The Hodge-Tate decomposition theorem for Abelian Varieties over $p$-adic fields

following J.M. Fontaine

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Introduction

0.1. For a smooth complex projective variety $X$ or, more generally, a compact Kähler manifold $X$, a fundamental result is the so-called “Hodge decomposition” of its singular cohomology with complex coefficients. More precisely, we have a decomposition of the cohomology groups

$$ H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^p(X, \Omega^q_X) $$

where $\Omega^q_X$ is the sheaf of holomorphic $q$-differential forms on $X$. This decomposition behaves well with respect to the action of the Galois group of $\mathbb{C}$ over $\mathbb{R}$: if we denote by $\sigma$ the complex conjugation, i.e. the unique non trivial element of $\text{Gal(\mathbb{C}/\mathbb{R})}$, then $\sigma$ acts on $H^n(X, \mathbb{C})$ and transforms a holomorphic $q$-form in an anti-holomorphic $q$-form, inducing a map on the cohomology groups that satisfies $H^p(X, \Omega^q_X) = H^q(X, \Omega^p_X)$.

If $X$ is an abelian variety over $\mathbb{C}$, the Hodge decomposition (1) reduces to give the following canonical isomorphism

$$ H^1(X, \mathbb{C}) = H^0(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X), $$

since the cup-product pairings identify $H^r(X, \mathbb{C})$ with the $r$-th exterior power of $H^1(X, \mathbb{C})$ and (see [Ser59, VII, Th. 10])

$$ H^q(X, \Omega^p_X) = \bigwedge^q H^1(X, \mathcal{O}_X) \otimes \bigwedge^p H^0(X, \Omega^1_X). $$

0.2. In the late sixties, Tate asked if a similar result could hold for the $p$-adic étale cohomology of a proper and smooth variety over a complete discrete valuation field $K$ of characteristic 0 and perfect residue field of characteristic $p > 0$. In [Tat67], he established a “Hodge-like” decomposition for an abelian variety with good reduction over $K$, after extending the scalars to the $p$-adic completion of an algebraic closure of $K$.

More precisely, let $\mathcal{O}_K$ be the valuation ring of $K$, $S = \text{Spec}(\mathcal{O}_K)$, $\eta$ the generic point of $S$ and $\eta$ the geometric point corresponding to an algebraic closure $\overline{K}$ of $K$. Let $\mathcal{O}_C$ be the $p$-adic completion of $\mathcal{O}_K$, $\mathcal{C}$ its fraction field. Let $G_K = \text{Gal}(\overline{K}/K)$ be the absolute Galois group of $K$. For every $r \in \mathbb{N}$, we denote by $\mathcal{C}(r)$ the Galois module $\mathcal{C}$ twisted by the action of the power of the $p$-adic cyclotomic character $\chi_p$ and by $\mathcal{C}(-r)$ its dual. Let $X$ be an abelian variety over $\eta$ with good reduction. Tate proved the existence of a canonical $G_K$-equivariant isomorphism

$$ \mathcal{C} \otimes_{\mathcal{Q}_p} H^1_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Q}_p) \cong H^0(X, \Omega^1_{X/\eta}) \otimes_K \mathcal{C}(-1) \oplus H^1(X, \mathcal{O}_X) \otimes_{\mathcal{K}} \mathcal{C}, $$

now called the Hodge-Tate decomposition.

We know that there is a canonical isomorphism

$$ H^1_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}_p) \cong \text{Hom}_{\mathbb{Z}_p}(T_p(X_{\overline{\eta}}), \mathbb{Z}_p), $$

where
where $T_p(X_\eta)$ is the $p$-adic Tate module of the abelian variety $X_\eta$. In this case, (3) is equivalent to the existence of canonical isomorphisms

\begin{align*}
H^1(X, \mathcal{O}_X) &\xrightarrow{\sim} \text{Hom}_{\mathcal{O}[G_K]}(T_p(X_\eta), \mathbb{C}) \\
H^0(X, \Omega^1_{X/\eta}) &\xrightarrow{\sim} \text{Hom}_{\mathcal{O}[G_K]}(T_p(X_\eta), \mathbb{C}(1)).
\end{align*}

The theorem was proved more generally in [Tat67] for $p$-divisible groups. Using the semi-stable reduction theorem, Raynaud proved in [SGA 7] (Exposé 9 Th. 3.6 and Prop. 5.6) the conjecture for all abelian varieties over $K$, while the proof for the most general statement was established in 1988 by Faltings in [Fal88].

In this mémoire we present a different proof, due to Fontaine [Fon82], of the theorem of Tate and Raynaud as a consequence of a sophisticated, although relatively elementary, analysis of the module of Kähler differentials $\Omega^{1}_{\mathcal{O}[X]/\mathcal{O}_K}$. The main advantage of this argument is that it avoids completely the notion of $p$-divisible group as well as the notion of Néron model and it does not involve the semi-stable reduction theorem.

We give an overview of the content of the different chapters.

0.3. Let $K$ be a complete discrete valuation field of characteristic 0, with perfect residue field of characteristic $p > 0$. In the first chapter, following [Fon04], we present some classical results of Tate and Sen: they rely on a fine analysis of the ramification in the cyclotomic $\mathbb{Z}_p$-extension of $K$, i.e. the unique $\mathbb{Z}_p$-extension $K_{\infty}$ of $K$ contained in the field generated over $K$ by all the $p^n$-th roots of 1.

Let $\mathfrak{m}_{K_{\infty}}$ be the maximal ideal of $\mathcal{O}_{K_{\infty}}$, $H_K = \text{Gal}(\overline{K}/K_{\infty})$, $\Gamma_K$ the quotient $G_K/H_K$. Let $L$ be the fraction field of the $p$-adic completion of $\mathcal{O}_{K_{\infty}}$. The crucial point is the fundamental theorem of Tate 1.2.6, that states that for every finite extension $M$ of $K_{\infty}$, we have $\text{Tr}_{M/K_{\infty}}(\mathcal{O}_M) \supseteq \mathfrak{m}_{K_{\infty}}$. Using this result, we will show that $L^{\Gamma_K} = \mathbb{C}^{G_K} = K$ and that we have an isomorphism, for every $h \in \mathbb{N}$

$$H^1_{\text{cont}}(\Gamma_K, \text{GL}_h(K_{\infty})) \cong H^1_{\text{cont}}(G_K, \text{GL}_h(\mathbb{C})).$$

Furthermore, we prove that $H^0_{\text{cont}}(G_K, \mathbb{C}(1)) = H^1_{\text{cont}}(G_K, \mathbb{C}(1)) = 0$.

In the next section we study the category of $\mathbb{C}$-representations of $G_K$, that is the category of finite dimensional $\mathbb{C}$-vector spaces equipped with a continuous and semi-linear action of $G_K$. They form an abelian category, that we denote by $\text{Rep}_{\mathbb{C}}(G_K)$. In a similar way we define the notion of $L$-representation and $K_{\infty}$-representation of $\Gamma_K$. According to Sen, we have canonical $\otimes$-equivalences of categories of representations

$$\text{Rep}_{\mathbb{C}}(G_K) \xrightarrow{\sim} \text{Rep}_L(\Gamma_K) \rightarrow \text{Rep}_{K_{\infty}}(\Gamma_K),$$

that can be described as follows.

By a first theorem of Sen, every $\mathbb{C}$-representation of $G_K$, the $\mathbb{C}$-linear morphism $\mathbb{C} \otimes_L W^{H_K} \rightarrow W$ is an isomorphism. Hence the functor $W \mapsto W^{H_K}$ is a $\otimes$-equivalence between $\text{Rep}_{\mathbb{C}}(G_K)$ and $\text{Rep}_L(\Gamma_K)$.

Let $X \in \text{Rep}_L(G_K/H_K)$ and let $X_f$ be the $K_{\infty}$-vector space obtained by taking the union of all finite dimensional $K$-subspaces of $X$ that are stable by $G_K$. A second theorem of Sen proves that the functor $X \mapsto X_f$ defines a $\otimes$-equivalence between $\text{Rep}_{K_{\infty}}(G_K/H_K)$ and $\text{Rep}_L(G_K/H_K)$, quasi-inverse of the functor $Y \mapsto Y \otimes_{K_{\infty}} L$. 

INTRODUCTION

Let $Y \in \mathbf{Rep}_{K_{\infty}}(\Gamma_K)$. We will prove that there exists a unique endomorphism $s$ of the $K_{\infty}$-vector space $Y$ such that, for every $y \in Y$, there exists an open subgroup $\Gamma_y$ of $\Gamma_K$ such that

$$\gamma(y) = \exp(\log \chi_p(\gamma)s)(y)$$

for every $\gamma \in \Gamma_y$. The endomorphism $s$ is now called the Sen endomorphism of $Y$. We will see that $s$ provides enough information to classify the representations up to isomorphisms. We conclude the chapter by giving the abstract definition of Hodge-Tate representations.

0.4. In the second chapter we give the proof of Fontaine of the Theorem of Tate and Raynaud. Let $K$ be as in 0.3. Let $\Omega^1_{\mathcal{O}_K/\mathcal{O}_K}$ be the module of $\mathcal{O}_K$-differentials of $\mathcal{O}_K$. The first part of the chapter is dedicated to the study of this Galois module: we will construct a surjective, $G_K$-equivariant and $\mathcal{O}_K$-linear morphism

$$\xi: \overline{K} \otimes T_p(\mathbb{G}_m) \to \Omega^1_{\mathcal{O}_K/\mathcal{O}_K}$$

where $T_p(\mathbb{G}_m)$ denotes the $p$-adic Tate module of the multiplicative group over $\overline{K}$. The kernel of $\xi$ is given by $a \otimes T_p(\mathbb{G}_m)$, where

$$a = \{ a \in \overline{K} \mid v(a) \geq -v(D) - \frac{1}{q-1} \}$$

and $D$ is the absolute different of $K$. By passing to the limit, we will get a $G_K$-isomorphism

$$\text{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^1_{\mathcal{O}_K/\mathcal{O}_K}) \cong \mathbf{C}(1).$$

This will be obtained as a particular case of more general results on Lubin-Tate formal groups, that hold also when $K$ is a complete discrete valuation field of characteristic $p > 0$ and perfect residue field.

Let $X$ be an abelian variety over $\eta$. In section 2.4, we will use the results presented so far to give Fontaine’s proof of the decomposition (4). The idea goes as follows: the theorem can be reduced to showing the existence of a $K$-linear injective morphism

$$H^0(X, \Omega^1_{X/\eta}) \to \text{Hom}_{\mathbb{Z}_p[G_K]}(T_p(X), \mathbf{C}(1)).$$

The first step is to consider a proper model $\mathcal{X}/S$ of finite type for the abelian variety $X/\eta$. The group scheme structure on $X$ induces a group structure on the set $\mathfrak{X}(\mathcal{O}_K)$, identified with $X(\overline{K})$, and the translation action of $X(\overline{K})$ induces a morphism

$$\hat{\theta} = \hat{\theta}_{x,x,r}: p^r H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/S}) \to \text{Hom}_{\mathbb{Z}[G_K]}(X(\overline{K}), \Omega^1_{\mathcal{O}_K/\mathcal{O}_K})$$

for a suitable non negative integer $r$. More precisely, given $\omega \in p^r H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/S})$, we set $\hat{\theta}(\omega)$ to be the $\mathbb{Z}[G_K]$-linear morphism

$$\hat{\theta}(\omega): u \mapsto u^*(\omega).$$

Let $V_p(X) = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], X(\overline{K}))$. By composing with

$$\text{Hom}_{\mathbb{Z}[G_K]}(X(\overline{K}), \Omega^1_{\mathcal{O}_K/\mathcal{O}_K}) \to \text{Hom}_{\mathbb{Z}[G_K]}(V_p(X), \text{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^1_{\mathcal{O}_K/\mathcal{O}_K}))$$

and extending the scalars to $K$, we get a $K$-linear map that eventually restricts to

$$\theta = \theta_{x,x,r}: H^0(X, \Omega^1_{X/\eta}) \to \text{Hom}_{\mathbb{Z}_p[G_K]}(T_p(X), \text{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^1_{\mathcal{O}_K/\mathcal{O}_K})).$$
This is the required injective morphism (6), if we take into account the isomorphism (5). It does not depend on the choice of $r$ and of $X$.

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Io sono qui
sono venuto a suonare
sono venuto a danzare
e di nascosto ad amare...

(P. Conte)
CHAPTER 1

C-representations: the theory of Tate and Sen

1.1. Review of group cohomology

1.1.1. Let $G$ be a topological group. Let $M$ be a topological $G$-module, i.e. a topological abelian group endowed with a liner and continuous action of $G$. Let $C^n_{cont}(G, M)$ be the group of continuous $n$-cochains of $G$ with values in $M$. Let

$$d_n: C^n_{cont}(G, M) \to C^{n+1}_{cont}(G, M)$$

be the boundary map

$$d_n f(g_1, \ldots, g_{n+1}) = g_1 f(g_2, \ldots, g_{n+1}) + \sum_{j=1}^{n} (-1)^j f(g_1, \ldots, g_j g_{j+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n).$$

The sequence $C^*_{cont}(G, M)$ is a cochain complex. We denote by $H^n_{cont}(G, M)$ the $n$-th cohomology group of this complex: it is called the $n$-th continuous cohomology group of $G$ with coefficients in $M$.

1.1.2. Given a short exact sequence of topological $G$-modules

$$0 \to M \to M' \to M'' \to 0$$

we have a six-terms-long exact sequence

$$0 \to M^G \to M'^G \to M''^G \to H^1_{cont}(G, M) \to H^1_{cont}(G, M') \to H^1_{cont}(G, M'').$$

1.1.3. We can still define the groups $H^0$ and $H^1$ even when we drop the abelian hypothesis on $M$, as in [Ser62], Appendix to chap. VII. Let $M$ be a topological group, written multiplicatively, endowed with a continuous action of $G$. $H^0_{cont}(G, M)$ is defined as the group $M^G$ of elements of $M$ fixed by $G$. We denote by $Z^1_{cont}(G, M)$ the subset of the set of continuous functions of $G$ into $M$ such that

$$f(g_1 g_2) = f(g_1) g_1(f(g_2))$$

for $g_1, g_2 \in G$: we call $f \in Z^1_{cont}(G, M)$ a continuous cocycle. We say that two cocycles $f$ and $f'$ are cohomologous and write $f \sim f'$ if there exists $a \in M$ such that

$$f'(g) = a^{-1} f(g) g(a)$$

for every $g \in G$. This defines an equivalence relation on the set of cocycles. The quotient set has a structure of pointed set: it contains a distinguished element equal to the class of the unit cocycle $f(g) = 1$ for every $g \in G$. We denote its class by 1. We denote $Z^1_{cont}(G, M)/\sim$ by $H^1_{cont}(G, M)$ and we call it the cohomology set of $G$ with values in $M$. This definition coincides (if we retain just the structure of pointed sets) with the usual one in the abelian case.
1.2. STATEMENT OF THE THEOREMS OF TATE AND SEN

1.1.4. Let $G$ be a topological group and let $H$ be a closed normal subgroup of $G$. Any topological $G$-module $M$ (abelian or not) can be regarded as $H$-module, as well as $M^H$ can be regarded as $G/H$-module. Then we can naturally define the restriction map
\[ \text{res}: H^1_{\text{cont}}(G, M) \to H^1_{\text{cont}}(H, M) \]
and the inflation map
\[ \text{Inf}: H^1_{\text{cont}}(G/H, M^H) \to H^1_{\text{cont}}(G, M). \]
One has the following inflation-restriction exact sequence of pointed sets (resp. of abelian groups if $M$ is abelian):
\[ 1 \to H^1_{\text{cont}}(G/H, M^H) \overset{\text{Inf}}{\longrightarrow} H^1_{\text{cont}}(G, M) \overset{\text{res}}{\longrightarrow} H^1_{\text{cont}}(H, M). \]
There is a direct proof, valid for the abelian as well as for the non abelian case, in [Ser62], chap. VII, §6.

1.2. Statement of the theorems of Tate and Sen

1.2.1. Let $K$ be a complete discrete valuation field of characteristic 0, with perfect residue field of characteristic $p > 0$. We fix an algebraic closure $\overline{K}$ of $K$ and we denote by $G_K$ the Galois group of $K$ over $K$. We denote by $O_K$ the ring of integers of $K$ and by $O_{\overline{K}}$ the ring of integers of $\overline{K}$. Let $O_C$ be the $p$-adic completion of $O_{\overline{K}}$, $C$ its field of fractions. We denote by $v_p$ the valuation of $C$ extending the valuation of $K$ normalized by $v_p(p) = 1$, and by $|.|$ the $p$-adic absolute value.

For any subfield $M$ of $C$, we denote by $O_M$ its valuation ring and by $m_M$ the maximal ideal of $O_M$. If $M$ is a finite extension of $K$ we denote by $v_M$ the unique valuation of $C$ normalized by $v_M(M^\times) = \mathbb{Z}$ and by $e_M = v_M(p)$ the absolute ramification index of $M$.

1.2.2. Let $\chi_p$ be the cyclotomic character of $K$, i.e. the continuous homomorphism
\[ \chi_p: G_K \to \mathbb{Z}_p^\times \]
that gives the action of $G_K$ on the group of units of order a power of $p$. Let $\log$ be the $p$-adic logarithm, $\log: \mathbb{Z}_p^\times \to \mathbb{Z}_p$. We denote by $H_K$ its kernel and by $\Gamma_K$ the quotient $G_K/H_K$. Notice that $\Gamma_K \cong \mathbb{Z}_p$ as abelian groups.

Let $K_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $K$: it is the unique $\mathbb{Z}_p$ extension of $K$ contained in the subfield of $K$ generated by the roots of unity of order a power of $p$. By construction we have that $H_K = \text{Gal}(\overline{K}/K_\infty)$ and $\Gamma_K = \text{Gal}(K_\infty/K)$. Let $L$ be the closure of $K_\infty$ in $C$.

The goal of the first part of this chapter is to present the proof of the following theorems (originally due to Tate and Sen).

1.2.3. Theorem. We have:
   i) $H^0_{\text{cont}}(H_K, C) = C^{H_K} = L$;
   ii) For every $n \geq 1$, $H^1_{\text{cont}}(H_K, \text{GL}_h(C)) = 1$.

   As a corollary, we have $C^{G_K} = L^{G_K}$ and $H^1_{\text{cont}}(G_K, \text{GL}_h(C)) = H^1_{\text{cont}}(\Gamma_K, \text{GL}_h(L))$. Indeed, $C^{G_K} = (C^{H_K})^{G_K} = L^{G_K}$ while the second statement follows from the inflation-restriction exact sequence (1.1.4.1).

1.2.4. Theorem. We have:
1.3. THE PROOF OF TATE’S THEOREM 1.2.6

i) \( H^0_{\text{cont}}(G_K, C) = H^0_{\text{cont}}(\Gamma_K, L) = K \);

ii) For every \( h \geq 1 \), the map

\[
H^1_{\text{cont}}(\Gamma_K, \text{GL}_h(K_\infty)) \to H^1_{\text{cont}}(\Gamma_K, \text{GL}_h(L)) = H^1_{\text{cont}}(G_K, \text{GL}_h(C))
\]

induced by \( \text{GL}_h(K_\infty) \subset \text{GL}_h(C) \) is bijective.

1.2.5. The proof of theorems 1.2.3 and 1.2.4 relies on the following important result of Tate, whose proof is a consequence of a detailed analysis of a ramified \( \mathbb{Z}_p \)-extension of \( K \) (not necessarily the cyclotomic \( \mathbb{Z}_p \)-extension of 1.2.2).

1.2.6. THEOREM (Tate, cf. [Fon04], Théorème 1.8). We keep the notations of 1.2.1. Let \( K_\infty \) be a ramified \( \mathbb{Z}_p \) extension of \( K \) and let \( M \) be a finite extension of \( K_\infty \). Let \( \text{Tr}_{M/K_\infty} : M \to K_\infty \) be the trace map. Then \( \text{Tr}_{M/K_\infty}(O_M) \supseteq M_{K_\infty} \).

1.3. The proof of Tate’s Theorem 1.2.6

1.3.1. Let \( K \) be as in 1.2.1. Let \( E \) be a finite extension of \( K \), \( J \) the Galois group \( \text{Gal}(E/K) \), \( \alpha \in O_E \) such that \( O_E = O_K[\alpha] \) [Ser62, chap. III, Prop. 12]. We denote by \( J_i \) the \( i \)-th higher ramification group of \( K \) of \( J \) [Ser62, chap. IV, §1]. We have

\[ J_i = \{ g \in J | i_J(g) \geq i + 1 \} \]

where \( i_J(g) = v_E((g - 1)\alpha) \) for every \( g \in J \). We call the integers \( i \) such that \( J_i \neq J_{i+1} \) the ramification numbers of the extension \( E/K \).

1.3.2. PROPOSITION. Let \( E \) be a cyclic ramified extension of \( K \) of degree \( p \). Let \( i \) be the unique ramification number of the extension \( E/K \). Then we have \( i \leq \frac{\varepsilon_E}{p-1} \) and, for every \( x \in E \), \( v_E(\text{Tr}_{E/K}(x)) \geq v_E(x) + (p-1)i \).

PROOF. Let \( \tau \) be a generator of \( J = \text{Gal}(E/K) \). We have, for every \( x \in E \), \( v_E((\tau - 1)x) \geq v_E(x) + i \), and the equality holds if and only if \( v_E(x) \) is prime to \( p \). Let \( P(T) \in \mathbb{Z}[T] \) be a polynomial such that

\[
(1.3.2.1) \quad pP(T) = \sum_{j=0}^{p-1} T^j - (T - 1)^{p-1}.
\]

Hence, for every \( x \in E \), we have

\[
(1.3.2.2) \quad \text{Tr}_{E/K}(x) = (\tau - 1)^{p-1}(x) + pP(\tau)(x)
\]

and

\[
(1.3.2.3) \quad v_E(pP(\tau)(x)) = e_E + v_E(x),
\]

since

\[
pP(\tau)(x) = px + \sum_{j=1}^{p-1}(1 + \tau + \ldots + \tau^{j-1})(1 - \tau)(x) - (1 - \tau)^{p-1}(x).
\]

Suppose that \( p \) divides \( i \) and let \( \pi \in E \) such that \( v_E(\pi) = 1 \). We have \( v_E((\tau - 1)^{p-1}(\pi)) = (p-1)i + 1 \) and \( v_E(pP(\tau)(\pi)) = e_E + 1 \) (by (1.3.2.3)), that are both prime to \( p \) (as \( e_E \) is divisible by \( p \)). On the other hand, \( v_E(\text{Tr}_{E/K}(\pi)) = pV_K(\text{Tr}_{E/K}(\pi)) \) is divisible by \( p \). Therefore we have the equality \( e_E + 1 = (p - 1)i + 1 \) (using (1.3.2.2)).
Suppose that \( p \) does not divide \( i \) and let \( y \in E \) such that \( v_E(y) = i \). We have

\[
v_E((\tau - 1)^{p-1}(y)) = (p-1)i + i = pi,
\]
while \( v_E(pP(\tau)(y)) = e_E + i \) is prime to \( p \). As we have again that \( v_E(\text{Tr}_E/K(y)) \) is divisible by \( p \), we must have \( pi < e_E + i \).

By (1.3.2.2) we have, in both cases,

\[
v_E(\text{Tr}_E/K(x)) \geq v_E(x) + \min\{(p-1)i, e_E\} \geq v_E(x) + (p-1)i
\]
for every \( x \in E \).

1.3.3. LEMMA. Let \( m, n \) be integers verifying \( n \geq m - 1 \geq 0 \). Let \( i_0, i_1, \ldots, i_{m-1} \) be integers verifying \( i_r \equiv i_{r-1} \mod p^r \) for \( 1 \leq r \leq m - 1 \). Then the integers \( j + iv_p(j) \) for \( j \in \mathbb{Z} \) verifying \( 0 < j < p^n \) and \( iv_p(j) < m \) are all distinct mod \( p^n \).

PROOF. Suppose, by contradiction, that there exist \( j, j' \in \mathbb{Z} \) as above and verifying \( j + iv_p(j) = j' + iv_p(j') \). Let \( s = v_p(j), s' = v_p(j') \). We can suppose \( s < s' \), so that \( 0 \leq s \leq m - 2 \). But then \( v_p(j' - j) = s \), while

\[
v_p(i_s - i_{s'}) + p^ns \geq \min\{s + 1, n\} = s + 1,
\]
which is a contradiction, as \( j' - j = (i_s - i_{s'}) + p^ns \).

