1\\0&0&1\end{bmatrix}z\right)e(-m_1\xi_1-m_2\xi_2)d\xi_1d\xi_2d\xi_3.$$

Since

$$\begin{bmatrix}1&\xi_2&\xi_3\\0&1&\xi_1\\0&0&1\end{bmatrix}z=\begin{bmatrix}1&\xi_2&\xi_3\\0&1&\xi_1\\0&0&1\end{bmatrix}\begin{bmatrix}1&x_2&x_3\\0&1&x_1\\0&0&1\end{bmatrix}\begin{bmatrix}y_1y_2&0&0\\0&y_1&0\\0&0&1\end{bmatrix}$$
$$=\begin{bmatrix}y_1y_2&(x_2+\xi_2)y_1&\xi_3+\xi_2x_1+x_3\\0&y_1&\xi_1+x_1\\0&0&1\end{bmatrix},$$

we can make a change of variables

$$\xi_1\to\xi_1-x_1,$$

$$\xi_2\to\xi_2-x_2,$$

$$\xi_3\to\xi_3-x_3-\xi_2x_1$$

to obtain

$$F_{m_1,m_2}(z)=e(x_1m_1+x_2m_2)\int_0^1\int_0^1\int_0^1 P_{n_1,n_2}\left(\begin{bmatrix}y_1y_2&y_1\xi_2&\xi_3\\0&y_1&\xi_1\\0&0&1\end{bmatrix}\right)e(-m_1\xi_1-m_2\xi_2)d\xi_1d\xi_2d\xi_3$$

$$=e(x_1m_1+x_2m_2)\sum_{\gamma\in\Gamma_{3,\infty}\backslash\Gamma_3/U}\sum_{u\in U}\int_0^1\int_0^1\int_0^1 I_{\nu_1,\nu_2}E_{n_1,n_2}\left(\gamma u\begin{bmatrix}y_1y_2&y_1\xi_2&\xi_3\\0&y_1&\xi_1\\0&0&1\end{bmatrix}\right)$$

$$\times e(-m_1\xi_1 - m_2\xi_2)d\xi_1 d\xi_2 d\xi_3.$$

Note that each element $u \in U$, $u \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ changes signs of some variables ξ_i, $1 \leq i \leq 3$. So the following substitution

$$\xi_i \to \epsilon_i \xi_i,\ i = 1, \dots, 3$$

leads to

$$F_{m_1,m_2}(z) = e(x_1 m_1 + x_2 m_2) \sum_{\epsilon_1,\epsilon_2=\pm 1} \sum_{\gamma \in \Gamma_{3,\infty} \backslash \Gamma_3 / U} \int_0^1 \int_0^1 \int_0^1 I_{\nu_1,\nu_2}$$

$$\times E_{n_1,n_2}\left(\gamma \begin{bmatrix} y_1 y_2 & y_1\xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix}\right) e(-m_1\epsilon_1\xi_1 - m_2\epsilon_2\xi_2)d\xi_1 d\xi_2 d\xi_3.$$

Let us denote $k_1 = m_1\epsilon_1$, $k_2 = m_2\epsilon_2$ and

$$L = \sum_{\gamma \in \Gamma_{3,\infty} \backslash \Gamma_3 / U} \int_0^1 \int_0^1 \int_0^1 I_{\nu_1,\nu_2} E_{n_1,n_2}\left(\gamma \begin{bmatrix} y_1 y_2 & y_1\xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix}\right) e(-k_1\xi_1 - k_1\xi_2)d\xi_1 d\xi_2 d\xi_3.$$

Now we apply results of the section 2.4 to modify the given sum. Note that

$$\Gamma_3 = G_3 \cap \Gamma_3 = (\cup_{w \in W_3} G_w) \cap \Gamma_3$$

by Bruhat decomposition 2.15. So that

$$\Gamma_{3,\infty} \setminus \Gamma_3 / U = \cup_{w \in W_3} \Gamma_{3,\infty} \setminus \Gamma_3 \cap G_w / U.$$

Let

$$e_{n_1,n_2}\left(\begin{bmatrix} 1 & x_2 & * \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix}\right) = e(n_1 x_1 + n_2 x_2).$$

Then, using proposition 2.4.15 and property 2.12 , we have

$$L = \sum_{w \in W_3} \sum_{\substack{\gamma \in R_w \\ \gamma = b_1 w d b_2 \\ b_1, b_2 \in G_\infty, d \in D}} e_{n_1,n_2}(b_1) \sum_{t \in \Gamma_w} \int_0^1 \int_0^1 \int_0^1 I_{\nu_1,\nu_2}$$

$$\times E_{n_1,n_2}\left(w d b_2 t \begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \right) e(-k_1\xi_1 - k_1\xi_2) d\xi_1 d\xi_2 d\xi_3.$$

