Geometry of Curves over a Discrete Valuation Ring

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Introduction

We are going to explain a theorem proven by Deligne-Mumford, so called Stable Reduction Theorem. Roughly speaking, we extend an algebraic variety V defined over a number field K to a scheme ν over the ring of integer O_K of K, while trying to preserve as many good properties of V as possible. The reduction of V modulo a maximal ideal p is the fiber ν over the point of $Spec\, O_K$ corresponding to p.
The first part is devoted to some definitions and facts about elliptic curves, as a special case. There, we wish to explain the criterion of Neron-Ogg-Shafarevich on reduction of an elliptic curve.
Then in the second part, by introducing models of a curve, we get back to the notion of reduction in a more general setting. In the third part, we show how to construct étale cohomology and the sheaf of vanishing cycles which helps us to investigate on the behavior of a scheme around its singularities and eventually in the last part, we give a purely cohomolgical proof of the stable reduction theorem and its local version.

Contents

Chapter 1

Required Algebraic Geometry

We begin with some basic definitions and facts [1] which are necessary for study the geometry of curves.

1.1 Varieties and Curves

Definition 1.1.1. Affine n-space (over a field K) is the set of n-tuples

$$\mathbb{A}^n(\bar{K}) = \left\{P = (x_1, ..., x_n) \in \mathbb{A}^n : x_i \in \bar{K}\right\}.$$

Also, the set of K-rational points in $\mathbb{A}^n$ is the set

$$\mathbb{A}^n(K) = \{P = (x_1, ..., x_n) \in \mathbb{A}^n : x_i \in K\}.$$

Notice that the Galois group $G_{\bar{K}/K}$ acts on $\mathbb{A}^n$ for $\sigma \in G_{\bar{K}/K}$ and $P \in \mathbb{A}^n$

$$P^\sigma = (x_1^\sigma, ..., x_n^\sigma).$$

Then $\mathbb{A}^n(K)$ may be characterized by

$$\mathbb{A}^n(K) = \left\{P \in \mathbb{A}^n : P^\sigma = P \; for \; all \; \sigma \in G_{\bar{K}/K}\right\}.$$

Definition 1.1.2. Projective n-space (over K), denoted by $\mathbb{P}^n \; or \; \mathbb{P}^n(\bar{K})$ is the set of all $(n+1)$-tuples $(x_0, ..., x_n) \in \mathbb{A}^{n+1}$ such that at least one x_i is non-zero, modulo the equivalence relation given by

$$(x_0, ..., x_n) \sim (y_0, ..., y_n)$$

if there exists a $\lambda \in \bar{K}^* \; with \; x_i = \lambda y_i \; for \; all \; i$ The set of K-rational points in $\mathbb{P}^n$ is the set $\mathbb{P}^n(K) = \{[x_0, ..., x_n] \in \mathbb{P}^n : \; all \; x_i \in K\}$ where $[x_0, ..., x_n]$ is an equivalence class $\{(\lambda x_0, ..., \lambda x_n)\}$

[1] The elementary facts are not poven in this chapter. For the proofs refer to [**?**].

Definition 1.1.3. A (Projective) algebraic set is any set of the form

$$V_I = \{P \in \mathbb{P}^n : \ f(p) = 0 \ for\ all\ homogeneous\ f \in I\}.$$

in which I is a homogeneous ideal in $\bar{K}[X] = \bar{K}[X_0, ..., X_n]$. If V is a projective algebraic set, the (homogeneous) ideal of V, denoted $I(V)$, is the ideal in $\bar{K}[X]$ generated by

$$\left\{f \in \bar{K}[X] : \ f\ is\ hojmogeneous\ and\ f(p) = 0\ for\ all\ P \in V\right\}.$$

Definition 1.1.4. A projective algebraic set is called a (projective) variety if its homogeneous ideal $I(V)$ is a prime ideal in $\bar{K}[X]$.

Definition and Fact 1.1.5. Let V be an affine algebraic set with ideal $I(V)$, and consider Vas a subset of $\mathbb{P}^n$ via a map $V \subset A^n \xrightarrow{\varphi_i} P^n$.
The projective closure of V, denoted $\bar{V}$, is the projective algebraic set whose homogeneous ideal $I(\bar{V})$ is generated by $\{f^*(X) : \ f \in I(V)\}$.

1. Let V be an affine variety. Then $\bar{V}$ is a projective variety, and $V = \bar{V} \cap A^n$.
2. Let V be a projective variety. Then $V \cap A^n = \emptyset \ \ or \ \ V = \overline{V \cap A^n}$.
3. If an affine (respectively, projective) variety V is defined over K, then $\bar{V}$ (respectively, $V \cap A^n$) is defined over K.