1.3.4. PROPOSITION. Let \( n \) be an integer \( \geq 1 \), \( E \) a cyclic totally ramified extension of \( K \) of degree \( p^n \). Let \( \gamma \) be a generator of the Galois group \( \text{Gal}(E/K) \). Then

i) The extension \( E/K \) has exactly \( n \) distinct ramification numbers

\[
0 < i_0 < i_1 < \ldots < i_{n-1}.
\]

ii) For \( 1 \leq r \leq n - 1 \), we have \( i_r \equiv r_{n-1} \mod p^r \).

iii) For every \( y \in E^\times \) there exists \( \lambda \in K \) such that

\[
v_p(y - \lambda) \geq v_p((\gamma - 1)y) - \frac{1}{p-1}
\]

PROOF. Let \( K' \) (resp. \( E' \)) be the unique extension of degree \( p \) (resp. of degree \( p^{n-1} \)) of \( K \) contained in \( E \). We argue by induction on \( n \). The ramification numbers of \( E/K' \) are \( i_1, i_2, \ldots, i_{n-1}, i_0 \), since the lower numbering is compatible with the passage to subgroups. Using [Ser62, chap. IV, Prop. 3], we get that for \( n \geq 2 \), the ramification numbers of \( E'/K \) are \( i_0, i_1, \ldots, i_{n-2}, i_0 \), and i) follows.

Let \( \pi \) be a uniformizer of \( E \), so that \( v_E(\pi) = 1 \). Let \( J = \text{Gal}(E/K) \). For every \( r \in \mathbb{N} \) verifying \( 1 \leq r < p^n \), we have \( i_j(\gamma^r) = iv_p(r) \) and \( v_E(\gamma^r - 1)(\pi) = iv_p(r) + 1 \). For every \( s \in \mathbb{Z} \) verifying \( v_p(s) < n \), there exists \( \pi_s \in E \) such that \( v_E(\pi_s) = s \) and \( v_E((\gamma - 1)(\pi_s)) = s + iv_p(s) \).

Indeed, set \( \pi_0 \) and define, for every \( 1 \leq r < p^n \), \( \pi_r = \pi \gamma(\pi) \cdots \gamma^{r-1}(\pi) \). Then \( v_E(\pi_r) = r \) and \( (\gamma - 1)(\pi_r) = \pi \gamma(\pi) \cdots \gamma^{r-1}(\pi)(\gamma^r(\pi) - \pi)/\pi \), so that \( v_E((\gamma - 1)(\pi_r)) = r + iv_p(r) \). For \( s \geq p^n \), let \( r \) be the remainder of the division of \( s \) by \( p^n \). Then there exists \( \lambda_s \in K \) such that \( v_E = s - r \), and we can take \( \pi_s = \lambda_s \pi_r \). By substituting \( K \) with \( K' \), we see that for every \( s \in \mathbb{Z} \) verifying \( v_p(s) < n - 1 \), there exists \( z_s \in E \) such that \( v_E(z_s) = s \) and \( v_E((\gamma^p - 1)(z_s)) = s + iv_p(s + 1) \).

We show ii) by induction on \( n \). For \( n = 1 \) there is nothing to prove, so we can assume \( n \geq 2 \). The induction hypothesis applied to the extension \( E'/K \) shows that

\[
i_r \equiv i_{r-1} \mod p^r \quad \text{for} \quad 1 \leq r \leq n - 2.
\]
On the other hand, the induction hypothesis applied to $E/K'$ shows that $i_{n-1} \equiv i_{n-2} \mod p^{n-2}$. Let $s = i_{n-2} - i_{n-1}$. To conclude we need to show that $v_p(s) \neq n - 2$.

We argue by contradiction. Let $z_s$ be as above, so that $v_E((\gamma^p - 1)(z_s)) = s + i_{n-1} = i_{n-2}$. Let $x_s = (1 + \gamma + \gamma^2 + \ldots + \gamma^{p-1})(z_s)$. By (1.3.2.1) and (1.3.2.3) we have $v_E(x_s) > s$ and $v_E((\gamma - 1)(x_s)) = v_E((\gamma^p - 1)(z_s)) = i_{n-2}$. Since the extension $E/K$ is totally ramified of degree $p^n$, $\{\pi_r\}_{1 \leq r < p^n}$ is a basis of $E$ over $K$. Write $x_s = \sum_{r=0}^{p^n-1} b_r \pi_r$ for $b_r \in K$. Hence

$$v_E(x_s) = \min_{0 \leq r < p^n} \{p^nv_K(b_r) + r\}$$

so that $p^nv_K(b_r) + r > s$ for every $r$. As $(\gamma - 1)(x_s) = \sum_{r=1}^{p^n} b_r(\gamma - 1)(\pi_r)$, if $v_p(r) = n - 1$ we have $v_E(b_r(\gamma - 1)(\pi_r)) > s + i_{n-1} = i_{n-2}$. By 1.3.3 (for $m = n - 1$) we have

$$i_{n-2} = v_E((\gamma - 1)(x_s)) = \min_{0 \leq r < p^n; v_p(r) < i_{n-1}} \{p^nv_K(b_r) + r + i_{v_p(r)}\}.$$

Therefore there exists $r$ such that $i_{n-2} \equiv r + i_{v_p(r)} \mod p^r$, which is impossible as

$$v_p(i_{n-2} - i_{v_p(r)}) \geq v_p(r) + 1$$

by (1.3.4.2).

We finally prove iii). For $y \in E^x$ we have $y = \sum_{r=0}^{p^n-1} b_r \pi_r$, $b_r \in K$ and we can take $\lambda = b_0$. Indeed, there exists a unique $r_0$, $0 < r_0 < p^n$, such that $v_E(y - \lambda) = v_E(b_0 \pi_{r_0})$. By 1.3.3 (for $m = n$) we have

$$v_E((\gamma - 1)(y)) = \min_{0 \leq r < p^n} \{v_E(b_r \pi_r + i_{v_p(r)}) \leq v_E(y - \lambda) + i_{n-1}$$

so $v_E(y - \lambda) \geq v_E((\gamma - 1)(y)) - i_{n-1}$. Hence

$$v_p(y - \lambda) \geq v_p((\gamma - 1)(y)) - \frac{i_{n-1}}{e_E} \geq v_p((\gamma - 1)(y)) - \frac{1}{p - 1}$$

by 1.3.2 applied to the extension $E/E'$.

**1.3.5. Proposition.** Let $n$ be an integer $\geq 1$, $E$ a cyclic totally ramified extension of $K$ of degree $p^n$. Then for every $x \in E$ we have

$$v_p(\text{Tr}_{E/K}(x)) \geq v_p(x) + \frac{n(p - 1)}{pe_K}$$

**Proof.** Let $i_0 < i_1 < \ldots < i_{n-1}$ be the ramification numbers of the extension $E/K$. From 1.3.2 we deduce that

$$v_p(\text{Tr}_{E/K}(x)) \geq v_p(x) + \frac{(p - 1)}{e_K} \left(\frac{i_0}{p} + \frac{i_1}{p^2} + \ldots + \frac{i_{n-1}}{p^n}\right).$$

and the result follows, since by 1.3.4 ii) we have $i_r \geq p^r$ for every $r$.

**1.3.6.** Let $K_\infty$ be a ramified $\mathbb{Z}_p$ extension of $K$. For every $r \in \mathbb{N}$, we denote by $K_r$ the unique extension of degree $p^r$ of $K$ contained in $K_\infty$. If $\Gamma_K = \text{Gal}(K_\infty/K)$, we denote by $\Gamma_r$ the Galois group $\text{Gal}(K_\infty/K_r)$. We fix a topological generator $\gamma_0$ of $\Gamma_K$ and we let $\gamma_r = \gamma_0^{p^r}$ be a topological generator of $\Gamma_r$.

By 1.3.4, there exists a unique non negative integer $r_0 \geq 0$ and a strictly increasing sequence of positive integers

$$i_0 < i_1 < \ldots < i_{r-1} < i_r < \ldots$$
such that \( K_{r_0} \) is the maximal unramified extension of \( K \) contained in \( K_\infty \) and that, for every \( r > r_0 \), the ramification numbers of the extension \( K_r/K_{r_0} \) are precisely \( i_0, i_1, \ldots, i_{r-r_0-1} \). Moreover, we have \( i_r \equiv i_{r-1} \mod p^r \). The sequence \((i_r)_{r\in\mathbb{N}}\) is called the sequence of ramification numbers of the extension \( K_\infty/K \).

1.3.7. Let \( F \) be a finite Galois extension of \( K \) such that \( F \cap K_\infty = K \). For every \( r \geq 0 \), let \( F_r = K_rF \) and \( F_\infty = K_\infty F \). Let \( J = \text{Gal}(F_\infty/K_\infty) \), \( J_r = \text{Gal}(F_r/K_r) \). Let \( \varpi_r \) be the canonical isomorphism \( J \xrightarrow{\cong} J_r \). For \( \tau \in J \), we set \( i_r(\tau) = i_{J_r}(\varpi_r(\tau)) \).

1.3.8. Proposition. Under the assumptions of 1.3.7, for every \( \tau \in J \) the sequence \( \{i_r(\tau)\} \) is stationary.

Proof. Up to replacing \( K \) with \( K_m \) for a sufficiently large \( m \), we can suppose that the extension \( F_\infty/F \) is totally ramified. Let \((j_r)\) be the sequence of ramification numbers of this extension. Using [Ser62, chap. IV, Prop. 3] we have

\[
i_r(\tau) = \begin{cases} i_{r+1}(\tau) & \text{if } i_{r+1}(\tau) \leq j_r \\ \frac{1}{p}(i_{r+1}(\tau) + (p-1)j_r) & \text{if } i_{r+1}(\tau) > j_r \end{cases}
\]

or, equivalently,

\[(1.3.8.1) \quad i_{r+1}(\tau) = \begin{cases} i_r(\tau) & \text{if } i_r(\tau) \leq j_r \\ \frac{p}{p_r}(i_r(\tau) - (p-1)j_r) & \text{if } i_r(\tau) > j_r. \end{cases}\]

Therefore we have to show that there exists \( r \) such that \( i_r(\tau) \leq j_r \). Otherwise we would have \( i_r(\tau) > j_r \), so that \( i_{r+1}(\tau) = p\frac{i_r(\tau) + (p-1)j_r}{p_r} \) by (1.3.8.1). Hence, by induction,

\[
i_r(\tau) = p^ri_0(\tau) - (p-1)(j_{r-1} + pj_{r-2} + \ldots + p^{r-1}j_0)
\]

so that

\[
j_0 + \frac{j_1 - j_0}{p} + \frac{j_2 - j_1}{p^2} + \ldots + \frac{j_r - j_{r-1}}{p^r} < i_0(\tau).
\]

The right-hand term is independent from \( r \), but the left-hand term is \( \geq r+1 \), since it is the sum of \( r+1 \) integers \( \geq 1 \) by 1.3.4, which is a contradiction.

1.3.9. Let \( E \) be a finite extension of \( K \). Let \( r \) be the unique integer such that \( m^r_K = D_{E/K} \cap \mathcal{O}_K \). We have \( m^r_K D_{E/K}^{-1} \subset \mathcal{O}_E \). Let \( \{a_1, \ldots, a_d\} \) be a basis of \( \mathcal{O}_E \) over \( \mathcal{O}_K \), \( \{a^*_1, \ldots, a^*_d\} \) the dual basis with respect to the trace form \( \text{Tr}: E \times E \to K \), \( b \) a generator of \( m^r_K \). Then \((ba_i)a^*_i \in \mathcal{O}_E \) and \( \text{Tr}_{E/K}((ba_i)a^*_i) = b \) for every \( 1 \leq i \leq d \). As \( \text{Tr}_{E/K}(\mathcal{O}_E) \) is an ideal of \( \mathcal{O}_K \), we deduce that

\[(1.3.9.1) \quad m^r_K \subset \text{Tr}_{E/K}(\mathcal{O}_E).
\]

Proof of 1.2.6. Up to replacing \( M \) with a finite extension, we can suppose that \( M \) is a Galois extension of \( K_\infty \). Up to replacing \( K \) with a finite extension contained in \( K_\infty \), we can suppose that \( M = K_\infty F \), for a finite Galois extension \( F \) of \( K \) such that \( K_\infty \cap F = K \). Using the notations of 1.3.8, we have by [Ser62, chap. IV, Prop. 4]

\[
v_{F_r}(D_{F_r/K_r}) = \sum_{\tau \in J, \tau \neq 1} i_r(\tau)
\]

for every \( r \in \mathbb{N} \). By 1.3.8 there exist an integer \( r_0 \) and a constant \( c \geq 0 \) such that \( v_{F_r}(D_{F_r/K_r}) = c \) for \( r \geq r_0 \).
Let $e$ be the ramification number of $F_r/K_r$ for every $r \geq r_0$. Let $n \in \mathbb{N}$ be the smallest integer such that $en \geq c$. We have
\[ m^n_{K_r} \subseteq \text{Tr}_{F_r/K_r}(O_{F_r}) \subseteq \text{Tr}_{M/K_\infty}(O_M). \]

The first inclusion follows from (1.3.9.1). For the second inclusion, notice that $M = FK_\infty$ and that $J = \text{Gal}(M/K_\infty)$ is isomorphic to $J_r = \text{Gal}(F_r/K_r)$. Hence, for $x \in O_F \subset O_M$, we have
\[ \text{Tr}_{F_r/K_r}(x) = \sum_{g \in J_r} g(x) = \sum_{\tau \in \tau} \tau(x) = \text{Tr}_{M/K_\infty}(x) \]
using the isomorphism $\omega_r$.

Since $v_p(m^n_{K_r}) = n/e_{K_r}$ goes to 0 as $r$ goes to $\infty$, we have that $\cup_{r \geq r_0} m^n_{K_\infty}$ and we conclude that $\text{Tr}_{M/K_\infty}(O_M) \supseteq m^n_{K_\infty}$. \qed

### 1.4. The cohomology of $\text{Gal}(\overline{K}/K_\infty)$: the proof of Theorem 1.2.3

1.4.1. We keep the notations of 1.2.1—1.2.2. Let $M$ be a finite Galois extension of $K_\infty$ and let $J = \text{Gal}(M/K_\infty)$ be the Galois group of $M$ over $K_\infty$.

1.4.2. **Lemma.** Let $c$ be a real number $> 1$. For every $\lambda \in M$ there exists $a \in K_\infty$ such that
\[ |\lambda - a| < c \sup_{g \in J} |(g - 1)\lambda| \]

**Proof.** By 1.2.6 the elements in $\text{Tr}_{M/K_\infty}(O_M)$ have arbitrary small valuation. Therefore, we can find $y \in O_M$ such that $x = \text{Tr}_{M/K_\infty}(y)$ satisfies $|x| > \frac{1}{c}$. Let $\mu = \frac{\lambda y}{x}$ and let $a = \text{Tr}_{M/K_\infty}(\mu)$. We have:
\[ a = \frac{\text{Tr}(y\lambda)}{x} = \frac{1}{x} \sum_{g \in J} g(y)g(\lambda) = \lambda + \frac{1}{x} \sum_{g \in J} g(y)(g - 1)\lambda \]
and
\[ |\lambda - a| \leq \sup_{g \in J} \left| \frac{1}{x} g(y)(g - 1)\lambda \right| < c \sup_{g \in J} |(g - 1)\lambda| \]
as $|g(y)| \leq 1$, being $y \in O_M$. \qed

**Proof of part i) of 1.2.3.** Let $\lambda \in H^0_{\text{cont}}(H_K, C) = C^{H_K}$ and write $\lambda$ as limit of a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \mathcal{K}$ such that $|\lambda - \lambda_n| < p^{-n}$. As $\lambda$ is fixed by $H_K$ we have, for every $h \in H_K$,
\[ \text{(1.4.2.1)} \quad |(h - 1)\lambda_n| = |h(\lambda - \lambda_n) + (\lambda - \lambda_n)| \leq |h\lambda - \lambda_n| = |(\lambda - \lambda_n)| < p^{-n}. \]

For every $n \in \mathbb{N}$, let $M_n$ be a finite Galois extension of $K_\infty$ containing $\lambda_n$. Let $J_n = \text{Gal}(M_n/K_\infty)$. By (1.4.2.1) we have $|(g - 1)\lambda_n| < p^{-n}$ for every $g \in J_n$ (as $J_n \leq H_K$). By 1.4.2 with $c = p$, we have that there exists $a_n \in K_\infty$ such that $|\lambda_n - a_n| < p^{1-n}$. Hence the sequences $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\{a_n\}_{n \in \mathbb{N}}$ have the same limit $\lambda$. Hence $\lambda \in L$. \qed

1.4.3. Let $M_h(O_C)$ be the ring of $h \times h$ square matrices with coefficients in $O_C$. We equip $M_h(O_C)$ with the $p$-adic topology. Let $|.|$ be the $p$-adic absolute value on $M_h(O_C)$: we have $|A| \leq p^{-r}$ if and only if $A \in p^r M_h(O_C)$.

1.4.4. **Lemma.** Let $H$ be an open subgroup of $H_K$ and let $m$ be an integer $\geq 2$. Let $f_m \in Z^1_{\text{cont}}(H, \text{GL}_h(C))$ be a continuous cocycle verifying $|f_m(s) - 1| \leq p^{-m}$ for every $s \in H$. Then there exists $b_m \in \text{GL}_h(C)$ with $|b_m - 1| \leq p^{1-m}$ such that the continuous cocycle $f_{m+1}$ defined by
\[ f_{m+1}(s) = (b_m)^{-1} f_m(s) b_m \]

satisfies $|f_{m+1}(s) - 1| \leq p^{-m-1}$ for every $s \in H$. \[ \square \]
1.4. THE COHOMOLOGY OF Gal(\(\overline{K}/K_\infty\)): THE PROOF OF THEOREM 1.2.3

PROOF. We can reduce to the case \(H = H_K\). Indeed, if \(K'_\infty = \overline{K}^H\), we can find a finite Galois extension \(K'\) of \(K\) such that \(K'_\infty\) is a ramified \(\mathbb{Z}_p\) extension of \(K'\).

Being \(f_m\) continuous, we can find an open normal subgroup \(N\) of \(H_K\) such that \(|f_m(g) - 1| \leq p^{-m-2}\) for \(g \in N\) (it’s enough to take for \(N\) the pre-image of the open ball of radius \(p^{-m-2}\) and center 1). Let \(J = H_K/N\) and let \(M = \overline{K}^N\) be the corresponding finite Galois extension of \(K_\infty\). By Theorem 1.2.6, there exists \(y \in \mathcal{O}_M\) such that \(\sum_{\tau \in J} \tau(y) = p\). If \(T\) is a system of representatives for \(J\) in \(H_K\), we let

\[
b_m = \frac{1}{p} \sum_{g \in T} f_m(g)g(y).
\]

As \(f_m(g) \in 1 + p^m M_h(\mathcal{O}_C)\), we can write \(f_m(g) = 1 + p^m a_m(g)\) for \(a_m(g) \in M_h(\mathcal{O}_C)\), so that

\[
b_m = \frac{1}{p} \sum_{g \in T} (1 + p^m a_m(g))g(y) = \frac{1}{p} \sum_{g \in T} g(y) + p^{m-1} \sum_{g \in T} a_m(g)g(y).
\]

Hence \(b_m \in 1 + p^{m-1} M_h(\mathcal{O}_C)\). In particular, \(b_m \in \text{GL}_h(\mathcal{O}_C)\). For every \(s \in H_K\) we have

\[
s(b_m) = \frac{1}{p} \sum_{g \in T} s(f_m(g))((sg)(y)) = \frac{1}{pf_m(s)} \sum_{g \in T} f_m(sg)((sg)(y)).
\]

By the cocycle condition we also have \(f_m(sg) \equiv f_m(g) \pmod{p^{m+2}}\) when \(s \in N\) and \(g \in H_K\), and (1.4.4.1) implies

\[
s(b_m) \equiv f_m(s)^{-1} b_m \pmod{p^{m+1}}
\]
i.e. \(f_{m+1} = (b_m)^{-1} f_m(s) b_m \equiv 1 \pmod{p^{m+1}}\).

PROOF OF PART ii) of 1.2.3. Let \(f \in Z^1_{\text{cont}}(H_K, \text{GL}_h(\mathbb{C}))\). Being \(f\) continuous, we can find an open normal subgroup \(N\) of \(H_K\) such that \(|f(s) - 1| \leq p^{-2}\) for every \(s \in N\) (notice that if \(f\) is a cocycle, then \(f(1) = 1\), so that the inverse image of an open ball centred in 1 is not empty). Let \(f_2\) be the restriction of \(f\) to \(N\). By 1.4.4 we can find a sequence \(\{f_m\}_{m \geq 2}\) of continuous cocycles verifying \(|f_m(s) - 1| \leq p^{-m}\) for every \(s \in N\) and a sequence \(\{b_m\}_{m \geq 2} \subseteq \text{GL}_h(\mathbb{C})\) verifying \(|b_m - 1| \leq p^{1-m}\) such that

\[
f_{m+1}(s) = b_m^{-1} f_m(s) b_m
\]
for every \(s \in N\). Let \(\{\beta_m = \prod_{k=2}^{m} b_k\}\) the sequence of products. Then, for every \(s \in N\),

\[
f_{m+1}(s) = \beta_m^{-1} f(s) \beta_m.
\]

Let \(b\) be the limit of the sequence \(\{\beta_m\}_{m \geq 2}\); since \(\lim_{m \to \infty} f_m = 1\), \(b\) is an element of \(\text{GL}_h(\mathbb{C})\) satisfying \(1 = b^{-1} f(s) b\) for every \(s \in N\). In other words, the restriction of \(f\) to \(N\) is cohomologous to the trivial cocycle. The inflation-restriction exact sequence (1.1.4.1) implies that \(f\) is in the image of \(H^1_{\text{cont}}(H_K/N, (\text{GL}_h(\mathbb{C}))^N)\). But \(H_K/N\) is the galois group \(J\) of the finite Galois extension \(\mathbb{C}^N/\mathbb{C}^{H_K}\) and

\[
H^1_{\text{cont}}(J, (\text{GL}_h(\mathbb{C}))^N) = H^1(J, \text{GL}_h(\mathbb{C}^N))
\]
which is trivial by Hilbert’s Theorem 90 [Ser62, chap. X, Prop. 3].
1.5. The cohomology of $\text{Gal}(K_\infty/K)$: the proof of Theorem 1.2.4

1.5.1. Throughout this section, we denote by $K_\infty$ a ramified $\mathbb{Z}_p$ extension of $K$. We keep the notations of 1.3.6. We say that the $\mathbb{Z}_p$ extension $K_\infty/K$ is regular if it is totally ramified and if the sequence $(i_r)_{r \in \mathbb{N}}$ of ramification numbers verifies

$$i_r - i_{r-1} = p^r e_K$$

for every $r \geq 1$.

We say that the extension $K_\infty/K$ is potentially regular if there exists $r_0 \geq 0$ such that $K_\infty/K_{r_0}$ is regular. In this case, for every $r \geq r_0$, $K_\infty/K_r$ is regular.

1.5.2. Lemma ([Fon04, Prop. 1.11]). The cyclotomic $\mathbb{Z}_p$ extension of $K$ considered in 1.2.2 is potentially regular.

1.5.3. Lemma ([Fon04, Prop. 1.12]). Let $F$ be a finite extension of $K$. Then a $\mathbb{Z}_p$ extension $K_\infty/K$ is potentially regular if and only if $FK_\infty/F$ is potentially regular.

1.5.4. For every $r \in \mathbb{N}$, let $\text{Tr}_{K_r/K} : K_r \to K$ be the trace map. For $x \in K_\infty$, let $r \in \mathbb{N}$ such that $x \in K_r$; let

$$t_K(x) = \frac{1}{p^r} \text{Tr}_{K_r/K}(x).$$

The map $t_K : K_\infty \to K$ does not depend on the choice of $r$: it’s a projector from the $K$-vector space $K_\infty$ to its subspace $K$. Indeed, let $x \in K_r \subseteq K_r'$. We have

$$\frac{1}{p^r} \text{Tr}_{K_r'/K}(x) = \frac{1}{p^r} \left( \frac{1}{p^{r'-r}} \text{Tr}_{K_r/K_r'}(\text{Tr}_{K_r'/K_r}(x)) \right) = \frac{1}{p^r} \left( \frac{1}{p^{r'-r}} \sum_i \text{Tr}_{K_r/K}(\gamma^i(x)) \right)$$

where $\gamma$ is a generator of $\text{Gal}(K_r'/K_r) \leq \text{Gal}(K_r'/K)$, so that $\text{Tr}_{K_r/K}(\gamma^i(x)) = \text{Tr}_{K_r/K}(x)$, repeated exactly $p^{r'-r}$ times.

1.5.5. Proposition ([Fon04, Prop. 1.13]). Suppose that $K_\infty/K$ is regular. Then there exists $c \in \mathbb{R}_{>0}$ such that for every $x \in K_\infty$ we have

$$|t_K(x) - x| \leq c.|(\gamma_0 - 1)x|.$$

1.5.6. Proposition. Let $K_\infty/K$ be a potentially regular $\mathbb{Z}_p$ extension. Then the map $t_K : K_\infty \to K$ is continuous. If $\hat{t}_K : L \to K$ denotes the extension of $t_K$ by continuity and $L_0$ denotes the kernel of $\hat{t}_K$, we have a decomposition $L = K \oplus L_0$. The operator $\gamma_0 - 1$ is bijective on $L_0$, with a continuous inverse.