According to the definition 2.4.13, there are six types of groups Γ_w associated to different elements $w \in W_3$. We can treat them case by case in order to apply the action of $t \in \Gamma_w$ to the matrix $\begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix}$ and change the domain of integration. Consider, for instance,

$$\Gamma_{w_5} = \left\{ \begin{bmatrix} 1 & m & l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, m, l \in \mathbb{Z} \right\} \text{ with } w_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} 1 & m & l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 y_2 & y_1 \xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} y_1 y_2 & y_1(\xi_2 + m) & \xi_3 + m\xi_1 + l \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us make the following change of variables

$$\xi_1 \to \xi_1,$$

$$\xi_2 \to \xi_2 - m,$$

$$\xi_3 \to \xi_3 - m\xi_1 - l.$$

Summing over all elements in Γ_{w_5} (equivalently, summing over all $m, l \in \mathbb{Z}$), the domain of integration in the space $\xi_3 \times \xi_2 \times \xi_1$ is

$$[-\infty, +\infty] \times [-\infty, +\infty] \times [0,1].$$

In general,

$$L = \sum_{w \in W_3} \sum_{\substack{\gamma \in \Gamma_{3,\infty} \backslash \Gamma_3 \cap G_w / U\Gamma_w \\ \gamma = b_1 w d b_2}} e_{n_1,n_2}(b_1) \int_{\Omega_w} I_{\nu_1,\nu_2}$$

$$\times E_{n_1,n_2}\left(wdb_2 \begin{bmatrix} y_1y_2 & y_1\xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \right) e(-k_1\xi_1 - k_1\xi_2) d\xi_1 d\xi_2 d\xi_3,$$

where in the space $\xi_3 \times \xi_2 \times \xi_1$

$$\Omega_{w_1} = [0,1] \times [0,1] \times [0,1] \text{ with } w_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\Omega_{w_2} = [0,1] \times [-\infty, +\infty] \times [0,1] \text{ with } w_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\Omega_{w_3} = [0,1] \times [0,1] \times [-\infty, +\infty] \text{ if } w_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\Omega_{w_4} = [-\infty, +\infty] \times [0,1] \times [-\infty, +\infty] \text{ with } w_4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\Omega_{w_5} = [-\infty, +\infty] \times [-\infty, +\infty] \times [0,1] \text{ with } w_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\Omega_{w_6} = [-\infty, +\infty] \times [-\infty, +\infty] \times [-\infty, +\infty] \text{ if } w_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Next we apply

$$b_2 = \begin{bmatrix} 1 & \beta_2 & \beta_3 \\ 0 & 1 & \beta_1 \\ 0 & 0 & 1 \end{bmatrix}$$

to

$$\begin{bmatrix} y_1y_2 & y_1\xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix}$$

and make a change of variables

$$\xi_1 \to \xi_1 - \beta_1,$$

$$\xi_2 \to \xi_2 - \beta_2,$$

$$\xi_3 \to \xi_3 - \beta_2\xi_1 - \beta_3.$$

Then,

$$L = I_{\nu_1,\nu_2}(z)E_{n_1,n_2}(y_1,y_2)\int_0^1\int_0^1\int_0^1 e((n_1-k_1)\xi_1 + (n_2-k_2)\xi_2)d\xi_1 d\xi_2 d\xi_3$$

$$+ \sum_{\substack{w\in W_3 \\ w\neq w_1}} \sum_{\substack{\gamma\in\Gamma_{3,\infty}\backslash G_w\cap\Gamma_3/U\Gamma_w \\ \gamma=b_1wdb_2}} e_{n_1,n_2}(b_1)e_{k_1,k_2}(b_2)$$

$$\times \int_{\Omega_{w,b_2}} I_{\nu_1,\nu_2}(z)E_{n_1,n_2}\left(wd\begin{bmatrix} y_1y_2 & y_1\xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix}\right) e(-k_1\xi_1 - k_2\xi_2)d\xi_1 d\xi_2 d\xi_3.$$