Definition 1.1.6. Let V_1 and $V_2 \subset P^n$ be projective varieties. A *rational map* from V_1 to V_2 is a map of the form

$$\varphi : V_1 \to V_2$$
$$\varphi = [f_0, ..., f_n]$$

where $f_0, ..., f_n \in \bar{K}(V_1)$ have the property that for every point $P \in V_1$ at which $f_0, ..., f_n$ are all defined, $\varphi(P) = [f_0(P), ..., f_n(P)] \in V_2$

Definition 1.1.7. A rational map $\varphi = [f_0, ..., f_n] : V_1 \to V_2$ is *regular* at $P \in V_1$ if there is a function $g \in \bar{K}(V_1)$ such that

1. each gf_i is regular at P; and
2. for some i, $(gf_i)(P) \neq 0$.

Definition and Fact 1.1.8. Let V be a variety, $P \in V$ and $f_1, ..., f_m \in \bar{K}[X]$ a set of generators for $I(V)$. Then V is *non-singualr (or smooth)* at P if $m \times n$ matrix $(\partial f_i / \partial X_j(P))_{1 \leq i \leq m, 1 \leq j \leq n}$ has rank $n - \dim(V)$. If V is non-singualr at every point, then we say that V is non-singualr (or smooth).

P is non-singualr if and only if $\dim_{\bar{K}} M_p/M_P^2 = \dim V$ where
$M_P = \{f \in \bar{K}[V] : \ f(P) = 0\}$.

Note: Here by a *curve* we will mean a projective variety of dimension 1.

Definition and Fact 1.1.9. Let C be a curve and $P \in C$ a smooth point. Then $\bar{K}[C]_P$ is a discrete valuation ring. The *(normalized) valuation* on $\bar{K}[C]_P$ is given by

$$ord_p : \bar{K}[C]_P \to \{0, 1, 2, ...\} \cup \{\infty\}$$
$$ord_p(f) = \max\{d \in \mathbb{Z} : \ f \in M_P^d\}$$

Using $ord_p(f/g) = ord_p(f) - ord_p(g)$, we extend ord_p to $\bar{K}(C)$, so $ord_p : \bar{K}(C) \to \mathbb{Z} \cup \{\infty\}$. A *uniformizer* for C at P is a function $t \in \bar{K}(C)$ with $ord_p(t) = 1$. (i.e. a generator for M_P).

Definition 1.1.10. Let C and P be as above and $f \in \bar{K}(C)$. Then order of f at P is $ord_P(f)$. If $ord_P(f) > 0$, then f has a zero at P; if $ord_P(f) < 0$, then f has a pole at P. If $ord_P(f) \geq 0$, then f is regular (or defined) at P.

1.2 Maps between curves

Fact 1.2.1.

1. Let C be a curve, $V \subset P^N$ a variety, $P \in C$ a smooth point, and $\varphi : C \to V$ a rational map. Then φ is regular at P. In particular, if C is smooth, then φ is a morphism.
2. Let $\varphi : C_1 \to C_2$ be a morphism of curves. Then φ is either constant or surjective.

Fact 1.2.2. Let C_1/K and C_2/K be curves and $\varphi : C_1 \to C_2$ a non-constant rational maps defined over K. Then composition with φ induces an injection of function fields fixing K,

$$\varphi^* : K(C_2) \to K(C_1)$$
$$\varphi^* f = f \circ \varphi$$

Definition 1.2.3. Let $\varphi : C_1 \to C_2$ be a map of curves defined over K. If φ is constant, we define the *degree* of φ to be 0; otherwise we say that φ is finite, and define its degree by $\deg \varphi = [K(C_1) : \varphi^*(C_2)]$. We say that φ is separable (inseparable, purely inseparable) if the extension $K(C_1)/\varphi^* K(C_2)$ has the corresponding property, and we denote the separable and inseparable degrees of the extension by $\deg_s \varphi$ and $\deg_i \varphi$.

Definition 1.2.4. Let $\varphi : C_1 \to C_2$ be a non-constant map of smooth curves, and let $P \in C_1$. The ramification index of φ at P, denoted $e_\varphi(P)$ is given by $e_\varphi(P) = ord_P(\varphi^* t_{\varphi(P)})$ where $t_{\varphi(P)} \in K(C_2)$ is a uniformizer at $\varphi(P)$. Note that $e_\varphi(P) \geq 1$. It is said that φ is unramified at P, if $e_\varphi(P) = 1$; and φ is unramified if it is unramified at every point C_1.