Proof. Let $r_0$ be an integer such that the extension $K_\infty/K_{r_0}$ is regular. We have

$$t_K = p^{-r_0} \text{Tr}_{K_{r_0}/K} \circ t_{K_{r_0}}$$

by transitivity of the norm maps: $p^{-r_0} \text{Tr}_{K_{r_0}/K}$ is clearly continuous (being $K_{r_0}/K$ finite) and $t_{K_{r_0}}$ is continuous by 1.5.5.

For the second assertion, suppose firstly that $K_\infty/K$ is regular. If $x \in K$, then $\hat{t}_K(x) = x$, so that $\hat{t}_K^2 = \hat{t}_K$ and we can write $L$ as sum $K \oplus L_0$. For every $x \in L$ we clearly have $(\gamma_0 - 1)(x) \in L_0$ and, in particular, $(\gamma_0 - 1)(L_0) \subset L_0$. Let $K_{\infty,0} = K_\infty \cap L_0$ and let, for every $r \in \mathbb{N}$, $K_{r,0} = K_r \cap L_0$: with this notation $K_{\infty,0}$ is the union of $K_{r,0}, r \in \mathbb{N}$ ($K_{r,0} \subset K_{r+1,0} \subset \ldots$) and $L_0$ is the closure of $K_{\infty,0}$ in $L$. As the operator $\gamma_0 - 1$ is injective (hence bijective) on every
finite-dimensional $K$-vector space $K_{r,0}$, it is also bijective on their union $K_{\infty,0}$. Let $\varrho$ be its inverse. For every $y \in K_{\infty,0}$, as $t_K(\varrho(y)) = \hat{t}_K(\varrho(y)) = 0$, we have by 1.5.5

$$|\varrho(y)| \leq c|y|$$

and $\varrho$ is continuous. We can extend it to a continuous map, denoted again by $\varrho$, from $L_0$ to itself, which is a continuous inverse of $\gamma_0 - 1$.

For the general case, let $r_0$ be an integer such that the extension $K_\infty / K_{r_0}$ is regular and let $\hat{t}_{K_{r_0}}$ be the continuous extension of $t_{K_{r_0}}$ to $L$. Let $L_{r_0}$ be its kernel, $\varrho_{r_0} : L_{r_0} \to L_{r_0}$ the inverse of the restriction of $\gamma_{r_0} - 1$. We have

$$L = K \oplus L_0 = K_{r_0} \oplus L_{r_0}$$

and, since $L_{r_0} \subset L_0$, we can write

$$L_0 = L_0 \cap K_{r_0} \oplus L_{r_0}.$$  

The map $\gamma_0 - 1$ is injective on $L_0$, as $L_0 \cap K = 0$. Since $K_{r_0}$ is a finite-dimensional $K$-vector space, $L_0 \cap K_{r_0}$ is of finite dimension over $K$, so that $\gamma_0 - 1$ is bijective with continuous inverse on it. As

$$\gamma_{r_0} - 1 = \gamma_{r_0}^0 - 1 = (\gamma_0 - 1)A(\gamma_0)$$

for $A \in Z[\gamma_0]$, we see that $\gamma_0 - 1$ is bijective on $L_{r_0}$, with continuous inverse $A(\gamma_0)\varrho_{r_0}$. \hfill \Box

1.5.7. **Proposition.** Suppose that $K_\infty / K$ is regular. Let $\lambda$ be a principal unit of $O_K$ (i.e. $|\lambda - 1| < 1$) but not a root of unity, then $\gamma_0 - \lambda$ is bijective with continuous inverse on $L$.

**Proof.** Since $\gamma_0 - \lambda$ is obviously bijective on $K$ if $\lambda \neq 1$, we can use the decomposition $L = K \oplus L_0$ and prove the statement for $L_0$. Let $\varrho$ be the inverse of $\gamma_0 - 1$. We have:

$$(1.5.7.1) \quad \varrho \circ (\gamma_0 - \lambda) = \varrho \circ ((\gamma_0 - 1) - \lambda + 1) = 1 - (\lambda - 1)\varrho.$$  

Let $c$ be the constant in 1.5.5. If $|\lambda - 1|c < 1$, we have $|(\lambda - 1)\varrho(y)| \leq |y|$ for all $y \in L_0$ (see the proof of 1.5.6), and consequently $1 - (\lambda - 1)\varrho$ is an automorphism of $L_0$, with inverse given by the (convergent) geometric series

$$\sum_{r \geq 0}[(\lambda - 1)\varrho]^r.$$  

Hence, by (1.5.7.1), $\gamma_0 - \lambda$ has a continuous inverse on $L_0$. If $|\lambda - 1|c \geq 1$, we replace $\gamma_0$ by $\gamma_r = \gamma_0^r$ and $\lambda$ by $\lambda^r$, where $r$ is large so large that $|\lambda^r - 1|c < 1$ (notice that such $r$ exists, since $\lambda = 1 + x$, where $v(x) \geq 1$). We then replace $K$ by $K_r$, so that $\gamma_r - \lambda^r$ has a continuous inverse on $L_0$. Hence the map

$$(\gamma_0 - \lambda)^r - \gamma_r - \lambda^r$$

has a continuous inverse, so the same is true for $(\gamma_0 - \lambda)^r$ and hence for $(\gamma_0 - \lambda)$ too. \hfill \Box

1.5.8. **Remark.** Using exactly the same argument as in the proof of 1.5.6, we can prove 1.5.7 assuming only that $K_\infty / K$ is potentially regular.
1.5.9. From now on, we suppose that the $\mathbb{Z}_p$ extension $K_\infty/K$ is potentially regular. We denote by $L$ the closure of $K_\infty$ in $C$, $\Gamma_K = \text{Gal}(K_\infty/K)$, $H_L = \text{Gal}(\overline{K}/K_\infty)$. We will prove Theorem 1.2.4 as a particular case of the same statement for any potentially regular $\mathbb{Z}_p$-extension.

**Proof of part i) of 1.2.4.** It’s an immediate consequence of 1.5.6. Indeed we have

$$H^0_{\text{cont}}(\Gamma_K, L) = L^{\Gamma_K} = \{ x \in L \mid (\gamma_0 - 1)x = 0 \} = \text{Ker}(\gamma_0 - 1),$$

but $L = K \oplus L_0$ and $\gamma_0 - 1$ is bijective on $L_0$, so that $\text{Ker}(\gamma_0 - 1) = K$. □

1.5.10. **Theorem ([Sen80, Prop. 3]).** Let $V$ be a finite dimensional $K$-vector space, $V \subset L$. If $V$ is stable by $\gamma_0$, then $V \subset K_\infty$.

**Proof.** Let $u \in \text{End}_K(V)$ be the restriction of $\gamma_0$ to $V$ and let $f_u(T)$ be its characteristic polynomial: we can reduce to the case $f_u(T)$ has all its roots in $K$. Indeed, let $K'$ be the extension of $K$ obtained by adding the roots of $f_u(T)$ in $\overline{K}$. Let $K'\infty = K'K_\infty$. Then the extension $K_\infty'/K'$ is potentially regular (see Remark 1.5.3) and we can substitute $K$ by $K'$, $V$ by $K' \otimes K$ and so on. Moreover, we can suppose that $u$ has only one eigenvalue, say $a$, by taking the decomposition of $V$ as direct sum of its generalized eigenspaces.

Let $v$ be a non zero eigenvector of $u$. We have $\gamma_0(v) = av$, so that $\gamma_0^p(v) = a^p v$. We have that $|(\gamma_0 - 1)x| \leq |x|$, being the action of $\Gamma_K$ on $L$ continuous, so that $a$ must be a principal unit (i.e. congruent to 1 mod $p$). By 1.5.7 $a$ must be a root of unity (cfr [Tat67], Prop. 7).

Up to replacing $K$ by a finite extension contained in $K_\infty$, we can suppose that $a = 1$. Up to replacing $V$ by $V + K$ (if $V$ does not contain $K$), we may assume that $V = K \oplus V'$, with $V' \subset L_0 = \text{Ker} \gamma_0$. But then $\gamma_0 - 1$ is bijective on $L_0$, so that $V' = 0$ and $V = K \subset K_\infty$. □

**Proof of part ii) of 1.2.4.** Let $\iota$ be the map

$$\iota : H^1_{\text{cont}}(\Gamma_K, \text{GL}_h(K_\infty)) \to H^1_{\text{cont}}(\Gamma_K, \text{GL}_h(L))$$

We first prove that $\iota$ is injective: let $f, f' \in Z^1_{\text{cont}}(\Gamma_K, \text{GL}_h(K_\infty))$ be two continuous cocycles that become cohomologous in $\text{GL}_h(L)$. Then there exists $b \in \text{GL}_h(L)$ such that

$$f'(\gamma_0) = b^{-1}f(\gamma_0)\gamma_0(b)$$

(1.5.10.1)

and it’s enough to show that $b \in \text{GL}_h(K_\infty)$. We can rewrite (1.5.10.1) as

$$\gamma_0(b) = f(\gamma_0)^{-1}b f'(\gamma_0).$$

(1.5.10.2)

Let $K'$ be the extension of $K$ generated by the coefficients of $f(\gamma_0)$ and $f'(\gamma_0)$: it is a finite extension of $K$ contained in $K_\infty$. Let $V$ be the $K'$-vector space generated by the coefficients of $b$: it’s a finite dimensional $K'$-vector space, contained in $L$, and (1.5.10.2) shows that it is stable by $\gamma_0$. Being $V$ closed in $L$, we can apply Theorem 1.5.10 to get $V \subset K_\infty$, so that $b \in \text{GL}_h(K_\infty)$.

To prove the surjectivity we need an auxiliary technical result:

1.5.11. **Lemma.** For every matrix $A \in M_h(L)$, let $v(A)$ be the minimum of the $p$-adic valuations of its coefficients. Let $r$ be an integer such that the extension $K_\infty/K_r$ is regular and let $m$ be an integer $\geq 5$. Let $A_m \in \text{GL}_h(L)$, $X_m \in \text{GL}_h(K_r)$ be matrices verifying

$$v(A_m - 1) \geq \frac{3p}{p - 1}, \quad v(A_m - X_m) \geq \frac{mp}{p - 1}.$$
Then there exist $B_m \in \text{GL}_h(L)$ verifying $v(B_m - 1) \geq \frac{(m-2)p}{p-1}$ and $X_m \in \text{GL}_h(K_r)$ such that the matrix

$$A_{m+1} = B_{m}^{-1}A_{m}\gamma_r(B_m)$$

verifies $v(A_{m+1} - 1) \geq \frac{3p}{p-1}$ and $v(A_{m+1} - X_{m+1}) \geq \frac{p(m+1)}{p-1}$.

The proof of the lemma is a direct computation similar to 1.4.4, using 1.5.6, and we omit it. See [Fon04, Lemme 1.17].

We can now prove that $\iota$ is surjective: let $f \in Z_{\text{cont}}^1(\Gamma_K, \text{GL}_h(L))$. Being $f$ continuous, there exists an integer $r$ — that we can choose big enough so that the extension $K_\infty/K_r$ is regular — such that $v(f(\gamma_r) - 1) \geq \frac{5p}{p-1}$. Let $a_5 = f(\gamma_r)$ and let $x_5 = 1$. Using the previous lemma, we can produce three sequences of matrices: \{a_m\}_{m \geq 5} and \{b_m\}_{m \geq 5} in $\text{GL}_h(L)$ and \{x_m\}_{m \geq 5} in $\text{GL}_h(K_r)$ such that, for every $m \geq 5$:

$$v(a_m - 1) \geq \frac{3p}{p-1};$$

$$v(a_m - x_m) \geq \frac{mp}{p-1};$$

$$v(b_m - 1) \geq \frac{(m-2)p}{p-1};$$

$$a_{m+1} = b_{m}^{-1}a_m\gamma_r(b_m).$$

The sequence $\{\beta_m = \prod_{k=5}^m b_k\}_{m \geq 5}$ converges to a matrix $b \in \text{GL}_h(L)$ and the sequences $\{a_m\}_{m \geq 5}$ and $\{x_m\}_{m \geq 5}$ both converge to the same limit $x \in \text{GL}_h(K_r)$ and we have

$$x = b^{-1}f(\gamma_r)\gamma_r(b).$$

Let $f'$ be the continuous cocycle, cohomologous to $f$, defined by $f'(\gamma) = b^{-1}f(\gamma)\gamma_r(b)$ for every $\gamma \in \Gamma_K$: by construction we have $f'(\gamma_r) = x \in \text{GL}_h(K_r)$. For every $\gamma \in \Gamma_K$, $\gamma_r\gamma = \gamma\gamma_r$, so that

$$f'(\gamma)\gamma(f'(\gamma_r)) = f'(\gamma_r)\gamma_r(f'(\gamma))$$

or, equivalently

$$\gamma_r(f'(\gamma)) = f'(\gamma_r)^{-1}f'(\gamma)\gamma(f'(\gamma_r)) = x^{-1}f'(\gamma)\gamma(x).$$

Hence, the $K_r$ subspace $V$ of $L$ generated by the coefficients of $f'(\gamma)$ is stable by $\gamma_r$. Since $V$ is finite dimensional over $K$ we can use again Theorem 1.5.10 to deduce $f'(\gamma) \in \text{GL}_h(K_\infty)$ for every $\gamma \in \Gamma_K$, i.e. $f$ is cohomologous to a cocycle with values in $\text{GL}_h(K_\infty)$ and it is therefore in the image of $\iota$. \hfill $\square$

1.5.12. Corollary. We have $H_{\text{cont}}^0(\Gamma_K, L_0) = H_{\text{cont}}^1(\Gamma_K, L_0) = 0$.

Proof. Indeed, $L_0^{\Gamma_K} = 0$ as we have seen in the proof of part i) of 1.2.4. Let $f \in Z_{\text{cont}}(\Gamma_K, L_0)$ be a cocycle. Being $f$ continuous, it is determined by $f(\gamma_0)$ and under this identification the group $B_{\text{cont}}^1(\Gamma_K, L_0)$ of continuous coboundaries is a subgroup of the image of $\gamma_0 - 1$. Hence $H_{\text{cont}}^1(\Gamma_K, L_0) \subset \text{Coker}(\gamma_0 - 1) = 0$ by 1.5.6. \hfill $\square$
1.6. Galois Representations

1.6.1. Let $G$ be a topological group and let $F$ be a field endowed with a linear topology and a continuous action of $G$, compatible with the field structure. A finite-dimensional $F$-vector space $V$ endowed with a semi-linear action of $G$ is called an $F$-representation of $G$. We form a category, denoted $\text{Rep}_F(G)$, with morphisms given by the $G$-equivariant maps.

We call unit representation the field $F$ with the given action of $G$. If $V \in \text{Rep}_F(G)$ we call the dual representation of $V$ the $F$-vector space $V^*$ (dual of $V$) with the action $g(\varphi(v)) = g(\varphi(g^{-1}(v)))$ for every $g \in G$, $v \in V$, $\varphi \in V^*$. Finally, given $V_1, V_2 \in \text{Rep}_F(G)$, we can form the tensor product representation $V_1 \otimes V_2$ where the action of $G$ is given by $g(v_1 \otimes v_2) = g(v_1) \otimes g(v_2)$ for every $g \in G, v_1 \in V_i$ ($i = 1, 2$). If $E = F^G$ is the subfield of $F$ fixed by $G$, the category $\text{Rep}_F(G)$ is a Tannakian category over $E$.

1.6.2. Proposition. For every $V \in \text{Rep}_F(G)$, the $F$-linear morphism

$$g_F(V) : F \otimes_E V^G \to V$$

induced by the inclusion $V^G \subset V$ is injective.

Proof. By contradiction, let $m$ be the smallest positive integer such that there exist $v_1, v_2, \ldots, v_m \in V^G$ linearly independent over $E$ but not over $F$. By the minimality of $m$, there exist $a_1 = 1, \ldots, a_m \in F^\times$ such that $\sum_{i=1}^m a_i v_i = 0$. For every $g \in G$ we have

$$0 = g\left(\sum_{i=1}^m a_i v_i\right) = v_1 + \sum_{i=2}^m g(a_i) v_i$$

so that $\sum_{i=2}^m (g(a_i) - a_i) v_i = 0$. Hence, again by the minimality of $m$, $g(a_i) - a_i = 0$ for every $i = 2, \ldots, m$, i.e. $a_i \in E$, that contradicts the independence of the $v_i$'s over $E$. \qed
1.6.3. Remark. We can prove in a similar way the following strengthened version of 1.6.2. Let $B$ be an integral $E$-algebra endowed with a linear topology and a continuous action of $G$, compatible with the ring structure. Suppose that $B^G = \text{Frac}(B)^G = E$. Then for every $V \in \text{Rep}_F(G)$, the $F$-linear morphism
\[
\varrho_{B,F}(V) : B \otimes_E (B \otimes_F V)^G \to B \otimes_F V
\]
is injective.

1.6.4. We say that $V \in \text{Rep}_F(G)$ is trivial if $V \cong F^n$ for some $n \in \mathbb{N}$ (isomorphism as $F$-representations of $G$). By 1.6.2, we see that $V$ is trivial if and only if the map $\varrho_F(V)$ is bijective or, equivalently, if and only if we have the equality $\dim_E(V^G) = \dim_F V$.

1.6.5. We keep the notations of 1.2.1—1.2.2: $K_\infty$ is the cyclotomic $\mathbb{Z}_p$ extension of $K$ contained in $\bar{K}$, $L = \text{Frac}(\hat{O}_{K_\infty})$, the completion taken with respect to the $p$-adic topology. For every $r \in \mathbb{N}$ we denote by $N_r$ the unique extension of degree $p^r$ over $K$ contained in $K_\infty$. We have $H_K = \text{Gal}(\bar{K}/K_\infty)$ and $\Gamma = \Gamma_K$ is the Galois group $\text{Gal}(K_\infty/K)$. If $\gamma_0$ is a topological generator of $\Gamma$, $\gamma_r = \gamma_0^p$ is a topological generator of $\Gamma_r = \text{Gal}(K_\infty/K_r)$.

We naturally have two $\otimes$-functors
\[
\begin{align*}
\text{Rep}_{K_\infty}(\Gamma) & \to \text{Rep}_L(\Gamma) \\
V & \mapsto L \otimes_{K_\infty} V
\end{align*}
\]
and
\[
\begin{align*}
\text{Rep}_L(\Gamma) & \to \text{Rep}_C(G_K) \\
W & \mapsto C \otimes_L W.
\end{align*}
\]
The object of the theory of Sen is to construct two functors in the opposite direction defining $\otimes$-equivalences of categories
\[
\text{Rep}_{K_\infty}(\Gamma) \not\simeq \text{Rep}_L(\Gamma) \not\simeq \text{Rep}_C(G_K).
\]

1.6.6. Theorem ([Sen80, Th. 2]). Every $C$-representation of $H_K$ is trivial.

Proof. By 1.6.2 we have to show that, for every $W \in \text{Rep}_C(H_K)$, the map $\varrho_C(W)$ is bijective. Let $\{w_1, \ldots, w_h\}$ be a $C$-basis of $W$. We can define a continuous cocycle $f : H_K \to \text{GL}_h(C)$ by the assignment $g \mapsto M_g$, where $M_g$ is the matrix representing the action of $g$ on $W$ in the basis $\{w_1, \ldots, w_h\}$, so that the $i$-th column is given by the coefficients of $g(w_i)$. Let $b$ be the matrix of base-change for another basis of $W$: the corresponding cocycle is given by the formula $f'(g) = bf(g)b^{-1}$, so that $f$ and $f'$ are cohomologous and the map does not depend on the choice of $\{w_1, \ldots, w_h\}$. By 1.2.3 (ii), $H^1_{\text{cont}}(H_K, \text{GL}_h(C))$ is trivial, so that we can choose a basis formed by elements $\{w_i\}_{i=1}^h$ fixed by $H_K$. Hence, given $w = \sum_{i=1}^h b_i w_i \in W$, we have $w \in W^{H_K}$ if and only $b_i \in C^{H_K} = L$ (by 1.2.3 (i)). Therefore $W^{H_K}$ is the $L$-vector space of basis $\{w_i\}_{i=1}^h$ and the statement follows.

1.6.7. Corollary. The functor $W \mapsto W^{H_K}$ defines a $\otimes$-equivalence between the category $\text{Rep}_C(G_K)$ and the category $\text{Rep}_L(\Gamma)$, quasi-inverse of the functor (1.6.5.2).
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Proof. By 1.6.6, the functor $W \mapsto W^{H_K}$ defines a $\otimes$-equivalence between $\text{Rep}_C(H_K)$ and the category of finite-dimensional $L$-vector spaces, where a quasi inverse given by

$$X \mapsto C \otimes_L X.$$  

If $W \in \text{Rep}_C(G_K)$, $W^{H_K}$ is naturally an $L$-representation of $\Gamma = G_K/H_K$ and $C \otimes_L W^{H_K}$ is isomorphic to $W$ as (trivial) representation of $H_K$, but also as representation of $G_K$. If $Y \in \text{Rep}_L(\Gamma)$, $(C \otimes_L Y)^{H_K} \cong C^{H_K} \otimes_L Y = Y$, by definition of the action of $G_K$ on a tensor product. \hfill $\square$

1.6.8. Let $V \in \text{Rep}_{K_\infty}(\Gamma)$ and let $\{v_1, \ldots, v_h\}$ be a $K_\infty$-basis of $V$ as vectors space. Let $M_0$ be the matrix representing the action of $\gamma_0$ on $V$ in the basis $\{v_i\}$. Let $K_r$ be the field generated over $K$ by the coefficients of $M_0$: the integer $r$ is called the degree of the basis $\{v_1, \ldots, v_h\}$. Since $K_r$ is complete and the action of $\Gamma$ over $V$ is continuous, the $K_r$-vector space generated by $\{v_1, \ldots, v_h\}$ and contained in $V$ is stable for $\Gamma$.

1.6.9. Theorem (\cite[Th. 3]{Sen80}). Let $X \in \text{Rep}_{L}(\Gamma)$. Let $X_f$ be the union of the sub-$K$-vector spaces of finite dimension of $X$ that are stable by $\Gamma$. The $L$-linear map

$$L \otimes_{K_\infty} X_f \to X$$

induced by the inclusion $X_f \subset X$ is bijective.

Proof. As in the proof of 1.6.6, we fix a basis $\{x_1, \ldots, x_h\}$ of $X$ over $L$ and we consider the continuous cocycle $f: \Gamma \to \text{GL}_h(L)$ that maps $\gamma \in \Gamma$ to $M_\gamma$, where $M_\gamma$ represents the action of $\gamma$ on $X$ in the basis $\{x_1, \ldots, x_h\}$: $f$ does not depend on the choice of the basis. By 1.2.4 (ii), the map

$$H^1_{\text{cont}}(\Gamma, \text{GL}_h(K_\infty)) \to H^1_{\text{cont}}(\Gamma, \text{GL}_h(L))$$

is surjective, so we can suppose that $f$ takes value in $\text{GL}_h(K_\infty)$. In other words, we can choose the $x_i$’s such that the sub-$K_\infty$-vector space $Y$ of $X$ is stable for $\Gamma$; in particular $Y \in \text{Rep}_{K_\infty}(\Gamma)$. Since the $L$-linear map $L \otimes_{K_\infty} Y \to X$ induced by the inclusion $Y \subset X$ is clearly bijective, to complete the proof of the theorem it is enough to show that $Y = X_f$.

First of all, we have $Y \subset X_f$. Indeed, let $r$ be the degree of the basis $\{x_1, \ldots, x_h\}$. For every $s \geq r$, the $K_s$-vector space generated by the $x_i$’s is of finite dimension over $K$, stable by $\Gamma$ and $Y$ is clearly equal to the union of those space.

Let $x \in X_f$, $x = \sum_{i=1}^h c_i x_i$ with $c_i \in L$. For every $\gamma \in \Gamma$, $\gamma(x) = \sum_{i=1}^h c_i(\gamma)x_i$, for suitable coefficients $c_i(\gamma) \in L$. Let $V$ be the $K_r$-subspace of $L$ generated by $c_i(\gamma)$ for $i = 1, \ldots, h$ and $\gamma \in \Gamma$ is of finite dimension over $K$. Write $(a_{i,j}(\gamma))_{1 \leq i,j \leq h}$ for the matrix $M_\gamma$. Then $(a_{i,j}(\gamma))_{1 \leq i,j \leq h} \in \text{GL}_h(K_r)$ and

$$\gamma(x) = \sum_{i=1}^h (c_i(\gamma)) x_i = \sum_{i=1}^h c_i a_{i,j}(\gamma)x_i$$

so that $V$ is also the $K_r$ vector space generated by $\gamma(c_i)$ for $i = 1, \ldots, h$ and $\gamma \in \Gamma$. It is therefore stable by $\Gamma$ and, being finite-dimensional, it is contained in $K_\infty$ by 1.5.10. Hence $c_i \in K_\infty$ and $x \in Y$. \hfill $\square$

1.6.10. Corollary. The functor $X \mapsto X_f$ defines a $\otimes$-equivalence between the category $\text{Rep}_L(\Gamma)$ and the category $\text{Rep}_{K_\infty}(\Gamma)$, quasi-inverse of the functor (1.6.5.1).
1.7. The study of $\text{Rep}_{K_\infty}(\Gamma)$

1.7.1. Theorem ([Sen80, Th. 4]). Let $Y \in \text{Rep}_{K_\infty}(\Gamma)$. There exists a unique $K_\infty$-linear endomorphism $s$ of $Y$ such that, for every $y \in Y$, there is an open subgroup $\Gamma_y$ of $\Gamma$ satisfying

$$\gamma(y) = \exp(\log \chi_p(\gamma).s)(y)$$

for every $\gamma \in \Gamma_y$. Moreover, the characteristic polynomial of $s$ has coefficients in $K$.