According to Bruhat decomposition and propositions 2.4.7-2.4.21, the domain of integration Ω_{w,b_2} is given by

$$\Omega_{w_2,b_2} = [0,1]\times[-\infty,+\infty]\times[0,1],$$

$$\Omega_{w_3,b_2} = [0,1]\times[0,1]\times[-\infty,+\infty],$$

$$\Omega_{w_4,b_2} = [-\infty,+\infty]\times[\frac{\alpha C_2B_1}{A_2}, 1+\frac{\alpha C_2 B_1}{A_2}]\times[-\infty,+\infty],$$

$$\Omega_{w_5,b_2} = [-\infty,+\infty] \times [-\infty,+\infty] \times [\frac{-b_{2,2}B_2}{A_1}, 1-\frac{b_{2,2}B_2}{A_1}],$$

$$\Omega_{w_6,b_2} = [-\infty,+\infty] \times [-\infty,+\infty] \times [-\infty,+\infty].$$

The next step is to modify the integral $\int_{\Omega_{w,d_2}}$. Since $wd \begin{bmatrix} y_1y_2 & y_2\xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix}$ belongs to the generalized upper-half space $\mathbb{H}^3$, we consider

$$\begin{bmatrix} y_1'y_2' & y_2'x_2 & x_3 \\ 0 & y_1' & x_1 \\ 0 & 0 & 1 \end{bmatrix} \equiv wd \begin{bmatrix} y_1y_2 & y_2\xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix} \mod \left(\mathbf{O}_3(\mathbb{R}) \cdot \mathbb{R}^\times\right).$$

Let

$$\xi_4 = \xi_1\xi_2 + \xi_3,$$

$$Z_3 = \xi_3^2 + y_1^2\xi_2^2 + y_1^2y_2^2,$$

$$Z_4 = \xi_4^2 + y_2^2\xi_1^2 + y_1^2y_2^2.$$

Then the values of x_1', x_2', x_2', y_1', y_2' are as follows.

- If $w = w_2$, then

$$x_1' = B_2\xi_3,$$

$$x_2' = \frac{-\xi_2}{B_2^2(\xi_2^2+y_2^2)},$$

$$x_3' = \frac{\xi_1}{B_2},$$

$$y_1' = B_2y_1(\xi_2^2+y_2^2)^{0.5},$$

$$y_2' = \frac{y_2}{B_2^2(\xi_2^2+y_2^2)}.$$

- If $w = w_3$, then

$$x_1' = \frac{-\xi_1}{B_1^2(\xi_1^2+y_1^2)},$$

$$x_2' = B_1(\xi_1\xi_2 - \xi_3),$$

$$x_3' = \frac{\xi_1\xi_3 + \xi_2y_1^2}{B_1(\xi_1^2+y_1^2)},$$

$$y_1' = \frac{y_1}{B_1^2(\xi_1^2 + y_1^2)},$$
$$y_2' = B_1 y_2(\xi_1^2 + y_1^2)^{0.5}.$$

- If $w = w_4$, then

$$x_1' = \frac{A_2}{B_1^2}\frac{(\xi_1\xi_3 + \xi_2 y_1^2)}{\xi_1^2 + y_1^2},$$
$$x_2' = \frac{-B_1\xi_4}{A_2^2 Z_4},$$
$$x_3' = \frac{\xi_1}{B_1 A_2(\xi_1^2 + y_1^2)},$$
$$y_1' = \frac{A_2 y_1 Z_4^{0.5}}{B_1^2(\xi_1^2 + y_1^2)},$$
$$y_2' = \frac{B_1 y_2(\xi_1^2 + y_1^2)^{0.5}}{A_2^2 Z_4}.$$

- If $w = w_5$, then

$$x_1' = \frac{B_2\xi_3}{A_1^2 Z_3},$$
$$x_2' = \frac{A_1}{B_2^2}(\xi_1 - \frac{\xi_2\xi_3}{\xi_2^2 + y_2^2}),$$
$$x_3' = \frac{\xi_1\xi_3 + \xi_2 y_1^2}{A_1 B_2 Z_3},$$
$$y_1' = \frac{B_2 y_1(\xi_2^2 + y_2^2)^{0.5}}{A_1^2 Z_3},$$
$$y_2' = \frac{A_1 y_2 Z_3^{0.5}}{B_2^2(\xi_2^2 + y_2^2)}.$$