Fact 1.2.5. Let $\varphi : C_1 \to C_2$ be a non-constant map of smooth curves.

1. For every $Q \in C_2$, $\sum_{P \in \varphi^{-1}(Q)} e_\varphi(P) = \deg \varphi$
2. For all but finitely many $Q \in C_2$, $\#\varphi^{-1}(Q) = \deg_s \varphi$
3. Let $\psi : C_2 \to C_3$ be another non-constant map. Then for all $P \in C_1$, $e_{\psi\circ\varphi}(P) = e_\varphi(P) e_\psi(\varphi P)$

Hence, a map $\varphi : C_1 \to C_2$ is unramified if and only if $\#\varphi^{-1}(Q) = \deg(\varphi)$ for all $Q \in C_2$.

1.3 The Frobenius Map

Definition 1.3.1. Suppose that $Char(K) = P > 0$ and let $q = p^r$. For any polynomial $f \in K[X]$, let $f^{(q)}$ be the polynomial obtained from f by raising each coefficient of f to the q^{th} power. Then for any curve C/K we can define a new curve $C^{(q)}/K$ by describing its homogeneous ideal as follow:

$$I(C^{(q)}) = \text{ideal generated by } f^{(q)} : f \in I(C).$$

Furthermore there is a natural map from C to $C^{(q)}$, called q^{th} - power Frobenius morphism, given by

$$\varphi : C \to C^{(q)}$$
$$\varphi([x_0, ..., x_n]) = [x_0^q, ..., x_n^q]$$

Fact 1.3.2. Let K be a field of characteristic $p > 0$, $q = p^r$, C/K a curve, and $\varphi : C \to C^{(q)}$ the q^{th}-power Frobenius morphism described above.

1. $\varphi^* K(C^{(q)}) = K(C)^q$ $(= \{f^q : f \in K(C)\})$.
2. φ is purely inseparable.
3. $\deg \varphi = q$.

1.4 Divisor group of a curve

Definition 1.4.1. The *divisor group* of a curve C, denoted $Div(C)$, is the free abelian group generated by the points of C. Thus a divisor $D \in Div(C)$ is a formal sum

$$D = \sum_{P \in C} n_P(P)$$

where $n_P \in \mathbb{Z}$ and $n_P = 0$ for all but finitely many $P \in C$. the degree of D is defined by $\deg D = \sum_{P \in C} n_P$.

The divisors of degree 0 form a subgroup of $Div(C)$, which we denote by

$$Div^0(C) = \{D \in Div(C) : degD = 0\}$$

Assume now that the curve C is smooth, and let $f \in \bar{K}(C)^*$. Then we can associate to f the divisor $div(f)$ given by $div(f) = \sum_{P \in C} ord_P(f)(P)$.

Definition 1.4.2. A divisor $D \in Div(C)$ is *principal* if it has the form
$D = div(f)$ for some $f \in \bar{K}(C)^*$. Two divisors D_1 and D_2 are *linearly equivalent*, denoted $D_1 \sim D_2$, if $D_1 - D_2$ is principal. The *divisor group (or Picard group)* of C, denoted $Pic(C)$, is the quotient of $Div(C)$ by the subgroup of principal divisors. We let $Pic_K(C)$ be the subgroup of $Pic(C)$ fixed by $G_{\bar{K}/K}$.

$$Pic^0(C) = \frac{Div^0(C)}{< subgroup\ of\ principal\ divisors >}$$

1.5 Differential Forms on Curves

Definition 1.5.1. Let C be a curve. the space of *(meromorphic) differential forms* on C, denoted Ω_C is the $\bar{K}(C)$-vector space generated by symbols of the form dx for $x \in \bar{K}(C)$, subject to the usual relations:

1. $d(x + y) = dx + dy$ for all $x, y \in \bar{K}(C)$;
2. $d(xy) = xdy + ydx$ for all $x, y \in \bar{K}(C)$;
3. $da = 0$ for all $a \in \bar{K}$.

Let $\varphi : C_1 \to C_2$ be a non-constant map of curves. Then the natural map $\varphi^* : \bar{K}(C_2) \to \bar{K}(C_1)$ induces a map on differentials

$$\varphi^* : \Omega_{C_2} \to \Omega_{C_1}$$
$$\varphi^*(\sum f_i dx_i) = \sum (\varphi^* f_i) d(\varphi^* x_i).$$

Fact 1.5.2.