Proof. Let $\{y_1, \ldots, y_h\}$ be a $K_\infty$ basis of $Y$. We first prove the uniqueness of $s$. Let $s, s'$ be two endomorphisms of $Y$ having the required properties. Then there exists an open subgroup $\Gamma_r$ of $\Gamma$ such that for every $\gamma \in \Gamma_r$

$$\gamma(y_i) = \exp(\log \chi_p(\gamma).s)(y_i) = \exp(\log \chi_p(\gamma).s')(y_i)$$

for $i = 1, \ldots, h$. Hence $\exp(\log \chi_p(\gamma).s) = \exp(\log \chi_p(\gamma).s')$ for every $\gamma \in \Gamma_r$ and $s = s'$.

Let $r_0$ be the degree of the basis $\{y_1, \ldots, y_h\}$, $Y'$ the $K_{r_0}$-sub-vector space of $Y$ generated by the $y_i$'s and stable by $\Gamma$: $\Gamma_{r_0}$ acts linearly on $Y'$ (since $\Gamma_{r_0}$ fixes $K_{r_0}$) and the action on the $y_i$'s is given by a continuous homomorphism $\Gamma_{r_0} \rightarrow \text{GL}_h(K_{r_0})$, $\gamma \mapsto M_{\gamma}$. For $\gamma$ sufficiently close to 1 (but different from 1), $M_{\gamma}$ is close to $I_{h}$ in $\text{GL}_h(K_{r_0})$ and the series $\log(M_{\gamma})$ converges to an endomorphism $\log(\gamma) \in \text{End}_{K_{r_0}}(Y')$. The endomorphism $s_0 = \frac{\log \gamma}{\log \chi_p(\gamma)}$ does not depend on the choice of $\gamma$. Indeed, let $\gamma_0$ be a topological generator of $\Gamma$. Let $\gamma = \gamma_0^i$ and let $\gamma' = \gamma_0^{i'}$ be another element in $\Gamma$ such that $\log \gamma'$ is defined. Then

$$\log(\gamma') = \log(\gamma_0^{i'} \log \chi_p(\gamma')) = \log \chi_p(\gamma')\log \gamma_0$$

so that the quotient $\frac{\log(\gamma')}{\log \chi_p(\gamma')} = \frac{\log(\gamma)}{\log \chi_p(\gamma)}$ is independent from $\gamma$.

Let $s$ be the unique $K_\infty$ endomorphism of $Y$ that restricts to $s_0$ on $Y'$.

1.7.2. Lemma. There exists $r \geq r_0$ such that the endomorphism $\exp(\log \chi_p(\gamma).s)$ of $Y$ is well defined for every $\gamma \in \Gamma_r$.

We postpone the proof of the lemma. Writing out the definition of $s$, for every $y \in Y'$, we have

$$\gamma(y) = \exp(\log \chi_p(\gamma).s)(y).$$

For a general $y = \sum_{i=1}^h c_iy_i$ with $c_i \in K_\infty$, the formula is satisfied if $\gamma \in \Gamma_y = \Gamma' \cap \Gamma_r$, where $\Gamma'$ is an open subgroup of $\Gamma$ which fixes all the $c_i$'s. This proves the existence part of the theorem.

Let $M$ be the matrix of $s$ in the basis $\{y_1, \ldots, y_h\}$. For every $\gamma \in \Gamma_r$ we have

$$(\gamma(y_1), \ldots, \gamma(y_h)) = \exp(\log \chi_p(\gamma)M)(y_1, \ldots, y_h).$$

As $\gamma_0 = \gamma_0\gamma$ we have, for every $\gamma \in \Gamma_r$

$$(\gamma_0(\gamma(y_1)), \ldots, \gamma_0(\gamma(y_h))) = \exp(\log \chi_p(\gamma)\gamma_0(M))(y_1, \ldots, y_h)$$

so that $M$ and $\gamma_0(M)$ are similar, that implies that the characteristic polynomial of $s$ is fixed by $\gamma_0$, i.e. it’s coefficients are in $K$. □
1.7. THE STUDY OF $\text{Rep}_{K_\infty}(\Gamma)$

Proof of Lemma 1.7.2. It’s enough to show that there exists an open subgroup $\Gamma_r$ of $\Gamma$ such that the series

$$\exp(\log \chi_p(\gamma).s) = \sum_{n \geq 0} \frac{(\log(\chi_p(\gamma)))^n}{n!} s^n$$

converges in the ring $\text{End}_{K_\infty}(Y)$ for every $\gamma \in \Gamma_r$.

Let $\{y_1, \ldots, y_h\}$ be a $K_\infty$-basis of $Y$. For every $b \in \mathbb{Q}$, let $Y_b$ be the $\mathcal{O}_{K_\infty}$-sub module of $Y$ defined by

$$Y_b = \left\{ \sum_{i=1}^h c_i y_i \in Y \mid v_p(c_i) \geq b \right\}.$$ 

Let $a \in \mathbb{Q}$ be such that $s(Y_0) \subseteq Y_a$. Recall that (see [NS99, chap. II, Prop. 5.5])

$$\log \mathbb{Z}_p^\times = \left\{ \begin{array}{ll} p\mathbb{Z}_p & \text{if } p \neq 2 \\ p^2\mathbb{Z}_p & \text{if } p = 2. \end{array} \right.$$ 

Let $r_K$ be the unique integer such that $\log \chi_p(\Gamma_K) = p^{r_K} \mathbb{Z}_p$: we have $r_K \geq 1$ if $p \neq 2$ (resp. $r_K \geq 2$ if $p = 2$) and the equality holds if and only if $K$ is absolutely unramified, i.e. $v_K(p) = e_K = 1$ (see [NS99, chap. II, Prop. 5.4.5]). Let $r$ be the smallest non-negative integer such that $r + r_K + a > \frac{1}{p}$. Then for every $n \in \mathbb{N}$ and $\gamma \in \Gamma_r$ we have

$$\frac{(\log(\chi_p(\gamma)))^n}{n!} s^n(Y_0) \subseteq Y_{n(r+r_K-a)}$$

as $v_p(n!) = \frac{1}{p^{r_K}} \sum_{i=1}^r a_i(p^i - 1)$ if $n = \sum_{i=0}^r a_i p^i$, $0 \leq a_i < p$ is the $p$-adic expansion of $n$ ([NS99, chap. II, Lemma 5.6]). Therefore the series $\exp(\log \chi_p(\gamma).s)$ converges. \hfill $\square$

1.7.3. Let $E$ be any field. We denote by $S_E$ the category whose objects are couples $(Y, s)$, where $Y$ is a finite-dimensional $E$-vector space and $s \in \text{End}_E(Y)$, and morphisms $f : (Y_1, s_1) \to (Y_2, s_2)$ are $E$-linear maps from $Y_1$ to $Y_2$ such that $s_2 \circ f = f \circ s_1$.

We set the unit object to be $(E, 0)$ and we define the tensor product $(Y_1, s_1) \otimes (Y_2, s_2)$ by $(Y_1 \otimes_E Y_2, s_1 \otimes id_{Y_2} + id_{Y_1} \otimes s_2)$. The dual of $(Y, s)$ is $(Y^*, -s^t)$ where $Y^*$ is the dual vector space of $Y$ and $s^t$ is the transpose homomorphism of $s$. With these definitions $S_E$ has a structure of Tannakian category over $E$.

1.7.4. Let $E$ be a field containing $K_\infty$. Let $Y \in \text{Rep}_{K_\infty}(\Gamma)$. Let $Y_E = E \otimes_{K_\infty} Y$ and let $s_E$ be the $E$-endomorphism of $Y_E$ deduced by scalar extension from the endomorphism $s$ of 1.7.1. We have therefore defined a $\otimes$-functor

$$Y \mapsto (Y_E, s_E)$$

from $\text{Rep}_{K_\infty}(\Gamma)$ to $S_E$.

1.7.5. Theorem. In the notations of 1.7.4, let $Y_1, Y_2 \in \text{End}_{K_\infty}(\Gamma)$. The canonical $E$-linear map

$$E \otimes_K \text{Hom}_{\text{Rep}_{K_\infty}(\Gamma)}(Y_1, Y_2) \to \text{Hom}_{S_E}((Y_1, s_{1,E}), (Y_2, s_{2,E}))$$

is an isomorphism.
1.7. THE STUDY OF $\Rep_{K_\infty}(\Gamma)$

**Proof.** We can reduce to the case $Y_1 = K_\infty$. Indeed we have the following canonical isomorphisms:

$$\Hom_{\Rep_{K_\infty}(\Gamma)}(Y_1, Y_2) = \Hom_{\Rep_{K_\infty}(\Gamma)}(K_\infty, Y_1^* \otimes Y_2)$$

$$\Hom_{S_E}(Y_1^*, Y_2^*) = \Hom_{S_E}(E, Y_1^* \otimes Y_2^*).$$

We put $Y = Y_2$. For every $\xi \in \Hom_{\Rep_{K_\infty}(\Gamma)}(K_\infty, Y)$, the map $\xi \mapsto (1)$ allow us to identify the $K$-vector space $\Hom_{\Rep_{K_\infty}(\Gamma)}(K_\infty, Y)$ with $H^0_{cont}(\Gamma, Y) = Y^\Gamma$. Moreover, we can identify $\Hom_{S_E}(E, Y_E)$ with $\Ker s_E$. Indeed, if $\varphi: (E, 0) \to (Y_E, s_E)$ is a $S_E$-morphism, then $s_E \circ \varphi = \varphi \circ 0 = 0$, so that $\varphi(1) \in \Ker s_E$. We are therefore reduced to prove that the canonical map

$$\varrho: E \otimes_K Y^\Gamma \to \Ker s_E$$

is bijective. By definition of $s_E$, we see that it is enough to prove the statement for $E = K_\infty$. Up to replacing $Y$ by $\Ker s$, we can assume $s = 0$, $\Ker s = Y$. By 1.6.2 $\varrho$ is injective. We fix a $K_\infty$-basis $\{y_1, \ldots, y_h\}$ of $Y$. Let $r_0$ be its degree. Being $s = 0$, by 1.7.1, there exists $r$ —that we may assume $r \geq r_0$— such that $\gamma(y_i) = y_i$ for $i = 1, \ldots, r$ and $y_i \in \Gamma$. Let $Y_r$ be the $K_r$-sub-vector space of $Y$ generated by $y_1, \ldots, y_r$: by construction, $Y_r$ is stable by $\Gamma$, that acts on it by means of the finite quotient $\Gal(K_r/K)$. As in the proof of 1.6.6, we can define a 1-cocycle $f: \Gal(K_r/K) \to GL_n(K_r)$ describing the action of $\Gal(K_r/K)$ on $Y_r$ with respect to $\{y_1, \ldots, y_r\}$. By Hilbert’s Theorem 90 [Sen62, chap. X, Prop. 3], we have

$$H^1(\Gal(K_r/K), GL_n(K_r)) = 1.$$

Hence we can assume that $\Gal(K_r/K)$ acts trivially on $y_1, \ldots, y_r$, so that $\Gamma$ fixes a basis of $Y$ and the map $\varrho$ is therefore surjective.

**1.7.6. Lemma.** Let $E$ be a field and let $Z_1, Z_2$ be finite-dimensional $E$-vector spaces. Let $E_0$ be an infinite subfield of $E$, $L$ a sub-$E_0$-vector space of the $E$-vector space $L_E(Z_1, Z_2)$ of $E$-linear applications from $Z_1$ to $Z_2$. The $E$-vector space $L_E = E \otimes L$ contains an isomorphism if and only if $L$ already contains one.

**Proof.** Let $f \in L_E$ be an isomorphism, $f: Z_1 \to Z_2$. Let $\{f_1, \ldots, f_n\}$ be an $E$-basis of $L_E$ formed by elements of $L$. Let $h$ be the dimension $\dim_E Z_1 = \dim_E Z_2$ and fix an $E$-basis of $Z_1$ and an $E$-basis of $Z_2$. For $j = 1, \ldots, n$, let $A_j \in M_h(E)$ be the matrix of $f_j$ with respect to those basis. Let $P(X_1, \ldots, X_n)$ be the polynomial

$$P(X_1, \ldots, X_n) = \det(X_1 A_1 + X_2 A_2 + \ldots + X_n A_n) \in E[X_1, \ldots, X_n].$$

If $f = \sum_{i=1}^n \lambda_i f_i$, $\lambda_i \in E$, we have $P(\lambda_1, \ldots, \lambda_n) \neq 0$, so that $P$ is not identically zero. Being $E_0$ an infinite field, there exist $\mu_1, \ldots, \mu_n \in E_0$ such that $P(\mu_1, \ldots, \mu_n) \neq 0$ and the element $\sum_{i=1}^n \mu_i f_i$ is an element of $L$, isomorphism of $Z_1$ over $Z_2$.

**1.7.7. Corollary.** Two $K_\infty$-representations of $\Gamma$, $Y_1$ and $Y_2$, are isomorphic in $\Rep_{K_\infty}(\Gamma)$ if and only if $(Y_{1,E}, s_{1,E})$ and $(Y_{2,E}, s_{2,E})$ are isomorphic in $S_E$.

**1.7.8.** Let $W \in \Rep_C(G_K)$. Then we dispose of the $L$-representation of $\Gamma$ $W^{H_K}$ and of the $K_\infty$-representation of $\Gamma$ $(W^{K_\infty})_f$. We denote by $\Delta_{\Sen}(W)$ the object of $S_{K_\infty}$ formed by the $K_\infty$-vector space underlying $(W^{K_\infty})_f$ and by the endomorphism $s_{W,f}$ defined in 1.7.1.

$\Delta_{\Sen}$ defines a faithful $\otimes$-functor

$$\Delta_{\Sen}: \Rep_C(G_K) \to S_{K_\infty}.$$
By 1.7.7 we see that the knowledge of $\Delta_{\text{Sen}}(W)$ determines — up to isomorphisms — $W$ as $\mathbb{C}$-representation of $G_K$.

1.8. Classification of $\mathbb{C}$-representations

1.8.1. We keep the notations of 1.2.1—1.2.2. Let $W$ be a $\mathbb{C}$-representation of $G_K$. In the notations of 1.7.8, we call Sen weights of $W$ the eigenvalues of the endomorphism $s_{W,f}$ in $\overline{K}$. By 1.7.1, the characteristic polynomial of $s_{W,f}$ has coefficients in $K$. Hence the set of Sen weights of $W$ is stable by $G_K$.

Let $X$ be a subset of $\overline{K}$ which is stable by $G_K$. We say that a $\mathbb{C}$-representation $W$ of $G_K$ is of type $S_X$ if its Sen weights are in $X$. We say that $W$ is of type $S^w_X$ if it is of type $S_X$ and if $s_{W,f}$ is semi-simple.

1.8.2. We denote by $\mathcal{C}(K)$ the set of the orbists of $\overline{K}$ for the action of $G_K$. For every indecomposable object $W$ in $\text{Rep}_\mathbb{C}(G_K)$, there exists a unique $A \in \mathcal{C}(K)$ such that $W$ is of type $S_A$.

Let $W$ be an indecomposable object of type $S_A$. We can write its Sen endomorphism $s_{W,f}$ as $s_0, s_u = s_u, s_0$, with $s_0$ semi-simple and $s_u$ unipotent. Let $V$ be the $K_\infty$-vector space underlying $(\mathbb{W}^H_K)_f$, $\nu$ the vector space $V \otimes_{K_\infty} K$. We denote again by $s_0$ the endomorphism of $\nu$ deduced by scalar extension. Then we have:

i) a decomposition of $\nu$ as a direct sum of the eigenspaces of $s_0$;

ii) a nilpotent endomorphism $\log s_u$ of $V$.

1.8.3. The $\mathbb{C}$-representations of $G_K$ of type $S_{\{0\}}$ correspond to representations of the additive group $\mathbb{G}_a$. Indeed, to give an action of the additive group $\mathbb{G}_a$ over a $K_\infty$-vector space $V$ comes down to give a nilpotent endomorphism $\nu$ of $V$ (so that $\lambda \in K_\infty = \mathbb{G}_a(K_\infty)$ acts over $V$ via $\exp(\lambda \nu)$). Let $K_\infty[\log t]$ be the algebra of polynomials in the variable $\log t$ and coefficients in $K_\infty$. For every $d \geq 1$, we denote by $\mathbb{Z}_p[0; d]$ the sub $\mathbb{Z}_p$-module of $K_\infty[\log t]$ formed by the polynomials in $\log t$ of degree $< d$ with coefficients in $\mathbb{Z}_p$. Hence we see that, up to isomorphisms, there exists a unique indecomposable $\mathbb{C}$-representation of $G_K$ of type $S_{\{0\}}$ of dimension $d$ over $\mathbb{C}$, namely

$$C^K(0; d) = \mathbb{C} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[0; d]$$

where the nilpotent endomorphism $\nu$ is $-\frac{d}{\log t}$.

Notice that $C^K(0; d)$ is not simple, as $C^K(0; d) \supset C^K(0; d - 1) \supset \ldots \supset C^K(0; 1)$.

1.8.4. Let $W$ be a simple object of $\text{Rep}_\mathbb{C}(G_K)$ and let $A$ be the unique conjugacy class of $\overline{K}$ such that $W$ is of type $S_A$. Then, for every $d \geq 1$, we can define the indecomposable object of type $S_A$

$$W \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[0; d].$$

On the other hand, we see that a $\mathbb{C}$-representation $W'$ of $G_K$ is indecomposable of type $S_A$ if and only if there exists $d \in \mathbb{N}^*$ (necessarily unique) such that $W' \cong W \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[0; d]$. Then $W'$ is simple if and only if $d = 1$. 
1.8. CLASSIFICATION OF C-REPRESENTATIONS

1.8.5. We fix a topological generator \( \gamma_0 \) of \( \Gamma \). For every \( r \in \mathbb{N} \), let \( a_r \) be the \( \mathcal{O}_K \)-sub-module of \( K \)

\[
a_r = \{ \alpha \in K | v_p(\alpha) > -r_r + \frac{1}{p-1} \}
\]

where \( r_K \) is the integer defined in the proof of 1.7.2. Let \( A \in \mathbb{C}(K) \) and set \( P_A(X) = \prod_{\alpha \in A}(X - \alpha) \in K[X] \) be the minimal polynomial of any \( \alpha \in A \) over \( K \). Let \( K_A \subset K \) be the field \( K[X]/(P_A(X)) \) and denote by \( \beta \) the image of \( X \) in \( K_A \). Let \( d_A \) be the number of elements in \( A \). Let \( r_A \) be the smallest integer \( r \) such that an element \( \alpha \in A \) belongs to \( a_r \). By construction, it is the smallest \( r \in \mathbb{N} \) such that

\[
v_p(\beta \log \chi_p(\gamma)) = v_p(\beta) + v_p(\log(\chi_p(\gamma))) > \frac{1}{p-1}
\]

for every \( \gamma \in \Gamma_r \). We can therefore define a continuous homomorphism \( \rho_A : \Gamma_{r_A} \to K_A^\times \) by

\[
\rho_A(\gamma) = \exp(\beta \log \chi_p(\gamma)).
\]

We denote by \( M[A] \) the field \( K_A \) endowed with the linear and continuous action of \( \Gamma_{r_A} \) given by \( \rho_A \).

Let \( N[A] = K_A[\Gamma] \otimes K_A[\Gamma_{r_A}] \) \( M[A] \) be the induced \( K_A \)-linear representation of \( \Gamma \). It is a \( K_A \)-vector space of dimension \( \rho^A \), since \( \{ \gamma_0^i \otimes 1 \}_{0 \leq i < \rho^A} \) is a basis of \( N[A] \) over \( K_A \). We denote by \( N_\infty[A] = K_\infty \otimes_K N[A] \) the \( K_\infty \)-representation of \( \Gamma \) deduced by \( N[A] \) by scalar extension. We choose a simple sub-object of \( N_\infty[A] \) in \( \text{Rep}_{K_\infty}(\Gamma) \) and we denote it by \( K_\infty[A] \). We set \( C[A] \) to be the \( C \)-representation of \( G_K \) corresponding to \( K_\infty[A] \), i.e.

\[
C[A] = C \otimes_{K_\infty} K_\infty[A].
\]

1.8.6. Theorem. In the notations 1.8.5, let \( W \) be a \( C \)-representation of \( G_K \).

i) \( W \) is simple if and only if there exists \( A \in \mathbb{C}(K) \) such that \( W \cong C[A] \); then \( W \) is of type \( S_A^m \) and has dimension \( d_A \rho^A \) over \( C \), where \( s_A \) is an integer \( 0 \leq s_A \leq r_A \) verifying \( \dim_{K_\infty}(K_\infty[A]) = \dim_C(C[A]) \).

ii) \( W \) is indecomposable if and only if there exists \( A \in \mathbb{C}(K) \) such that \( W \cong C[A;d] = C[A] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(0;d) \); then \( W \) is of type \( S_A \) and has dimension \( d_A \rho^A \) over \( C \).

iii) There exist natural integers \( (h_{A,d}(W))_{A \in \mathbb{C}(K), d \in \mathbb{N}^*} \), almost all zero, uniquely determined, such that

\[
W \cong \bigoplus_{A \in \mathbb{C}(K), d \in \mathbb{N}^*} C[A;d]^{h_{A,d}(W)}
\]

1.8.7. Lemma ([Fon04, Prop. 2.12]). Let \( F \) be a field, \( E \) a subfield of \( F \), \( \overline{E} \) a separable closure of \( E \), \( G_E = \text{Gal}(\overline{E}/E) \), \( \eta : G_E \to \mathbb{Q}/\mathbb{Z} \) a continuous homomorphism and \( b \in F \). Let \( E' = \overline{E}^{\text{Ker}\eta} \), \( N \) the degree of the cyclic extension \( E'/E \), \( \sigma \) the generator of \( \text{Gal}(E'/E) \) such that \( \eta(\sigma) \equiv 1/N \mod \mathbb{Z} \). Let \( \Lambda_{E,F}(\eta, b) \) be the associative and unitary \( E' \otimes_E F \)-algebra generated by an element \( c \) satisfying

\[
(1.8.7.1) \quad c^N = 1 \otimes b; \quad c(u \otimes x) = (\sigma(u) \otimes x)c \quad \text{if } u \in E' \text{ and } x \in F.
\]

Then the algebra \( \Lambda_{E,F}(\eta, b) \) is a central simple algebra. The center of \( \Lambda_{E,F}(\eta, b) \) is \( F \) and of dimension \( N^2 \) over its center. \( \Lambda_{E,F}(\eta, b) \) isomorphic to an algebra of square matrices with coefficients in a skew field \( D_{E,F}(\eta, b) \).
1.8. CLASSIFICATION OF $C$-REPRESENTATIONS

Theorem 1.8.6 is then a consequence of the previous discussion and of the following

1.8.8. Proposition. In the notations of 1.8.5, let $\eta : G_K \to \mathbb{Q}/\mathbb{Z}$ be the unique character of $G_K$ that factors through $\Gamma$ and that maps $\gamma_0$ to $1/p^r$. Let $b = g_A(\gamma_0^{p^r})$. The $K_A$-algebra

$$E_A = \operatorname{End}_{\operatorname{Rep}_{K_{\infty}}(K)}(N_{\infty}[A])$$

is identified with $\Lambda_{K,K_A}(\eta,b)$. The skew field $D_A = D_{K,K_A}(\eta,b)$ has rank $p^{2s_A}$, where $s_A$ is an integer verifying $0 \leq s_A \leq r_A$. We have

$$\dim_{K_{\infty}}(K_{\infty}[A]) = \dim_{C}[A] = d_Ap^{s_A}.$$ 

Moreover, $C[A]$ is a simple object of $\operatorname{Rep}_C(G_K)$ of type $S_A$ and

$$\operatorname{End}_{\operatorname{Rep}_{K_{\infty}}(K)}(K_{\infty}[A]) = \operatorname{End}_{\operatorname{Rep}_C(G_K)}(C[A]) = D_A.$$ 

Proof. Let $M_{\infty}[A] = K_{\infty} \otimes_K M[A]$. For every $s \in \mathbb{N}$ we set $M_s[A] = K_s \otimes_K M[A]$ and $N_s[A] = K_s \otimes_K N[A]$. We have then the following inclusions:

$$M[A] \subset M_s[A] \subset M_{\infty}[A]$$

$$\cap \cap \cap \cap$$

$$N[A] \subset N_s[A] \subset N_{\infty}[A].$$

To simplify the notation, we set $r = r_A$. For every $s \geq r$ we have the topological generator $\gamma_s = \gamma_0^{p^s}$ of $\Gamma_s \subset \Gamma_r$. By construction, $\gamma_s$ acts on $M[A]$ by multiplication with the element $b^{p^{s-r}} = \exp(\beta \log(\chi_0(\gamma_0^{p^s})))$.