- If $w = w_6$, then

$$x_1' = \frac{-A_2}{A_1^2}\frac{(\xi_1\xi_3 + \xi_2 y_1^2)}{Z_3},$$
$$x_2' = \frac{-A_1}{A_2^2}\frac{(\xi_2\xi_4 + \xi_2 y_2^2)}{Z_4},$$
$$x_3' = \frac{\xi_3}{A_1 A_2 Z_3},$$

$$y_1' = \frac{A_2 y_1 Z_4^{0.5}}{A_1^2 Z_3},$$

$$y_2' = \frac{A_1 y_2 Z_3^{0.5}}{A_2^2 Z_4}.$$

Given results can be verified by direct calculations. Consider, for example, case $w = w_5$. Let $r_1 = \sqrt{\xi_2^2 + y_2^2}$, $r_2 = \sqrt{\xi_2^2 y_1^2 + y_1^2 y_2^2 + \xi_3^2}$, then there are

$$O = \begin{bmatrix} \frac{-\xi_2}{r_1} & \frac{y_2}{r_1} & 0 \\ \frac{-\xi_3 y_2}{r_1 r_2} & \frac{-\xi_2 \xi_3}{r_1 r_2} & \frac{r_1 y_1}{r_2} \\ \frac{y_1 y_2}{r_2} & \frac{y_1 \xi_2}{r_2} & \frac{\xi_3}{r_2} \end{bmatrix} \in \mathbf{O}_3(\mathbb{R})$$

and

$$R = \begin{bmatrix} A_1 r_2^2 & 0 & 0 \\ 0 & A_1 r_2^2 & 0 \\ 0 & 0 & A_1 r_2^2 \end{bmatrix} \in \mathbb{R}^\times$$

such that

$$\begin{bmatrix} y_1' y_2' & y_2' x_2 & x_3 \\ 0 & y_1' & x_1 \\ 0 & 0 & 1 \end{bmatrix} OR = w_5 d \begin{bmatrix} y_1 y_2 & y_2 \xi_2 & \xi_3 \\ 0 & y_1 & \xi_1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now summing up all the results we immediately obtain the statement of the theorem in case $w = w_1, w_2, w_3$. The remained three cases involve some more computations. Let us consider for instance the case $w = w_5$:

$$I_{\nu_1,\nu_2}(z) = A_1^{-3\nu_1} B_2^{-3\nu_2} Z_3^{\frac{-3\nu_1}{2}} (\xi_2^2 + y_2^2)^{\frac{-3\nu_2}{2}}$$

and according to 2.33

$$E_{n_1,n_2}(z) = E_{n_1,n_2}(x_1' + iy_1', x_2' + iy_2').$$

Then

$$\int_{\Omega_{w_5,b_2}} = A_1^{-3\nu_1} B_2^{-3\nu_2} \int_{-\frac{b_{2,2}B_2}{A_1}}^{1-\frac{b_{2,2}B_2}{A_1}} e((\frac{n_2 A_1}{B_2^2} - k_1)\xi_1) d\xi_1$$

$$\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Z_3^{\frac{-3\nu_1}{2}}(\xi_2^2+y_2^2)^{\frac{-3\nu_2}{2}} E_{n_1,n_2}\left(\frac{B_2}{A_1^2}\left(\frac{\xi_3+iy_1(\xi_2^2+y_2^2)^{1/2}}{Z_3}\right), \frac{A_1}{B_2^2}\left(\frac{-\xi_2\xi_3+iy_2Z_3^{1/2}}{(\xi_2^2+y_2^2)}\right)\right)$$
$$\times e(-k_2\xi_2)d\xi_2 d\xi_3.$$

If $\frac{n_2A_1}{B_2^2}\neq k_1$,

$$\int_{-\frac{b_{2,2}B_2}{A_1}}^{1-\frac{b_{2,2}B_2}{A_1}} e((\frac{n_2A_1}{B_2^2}-k_1)\xi_1)d\xi_1 = \frac{1}{2\pi i}(\frac{n_2A_1}{B_2^2}-k_1)^{-1}e(b_{2,2}(\frac{k_1B_2}{A_1}-\frac{n_2}{B_2}))(e(\frac{n_2A_1}{B_2^2}-k_1)-1).$$