1. Let $\varphi : C_1 \to C_2$ be a non-constant map of curves. Then φ is separable if and only if the map $\varphi^* : \Omega_{C_2} \to \Omega_{C_1}$ is injective (equivalently, non-zero).
2. Let $P \in C$, and let $t \in \bar{K}(C)$ be a uniformizer at P. For every $\omega \in \Omega$, there exists a unique function $g \in \bar{K}(C)$, depending on ω and t, such that $\omega = gdt$. we denote g by ω/dt.
3. The quantity $ord_P(\omega/dt)$ depends only on ω and P, independent of the choice of uniformizer t. We call this valuse the order of ω at P, and denote it by $ord_P(\omega)$.
4. For all but finitely many $P \in C$, $ord_P(\omega) = 0$.

Definition 1.5.3. Let $\omega \in \Omega_C$. The *divisor associated* to ω is

$$div(\omega) = \sum_{P \in C} ord_P(\omega)(P) \in Div(C)$$

A differential ω is *regular (or holomorphic)* if $ord_P(\omega) \geq 0$ for all $P \in C$. It is non-vanishing if $ord_P(\omega) \leq 0$ for all $P \in C$.

Definition 1.5.4. The *canonical divisor* on C is the image in $Pic(C)$ of $div(\omega)$ for any non-zero differential $\omega \in \Omega_C$. Any divisor in this divisor class is called canonical divisor.

Definition 1.5.5. A divisor $D = \sum n_P(P) \in Div(C)$ is *positive (or effective)*, denoted by $D \geq 0$ if $n_P \geq 0$ for every $P \in C$. Similarly, if $D_1, D_2 \in Div(C)$, then we write $D_1 \geq D_2$ to indicate that $D_1 - D_2$ is positive.

Definition 1.5.6. Let $D \in Div(C)$. We associate to D the set of functions

$$L(D) = \{f \in \bar{K}(C)^* : div(f) \geq -D\} \cup \{0\}$$

It is easily seen that $L(D)$ is a finite dimensional $\bar{K}$-vector space, and denote its dimension by $l(D) = dim_{\bar{K}} L(D)$.

Reimann-Roch Theorem for curves

Theorem 1.5.7. Let C be a smooth curve and K_C a canonical divisor on C. There is an integer $g \geq 0$, called the *genus* of C, such that for every divisor $D \in Div(C)$,

$$l(D) - l(K_C - D) = \deg D - g + 1$$

1.6 Weierstrass Equations

Here we are going to study *elliptic curves*, which are curves of genus 1 having a special basepoint. It can be shown that every such curve can be written as the locus in the projective plan $\mathbb{P}^2$ of a cubic equation with only one point (the basepoint) on the line at ∞; i.e., an equation of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

.

Here $O = [0, 1, 0]$ is the basepoint and $a_1, ..., a_6 \in \bar{K}$. These equations are called *weierstrass equations*.

1.6.1 Discriminant and j-invariant of a Weierstrass equation

In order to ease notations, it is better to write the Weierstrass equation for our elliptic curve using non-homogeneous coordinators $x = X/Z$ and $y = Y/Z$, then

$$E : \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

We always remember that there is the extra point $O = [0, 1, 0]$ in infinity. Now put,

$$b_2 = a_1^2 + 4a_2$$
$$b_4 = 2a_4 + a_1a_3$$
$$b_6 = a_3^2 + 4a_6$$

The quantity $\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$ is called the *discriminant*, and $j = c_4^3/\Delta$ is called the *j-invariant* of the given Weierstrass equation.

$$\omega = dx/(2y + a_1x + a_3) = dy/(3x^2 + 2a_2x + a_4 - a_1y).$$

is the *invariant differential* associated with the Weierstrass equation.

1.6.2 Some Basic Facts about Weierstrass equations

1. The curve given by a Weierstrass equation is classified as follows,

   (a) It is non-singualr if and only if $\Delta \neq 0$.

   (b) It has a node if and only if $\Delta = 0$ and $c_4 \neq 0$.

   (c) It has a cusp if and only if $\Delta = c_4 = 0$.