Let $f \in \operatorname{End}_{\operatorname{Rep}_{K_s}(\Gamma_s)}(M[A])$. Then for every $\gamma \in \Gamma_s$, we have $f(\gamma(x)) = \gamma f(x)$ if and only if $f(\gamma_0^{p^s}(x)) = \gamma_0^{p^s} f(x)$, i.e. $f$ satisfies $f(b^{p^{s-r}} x) = b^{p^{s-r}} f(x)$ for every $x \in K[A]$. But we have $K_A = K(b^{p^{s-r}})$, since

$$\beta = \frac{\log b^{p^{s-r}}}{\log(\chi_0(\gamma_0^{p^s}))}$$

and $\log(\chi_0(\gamma_0^{p^s})) \in K^\times$. Hence, the natural injection $K_A \to \operatorname{End}_{\operatorname{Rep}_{K_s}(\Gamma_s)}(M[A])$ is an isomorphism.

Let $\{e_1, \ldots, e_d\}$ be a basis of $K_A$ over $K$, seen as ring of endomorphisms $\operatorname{End}_{\operatorname{Rep}_{K_s}(\Gamma_s)}(M[A])$. Let $f \in \operatorname{End}_{\operatorname{Rep}_{K_s}(\Gamma_s)}(M_{\infty}[A])$. Then there exists $s \geq r$ such that $f(M_s[A]) \subset M_s[A]$. Since $f$ is $K_{\infty}$-linear, we also have $f(M_s[A]) \subset M_s[A]$, so that the restriction $f_s$ of $f$ to $M_s[A]$ is an element of $\operatorname{End}_{\operatorname{Rep}_{K_s}(\Gamma_s)}(M_s[A])$. Since $\Gamma_s$ acts trivially on $K_s$, we have

$$\operatorname{End}_{\operatorname{Rep}_{K_s}(\Gamma_s)}(M_s[A]) = \operatorname{End}_{\operatorname{Rep}_{K_s}(\Gamma_s)}(M_{\infty}[A]) = K_s \otimes_K K_A.$$ 

We can therefore find $\lambda_1, \ldots, \lambda_d \in K_s$ such that $f_s$, as element of $\operatorname{End}_{\operatorname{Rep}_{K_s}(\Gamma_s)}(M_s[A])$, can be written as $f_s = \sum_{i=1}^d \lambda_i \otimes e_i$. Adding the further condition that $f_s$ commutes with the action of $\gamma_s$, we have $\gamma_s(\lambda_i) = \lambda_i$ for every $i = 1, \ldots, d$, i.e. $\lambda_i \in K_{\infty}$, so that

$$\operatorname{End}_{\operatorname{Rep}_{K_{\infty}}(\Gamma_s)}(M_{\infty}[A]) = K_r \otimes_K K_A.$$ 

By construction, every element of $N_{\infty}[A]$ can be written in a unique way as $x = \sum_{i=0}^{r-1} \gamma_0^i(x_i)$, with $x_i \in M_{\infty}[A]$. Let $f \in E_A$. Then $f(x) = \sum_{i=0}^{r-1} \gamma_0^i(\varphi(x))$ where $\varphi$ is the restriction of $f$ to $M_{\infty}[A]$. Therefore the application

$$E_A \to \operatorname{Hom}_{\operatorname{Rep}_{K_{\infty}}(\Gamma_s)}(M_{\infty}[A], N_{\infty}[A]), \quad f \mapsto \varphi$$
1.9. Hodge-Tate representations

1.9.1. We keep the notations of 1.8.5. Let $W \in \text{Rep}_C(\mathcal{G}_K)$. We say that $W$ is deployed (fr. déployée) over $K$ if the Sen weights of $W$ are in $K$. Let $a^K_0 = a_0 \cap K$ be the fractional ideal of $\mathcal{O}_K$ formed by the elements of $p$-adic valuation $>-r_K + \frac{1}{p-1}$. Every simple $C$-representation of $G_K$ of type $S^K_0$ has dimension 1 over $C$ and the ring of its endomorphisms is reduced to $K$ (see 1.8.8).

Among the representations of type $S^K_a$ we have the representations of type $\overline{S}^m_a$. These latter are called $C$-representation of type Hodge-Tate (or simply $C$-representation Hodge-Tate). Thus $W \in \text{Rep}_C(\mathcal{G}_K)$ is Hodge Tate if it is semi-simple and its Sen weights are in $\mathbb{Z}$.

Let $V$ be a $p$-adic representation of $G_K$, i.e. $V \in \text{Rep}_{Q_p}(\mathcal{G}_K)$. By base-change we get the corresponding $C$-representation, namely

$$C \otimes_{Q_p} V \in \text{Rep}_C(\mathcal{G}_K).$$

We say that $V$ is Hodge-Tate if $C \otimes_{Q_p} V$ is Hodge-Tate.

1.9.2. We fix a generator $t$ of the Tate module $\mathbb{Z}_p(1) = T_p(\mathbb{G}_m)(\overline{K})$. For every $i \in \mathbb{N}$ we denote by $\mathbb{Z}_p(i)$ the $i$-th power $\mathbb{Z}_p(1)^{\otimes i}$ and by $\mathbb{Z}_p(-i)$ its $\mathbb{Z}_p$-dual. For every $\mathbb{Z}_p$-module $M$, we denote by $M(i)$ the $i$-th Tate twist of $M$, i.e. $M(i) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$. For $x \in M$ and $u \in \mathbb{Z}_p(i)$, we write $xu$ for $x \otimes u \in M(i)$. The map $x \mapsto xt^i$ is a $\mathbb{Z}_p$-linear bijection between $M$ and $M(i)$, depending on the choice of $t$.

The group $G_K$ acts over $\mathbb{Z}_p(i)$ for every $i \in \mathbb{Z}$: we have $g.u = \chi^g_p(u)g$ for every $g \in G_K$ and $u \in \mathbb{Z}_p(i)$. Similarly, if $M$ is a topological $\mathbb{Z}_p$-module endowed with a linear and continuous action of $G_K$, we have an induced linear and continuous action on $M(i)$. Namely, we have

$$g(xt^i) = \chi^g_p(g(x)t^i) \quad \text{for every } g \in G_K, x \in M.$$

We can therefore identify $C(i) = C \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$ with $C[i]$ defined in 1.8.5 for every $i \in \mathbb{Z}$. Indeed, for $A = \{i\}$ we have that $\Gamma$ acts on $K = K_A$ via $\varrho_A$, that turns out to be $\chi^i_p$. This identification is not canonical, depending on the choice of a generator $t$ of $\mathbb{Z}_p(1)$, but is $G_K$-equivariant. Similarly, for every $d \in \mathbb{N}$, $C[i];d]$ is isomorphic to $C(i) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(0;d)$.

Hence, by Fontaine’s classification theorem 1.8.6, for any Hodge-Tate object $W$ in $\text{Rep}_C(\mathcal{G}_K)$ there exist non negative integers $h_q(W)$, almost always zero and uniquely determined by $W$, such that

$$W \cong \sum_{q \in \mathbb{Z}} C(q)^{h_q(W)}.$$

The integer $h_q(W)$ is called the multiplicity of $q$ as a Hodge-Tate weight of $W$. 

is bijective. Let $c$ be the unique element of $E_A$ defined by $c(x) = \gamma_0(x)$ for every $x \in M[A]$ (so that $c(\lambda x) = \lambda c(x) = \lambda \gamma_0(x)$ for every $\lambda \in K_\infty, x \in M[A]$). Then, using (1.8.8.1), every element of $E_A$ can be written in a unique way as $\sum_{i=0}^{p^r-1} c^j$, for $j \in K_r \otimes_K K_A$. We see therefore that $E_A$ is an algebra over $K_r \otimes_K K_A$ generated by an element $c$ satisfying the conditions (1.8.7.1) of 1.8.7. As $c^{p^r} = b$, we have that the dimension of $E_A$ over its center $K_A$ is $p^{2r}$. The skew field $D_A$ has rank $p^{2r}$ over $K_A$ for a suitable 0 $\leq s_A \leq r$ and for any simple sub-object $K_\infty[A]$ we have therefore

$$\text{End}_{\text{Rep}_{K_\infty}(\Gamma)}(K_\infty[A]) = D_A$$

and $\dim_{K_\infty}(K_\infty[A]) = d_A p^{s_A}$. The statement for $C[A]$ is clear. 

The integer $h_q(W)$ is called the multiplicity of $q$ as a Hodge-Tate weight of $W$. 

The group $G_K$ acts over $\mathbb{Z}_p(i)$ for every $i \in \mathbb{Z}$: we have $g.u = \chi^g_p(u)g$ for every $g \in G_K$ and $u \in \mathbb{Z}_p(i)$. Similarly, if $M$ is a topological $\mathbb{Z}_p$-module endowed with a linear and continuous action of $G_K$, we have an induced linear and continuous action on $M(i)$. Namely, we have

$$g(xt^i) = \chi^g_p(g(x)t^i) \quad \text{for every } g \in G_K, x \in M.$$
1.9.3. Let $B_{HT} = C[t^{(1)}, 1/t^{(1)}]$ be the polynomial algebra in the variable $t^{(1)}$. Let $t$ be a generator of $\mathbb{Z}_p(1)$. Then $t = (\varepsilon_n)_{n \in \mathbb{N}}$ where $\varepsilon_n$ is a primitive $p^n$-th root of 1 in $K$ and $\varepsilon_{n+1}^p = \varepsilon_n$. For $p \neq 2$, we denote by $\pi_t$ the unique uniformizer of $\mathbb{Q}_p(\varepsilon_1)$ such that

$$(\pi_t)^{p-1} + p = 0, \quad v_p(\varepsilon_1 - 1 - \pi_t) \geq \frac{2}{p-1}.$$ 

If $p = 2$ we set $\pi_t = \varepsilon_2 - 1$. Then the map

$$\mathbb{Z}_p(1) = \mathbb{Z}_p t \to B_{HT}, \quad \lambda t \mapsto \lambda \pi_t t^{(1)}$$

is injective and commutes with the action of $G_K$. We can identify $B_{HT}$ with $C[t, t^{-1}] = \bigoplus_{i \in \mathbb{Z}} C(i)$. By 1.2.4, we have $B_{HT}^{G_K} = \text{Frac}(B_{HT})^{G_K} = K$.

1.9.4. For every $C$-representation $W$ of $G_K$, we set $D_{HT}(W) = (B_{HT} \otimes_C W)^{G_K}$. By 1.6.2—1.6.3, the canonical map

$$(1.9.4.1) \quad \varrho: B_{HT} \otimes_K D_{HT}(W) \to B_{HT} \otimes_C W$$

is injective and $\dim_C(D_{HT}(W)) \leq \dim_C(W)$. Therefore, the representation $W$ is Hodge-Tate if and only if $\dim_C(D_{HT}(W)) = \dim_C(W)$, that is if and only if (1.9.4.1) is an isomorphism.

1.9.5. Let $V \in \text{Rep}_{\mathbb{Q}_p}(G_K)$. Then the dimension inequality in 1.9.4 can be stated as

$$(1.9.5.1) \quad \sum_{i \in \mathbb{Z}} \dim_K(C(i) \otimes_{\mathbb{Q}_p} V)^{G_K} \leq \dim_C(C \otimes_{\mathbb{Q}_p} V) = \dim_{\mathbb{Q}_p} V.$$ 

$V$ is Hodge-Tate if and only if the equality holds in (1.9.5.1).
CHAPTER 2

The Hodge-Tate decomposition Theorem for Abelian Varieties

2.1. Lubin-Tate formal groups and differential modules

2.1.1. Let \( K \) be a complete discrete valuation field with perfect residue field \( k \) of characteristic \( p > 0 \), \( \mathcal{O}_K \) the ring of integers of \( K \). We fix a separable closure \( \overline{K} \) of \( K \) and we denote by \( G_K \) the absolute Galois group of \( \overline{K} \) over \( K \). Let \( \mathcal{O}_C \) be the \( p \)-adic completion of \( \mathcal{O}_K \) and let \( C \) be its field of fractions.

Let \( E \) and \( K_0 \) be discrete valuation fields and let \( E \to K_0 \to K \) an injective homomorphism such that \( E \) has finite residue field \( k_E \), a uniformizer of \( E \) is a uniformizer of \( K_0 \), \( K \) is a finite, separable and totally ramified extension of \( K_0 \). Namely,

i) If \( K \) has characteristic 0, we take for \( E \) any finite extension of \( \mathbb{Q}_p \) contained in \( K \). If \( \pi \) is a uniformizer of \( E \), then \( K_0 \) is the subfield of \( K \) obtained by adjoining \( \pi \) to the fraction field of the ring of Witt vectors \( W(k) \).

ii) If \( K \) has characteristic \( p \), we have \( E = k_E((T)) \subseteq k((T)) = K_0 = K \).

We fix a uniformizer \( \pi \) of \( E \). We denote by \( v \) the valuation of \( C \), extending the valuation of \( K \), normalized by \( v(\pi) = 1 \). Given any subfield \( L \) of \( C \), we denote by \( \mathcal{O}_L = \{ x \in L | v(x) \geq 0 \} \) its valuation ring, by \( U_L = \{ x \in L | v(x) = 0 \} \) the group of units of \( \mathcal{O}_L \) and by \( m_L = \{ x \in L | v(x) > 0 \} \) the maximal ideal. If \( I \) is a sub-\( \mathcal{O}_L \)-module of \( L \) which is free of rank 1, we denote by \( v(I) \) the valuation of a generator of \( I \).

2.1.2. Let \( \Gamma \in \mathcal{O}_K[[X,Y]] \) be a formal power series in the variables \( X \) and \( Y \) and coefficients in \( \mathcal{O}_K \). We say that \( \Gamma \) is a one-parameter commutative formal group law over \( \mathcal{O}_K \) if the following identities are satisfied:

(1) \( \Gamma(X,\Gamma(Y,Z)) = \Gamma(\Gamma(X,Y),Z) \) [associativity];
(2) \( \Gamma(X,0) = X, \Gamma(Y,0) = Y \);
(3) \( \Gamma(X,Y) = \Gamma(Y,X) \) [commutativity];

It follows immediately that there exist a unique \( G(X) \in \mathcal{O}_K[[X]] \) such that \( \Gamma(X,G(X)) = 0 \) and that \( \Gamma(X,Y) = X + Y \mod (X,Y)^2 \). If \( \Gamma \) and \( \Gamma' \) are one-parameter commutative formal group laws over \( \mathcal{O}_K \), a morphism from \( \Gamma \) to \( \Gamma' \) is a power series \( f \) in one variable over \( \mathcal{O}_K \) with no constant term such that \( \Gamma(f(X),f(Y)) = \Gamma'(f(X),f(Y)) \).

2.1.3. Let \( \Gamma \) be a one-parameter commutative formal group law over \( \mathcal{O}_K \) and let \( x,y \in m_K \). Then the series \( \Gamma(x,y) \) converges and its sum belongs to \( m_K \). Under this composition law, \( m_K \) is a group which we denote \( \Gamma(m_K) \). We put

\[
\Gamma(m_K) = \lim_{\substack{L/K \text{ finite} \\ \mathcal{O}_L \to \mathcal{O}_K}} \Gamma(m_L)
\]
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If we equip $\mathcal{O}_K[[T]]$ with the $T$-adic topology and we consider $\mathcal{O}_K$ with the $\pi$-adic topology, we have a canonical isomorphism

$$m_K \sim \text{Hom}_{\text{cont}, \mathcal{O}_K}(\mathcal{O}_K[[T]], \mathcal{O}_K), \quad x \mapsto \varphi_x(T \mapsto x),$$

the identification being compatible with the group structure induced by $\Gamma$. By passage to the inductive limit from the finite case we get

(2.1.3.1) $\Gamma(m_K) \sim \text{Hom}_{\text{cont}, \mathcal{O}_K}(\mathcal{O}_K[[T]], \mathcal{O}_K)$.

2.1.4. We equip $\mathcal{O}_K[[T]]$ with the $T$-adic topology. Let $\hat{\Omega}^1_{\mathcal{O}_K[[T]]}/\mathcal{O}_K$ be the module of continuous $\mathcal{O}_K$-differentials of $\mathcal{O}_K[[T]]$: it is a free $\mathcal{O}_K[[T]]$-module of basis $dT$. Let $\Gamma$ be a one-parameter commutative formal group law over $\mathcal{O}_K$. An invariant differential with respect to the formal group law $\Gamma$ is a differential form $\omega = \alpha(T)dT \in \hat{\Omega}^1_{\mathcal{O}_K[[T]]}/\mathcal{O}_K$ satisfying

(2.1.4.1) $\alpha(\Gamma(X,Y))d\Gamma(X,Y) = \alpha(X)dX + \alpha(Y)dY$

or, equivalently,

(2.1.4.2) $\alpha(\Gamma(X,Y))\Gamma_X(X,Y) = \alpha(X)$

where $\Gamma_X(X,Y)$ is the partial derivative of $\Gamma$ with respect to the first variable. We denote by $\omega_\Gamma$ the submodule of $\hat{\Omega}^1_{\mathcal{O}_K[[T]]}/\mathcal{O}_K$ of the invariant differentials. We say that $\alpha(X)dT \in \omega_\Gamma$ is normalized if $\alpha(0) = 1$.

2.1.5. PROPOSITION. We keep the assumptions of 2.1.4. There exists a unique normalized invariant differential with respect to the formal group law $\Gamma$, given by the formula

$$\omega = \frac{dT}{F_X(0,T)},$$

$\omega_\Gamma$ is a free $\mathcal{O}_K$-module of rank 1, generated by $\omega$.

PROOF. Suppose $\alpha(T)dT$ is an invariant differential on $\Gamma$. Putting $X = 0$ in (2.1.4.2) gives

$$\alpha(Y)\Gamma_X(0,Y) = \alpha(0)$$

as $\Gamma(0,Y) = Y$. Since $\Gamma_X(0,T) \equiv 1 \pmod{T}$, we see that $\Gamma_X(0,T)^{-1} \in \mathcal{O}_K[[T]]$. Hence $\alpha(T)$ is determined by $\alpha(0)$ and every invariant differential is of the form $a \omega$ with $a \in \mathcal{O}_K$ and

$$\omega = \Gamma_X(0,T)^{-1}dT.$$

Since $\omega$ is normalized, it only remains to show that it is invariant. To prove this, we differentiate the relation

$$\Gamma(X, \Gamma(Y,Z)) = \Gamma(\Gamma(X,Y), Z)$$

with respect to $X$ to obtain

$$\Gamma_X(X, \Gamma(Y,Z)) = \Gamma_X(\Gamma(X,Y), Z)\Gamma_X(X,Y).$$

Putting $X = 0$ gives the desired result. \qed
2.1.6. Let \( q \) be the cardinality of the residue field \( k_E \) and let \( \mathcal{F}_\pi \) be the set of formal power series \( f \in \mathcal{O}_E[[T]] \) such that \( f(T) \equiv \pi T \mod (T^2) \) and \( f(T) \equiv T^q \mod (\pi) \).

2.1.7. Theorem ([LT65, Th. 1 and 2]). (i) For each \( f \in \mathcal{F}_\pi \) there exists a unique \( F_f(X,Y) \in \mathcal{O}_E[[X,Y]] \) such that

\[
F_f(X,Y) \equiv X + Y \mod (X,Y)^2, \quad f(F_f(X,Y)) = F_f(f(X), f(Y)).
\]

The series \( F_f \) defines a one-parameter commutative formal group law over \( \mathcal{O}_E \).

(ii) For each \( a \in \mathcal{O}_E \) and \( f,g \in \mathcal{F}_\pi \) there exists a unique \([a]_{f,g}(T) \in \mathcal{O}_E[[T]]\) such that

\[
[a]_{f,g}(T) \equiv aT \mod (X,Y)^2 \quad \text{and} \quad f([a]_{f,g}(T)) = [a]_{f,g}(g(T)).
\]

The series \([a]_{f,g} \) is a formal homomorphism from \( F_g \) to \( F_f \).

(ii) The map \( a \mapsto [a] = [a]_{f,f} \) defines an isomorphism from \( \mathcal{O}_E \) to \( \text{End}_{\mathcal{O}_E}(F_f) \), inverse of the morphism \( \sum_{i \geq 1} \xi X^i \mapsto c_1 \). Under this isomorphism,

\[
[a]_f(T) = f(T).
\]

The \( F_f \)'s for \( f \in \mathcal{F}_\pi \) are canonically isomorphic by means of the isomorphisms \([1]_{f,g} \). We call any one-parameter commutative formal group law over \( \mathcal{O}_E \) of the form \( F_f \), for \( f \in \mathcal{F}_\pi \), a Lubin-Tate formal group over \( \mathcal{O}_E \).

2.1.8. Let \( f \in \mathcal{F}_\pi \) and let \( \Gamma = F_f \) be the corresponding Lubin-Tate formal group. By 2.1.7, \( \Gamma(\mathfrak{m}_\pi) \) is canonically equipped with an \( \mathcal{O}_E \)-module structure. For \( a \in \mathcal{O}_E \), \( x \in \mathfrak{m}_\pi \) we write \( a.x = [a]_f(x) \). For every \( n \geq 0 \), let

\[
\Gamma_{\pi^n}(\mathfrak{m}_\pi) = \{ x \in \Gamma(\mathfrak{m}_\pi) \mid \pi^n.x = 0 \}
\]

be the set of \( \pi^n \)-torsion points of \( \Gamma(\mathfrak{m}_\pi) \). It is naturally an \( \mathcal{O}_E/\pi^n\mathcal{O}_E \)-module. Moreover, the maps \( \Gamma_{\pi^n+1}(\mathfrak{m}_\pi) \to \Gamma_{\pi^n}(\mathfrak{m}_\pi) \) given by \( x \mapsto \pi.x \) are \( \mathcal{O}_E \)-linear and \( \Gamma_{\pi^0}(\mathfrak{m}_\pi) = 0 \). We call the projective limit

\[
T_{\pi}(\Gamma) = \varprojlim \Gamma_{\pi^n}(\mathfrak{m}_\pi)
\]

the Tate module of \( \Gamma \).

2.1.9. Proposition. Under the assumptions of 2.1.8, \( T_{\pi}(\Gamma) \) is a free \( \mathcal{O}_E \)-module of rank 1.

Proof. According to 2.1.7, we may choose \( f(X) = \pi X + X^q \). Firstly, we prove that \( \Gamma(\mathfrak{m}_\pi) \) is \( \pi \)-divisible. With this choice of \( f \), the map

\[
\Gamma(\mathfrak{m}_\pi) \xrightarrow{\pi} \Gamma(\mathfrak{m}_\pi)
\]

is given by \( x \mapsto \pi x + x^q \). For every \( \alpha \in \mathfrak{m}_\pi \), the polynomial \( f(X) - \alpha \) is separable and so solvable in \( \overline{K} \). All its solutions belong clearly to \( \mathfrak{m}_\pi \). To prove that \( T_{\pi}(\Gamma) \) is a free \( \mathcal{O}_E \)-module of rank 1, it’s enough to show that, for every \( n \geq 1 \), \( \Gamma_{\pi^n}(\mathfrak{m}_\pi) \) is isomorphic to \( \mathcal{O}_E/(\pi^n) \) as \( \mathcal{O}_E \)-module. We proceed by induction on \( n \). For \( n = 1 \), \( \Gamma_{\pi}(\mathfrak{m}_\pi) \) is the set of solutions of the equation \( f(X) = 0 \): it has therefore \( q \) elements and it is isomorphic to \( \mathcal{O}_E/(\pi) \). Consider the sequence

\[
0 \to \Gamma_{\pi}(\mathfrak{m}_\pi) \to \Gamma_{\pi^n}(\mathfrak{m}_\pi) \xrightarrow{\pi} \Gamma_{\pi^{n-1}}(\mathfrak{m}_\pi) \to 0.
\]
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Since $\Gamma(m_{\mathcal{K}})$ is $\pi$-divisible, $(2.1.9.1)$ is exact. By induction hypothesis, $\Gamma_{n-1}(m_{\mathcal{K}}) \cong \mathcal{O}_E/(\pi^{n-1})$, and the sequence $(2.1.9.1)$ cannot split, since $\Gamma_{n}(m_{\mathcal{K}})$ contains an element of order exactly $\pi^n$: it is enough to divide a generator of $\Gamma_{n-1}(m_{\mathcal{K}})$ by $\pi$. □

2.1.10. Let $\Gamma$ be a Lubin-Tate formal group law over $\mathcal{O}_E$. Let $u \in \Gamma(m_{\mathcal{K}})$: according to 2.1.3.1, $u$ corresponds to $\varphi_u \in \text{Hom}_{\text{cont},\mathcal{O}_E}(\mathcal{O}_E[[T]], \mathcal{O}_\pi)$.