So that

$$\int_{\Omega_{w_5,b_2}} = \mu\cdot A_1^{-3\nu_1}B_2^{-3\nu_2}e(b_{2,2}(\frac{k_1B_2}{A_1}-\frac{n_2}{B_2}))$$
$$\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Z_3^{\frac{-3\nu_1}{2}}(\xi_2^2+y_2^2)^{\frac{-3\nu_2}{2}} E_{n_1,n_2}\left(\frac{B_2}{A_1^2}\left(\frac{\xi_3+iy_1(\xi_2^2+y_2^2)^{1/2}}{Z_3}\right), \frac{A_1}{B_2^2}\left(\frac{-\xi_2\xi_3+iy_2Z_3^{1/2}}{(\xi_2^2+y_2^2)}\right)\right)$$
$$\times e(-k_2\xi_2)d\xi_2 d\xi_3,$$

where

$$\mu = 1 \text{ if } \frac{n_2A_1}{B_2^2}=k_1$$

and

$$\mu = \frac{1}{2\pi i}(\frac{n_2A_1}{B_2^2}-k_1)^{-1}(e(\frac{n_2A_1}{B_2^2}-k_1)-1), \text{ otherwise.}$$

Let $D_1=A_1$ and $D_2=B_2$, then

$$\sum_{\gamma} e_{n_1,n_2}(b_1)e_{k_1,k_2}(b_2)e(b_{2,2}(\frac{k_1B_2}{A_1}-\frac{n_2}{B_2})) = \sum_{\substack{C_1(\text{mod } D_1)\\ C_2(\text{mod } D_2)}} e\left(\frac{n_1C_2^*}{D_1D_2^{-1}}+\frac{k_2C_2}{D_2}+\frac{n_2C_1C_2^*}{D_2}\right),$$

where $(C_2,D_2)=1$, $(C_1,D_1D_2^{-1})=1$, $C_2C_2^*\equiv 1(\text{mod } D_2)$ and $C_1C_1^*\equiv 1(\text{mod } D_1D_2^{-1})$. By property 2.6.11, the later sum is zero unless $\frac{n_2D_1}{D_2^2}\in\mathbb{Z}$. On the other hand, if $\frac{n_2D_1}{D_2^2}\in\mathbb{Z}$, then the integral $\int_{\Omega_{w_5,b_2}}$ vanishes for $k_1\neq\frac{n_2A_1}{B_2^2}$. This leads to the result. Applying the same procedure, one can also obtain required expressions for Kloosterman sums and integrals J_w in case $w=w_4,w_6$.

□

2.8 SL(3) Kloosterman angles

Let us recall proposition 2.6.6 in case $n_1 = n_2 = m_1 = m_2 = 1$, $D_1 = p_1$, $D_2 = p_2$, where $p_1 \neq p_2$ are prime numbers.

Proposition 2.8.1. *If* $(p_1, p_2) = 1$, *then*

$$S(1,1,1,1;p_1,p_2) = S(p_2,1,p_1)S(p_1,1,p_2),$$

where $S(m,n,c)$ *is a classical Kloosterman sum.*

According to Weil's bound,

$$|S(p_2,1,p_1)| \leq 2\sqrt{p_1},$$

$$|S(p_1,1,p_2)| \leq 2\sqrt{p_2}.$$

Thus, there are unique Kloosterman angles $(\theta_{p_2,p_1}, \theta_{p_1,p_2})$ on $[0,\pi]\times[0,\pi]$ such that

$$S(p_2,1,p_1) = 2\sqrt{p_1}cos(\theta_{p_2,p_1})$$

and

$$S(p_1,1,p_2) = 2\sqrt{p_2}cos(\theta_{p_1,p_2}).$$

We associate a couple of angles $(\theta_{p_1,p_2}, \theta_{p_2,p_1})$ with $\mathbf{SL}_3(\mathbb{Z})$ Kloosterman sum $S(1,1,1,1;p_1,p_2)$.

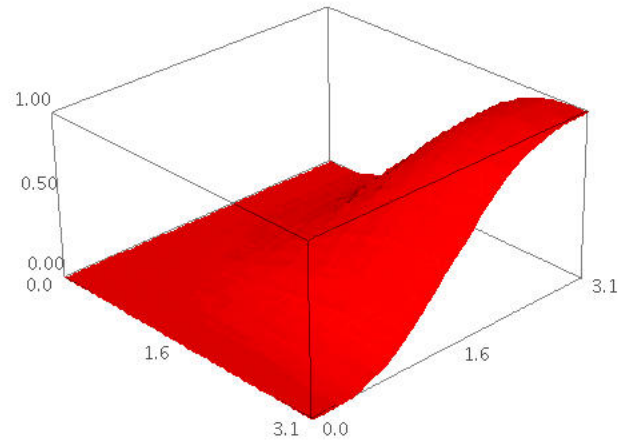
Conjecture 2.8.2. *Let* $P_1, P_2 \to \infty$, *then the set of Kloosterman angles*

$$\{(\theta_{p_2,p_1}, \theta_{p_1,p_2})\}_{\substack{p_1\leq P_1\\ p_2\leq P_2\\ p_1\neq p_2}}$$