2. Two elliptic curves are isomorphic (over $\bar{K}$) if and only if they have the same j-invariant.

3. Let $j_0 \in \bar{K}$. Then there exists an elliptic curve (defined over $K(j_0)$) with j-invariant equal to j_0.

1.6.3 The Group Law on points of a curve given by a Weierstrass equation

Let E be an elliptic curve given by a Weierstrass equation. We know that $E \subset P^2$ consists of the points $P = (x, y)$ satisfying the equation together with the point $O = [0, 1, 0]$ at infinity.
Let $L \subset P^2$ be a line. Then since the equation has degree three, L intersects E at exactly 3 points, taken the multiplicaties, by *Bezout's theorem*.

Let P,$Q \in E$ and L the line connecting P and Q (tangant line to E if $P = Q$), and R the third point of intersection of L and E. Let L' be the line connecting R and O. Then $P \oplus Q$ is the point such that L' intersects E at R, Q and $P \oplus Q$. This composition law makes E into an abelian group with the identity element O.

Notation

For $m \in \mathbb{Z}$ and $P \in E$, we let,

$$P = P + ... + P \ \ (m \ \ terms) \ for \ m > 0,$$

$$[0]P = O$$

$$[m]P = [-m](-P) \ \ for \ \ m < 0.$$

$$If \ \ m \neq 0, \ \ E[m] = \{P \in E : \ [m]P = O\} \ \ \text{(it is called m − torsion subgroup of } E\text{)}.$$

$$E_{tors} = \bigcup_{m=1}^{\infty} E[m] \ \ \text{(it is called the torsion subgroup of } E\text{)}.$$

1.6.4 Singularity of a curve given by a Weierstrass equation

We explained that if E, which is a curve given by a Weierstrass equation, is singular, then there are two possibilities for the singularity, namely a *node* or a *cusp* (determined by whether c_4 is equal to zero or not).

Let E be a curve given by a Weierstrass equation with discriminant $\Delta = 0$, so E has a singular point S. Then the composition law makes $E_n s$ (the set of non-singular points of E) into an abelian group. So we have two cases according to the quantity of c_4,

1. Suppose E has a node $c_4 \neq o$, and let

$$y = \alpha_1 x + \beta_1$$

and

$$y = \alpha_2 x + \beta_2$$

be the two distinct tangent lines to E at S. Then the map

$$E_{ns} \to \bar{K}^*$$
$$(x, y) \to \frac{y - \alpha_1 x - \beta_1}{y - \alpha_2 x - \beta_2}$$

is isomorphic of abelian groups.

2. Suppose E has a cusp (so $c_4 = 0$), and let $y = \alpha x + \beta$ be the tangent line to E as S. Then the map

$$E_{ns} \to \bar{K}^+$$
$$(x, y) \to \frac{x - x(S)}{y - \alpha x - \beta}$$

is an isomorphism of (additive) groups.

1.7 Decomposition, Inertia and Ramification

Here, we fix a base field K which is henselian with respect to a nonarchimedean valuation v or $| \ |$. We denote the valuation ring, the maximal ideal and the residue class field by o, ρ, κ, respectively. If $L|K$ is an algebraic extension, then the corresponding invariants are labelled ω, O, β, λ, respectively. An especially important role among these extensions is played by the unramified extensions, which are defined as follow.

Definition (Unramified Extension) 1.7.1. A finite extension $L|K$ is called *unramified* if the extension $\lambda|\kappa$ of the residue class field is separable and one has

$$[L : K] = [\lambda : \kappa]$$

. An arbitrary algebraic extension $L|K$ is called unramified if it is a union of finite unramified subextensions.

Fact 1.7.2. Let $L|K$ and $K'|K$ be two extensions inside an algebraic closure $\bar{K}|K$ and let $L' = LK'$. Then one has

$$L|K \quad unramified \Rightarrow L'|K' \quad unramified$$

Each subextension of an unramified extension is unramified. In particular, the composite of two unramified extensions of K is again unramified.

Definition (Maximal Unramified Extension) 1.7.3. Let $L|K$ be an algebraic extension. Then the composite $T|K$ of all unramified subextensions is called the *maximal unramified subextension* of $L|K$.

If the characteristic $p = \mathrm{char}(\kappa)$ of the residue class field is positive, then one has the following weaker notion accompanying that of an unramified extension.

Definition (Tamely Ramified Extension) 1.7.4. An algebraic extension $L|K$ is called *tamely ramified* if the extension $\lambda|\kappa$ of the residue class fields is separable and one has $([L : T], p) = 1$, in which T is the maximal unramified subextension of K in L. In the infinite case this latter condition is taken to mean that the degree of each finite subextension of $L|T$ is prime to p.