Let $\omega = \alpha(T)dT \in \Omega^1_{\mathcal{O}_E[[T]]/\mathcal{O}_E}$ be a continuous differential form. We denote by $u^*(\omega)$ the pull-back $\varphi_u(\alpha(T))d\varphi_u(T)$: it is a well defined element in $\Omega^1_{\mathcal{O}_\pi/\mathcal{O}_K}$. Indeed, by construction, the $\mathcal{O}_E$-linear and continuous morphism $\varphi_u$ factors through a finite extension $L/K$.

$$\varphi_u : \mathcal{O}_E[[T]] \to \mathcal{O}_L$$

where $u = \varphi_u(T) \in m_L \subset m_{\mathcal{K}}$. Since $L$ is complete, $\varphi_u(\alpha(T)) = \alpha(u)$ converges in $\mathcal{O}_L$ and we can consider $\alpha(u)du$ as an element in $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$. We denote by $u^*(\omega)$ its image by the canonical map

$$\Omega^1_{\mathcal{O}_L/\mathcal{O}_K} \to \Omega^1_{\mathcal{O}_\pi/\mathcal{O}_K}.$$ 

Restricting to the sub-module of invariant differentials, we have a map:

$$\langle , \rangle : \Gamma(m_{\mathcal{K}}) \times \omega_{\Gamma} \to \Omega^1_{\mathcal{O}_\pi/\mathcal{O}_K}; (u, \omega) \mapsto \langle u, \omega \rangle = u^*(\omega).$$

2.1.11. PROPOSITION. The pairing $\langle , \rangle$ is $\mathcal{O}_E$-bilinear and it is compatible with the action of $G_K$, i.e. for any $g \in G_K$, $u \in \Gamma(m_{\mathcal{K}})$, $\omega \in \omega_{\Gamma}$ we have $\langle g(u), \omega \rangle = g(\langle u, \omega \rangle)$.

PROOF. Indeed, for $u, u' \in \Gamma(m_{\mathcal{K}})$ and $\omega \in \omega_{\Gamma}$, $\langle u + u', \omega \rangle = \langle u, \omega \rangle + \langle u', \omega \rangle$ by $(2.1.4.1)$. The fact that $\langle au, \omega \rangle = a\langle u, \omega \rangle$ for any $a \in \mathcal{O}_E$, $\omega \in \omega_{\Gamma}$, $u \in \Gamma(m_{\mathcal{K}})$ follows from the identification of $\mathcal{O}_E$ with $\text{End}_{\mathcal{O}_E}(\Gamma)$ in 2.1.7. The linearity in the second variable and the compatibility with the action of $G_K$ are clear. □

2.1.12. Let $\Gamma$ be a Lubin-Tate formal group over $\mathcal{O}_E$. Let $G_K$ act trivially on $\omega_{\Gamma}$ and consider the $K$-vector space

$$K \otimes_{\mathcal{O}_K} T_{\pi}(\Gamma) \otimes_{\mathcal{O}_E} \omega_{\Gamma}.$$ 

By 2.1.9 and 2.1.5, it is a $K$-vector space of dimension 1, endowed with a semilinear continuous action of $G_K$.

Let $\alpha \in K \otimes_{\mathcal{O}_E} T_{\pi}(\Gamma) \otimes_{\mathcal{O}_E} \omega_{\Gamma}$. Then $\alpha$ can be written (in a non-unique way) as

$$\alpha = \frac{a}{\pi^r} \otimes u \otimes \omega$$

with $u = (u_n)_{n \in \mathbb{N}} \in T_{\pi}(\Gamma)$, $a \in \mathcal{O}_K$, $r \in \mathbb{N}$ and $\omega \in \omega_{\Gamma}$. It follows immediately from 2.1.11 and from the definition of $T_{\pi}(\Gamma)$ that the element $au^r_{\pi}(\omega)$ depends only on $\alpha$, so that the map

$$(2.1.12.1) \quad \xi_{K,\Gamma} : K \otimes_{\mathcal{O}_E} T_{\pi}(\Gamma) \otimes_{\mathcal{O}_E} \omega_{\Gamma} \to \Omega^1_{\mathcal{O}_{\pi}/\mathcal{O}_K}$$

$$\alpha = \frac{a}{\pi^r} \otimes u \otimes \omega \mapsto au^r_{\pi}(\omega)$$

is well defined, $\mathcal{O}_K$-linear and compatible with the action of $G_K$.

Let $D_{K/K_0}$ be the different of the extension $K/K_0$ and let $a_{K,\Gamma}$ be the $\mathcal{O}_K$-module

$$a_{K,\Gamma} = \left\{ a \in K \mid v(a) \geq -v(D_{K/K_0}) - \frac{1}{q-1} \right\}.$$
2.1.13. Theorem ([Fon82, Thm. 1]). Under the assumptions of 2.1.12, the map $\xi$ is surjective and

$$\text{Ker}(\xi) = a_{K,\Gamma} \otimes_{\mathcal{O}_E} T_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_\Gamma.$$

2.2. The proof of Theorem 2.1.13

2.2.1. Let $K$ be as in 2.1. For any field extension $L/K$, we denote by $\mathcal{D}_{L/K}$ the different of $L/K$ and by $d_{L/K} : \mathcal{O}_L \rightarrow \Omega^1_{L/K}$ the universal derivation.

2.2.2. Lemma. Let $K \subseteq M \subseteq L$ be a tower of finite and separable field extensions, $u$ the canonical map $\Omega^1_{M/K} \rightarrow \Omega^1_{L/K}$, and $\xi$ the map induced by the inclusion $\mathcal{O}_M \subseteq \mathcal{O}_L$. Then, for any $\omega \in \Omega^1_{M/K}$, we have:

$$v(\text{Ann}(u(\omega))) = \max\{0, v(\text{Ann}(\omega)) - v(D_{L/K})\}.$$

Proof. Let $b$ be a generator of $\mathcal{O}_L$ as an $\mathcal{O}_K$-algebra and let $\omega = ad_{L/K}b \in \Omega^1_{L/K}$ be a non-zero differential form. Since $\Omega^1_{L/K}$ is generated by $d_{L/K}b$ and is killed by $\mathcal{D}_{L/K}$, we have $v(\text{Ann}(\omega)) = v(D_{L/K}) - v(a)$. By definition $u(\omega) = ad_{L/M}b$, hence

$$v(\text{Ann}(u(\omega))) = \max\{0, v(D_{L/M}) - v(a)\}.$$ 

By [Ser62, chap. III, Prop. 8], we have

$$v(D_{L/M}) = v(D_{L/K}) - v(D_{M/K})$$

and we can conclude. \(\square\)

2.2.3. Lemma. Let $K \subseteq M \subseteq L$ be a tower of finite and separable field extensions. Let $\iota : \Omega^1_{M/K} \rightarrow \Omega^1_{L/K}$ be the map induced by the inclusion $\mathcal{O}_M \subseteq \mathcal{O}_L$. Then, for every $\omega \in \Omega^1_{M/K}$, we have

$$\text{Ann}_{\mathcal{O}_L}(\iota(\omega)) = \mathcal{O}_L \text{Ann}_{\mathcal{O}_M}(\omega).$$

Proof. It is enough to consider the case where $L/M$ is unramified or totally ramified. If $\mathcal{O}_L/\mathcal{O}_M$ is étale, then $\Omega^1_{M/K} \otimes_{\mathcal{O}_M} \mathcal{O}_L \cong \Omega^1_{L/K}$. By [EGA IV, 0.20.5.8] and the statement is clear.

Suppose now that $L/M$ is totally ramified. Let $b'$ be a uniformizer for $L$: it is a root of an Eisenstein polynomial $P(X) = \sum_{i=0}^n a_i X^i \in \mathcal{O}_M[X]$, with $a_0 = b$ a uniformizer for $M$. Let $\omega = ad_{M/K}b \in \Omega^1_{M/K}$ be a non-zero differential and let $a$ be its annihilator. Let $\iota(\omega) = ad_{L/K}b \in \Omega^1_{L/K}$ be the image of $\omega$. As $b = \sum_{i=1}^n a_i(b')^i$, we have

$$d_{L/K}b = (a_1 + 2a_2b' + \ldots + n(b')^{n-1})d_{L/K}b' = P'(b')d_{L/K}b',$$

so that $\iota(\omega) = aP'(b')d_{L/K}b'$. Hence $c \in \text{Ann}_{\mathcal{O}_L}(\iota(\omega))$ if and only if

$$v(caP'(b')) \geq v(D_{L/K}).$$

(2.2.3.1)

Since $D_{L/M} = (P'(b'))$ by [Ser62, chap. III, Cor. 2 to Prop. 11] and since $D_{L/K} = D_{L/M}D_{M/K}$ by [Ser62, chap. III, Prop. 8], (2.2.3.1) is equivalent to $v(c) \geq v(D_{M/K}) - v(a) = v(a)$, i.e.

$$\text{Ann}_{\mathcal{O}_L}(\iota(\omega)) = \mathcal{O}_L a.$$

\(\square\)
2.2.4. The modules $\Omega^1_{O_L/O_K}$ for $K \subseteq L$ varying in the set of finite and separable extensions of $K$ contained in $K$ form an inductive system and we have
\[
\lim_{\rightarrow} \Omega^1_{O_L/O_K} = \Omega^1_{O_K/O_K},
\]
that makes clear the fact that $\Omega^1_{O_K/O_K}$ is a torsion $O_K$-module. By 2.2.3, the canonical map $\Omega^1_{O_L/O_K} \rightarrow \lim_{\rightarrow} \Omega^1_{O_L/O_K}$ is injective.

Let $\omega \in \Omega^1_{O_K/O_K}$, $L$ a finite and separable extension of $K$ such that $\omega \in \Omega^1_{O_L/O_K} \subset \Omega^1_{O_L/O_K}$, $a$ the annihilator $\text{Ann}_{O_L}(\omega) \subset O_L$. Then the annihilator $\text{Ann}(\omega)$ of $\omega$ in $\Omega^1_{O_K/O_K}$ is simply given by $O_Ka$: in particular $\text{Ann}(\omega)$ is a principal ideal of $O_K$ and its valuation is the valuation of $a$.

2.2.5. Lemma. Let $\omega, \omega' \in \Omega^1_{O_K/O_K}$. Then we have $\text{Ann}(\omega) \subseteq \text{Ann}(\omega')$ if and only if there exists $c \in O_K$ such that $\omega' = c\omega$.

Proof. It is clear that $\omega' = c\omega$ for some $c \in O_K$ implies the inclusion between the annihilators.

Assume $\text{Ann}(\omega) \subseteq \text{Ann}(\omega')$. The case $\omega' = 0$ is trivial, so we can assume $\omega'$ and $\omega$ both non-zero: indeed $\omega' \neq 0$ implies $\text{Ann}(\omega')$ — and a fortiori $\text{Ann}(\omega)$ — different from $O_K$, so that also $\omega$ is non-zero. Let $L$ be a finite and separable extension such that $\omega, \omega' \in \Omega^1_{O_L/O_K}$. If $b$ is a uniformizer of $L$, we can write $\omega = adb$ and $\omega' = a'db$, with $a, a' \in O_L$.

As $\omega'$ and $\omega$ are both non-zero, we have $v(a) < v(D_{L/K})$ and $v(a') < v(D_{L/K})$, while $v(\text{Ann}(\omega)) = v(D_{L/K}) - v(a)$ and $v(\text{Ann}(\omega')) = v(D_{L/K}) - v(a')$. The assumption $\text{Ann}(\omega) \subseteq \text{Ann}(\omega')$ implies
\[
v(D_{L/K}) - v(a) \geq v(D_{L/K}) - v(a') \quad \text{hence} \quad v(a') \geq v(a)
\]
so that $a' \in aO_L$, i.e. there exists a $c \in O_K$ such that $\omega' = c\omega$. □

2.2.6. We consider again the notations of 2.1.12: $\Gamma$ is a Lubin-Tate formal group over $O_E$ and $T_{\pi}(\Gamma)$ is its Tate module. We fix a generator $(\pi_i)_{i \in \mathbb{N}}$ of $T_{\pi}(\Gamma)$ over $O_E$: for every $r \geq 1$, $\pi_r$ is a generator of the rank one $O_E/\pi^r$-module $\Gamma_{\pi^r}(\mathfrak{m}_K)$.

Let $E_r$ be the field $E[\pi_r]$. From [LT65, Theorem 2] and [CF67, VI, §3], we know that the field extensions $E_r = E[\Gamma_{\pi^r}(\mathfrak{m}_K)]$ of $E$ depend only on the uniformizer $\pi$ of $E$ and are totally ramified, finite, abelian Galois extensions of $E$. Moreover, $\pi_r$ is a uniformizer of $E_r$.

2.2.7. Proposition. For every $r \geq 1$ we have $v(D_{E_r/E}) = r - \frac{1}{q-1}$.

Proof. By [CF67, p. 152], we have:

i) the Galois group $\text{Gal}(E_r/E)$ is canonically isomorphic to the quotient
\[
U_E/U_E^{(r)} = U_E/(1 + \pi^r O_E);
\]

ii) $E_{E_r/E} = [E_r : E] = q^{-1}(q - 1)$;

Under the isomorphism $U_E/U_E^{(r)} \sim G = \text{Gal}(E_r/E)$, the subgroup $U_E^{(i)}/U_E^{(r)}$ maps onto the ramification group $G_{q^{-i}}$. Hence, from the filtration
\[
U_E/U_E^{(r)} \supset U_E^{(1)}/U_E^{(r)} \supset \cdots U_E^{(r)}/U_E^{(r)} = 1,
\]

we have:

i) $v(D_{E_r/E}) = \frac{1}{q-1}$;

ii) $[E_r : E] = q^{-1}(q - 1)$;

Under the isomorphism $U_E/U_E^{(r)} \sim G = \text{Gal}(E_r/E)$, the subgroup $U_E^{(i)}/U_E^{(r)}$ maps onto the ramification group $G_{q^{-i}}$. Hence, from the filtration
\[
U_E/U_E^{(r)} \supset U_E^{(1)}/U_E^{(r)} \supset \cdots U_E^{(r)}/U_E^{(r)} = 1,
\]

we have:

i) $v(D_{E_r/E}) = \frac{1}{q-1}$;

ii) $[E_r : E] = q^{-1}(q - 1)$;

Under the isomorphism $U_E/U_E^{(r)} \sim G = \text{Gal}(E_r/E)$, the subgroup $U_E^{(i)}/U_E^{(r)}$ maps onto the ramification group $G_{q^{-i}}$. Hence, from the filtration
\[
U_E/U_E^{(r)} \supset U_E^{(1)}/U_E^{(r)} \supset \cdots U_E^{(r)}/U_E^{(r)} = 1,
\]
we get that a complete set of ramification groups for the extension $E_r/E$ is given by

$$G = G_0;$$
$$G_1 = \ldots = G_{q-2} = G_{q-1};$$
$$G_q = \ldots = G_{q^2-1};$$
$$\ldots$$
$$1 = G_{q^r-1}.$$

The corresponding upper numbering is $G_i = G_{q^i-1}$ and

$$[G^0 : G^1] = q - 1 \quad [G^i : G^{i+1}] = q.$$

By [Ser62, chap. IV, Prop. 4], we have

$$v_{E_r}(\mathcal{D}_{E_r/E}) = \sum_{s \neq 1} i_G(s)$$

where $i_G(s) = v_{E_r}(s(\pi_r) - \pi_r)$ for $s \in G$. Moreover:

$$v_{E_r}(\mathcal{D}_{E_r/E}) = \sum_{i=0}^{r-1} \sum_{s \in G \setminus G^{i+1}} i_G(s)$$

and the function $i_G(s)$ is constant for $s \in G^i \setminus G^{i+1}$ and equal to $q^i$ for every $i$. For $i \geq 1$ we have that $\#G^i = q^r - 1$ and that $\#G^i \setminus G^{i+1} = (q - 1)q^{r-1}$, where $\#S$ denotes the cardinality of the (finite) set $S$. Hence:

$$\sum_{i=0}^{r-1} \sum_{s \in G \setminus G^{i+1}} i_G(s) = (q - 2)q^{r-1} + \sum_{i=1}^{r-1} q^i(q - 1)q^{r-1} = q^{r-1}(r(q - 1) - 1).$$

As $v(\mathcal{D}_{E_r/E}) = \frac{1}{v_{E_r/E}} v_{E_r}(\mathcal{D}_{E_r/E})$, we deduce that

$$v(\mathcal{D}_{E_r/E}) = \frac{1}{q^{r-1}(q - 1)} q^{r-1}(r(q - 1) - 1) = r - \frac{1}{q - 1}.$$

2.2.8. Corollary. Let $\omega_0$ be a generator of the module of invariant differentials $\omega_{\Gamma}$. Then for any non-negative integer $r$ we have:

$$v(\text{Ann}(\pi_r^*(\omega_0))) = \max \left\{ 0, r - \frac{1}{q - 1} - v(\mathcal{D}_{K/K_0}) \right\}$$

**Proof.** The statement is evident for $r = 0$ (since $u_0 = 0$), so we can assume $r \geq 1$. By passing to the limit in 2.2.2, we have

$$v(\text{Ann}(\nu(\omega))) = \max \{ 0, v(\text{Ann}(\omega)) - v(\mathcal{D}_{K/K_0}) \}$$

where $\nu$ is the canonical map $\nu : \Omega^1_{\mathcal{O}_{\mathcal{K}}/\mathcal{O}_{K_0}} \to \Omega^1_{\mathcal{O}_{\mathcal{K}}/\mathcal{O}_{K}}$. We can therefore assume that $K = K_0$.

Let $P_r$ be the minimal polynomial of $\pi_r$ over $E$: it is an Eisenstein polynomial. Since the uniformizer $\pi$ of $E$ is a uniformizer of $K$, then $K_r = K[\pi_r] = K \otimes_E E_r$ is a field extension of $K$, totally ramified, with $\pi_r$ as uniformizer.

Since $\mathcal{O}_{K_r} = \mathcal{O}_K[\pi_r]$, $d\pi_r$ generates $\Omega^1_{\mathcal{O}_{K_r}/\mathcal{O}_K}$ and we have:

$$v(\text{Ann}(d\pi_r)) = v(P_r^*(\pi_r)) = v(\mathcal{D}_{E_r/E}).$$
By 2.1.5, we know that $\omega_0$ is of the form $\alpha(T)dT$ with $\alpha(T)$ invertible in $O_E[[T]]$. Hence, for every $r \geq 1$,

$$\pi_r^*(\omega_0) = \alpha(\pi_r)d\pi_r$$

with $\alpha(\pi_r)$ unit. Therefore $v(\text{Ann}(\pi_r^*(\omega_0))) = v(D_{K_0}/E)$ and the statement follows from 2.2.7. \qed

**Proof of Theorem 2.1.13.** We first prove the surjectivity of the map $\xi$. Let $\omega_0$ be a generator of $\omega_\Gamma$ and let $u = (\pi_n)_{n \in \mathbb{N}}$ be a generator of $T_{\pi}(\Gamma)$. Let $\omega \in \Omega^1_{O_{\overline{\mathbb{K}}}/O_K}$ and let $r$ be an integer such that

$$v(\text{Ann}(\omega)) \leq r - \frac{1}{q-1} - v(D_{K/K_0}) \leq v(\text{Ann}(\pi_r^*(\omega_0)))$$

by 2.2.8. Hence $\text{Ann}(\omega) \supseteq \text{Ann}(\pi_r^*(\omega_0))$, so that there exists $c \in O_{\overline{\mathbb{K}}}$ such that $\omega = c\pi_r^*(\omega_0)$ (by 2.2.5) and

$$\omega = \xi\left(\frac{c}{\pi^r} \otimes u \otimes \omega_0\right),$$

proving the surjectivity of $\xi$.

We now determine the kernel: any element $\alpha \in \overline{\mathbb{K}} \otimes T_{\pi}(\Gamma) \otimes \omega_\Gamma$ can be written in a unique way as $a \otimes u \otimes \omega_0$, with $a \in \overline{\mathbb{K}}$. Let $r \in \mathbb{N}$ such that $r \geq \frac{1}{q-1} + v(D_{K/K_0})$ and such that $\pi^ra \in O_{\overline{\mathbb{K}}}$. The element $\alpha$ is in Ker($\xi$) if and only if $v(\text{Ann}(\xi(\alpha))) \leq 0$ (the annihilator taken in $\overline{\mathbb{K}}$). Hence

$$v(\pi^r a) \geq r - \frac{1}{q-1} - v(D_{K/K_0}),$$

so that $\alpha \in \text{Ker} \xi$ if and only if $\alpha \in a \otimes T_{\pi}(\Gamma) \otimes \omega_\Gamma$. \qed

**2.3. Consequences and corollaries**

**2.3.1.** We keep the assumptions of 2.1.12. Let $T_{\pi}(\Omega^1_{O_{\overline{\mathbb{K}}}/O_E})$ be the $\pi$-Tate module of the $O_E$-module $\Omega^1_{O_{\overline{\mathbb{K}}}/O_K}$, i.e.

$$T_{\pi}(\Omega^1_{O_{\overline{\mathbb{K}}}/O_E}) = \text{Hom}_{O_{\mathbb{K}}}(E/O_E, \Omega^1_{O_{\overline{\mathbb{K}}}/O_K})$$

and let $V_{\pi}(\Omega^1_{O_{\overline{\mathbb{K}}}/O_E})$ be the $E$-vector space

$$V_{\pi}(\Omega^1_{O_{\overline{\mathbb{K}}}/O_E}) = \text{Hom}_{O_{\mathbb{K}}}(E, \Omega^1_{O_{\overline{\mathbb{K}}}/O_K}).$$

**2.3.2. Corollary.** Let $\hat{a}$ be the $\pi$-adic completion of $a$. We have the following canonical isomorphisms of $O_{\overline{\mathbb{K}}}$-modules (resp. $O_{\mathbb{C}}$-modules, $C$-vector spaces)

(2.3.2.1) \quad $\Omega^1_{O_{\overline{\mathbb{K}}}/O_K} \cong (\mathbb{K}/a) \otimes_{O_{\mathbb{K}}} T_{\pi}(\Gamma) \otimes_{O_K} \omega_\Gamma,$

(2.3.2.2) \quad $T_{\pi}(\Omega^1_{O_{\overline{\mathbb{K}}}/O_E}) \cong \hat{a} \otimes_{O_{\mathbb{K}}} T_{\pi}(\Gamma) \otimes_{O_E} \omega_\Gamma,$

(2.3.2.3) \quad $V_{\pi}(\Omega^1_{O_{\overline{\mathbb{K}}}/O_K}) \cong C \otimes_{O_{\mathbb{K}}} T_{\pi}(\Gamma) \otimes_{O_E} \omega_\Gamma$

that commute with the action of $G_K$.