becomes equidistibuted with respect to Sato-Tate measure on $[0,\pi]\times[0,\pi]$. *Equivalently, for any* $I_1 \times I_2 = [a_1,b_1]\times[a_2,b_2] \in [0,\pi]\times[0,\pi]$,

$$lim_{\substack{P_1\to\infty,\\ P_2\to\infty}} \frac{\#\{p_1 \leq P_1,\ p_2\leq P_2,\ p_1\neq p_2,\ \theta(p_2,p_1)\in I_1,\ \theta(p_1,p_2)\in I_2\}}{\#\{p_1\leq P_1\}\times(\#\{p_2\leq P_2\}-1)}$$

$$= \mu_{ST}(I_1\times I_2) = \frac{4}{\pi^2}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\sin^2(\theta_1)\sin^2(\theta_2)d\theta_1 d\theta_2.$$



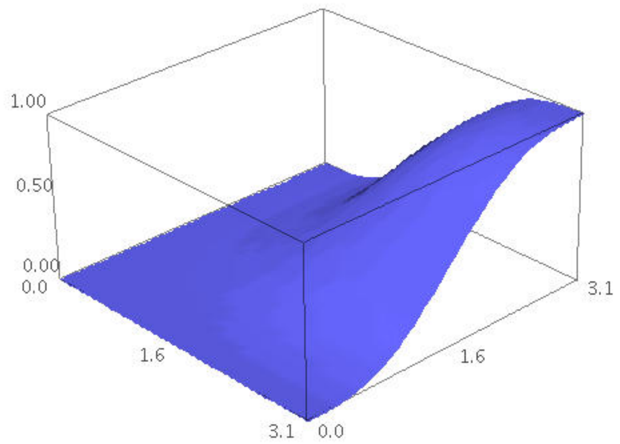


Figure 2.1: Cumulative distribution function for Kloosterman angles (red) and Sato-Tate cumulative distribution function (blue)

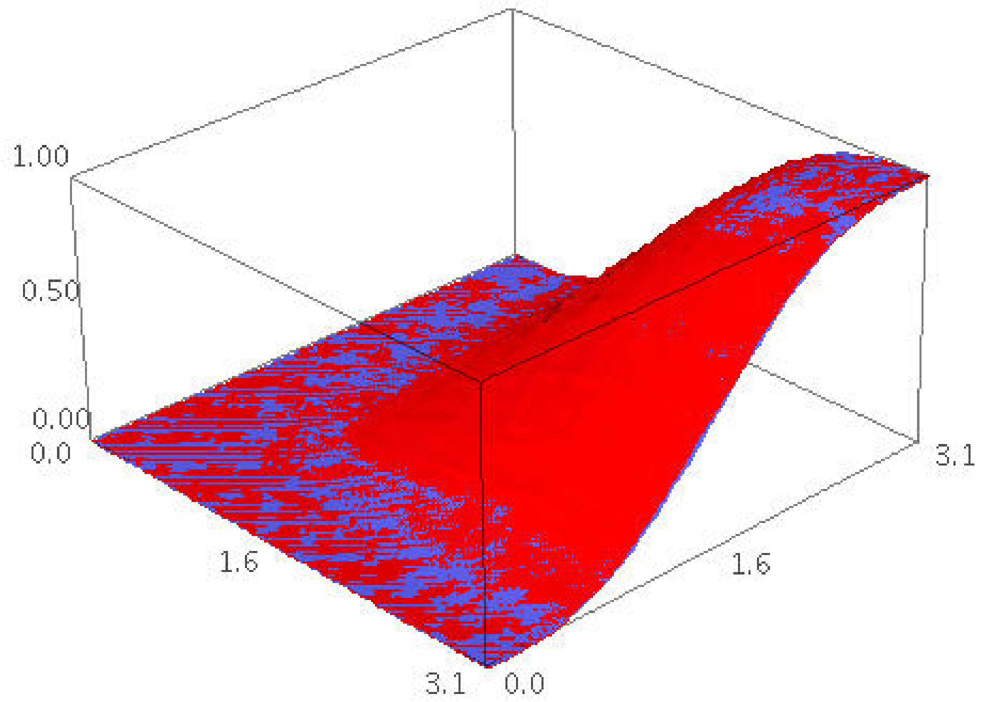


Figure 2.2: Cumulative distribution function for Kloosterman angles (red) and Sato-Tate cumulative distribution function (blue) in one plot

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