**Proof.** Isomorphism (2.3.2.1) simply follows from 2.1.13. As $E/O_E = \lim_{\leftarrow} (\frac{1}{\pi^n}O_E)/O_E$ we have:

$$T_{\pi}(\Omega^1_{O_{\overline{\mathbb{K}}}/O_E}) = \lim_{\leftarrow} \text{Hom}_{O_E}\left(\frac{1}{\pi^n}O_E/O_E, \Omega^1_{O_{\overline{\mathbb{K}}}/O_K}\right).$$
Moreover
\[ \text{Hom}_{\mathcal{O}_E} \left( \frac{1}{\pi^n} \mathcal{O}_E / \mathcal{O}_E, \Omega^1_{\mathcal{O}_F / \mathcal{O}_E} \right) \cong \left( \frac{1}{\pi^n} a / a \right) \otimes_{\mathcal{O}_E} T_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_T \]
using (2.3.2.1) together with the fact that \( T_\pi(\Gamma) \) and \( \omega_T \) are free rank one \( \mathcal{O}_E \)-modules (hence torsion-free) and that the morphisms are \( \mathcal{O}_E \)-linear. Therefore
\[ T_\pi(\Omega^1_{\mathcal{O}_F / \mathcal{O}_E}) = \lim \left( \frac{1}{\pi^n} a / a \right) \otimes_{\mathcal{O}_E} T_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_T = \hat{a} \otimes_{\mathcal{O}_E} T_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_T. \]
Finally, for (2.3.2.3) we write \( E = \lim \frac{1}{\pi^n} \mathcal{O}_E. \) As above we have:
\[ V_\pi(\Omega^1_{\mathcal{O}_F / \mathcal{O}_E}) = \text{Hom}_{\mathcal{O}_E}(E, \Omega^1_{\mathcal{O}_F / \mathcal{O}_E}) = \lim \text{Hom}_{\mathcal{O}_E} \left( \frac{1}{\pi^n} \mathcal{O}_E, \Omega^1_{\mathcal{O}_F / \mathcal{O}_E} \right). \]
To get the isomorphism with \( C \otimes_{\mathcal{O}_E} T_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_T, \) we use again (2.3.2.1). The morphisms \( \xi \) of Theorem 2.1.13 is compatible with the action of \( G_K, \) so isomorphisms (2.3.2.1), (2.3.2.2) and (2.3.2.3) commute clearly with the action of \( G_K. \) \( \square \)

2.3.3. Assume that \( K \) is of characteristic 0, that \( E = \mathbb{Q}_p \) and \( \pi = p, \) so that \( q = p \) and \( K_0 = \text{Frac}(W(k)). \) For this special case (see [LT65, §1, p. 380]), the Lubin-Tate formal group \( \Gamma \) over \( \mathbb{Q}_p \) is the formal multiplicative group \( \hat{G}_m, \) i.e. the completion along the unit section of the multiplicative group \( G_m \) over \( \mathbb{Z}_p. \) For \( f(T) = (1+T)^p - 1 \in \mathbb{Z}_p[[T]], \) the group law \( \Gamma = \Gamma_f(X,Y) \) is the power series \( X + Y + XY. \) By 2.1.5, we have a canonical generator of \( \omega_T, \) namely the unique normalized invariant differential form \( \omega_\pi = \frac{dT}{1+T}. \)

We can identify the Tate module \( T_p(\Gamma) \) with the points in \( \overline{K} \) of the Tate module of the multiplicative group \( G_m. \) More precisely we have, for any \( n \in \mathbb{N}, \)
\[ 1 \to \mu_{p^n}(\overline{K}) \to \overline{K}^* \xrightarrow{p^n} \overline{K}^* \to 1 \]
and \( T_p(G_m) \) is the projective limit \( \lim \mu_{p^n}(\overline{K}), \) where the transition maps are given by raising to the \( p \)-th power. As the map
\[ (2.3.3.1) \]
\[ a \mapsto 1 + a; \ m_{\overline{K}} \to 1 + m_{\overline{K}} \]
is an isomorphism between the group \( \Gamma(m_{\overline{K}}) \) and \( U^{(1)}_{\overline{K}} \) (with standard multiplication), the points of \( p^n\)-torsion with respect to the formal group law correspond to the point of \( p^n\)-torsion with respect to the standard multiplication in \( \overline{K}. \) Therefore
\[ T_p(\Gamma) = T_p(G_m) = \lim \mu_{p^n}(\overline{K}) \]
is the free \( \mathbb{Z}_p \)-module of rank 1 formed by the sequences \( (\varepsilon_n)_{n \in \mathbb{N}} \) of elements of \( \mathcal{O}_{\overline{K}} \) such that \( \varepsilon_0 = 1 \) and \( \varepsilon_{n+1} = \varepsilon_n. \)

Notice that, by definition, the character \( \chi: G_K \to \text{Aut}_{\mathbb{Z}_p}(T_p(\Gamma)) \cong \mathbb{Z}_p^\times \) giving the action of \( G_K \) on the Tate module of \( \Gamma \) is nothing else but the cyclotomic character \( \chi_p, \) giving the action of \( G_K \) on the group of units of order \( (a \text{ power of}) \) \( p. \)

2.3.4. For any \( \mathbb{Z}_p \)-module \( M \) endowed with a linear action of \( G_K \) and any \( i \in \mathbb{Z}, \) we write \( M(i) \) for the tensor product
\[ M \otimes_{\mathbb{Z}_p} T_p(G_m)^{\otimes i} \]
with the convention \( T_p(G_m)^{\otimes 0} = \mathbb{Z}_p \) and, for \( i > 0, \) \( T_p(G_m)^{\otimes -i} \) is the dual of \( T_p(G_m)^{\otimes i}. \)

In this setting, we can reformulate Theorem 2.1.13 in the following way:
2.3.5. **Theorem.** The map \( \xi: K(1) \to \Omega^1_{O_K/O_K} \) defined by

\[
p^{-r} a \otimes (\varepsilon_n)_{n \in \mathbb{N}} \mapsto a \cdot \frac{d\varepsilon_r}{\varepsilon_r}
\]

for \( a \in O_K, r \in \mathbb{N} \) is surjective with kernel \( a(1) \) and induces canonical isomorphisms:

\[
(2.3.5.1) \quad \Omega^1_{O_K/K} \cong (K/a)(1),
\]

\[
(2.3.5.2) \quad T_p(\Omega^1_{O_K/K}) = \text{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^1_{O_K/O_K}) \cong \hat{a}(1),
\]

\[
(2.3.5.3) \quad \nu_p(\Omega^1_{O_K/K}) = \text{Hom}_{\hat{a}}(\mathbb{Q}_p, \Omega^1_{O_K/O_K}) \cong \mathbb{C}(1).
\]

### 2.4. Applications to Abelian Varieties

2.4.1. Let \( K \) be a complete discrete valuation field of characteristic 0 with perfect residue field \( k \) of characteristic \( p > 0 \), \( O_K \) the valuation ring of \( K \), \( S = \text{Spec}(O_K) \). We note by \( \eta \) the generic point of \( S \) and by \( \eta \) a geometric point corresponding to an algebraic closure \( \bar{K} \) of \( K \). We denote by \( G_K \) the absolute Galois group of \( K \).

2.4.2. **Proposition ([EGA IV, 2.8.5]).** Let \( f: X \to S \) be a morphism of schemes and let \( X_\eta = f^{-1}(\eta) \) be the generic fibre of \( X \). Let \( \nu: X_\eta \to X \) be the canonical morphism. Let \( Z \) be a closed subscheme of \( X_\eta \). Then there exists a unique closed subscheme \( \tilde{Z} \) of \( X \), flat over \( S \) and such that \( \nu^{-1}(\tilde{Z}) = Z \).

The scheme \( \tilde{Z} \) is the schematic closure of \( Z \) by the composite morphism \( Z \to X_\eta \to X \), where the first arrow is the canonical injection; its underlying space is the closure in \( X \) of \( Z \).

2.4.3. From now on, let \( X \) be an abelian variety over \( K \) and let \( \varphi: X \to \mathbb{P}^n_K \) be a closed immersion. Let \( \nu: \mathbb{P}^n_K \to \mathbb{P}^n_{O_K} \) be the canonical morphism. By 2.4.2, there exists a unique scheme \( \tilde{X} \), flat and proper over \( S \), such that \( \nu^{-1}(\tilde{X}) = X \).

2.4.4. Let \( u: \text{Spec}(O_K) \to \tilde{X} \) and let \( \omega \in H^0(\tilde{X}, \Omega^1_{\tilde{X}/O_K}) \). We denote by \( u^*(\omega) \in \Omega^1_{\tilde{X}/O_K} \) the image of \( u^*\omega \) by the canonical \( O_K \)-linear map

\[
u^* \Omega^1_{X/O_K} \cong \Omega^1_{\tilde{X}/O_K}.
\]

In this way we obtain a pairing:

\[
(2.4.4.1) \quad H^0(\tilde{X}, \Omega^1_{\tilde{X}/O_K}) \times \tilde{X}(O_K) \to \Omega^1_{O_K/O_K}
\]

by

\[
(\omega, u) \mapsto \langle \omega, u \rangle = u^*(\omega).
\]

The map (2.4.4.1) is clearly \( O_K \)-linear in the first variable and it is compatible with the action of \( G_K \). More precisely, for any \( g \in G_K, \omega \in H^0(\tilde{X}, \Omega^1_{\tilde{X}/O_K}), u \in \tilde{X}(O_K) \) we have

\[
\langle \omega, g \cdot u \rangle = g(\langle \omega, u \rangle) = g(u^*(\omega)).
\]
2.4.5. By construction, we have the fibre product diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{q_1} & \mathfrak{X} \\
& \downarrow q_2 & \\
\eta & \rightarrow & S
\end{array}
\]

that allow us to identify \( H^0(X, \Omega^1_{X/K}) \) with \( K \otimes_{\mathcal{O}_K} H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \). Indeed, let \((U_i)_{i \in I}\) be an affine open covering of \(\mathfrak{X}\) and consider the canonical exact sequence:

\[
(2.4.5.1) \quad 0 \rightarrow H^0(U, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \rightarrow \bigoplus_i H^0(U_i, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \rightarrow \bigoplus_{i,j} H^0(U_{ij}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})
\]

where \(U_{ij} = U_i \cap U_j\). Since \(K\) is flat over \(\mathcal{O}_K\), the latter induces an exact sequence

\[
(2.4.5.2) \quad 0 \rightarrow H^0(U, \Omega^1_{\mathfrak{X}/\mathcal{O}_K} \otimes_{\mathcal{O}_K} K) \rightarrow \bigoplus_i H^0(U_i, \Omega^1_{\mathfrak{X}/\mathcal{O}_K} \otimes_{\mathcal{O}_K} K) \rightarrow \bigoplus_{i,j} H^0(U_{ij}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K} \otimes_{\mathcal{O}_K} K)
\]

On the other hand, \((U_i \cap X = U_i \otimes_{\mathcal{O}_K} K)_{i \in I}\) is an affine open covering of \(X\) and we have, for every \(i \in I\),

\[
H^0(U_i, \Omega^1_{\mathfrak{X}/\mathcal{O}_K} \otimes_{\mathcal{O}_K} K) = H^0(U_i \otimes_{\mathcal{O}_K} K, \Omega^1_{\mathfrak{X}/K}).
\]

Hence (2.4.5.2) implies that

\[
H^0(X, \Omega^1_{X/K}) = K \otimes_{\mathcal{O}_K} H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}).
\]

2.4.6. By the Valuative Criterion of Properness [EGA II, 7.3.8] we have a canonical identification of \(X(K)\) with \(\mathfrak{X}(\mathcal{O}_K)\): in this way \(\mathfrak{X}(\mathcal{O}_K)\) inherits a structure of abelian group, even though \(\mathfrak{X}\) is not a group scheme over \(S\).

2.4.7. PROPOSITION. Under the assumptions of 2.4.4, there exists a non negative integer \(r_0\) such that for every \(\omega \in p^u H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})\) and every \(u_1, u_2 \in \mathfrak{X}(\mathcal{O}_K) = X(K)\) we have:

\[
\langle \omega, u_1 + u_2 \rangle = \langle \omega, u_1 \rangle + \langle \omega, u_2 \rangle
\]

PROOF. Let \(\mathfrak{X}\) be an \(\mathcal{O}_K\)-model of \(X \times X\) over \(K\) such that the canonical projections \(p_1, p_2: X \times_X X \rightarrow X\) and the group multiplication \(m: X \times_X X \rightarrow X\) extend to maps from \(\mathfrak{X}\) to \(\mathfrak{X}\). We can construct \(\mathfrak{X}\) as follows: if \(\psi: X \times_X X \rightarrow \mathbb{P}^m_K\) is a projective embedding of the product \(X \times_X X\), we can consider the composite map

\[
X \times_X X \xrightarrow{id \times m} X \times_X X \times_X X \rightarrow \mathfrak{X} \times_S \mathfrak{X} \times_S \mathfrak{X}.
\]

Let \(\mathfrak{X}\) be schematic closure of the composite morphism, so that we have the diagram

\[
\begin{array}{ccc}
X \times_X X & \xrightarrow{id \times m} & X \times_X X \times_X X & \rightarrow & \mathfrak{X} \times_S \mathfrak{X} \times_S \mathfrak{X} \\
\sigma \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{X} & \rightarrow & \mathfrak{X} \times_S \mathfrak{X} \times_S \mathfrak{X} & \rightarrow & \mathbb{P}^m_K \rightarrow & S
\end{array}
\]

(2.4.7.1)

We get the required extensions

\[
p_1, p_2, m: \mathfrak{X} \rightarrow \mathfrak{X}
\]

by mean of the other projections.
We know ([BLR90, §4.2, Prop.1]) that the everywhere regular differential forms on $X$ are precisely the invariant forms, so that for any $\omega \in H^0(X, \Omega^1_X)$ we have:

$$m^*\omega - p_1^*\omega - p_2^*\omega = 0$$

in $H^0(X, \Omega^1_{X \times X/K})$. Let $\omega \in H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}_K})$ and consider the form $\omega' \in H^0(\mathcal{Y}, \Omega^1_{\mathcal{Y}/\mathcal{O}_K})$ defined by

$$\omega' = m^*\omega - p_1^*\omega - p_2^*\omega.$$

The natural map

$$(2.4.7.2) \quad H^0(\mathcal{Y}, \Omega^1_{\mathcal{Y}/\mathcal{O}_K}) \to H^0(X \times_K X, \Omega^1_{X \times X/K}) = K \times_{\mathcal{O}_K} H^0(\mathcal{Y}, \Omega^1_{\mathcal{Y}/\mathcal{O}_K})$$

corresponds to taking the pull-back of a differential form on $\mathcal{Y}$ via the map $\sigma$ of (2.4.7.1). Let $q_1$ be the canonical map $X \to \mathcal{X}$. Then, by definition, $m_\mathcal{X} \circ \sigma = q_1 \circ m$. Similarly,

$$p_1, X \circ \sigma = q_1 \circ p_1$$

$$p_2, X \circ \sigma = q_1 \circ p_2,$$

so that

$$1 \otimes \omega' = \sigma^*\omega' = m^*(q_1^*\omega) - p_1^*(q_1^*\omega) - p_2^*(q_1^*\omega) = 0.$$

The kernel of (2.4.7.2) is the torsion submodule of the $\mathcal{O}_K$-module $H^0(\mathcal{Y}, \Omega^1_{\mathcal{Y}/\mathcal{O}_K})$. Since $\mathcal{Y} \to S$ is proper and the sheaf of differentials $\Omega^1_{\mathcal{Y}/\mathcal{O}_K}$ is coherent, $H^0(\mathcal{Y}, \Omega^1_{\mathcal{Y}/\mathcal{O}_K})$ is of finite type. Therefore there exists an integer $r_0 \geq 0$ such that

$$p^{r_0}[H^0(\mathcal{Y}, \Omega^1_{\mathcal{Y}/\mathcal{O}_K})]_{\text{Tors}} = 0.$$

The restriction

$$(2.4.7.3) \quad p^{r_0}H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}_K}) \to p^{r_0}H^0(\mathcal{Y}, \Omega^1_{\mathcal{Y}/\mathcal{O}_K}), \quad \omega \mapsto \omega' = m^*\omega - p_1^*\omega - p_2^*\omega$$

vanishes.

Let $u_1, u_2 \in \mathcal{X}(\mathcal{O}_K)$ and denote by $u_{1,X}$ and $u_{2,X}$ the corresponding $\mathcal{K}$-points of $X$. Let $\nu_X$

$$\nu_X : \text{Spec}(\mathcal{K}) \xrightarrow{\Delta} \text{Spec}(\mathcal{K}) \times \text{Spec}(\mathcal{K}) \xrightarrow{u_{1,X} \times u_{2,X}} X \times_K X$$

and let $v \in \mathcal{Y}(\mathcal{O}_K)$ be the corresponding point of $\mathcal{Y}$. We have:

$$u_1 = p_{1,X} \circ v; \quad u_2 = p_{2,X} \circ v;$$

$$u_{1,X} = p_1 \circ \nu_X; \quad u_{2,X} = p_2 \circ \nu_X;$$

$$u_1 + u_2 = m_\mathcal{X} \circ v.$$

By (2.4.7.3), we get for any $\omega \in p^{r_0}H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}_K})$,

$$(u_1 + u_2)^*\omega = v^*(m_\mathcal{X}^*\omega) = v^*(p_{1,X}^*\omega + p_{2,X}^*\omega) = u_1^*\omega + u_2^*\omega.$$

$\square$
2.4.8. Let \( r \geq r_0 \) be a non negative integer such that \( p^*H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}_K}) \) is torsion free or, so that the restriction of the canonical map

\[
H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}_K}) \to K \otimes_{\mathcal{O}_K} H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}_K}) = H^0(X, \Omega^1_X/K)
\]

to \( p^*H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}_K}) \) is injective. We can restrict the map (2.4.4.1) to

\[
p^*H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}_K}) \to \mathcal{O}_K/\mathcal{O}_K.
\]

By 2.4.7, this pairing is \( \mathbb{Z}[G_K] \)-linear in the second variable. The associated homomorphism

\[
(2.4.8.1) \quad p^*H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}_K}) \to \text{Hom}_{\mathbb{Z}[G_K]}(X(\mathcal{K}), \Omega^1_{\mathcal{O}_K/\mathcal{O}_K})
\]

is \( \mathcal{O}_K \)-linear.

2.4.9. Let

\[
T_p(X) = T_p(X_\mathcal{K}) = \text{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, X(\mathcal{K}))
\]

be the \( p \)-adic Tate module of \( X \). Let \( V_p(X) = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], X(\mathcal{K})) \). We have a natural inclusion of \( T_p(X) \) in \( V_p(X) \): given any \( \alpha = (a_n)_{n \in \mathbb{N}} \in T_p(X) \) we can define a map \( \varphi_{\alpha} : \mathbb{Z}[p^{-1}] \to X(\mathcal{K}) \) by the assignment \( p^{-n} \mapsto a_n \) for \( n \geq 0 \). Let \( V_p(\Omega^1_{\mathcal{O}_K/\mathcal{O}_K}) \) be \( \text{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^1_{\mathcal{O}_K/\mathcal{O}_K}) \) as in 2.3. We have the isomorphism

\[
(2.4.9.1) \quad \text{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^1_{\mathcal{O}_K/\mathcal{O}_K}) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], \Omega^1_{\mathcal{O}_K/\mathcal{O}_K}).
\]

We can compose the \( \mathcal{O}_K \)-homomorphism (2.4.8.1) with the map:

\[
(2.4.9.2) \quad \text{Hom}_{\mathbb{Z}[G_K]}(X(\mathcal{K}), \Omega^1_{\mathcal{O}_K/\mathcal{O}_K}) \xrightarrow{\psi} \text{Hom}_{\mathbb{Z}[G_K]}(V_p(X), V_p(\Omega^1_{\mathcal{O}_K/\mathcal{O}_K}))
\]

to get

\[
p^*H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}_K}) \to \text{Hom}_{\mathbb{Z}[G_K]}(V_p(X), V_p(\Omega^1_{\mathcal{O}_K/\mathcal{O}_K}))
\]

and then, by extending the scalars to \( K \):

\[
\hat{\psi} = \hat{\psi}_{\mathcal{X}, X, r} : H^0(X, \Omega^1_{X/K}) = K \otimes_{\mathcal{O}_K} p^*H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}_K}) \to \text{Hom}_{\mathbb{Z}[G_K]}(V_p(X), V_p(\Omega^1_{\mathcal{O}_K/\mathcal{O}_K})).
\]

2.4.10. Remark. The map \( \psi \) in (2.4.9.2) is injective, as \( X(\mathcal{K}) \) is a \( p \)-divisible group (in the classical sense).

2.4.11. For any \( \omega \in H^0(X, \Omega^1_{X/K}) \) we can take the restriction of the morphism of \( \mathbb{Z}[G_K] \)-modules

\[
\hat{\psi}(\omega) : V_p(X) \to V_p(\Omega^1_{\mathcal{O}_K/\mathcal{O}_K})
\]

to \( T_p(X) \subset V_p(X) \to V_p(\Omega^1_{\mathcal{O}_K/\mathcal{O}_K}) \). By continuity, \( \hat{\psi}(\omega)|_{T_p(X)} \) is \( \mathbb{Z}_p \) linear and, in the end, we get a \( K \)-linear map:

\[
\hat{\psi}_{\mathcal{X}, X, r}^0 : H^0(X, \Omega^1_{X/K}) \to \text{Hom}_{\mathbb{Z}[G_K]}(T_p(X), V_p(\Omega^1_{\mathcal{O}_K/\mathcal{O}_K})) = \text{Hom}_{\mathbb{Z}[G_K]}(T_p(X), \mathbb{C}(1))
\]

since \( V_p(\Omega^1_{\mathcal{O}_K/\mathcal{O}_K}) \) is \( \mathbb{Z}[G_K] \)-isomorphic to \( \mathbb{C}(1) \) by Theorem 2.3.5.

2.4.12. Proposition. The restriction map

\[
\text{Hom}_{\mathbb{Z}[G_K]}(V_p(X), \mathbb{C}(1)) \to \text{Hom}_{\mathbb{Z}[G_K]}(T_p(X), \mathbb{C}(1))
\]

induced by the inclusion \( T_p(X) \subset V_p(X) \) is injective.
2.4. APPLICATIONS TO ABELIAN VARIETIES

Proof. Let $X[p^\infty]$ be the subgroup of $p$-primary torsion of $X(K)$. The quotient $D_p(X) = X(K)/X[p^\infty]$ is a uniquely $p$-divisible abelian group and we have a canonical isomorphism between $\text{Hom}_\mathbb{Z}(\mathbb{Z}[p^{-1}], D_p(X))$ and $D_p(X)$ given by

$$\varphi \mapsto \varphi(1), \quad x \in D_p(X) \mapsto (\varphi_x : 1 \mapsto x).$$

Therefore, the exact sequence

$$0 \to X[p^\infty] \to X(K) \to D_p(X) \to 0$$

leads to the exact sequence

$$(2.4.12.1) \quad 0 \to \text{Hom}_\mathbb{Z}(\mathbb{Z}[p^{-1}], X[p^\infty]) \to V_p(X) \to D_p(X) \to 0.$$ 

Moreover, we have a canonical isomorphism:

$$(2.4.12.2) \quad \text{Hom}_\mathbb{Z}(\mathbb{Z}[p^{-1}], X[p^\infty]) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(X).$$

Indeed, given any $\mathbb{Z}$-linear map $\varphi : \mathbb{Z}[p^{-1}] \to X[p^\infty]$, let $x_0 \in X[p^r]$ be $\varphi(1)$. Then for any $n \in \mathbb{N}$, $x_n = \varphi(1/p^n) \in X[p^{r+n}]$, with $px_n = x_{n-1}$, defining in this way the element $p^{-r} \otimes (p^rx_n)_{ne\mathbb{N}} \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(X)$: it is easy to check that the map is an isomorphism.

By applying $\text{Hom}_{\mathbb{Z}[G_K]}(-, \mathbb{C}(1))$ to (2.4.12.1) we get

$$(2.4.12.3) \quad 0 \to \text{Hom}_{\mathbb{Z}[G_K]}(D_p(X), \mathbb{C}(1)) \to \text{Hom}_{\mathbb{Z}[G_K]}(V_p(X), \mathbb{C}(1)) \xrightarrow{\alpha} \text{Hom}_{\mathbb{Z}[G_K]}(T_p(X), \mathbb{C}(1))$$

as

$$\text{Hom}_{\mathbb{Z}[G_K]}(T_p(X), \mathbb{C}(1)) = (\text{Hom}_{\mathbb{Z}[T_p(X), \mathbb{C}(1)]})^{G_K} = \text{Hom}_{\mathbb{Z}[G_K]}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(X), \mathbb{C}(1),$$

so that $\text{Hom}_{\mathbb{Z}[G_K]}(D_p(X), \mathbb{C}(1))$ is identified with the kernel of $\alpha$.

Since

$$X(K) = \bigcup_{L \supseteq K \text{ finite, Galois}} X(L) = \bigcup_{H \subseteq G_K \text{ open}} X(K)H,$$

also $D_p(X) = \bigcup (D_p(X)^H)$ for $H$ varying in the set of open normal subgroups of $G_K$. Given $f \in \text{Hom}_{\mathbb{Z}[G_K]}(D_p(X), \mathbb{C}(1))$ we have

$$f((D_p(X)^H) \subseteq (\mathbb{C}(1))^H = 0$$

by Tate’s Theorem (cfr. 1.5.15), for any open normal subgroup $H$ of $G_K$. Hence $f(D_p(X)) = \bigcup f((D_p(X)^H)) = 0$. \qed

2.4.13. Proposition. The maps $\hat{\varphi}^X$ and $\hat{\varphi}$ do not depend on the choice of $r$ and on the choice of the $\mathcal{O}_K$-model $\mathcal{X}$.

Proof. The $K$-linearity gives immediately the independence from $r$. It is clearly enough to check the independence of the map $\hat{\varphi}$ from the choice of $\mathcal{X}$. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two proper


In this situation we say that \( \mathfrak{x}_1 \) dominates \( \mathfrak{x}_2 \). The commutativity of the above diagram implies that

\[
\hat{\varrho}_{\mathfrak{x}_1}^0 : H^0(X, \Omega^1_{X/K}) \sim K \otimes_{\mathcal{O}_K} p^*H^0(\mathfrak{x}_1, \Omega^1_{\mathfrak{x}_1/\mathcal{O}_K}) \rightarrow \text{Hom}_{\mathbb{Z}[\mathbb{G}]}(V_p(X), V_p(\Omega^1_{\mathfrak{x}_1/\mathcal{O}_K}))
\]

also commutes, proving that \( \hat{\varrho}_{\mathfrak{x}_1, \mathfrak{x}_2}^0 = \hat{\varrho}_{\mathfrak{x}_1, \mathfrak{x}_1}^0 \). In the general case, if \( \mathfrak{x}_1 \) and \( \mathfrak{x}_2 \) are two models of \( X \), we can construct a third \( \mathcal{O}_K \)-model of \( X \), say \( \mathfrak{x}_3 \), forcing the existence of maps \( \mathfrak{x}_3 \xrightarrow{f_3} \mathfrak{x}_1 \) and \( \mathfrak{x}_3 \xrightarrow{f_{3,2}} \mathfrak{x}_2 \) extending the identity \( id_X \). Indeed, let \( \varphi : X \rightarrow \mathbb{P}^n_K \) be a projective embedding of \( X \). Arguing as in (2.4.7.1), we can consider the composite map

\[
X \xrightarrow{\Delta} \times_K X \times_K X \rightarrow \mathfrak{x}_1 \times_{\mathcal{O}_K} \mathfrak{x}_2,
\]

and we let \( \mathfrak{x}_3 \) be the schematic closure of the composite morphism.

\[\square\]

2.4.14. THEOREM. Let \( X \) be an abelian variety over \( K \). Then

\[
\varrho_X^0 : H^0(X, \Omega^1_{X/K}) \rightarrow \text{Hom}_{\mathbb{Z}[\mathbb{G}]}(T_p(X), \mathbb{C}(1))
\]

defined in 2.4.11 is an injective \( K \)-linear map, functorial in \( X \).

2.4.15. The same argument used in the proof of 2.4.13 allow us to prove that the map \( \varrho_X^0 \) just defined is actually functorial in \( X \): given any homomorphism of abelian varieties \( f : X \rightarrow Z \), it’s enough to choose two \( \mathcal{O}_K \)-models for \( X \) and \( Z \) respectively, say \( \mathfrak{x} \) and \( \mathfrak{z} \), such that \( f \) extends to a morphism \( f : \mathfrak{x} \rightarrow \mathfrak{z} \).

2.4.16. The map \( \varrho_X^0 \) is \( K \) linear by construction and functorial by 2.4.15. Since the restriction map \( \text{Hom}_{\mathbb{Z}[\mathbb{G}]}(V_p(X), \mathbb{C}(1)) \rightarrow \text{Hom}_{\mathbb{Z}[\mathbb{G}]}(T_p(X), \mathbb{C}(1)) \) is injective by 2.4.12, it’s enough to prove that \( \hat{\varrho} \) defined in (2.4.9) is injective. On the other hand, \( \hat{\varrho} \) is the scalar extension to \( K \) of the composition between the map (2.4.8.1) and the injective map \( \psi \) of (2.4.9.2). Hence, we are reduced to prove the following

2.4.17. PROPOSITION. The map

\[
p^*H^0(\mathfrak{x}, \Omega^1_{\mathfrak{x}/\mathcal{O}_K}) \rightarrow \text{Hom}_{\mathbb{Z}[\mathbb{G}]}(X(\overline{K}), \Omega^1_{\overline{K}/\mathcal{O}_K})
\]

defined in (2.4.8.1) is injective.

We dedicate section 2.5 to the proof of this result.
2.5. The Proof of Proposition 2.4.17

The independence from the choice of the \( \mathcal{O}_K \)-model \( \mathfrak{X} \) given by 2.4.13, allow us to use the following desingularization lemma:

2.5.1. Lemma. Let \( X \) be a projective variety over \( K \), of dimension \( d \). Let \( u \in X(K) \) be a regular point of \( X \). Then there exists a proper \( \mathcal{O}_K \)-model \( \mathfrak{X} \) of \( X \) such that if \( \overline{u} \) denotes the closed point in the closure of \( u \) in \( \mathfrak{X} \), the \( \mathfrak{m}_{\overline{u}} \)-adic completion of \( \mathcal{O}_{\mathfrak{X}, \overline{u}} \) is isomorphic to the ring of formal powers series in \( d \) variables over \( \mathcal{O}_K \).

Proof. Let \( \varphi \) be a closed immersion \( \varphi : X \to \mathbb{P}^n_K \), so that:
\[
X = \text{Proj}(K[X_0, \ldots, X_n]/I)
\]
for a homogeneous ideal \( I \) of \( K[X_0, \ldots, X_n] \). We choose homogeneous coordinates \( (X_0; \ldots; X_n) \) of \( \mathbb{P}^n_K \) so that \( u \) is the point \((1:0: \ldots:0)\): being \( u \) a regular point of \( X \), the Jacobian criterion implies — up to a variable reordering — that we can find homogeneous polynomials \( F_1, \ldots, F_{n-d} \) in \( I \), locally defining \( X \), such that the \((n-d) \times (n-d)\) minor
\[
(\frac{\partial F_i}{\partial X_{d+j}}(u))_{1 \leq i,j \leq n-d}
\]
of the Jacobian matrix at \( u \) is invertible. By a linear change of variables we can further assume that such minor is the identity matrix \( I_{n-d} \).

Let \( J \) be the homogeneous ideal of \( K[X_0, \ldots, X_n] \) generated by
\[
(2.5.1.2)
\]
If \( r_i = \deg F_i \), \( 1 \leq i \leq n-d \), we have
\[
(2.5.1.3)
F_i \equiv X_0^{r_i-1}X_{d+i} \pmod{J}, \quad \text{for } 1 \leq i \leq n-d.
\]
by (2.5.1.1) and (2.5.1.2). Let \( \pi \) be a uniformizer of \( \mathcal{O}_K \). We choose non negative integers \( s_i \) such that
\[
\pi^{s_i}F_i \in \mathcal{O}_K[X_0, \ldots, X_n], \quad \text{for } 1 \leq i \leq n-d.
\]
Let \( s \in \mathbb{N} \) such that \( s \geq s_i \) for every \( 1 \leq i \leq n-d \) and we set:
\[
(2.5.1.4)
X_0 = X_0^s, \quad X_i = \pi^{s_i}X_i^s \quad \text{for } 1 \leq i \leq d,
\]
\[
X_i = \pi^{s_i}X_i^s \quad \text{for } d+1 \leq i \leq n.
\]
With this choice, a straightforward computation shows that we can find \((n-d)\) homogeneous polynomials \( G_i \) in the variables \( X_i^s \) such that:
\[
(2.5.1.5)
F_i = \pi^sG_i,
G_i \equiv (X_0)^{r_i-1}X_{d+i} \pmod{\pi\mathcal{O}_K[X_0, \ldots, X_n]} \quad \text{for } 1 \leq i \leq n-d.
\]
We adopt the linear change of coordinates (2.5.1.4) in \( \mathbb{P}^n_K \) and consider the open immersion
\[
(2.5.1.6)
\mathbb{P}^n_K = \text{Proj}(K[X_0', \ldots, X_n']) \to \text{Proj}(\mathcal{O}_K[X_0', \ldots, X_n']) = \mathbb{P}^n_{\mathcal{O}_K}
\]
Let \( \mathfrak{X} \) be the schematic closure of \( X \to \mathbb{P}^n_{\mathcal{O}_K} \) via (2.5.1.6). Let \( \overline{u} \) be the closed point of the closure of \( u \) in \( \mathfrak{X} \). We place ourselves in the principal affine open neighbourhood of \( \overline{u} \) (resp. \( u \)) \( D_+(X_0') = \mathbb{P}^n_{\mathcal{O}_K} \setminus V_+(X_0') \) (resp. \( D_+(X_0') \cap \mathbb{P}^n_K \)), so to have affine coordinates \( x_i = X_i^s/X_0^s \).
Let $m_u \subset \mathcal{O}_{X,u}$ be the maximal ideal of the local ring of $X$ at $u$. The ring $\mathcal{O}_{X,u}$ is regular and local of dimension $d$. By construction, the $K$-vector space $m_u/m_u^2$ is generated by $x_1, \ldots, x_d$.

Let $m_\pi \subset \mathcal{O}_{X,\pi}$ be the maximal ideal of the local ring of $X$ at $\pi$. Let $I_0, \mathcal{O}_K$ be the ideal of $\mathcal{O}_K[x_1, \ldots, x_n]$ defining $x$ in $D_+(X'_0) = \mathcal{O}_K^\pi$. It is generated locally at $\pi$ by the de-homogenized polynomials $X'_0 - x_i G(X'_i)$, written in the variables $x_i$. Then $m_\pi$ is generated by $\pi$ together with the images of $x_1, \ldots, x_n$ modulo $I_0, \mathcal{O}_K$. The local ring $\mathcal{O}_{X,\pi}$ is a regular local ring of dimension $d + 1$. Indeed, $\mathcal{O}_{X,\pi}$ has dimension at least $d + 1$, since when we invert $\pi$ we obtain a ring of dimension $d$. The equality in the dimension and the regularity follow from the fact that $m_\pi/m_\pi^2$ is generated by $\pi, x_1, \ldots, x_d$ by (2.5.1.5).

We have $$\hat{\mathcal{O}}_{X,\pi} \cong \mathcal{O}_K[[x_1, \ldots, x_d]]$$
Indeed, any element of $\hat{\mathcal{O}}_{X,\pi}$ can be expanded as a power series in the $x_i$ with coefficients in $\mathcal{O}_K$, so we have a surjective map $$\mathcal{O}_K[[x_1, \ldots, x_d]] \to \hat{\mathcal{O}}_{X,\pi}$$
and we conclude by [EGA IV, 0.20.7.14.2], being $\mathcal{O}_K[[x_1, \ldots, x_d]]$ a regular local ring of dimension $d + 1 = \dim \mathcal{O}_{X,\pi} = \dim \hat{\mathcal{O}}_{X,\pi}$. 

2.5.2. Let $e \in X(K)$ be the unit section of $X$ and let $x$ be the proper $\mathcal{O}_K$-model of $X$ provided by Lemma 2.5.1, so that $$\hat{\mathcal{O}}_{x,\pi} = \mathcal{O}_K[[T_1, \ldots, T_g]]$$
where $g = \dim X$ and $\pi$ is the closed point of the closure of $e$ in $x$. Let $\hat{\mathcal{O}}_{x,\pi}/\mathcal{O}_K$ be the module of continuous $\mathcal{O}_K$-differentials of $\hat{\mathcal{O}}_{x,\pi}$, i.e. the separated completion of the $\mathcal{O}_{x,\pi}$-module of $\mathcal{O}_K$-differentials $\Omega^1_{\mathcal{O}_{x,\pi}/\mathcal{O}_K}$ (see [EGA IV, 0.20.7.14.2]). By [EGA IV, 0.20.4.5], we have the canonical isomorphism $$\hat{\mathcal{O}}_{x,\pi}/\mathcal{O}_K = \lim_{\leftarrow} \Omega^1_{\mathcal{O}_{x,\pi}/\mathcal{O}_K}/m_\pi^{n}\Omega^1_{\mathcal{O}_{x,\pi}/\mathcal{O}_K}.$$ If we take the composition with the (injective) canonical map

\[(2.5.2.1) \quad \Omega^1_{\mathcal{O}_{x,\pi}/\mathcal{O}_K} \to \hat{\mathcal{O}}_{x,\pi}/\mathcal{O}_K\]

we have an injective $\mathcal{O}_K$-linear morphism

\[(2.5.2.2) \quad p^*H^0(x, \Omega^1_{x,\mathcal{O}_K}) \to \hat{\mathcal{O}}_{x,\pi}/\mathcal{O}_K.\]

Indeed, a global section $\omega \in \Omega^1_{x,\mathcal{O}_K}$ is mapped to $0$ in the stalk $\Omega^1_{\mathcal{O}_{x,\pi}/\mathcal{O}_K}$ if and only if it is mapped to $0$ in $\Omega^1_{\mathcal{O}_{x,\pi}/\mathcal{O}_K}$, that implies $\omega = 0$, since the everywhere defined 1-form over an abelian variety are determined by the value in $e$.

2.5.3. We equip $\hat{\mathcal{O}}_{x,\pi}$ with the $m = (T_1, \ldots, T_g)$-adic topology and $\mathcal{O}_K$ with the $p$-adic topology. To give a continuous $\mathcal{O}_K$-linear map $f: \hat{\mathcal{O}}_{x,\pi} \to \mathcal{O}_K$ amounts to give $g$ elements $x_{fi}, \ldots, x_{fg}$ in the maximal ideal $m_\pi$ of $\mathcal{O}_K$. Therefore we have a canonical map

\[(2.5.3.1) \quad \hat{\mathcal{O}}_{x,\pi}/\mathcal{O}_K \to \text{Hom}_Z(\text{Hom}_{\mathcal{O}_K}(\hat{\mathcal{O}}_{x,\pi}, \mathcal{O}_K), \Omega^1_{\mathcal{O}_{x,\pi}/\mathcal{O}_K}).\]
2.5. THE PROOF OF PROPOSITION 2.4.17

Given by

\[ \omega = \sum_{i=1}^{d} \alpha_i(T_1, \ldots, T_g) dT_i \in \Omega_{\hat{O}_{X,K}/O_K}^1 \rightarrow (f \mapsto \sum_{i=1}^{d} \alpha_i(x_{f,1}, \ldots, x_{f,g}) dx_{f,i}) \]

as \( \alpha_i(x_{f,1}, \ldots, x_{f,g}) \) converges in \( \mathcal{O}_K \) for every \( i \) and \( f \).

Let \( \vartheta \) be the composition of (2.5.2.2) with (2.5.3.1):

\[ \vartheta: p^rH^0(\mathfrak{X}, \Omega_{\mathfrak{X}/\mathcal{O}_K}^1) \to \text{Hom}_\mathbb{Z}(\text{Hom}_{\text{cont}, \mathcal{O}_K}(\hat{\mathcal{O}}_{\mathfrak{X},\mathcal{O}_K}, \Omega_{\mathfrak{X}/\mathcal{O}_K}^1), \Omega_{\mathfrak{X}/\mathcal{O}_K}^1) \]

Using the natural inclusion

\[ \text{Hom}_{\text{cont}, \mathcal{O}_K}(\hat{\mathcal{O}}_{\mathfrak{X},\mathcal{O}_K}, \Omega_{\mathfrak{X}/\mathcal{O}_K}^1) \subset X(\mathcal{O}_K) = X(K) \]

we see that for every \( \omega \in p^rH^0(\mathfrak{X}, \Omega_{\mathfrak{X}/\mathcal{O}_K}^1) \), \( \vartheta(\omega) \) corresponds to the restriction to the subset \( \text{Hom}_{\text{cont}, \mathcal{O}_K}(\hat{\mathcal{O}}_{\mathfrak{X},\mathcal{O}_K}, \Omega_{\mathfrak{X}/\mathcal{O}_K}^1) \) of \( \langle \omega, - \rangle \in \text{Hom}_\mathbb{Z}(G_K)(X(K), \Omega_{\mathfrak{X}/\mathcal{O}_K}^1) \), image of \( \omega \) through (2.4.8.1). To complete the proof of 2.4.17 is therefore enough to establish the following

2.5.4. LEMMA. The canonical map

\[ \hat{\Omega}_{\mathfrak{X}/\mathcal{O}_K}^1 \to \text{Hom}_\mathbb{Z}(\text{Hom}_{\text{cont}, \mathcal{O}_K}(\hat{\mathcal{O}}_{\mathfrak{X},\mathcal{O}_K}, \Omega_{\mathfrak{X}/\mathcal{O}_K}^1), \Omega_{\mathfrak{X}/\mathcal{O}_K}^1) \]

is injective.

2.5.4 can be restated in the following purely algebraic form:

2.5.5. LEMMA. Let \( \omega = \sum_{i=1}^{d} \alpha_i(T_1, \ldots, T_g) dT_i \) be a formal power series in \( d \) variables with coefficients in \( \mathcal{O}_K \). be a non-zero continuous differential form. Then there exist \( x_1, \ldots, x_g \in m_K \) such that

\[ \sum_{i=1}^{d} \alpha_i(x_1, \ldots, x_g) dx_i \]

is a non-zero element of \( \Omega_{\mathfrak{X}/\mathcal{O}_K}^1 \).

PROOF. We first verify the statement for \( g = 1 \). Let \( \omega = \alpha(T) dT = \sum_{i \geq 0} a_i T^i dT \) with \( a_i \in \mathcal{O}_K \). Let \( v \) be the valuation of \( K \) normalized by \( v(K^\times) = \mathbb{Z} \) and let

\[ s = \inf_{i \in \mathbb{N}} v(a_i) \in \mathbb{N}. \]

As \( s \in \mathbb{N} \), there exists a smallest non negative integer \( i_0 \) satisfying \( v(a_{i_0}) = s \). Then, for any \( x \in m_K \) such that \( v(x) < \frac{1}{i_0} \) we have:

\[ v(\alpha(x)) = s + i_0 v(x) < s + 1. \]

It’s enough to choose \( x \) to be a uniformizer for a finite (ramified) extension \( L \) of \( K \), contained in \( K \) such that \( v(D_{L/K}) \geq s + 1 \). Then by 2.2.3 the annihilator of \( dx \) in \( \mathcal{O}_K \) is \( \mathcal{O}_K D_{L/K} \), so that \( \alpha(x) dx \) is not zero as element of \( \Omega_{\mathfrak{X}/\mathcal{O}_K}^1 \). □

The general case is a consequence of the following statement:
2.5.6. Lemma. Let \(\alpha_1, \ldots, \alpha_g\) be \(g\) formal power series in \(g\) variables, \(\alpha_i \in \mathcal{O}_K[[T_1, \ldots, T_g]]\) and suppose that at least one of them is non zero. Then there exist \(g\) formal power series \(\varphi_1, \ldots, \varphi_g\) in one variable \(T\) over \(\mathcal{O}_K\) with no constant terms such that

\[
\sum_{i=1}^{g} \alpha_i(\varphi_1, \ldots, \varphi_g)\varphi'_i
\]

is a non zero element of \(\mathcal{O}_K[[T]]\), where \(\varphi'_i\) denotes the formal derivative of \(\varphi_i\) with respect to the variable \(T\).

Proof. We look for the \(\varphi_i\)'s of the form \(\varphi_i = a_iT + b_iT^2\) with \(a_i, b_i \in \mathcal{O}_K\). Let \(\lambda = \sum_{i=1}^{g} \alpha_i(\varphi_1, \ldots, \varphi_g)\varphi'_i\); we have

\[
\lambda = \sum_{i=1}^{g} \alpha_i(a_1T + b_1T^2, \ldots, a_gT + b_gT^2)(a_i + 2b_iT).
\]

Write \(\alpha_i\) in the form \(\alpha_i = \sum_{m=0}^{\infty} \alpha_{i,m}\) with \(\alpha_{i,m}\) homogeneous of degree \(m\) in the variables \(T_1, \ldots, T_g\). If \(r\) is the smallest integer such that there exists \(j\) with \(\alpha_{j,r} \neq 0\), we have the following expansion for \(\lambda\):

\[
\lambda = \left( \sum_{i=1}^{g} a_i\alpha_{i,r}(a_1, \ldots, a_g) \right)T^r + \left( \sum_{i=1}^{g} a_i\alpha_{i,r+1}(a_1, \ldots, a_g) \right)
\]

\[
+ \sum_{j=1}^{g} 2b_j\alpha_{j,r}(a_1, \ldots, a_g) + \sum_{i,j} a_i b_j \frac{\partial \alpha_{i,r}}{\partial T_j}(a_1, \ldots, a_g)T^{r+1} + \ldots
\]

We now have three possibilities:

i) If \(F = \sum_{i=1}^{g} T_i\alpha_{i,r}(T_1, \ldots, T_g) \neq 0\), being \(\mathcal{O}_K\) infinite, we can find \(a_1, \ldots, a_g\) in \(\mathcal{O}_K\) such that \(F(a_1, \ldots, a_g) \neq 0\). For this choice of the \(a_i\)'s, \(\lambda \neq 0\) for any choice of the \(b_j\)'s.

ii) If \(F = 0\) we look at the next term in the expansion of \(\lambda\): if

\[
G = \sum_{i=1}^{g} T_i\alpha_{i,r+1}(T_1, \ldots, T_g) \neq 0,
\]

we can use again the fact that \(\mathcal{O}_K\) is infinite to find \(a_i\)'s such that \(G(a_1, \ldots, a_g) \neq 0\). If we set \(b_j = 0\) for every \(j\) we see that \(\lambda \neq 0\).

iii) If \(F = G = 0\), we have, by taking the derivative of \(F\) with respect to \(T_j\):

\[
(2.5.6.1) \quad \alpha_{j,r}(T_1, \ldots, T_g) + \sum_{i=1}^{g} T_i \frac{\partial \alpha_{i,r}}{\partial T_j}(T_1, \ldots, T_g) = 0
\]

for every \(1 \leq j \leq g\). Moreover

\[
\lambda = \left( \sum_{j=1}^{g} b_j \left( 2\alpha_{j,r}(a_1, \ldots, a_g) + \sum_{i=1}^{g} a_i \frac{\partial \alpha_{i,r}}{\partial T_j}(a_1, \ldots, a_g) \right) \right)T^{r+1} + \ldots
\]

so that if we substitute (2.5.6.1), we get

\[
\lambda = \left( \sum_{j=1}^{g} b_j \alpha_{j,r}(a_1, \ldots, a_g) \right)T^{r+1} + \ldots
\]
2.6. Connections with Tate’s conjecture

2.6.1. Let $K$ be as in 2.4.1, $X$ an abelian variety over $K$, $T_p(X) = T_p(X_π)$ the $p$-adic Tate module of $X$.

2.6.2. Theorem (Tate-Raynaud). Under the assumptions 2.6.1, there exist canonical, bijective, $K$-linear homomorphisms

$$g^1_X : H^1(X, \mathcal{O}_X) \to \text{Hom}_{Z_p[G_K]}(T_p(X), \mathcal{C}),$$

$$g^0_X : H^0(X, \Omega^1_X/K) \to \text{Hom}_{Z_p[G_K]}(T_p(X), \mathcal{C}(1))$$

where $g^0_X$ is the homomorphism defined in 2.4.11.

Proof. Let $g$ be the dimension of $X$. By 2.4.14 we have:

$$d = \dim_K(\text{Hom}_{Z_p[G_K]}(T_p(X), \mathcal{C}(1))) \geq \dim_K H^0(X, \Omega^1_X/K) = g.$$  

(2.6.2.1)

Equality holds in (2.6.2.1) if and only if $g^0_X$ is an isomorphism. Let $\hat{X}$ be the dual abelian variety of $X$. If we interchange the roles of $X$ and $\hat{X}$, we get from the injection

$$g^0_X : H^0(\hat{X}, \Omega^1_{\hat{X}/K}) \to \text{Hom}_{Z_p[G_K]}(T_p(\hat{X}), \mathcal{C}(1))$$

the inequality

$$d' = \dim_K(\text{Hom}_{Z_p[G_K]}(T_p(\hat{X}), \mathcal{C}(1))) \geq g.$$  

The Weil pairing

$$T_p(X) \times T_p(\hat{X}) \to Z_p(1)$$

is a perfect $Z_p$-linear pairing, compatible with the action of $G_K$ (see [Mum70, p. 186]). It induces a canonical isomorphism

$$(2.6.2.2) \quad T_p(X) \cong \text{Hom}_{Z_p}(T_p(\hat{X}), Z_p(1)).$$

Let $W = \text{Hom}_{Z_p}(T_p(X), \mathcal{C}(1))$ and $\hat{W} = \text{Hom}_{Z_p}(T_p(\hat{X}), \mathcal{C}(1))$. By (2.6.2.2) we have $W \cong T_p(\hat{X}) \otimes_{Z_p} \mathcal{C}$ and $\hat{W} \cong T_p(X) \otimes_{Z_p} \mathcal{C}$, so that there is a canonical non-degenerate $G_K$-pairing

$$(2.6.2.3) \quad W \times \hat{W} \to \mathcal{C}(1).$$

By (1.5.15), we have $H^0_{\text{cont}}(G_K, \mathcal{C}(1)) = H^1_{\text{cont}}(G_K, \mathcal{C}(1)) = 0$. By 1.6.2, $\hat{W}^{G_K} \otimes_K \mathcal{C}$ and $W^{G_K} \otimes_K \mathcal{C}$ are $\mathcal{C}$-subspaces of $\hat{W}$ and $W$. Since they are paired into $\mathcal{C}(1)^{G_K}$, they are orthogonal with respect to the pairing (2.6.2.3). Their dimensions are $d'$ and $d$ respectively, and by (1.9.5.1) we have $d + d' \leq 2g = \dim_C(T_p(X) \otimes_{Z_p} \mathcal{C})$, as required.

In order to get the morphism $g^1_X$ we use again duality for abelian varieties. First of all, recall that there is a canonical isomorphism between the tangent space at 0 to the dual abelian variety $\hat{X}$ and $H^1(X, \mathcal{O}_X)$ ([Mum70], Corollary 3, p. 130). Hence

$$H^1(X, \mathcal{O}_X) = \text{Hom}_K(H^0(\hat{X}, \Omega^1_{\hat{X}/K}), K).$$
The spaces $H^0(X, \Omega^1_{X/K})$ and $H^0(\hat{X}, \Omega^1_{\hat{X}/K})$ are mapped injectively onto subspaces of $W$ and $\hat{W}$ which are orthogonal with respect to the pairing to $C(1)$. Hence we have
\[
\text{Hom}_C(H^0(\hat{X}, \Omega^1_{\hat{X}/K}), C(1)) = W^{G_K},
\]
so that $H^1(X, \mathcal{O}_X) = W^{G_K} \otimes_K C(-1)$. But then
\[
W^{G_K} \otimes_K C(-1) = \text{Hom}_{\mathbb{Z}/p[G_K]}(T_p(X), C(1)) \otimes_K C(-1) \cong \text{Hom}_{\mathbb{Z}/p[G_K]}(T_p(X), C)
\]
providing the required isomorphism
\[
\varrho_X^1 : H^1(X, \mathcal{O}_X) \cong \text{Hom}_{\mathbb{Z}/p[G_K]}(T_p(X), C).
\]
Bibliography