As an immediate fact, if $L|K$ and $K'|K$ be two extensions inside the algebraic closure $\bar{K}|K$, and $L' = LK'$. Then we have

$$L|K \;\; tamelyramified \Rightarrow L'|K' \;\; tamelyramified$$

. Every subextension of a tamely ramified extension is tamely ramified. In particular, the composite of tamely ramified extensions is tamely ramified.

Definition (Maximal Tamely Ramified Extension) 1.7.5. Let $L|K$ be an algebraic extension. Then the composite $V|K$ of all tamely ramified subextensions is called the *maximal tamely ramified* subextension of $L|K$.

Definition (Wildly Ramified Extension) 1.7.6. Let $L|K$ be an algebraic extension. Let T and V be its maximal unramified and its maximal tamely ramified subextension, respectively. Then trivially, we have

$$K \subseteq T \subseteq V \subseteq L$$

. If $T = K$ we say that the extension $L|K$ is *totally (or purely) ramified*, and if $V \neq L$, we call it *wildly ramified* extension.

Definition 1.7.7. Let $A \to B$ be an injective homomorphism of discrete valuation rings. Let t (resp. π) denote a uniformizing parameter (= uniformizer) for A (resp. B). Let us recall that the *ramification index of* B over A is the integer $e_{B/A} \geq 1$ such that $tB = \pi^{e_{B/A}}B$ (we put $e_{B/A} = 1$ if $t = 0$). We say that B is *tamely ramified over* A if $B/\pi B$ is separable over A/tA, and if $e_{B/A}$ is prime to $\mathrm{char}(A/tA)$ when the latter is non-zero. Let us moreover suppose that $A \to B$ is finite, and that $L := \mathrm{Frac}(B)$ is separable over $K := \mathrm{Frac}(A)$. Let us recall that $\mathrm{Hom}_{\mathrm{A}}(B, A)$ can canonically be identified with the codifferent

$$W_{B/A} = \{\beta \in L |\; Tr_{L/K}(\beta B) \subseteq A\}$$

We are going to estimate the length of $W_{B/A}/B$ over B.

It is well-known that if A and B are discrete valuation rings and $A \to B$ is a finite injective homomorphism between them such that $\mathrm{Frac}(A) \to \mathrm{Frac}(B)$ is separable. Then

$$length_B(W_{B/A}/B) \geq e_{B/A} - 1$$

Furthermore, equality holds if and only if $A \to B$ is tamely ramified.

Definition (The scheme-theoretic definition of ramification) 1.7.8. Let $f : X \to Y$ be a finite morphism of normal projective curves over a field k. For any closed point $x \in X$, let e_x denote *the ramification index* of $O_{Y,f(x)} \to O_{X,x}$. We will say that f is *ramified at* x or that x is a *ramification point of* f if f is not etale at x (which is equivalent to $e_x \geq 2$ or $k(x)$ inseparable over $k(f(x))$).

It can be seen that the set of ramification points of f is finite and we call it *ramification locus of* f. Its image by f is called the *branch locus of* f.

We say that f is *tamely ramified at* x if $O_{Y,f(x)} \to O_{X,x}$ is tamely ramified.

The following theorem of Hurwitz's related the genus of X to that of Y in terms of the ramification indices of f.

Theorem (Hurwitz) 1.7.9. Let $f : X \to Y$ be a finite morphism of normal projective curves over k. We suppose that f is separable of degree n. Then we have an equality

$$2p_a(X) - 2 = n(2p_a(Y) - 2) + \sum_x (e'_x - 1)[k(x) : k],$$

where the sum is taken over the closed point $x \in X$, e'_x is an integer $\geq e_x$, and $e'_x = e_x$ if and only if f is tamely ramified at x.

As two immediate corollaries of Hurwitz's theorem;

If $f : X \to Y$ is a finite separable morphism of normal projective curves over k, with $p_a(X) \geq 0$. Then $p_a(X) \geq p_a(Y)$.

Now instead, let $f : X \to Y$ be a finite etale morphism of smooth, geometrically connected, projective curves and $g(Y) = 0$. Then f must already be an isomorphism.

1.7.1 Galois Theory of Valuations

Now, we would like to consider Galois extensions $L|K$ and study the effect of the Galois action on the extended valuations $\omega|v$. We know that in the case of a Galois extension $L|K$ of infinite degree, the main theorem of ordinary Galois theory, concerning the 1-1 correspondence between the intermediate fields of $L|K$ and the subgroups of the Galois group $G(L|K)$ ceases to hold; there are more subgroups than intermediate fields.
The correspondence can be salvaged, however, by considering a canonical topology on the group $G(L|K)$, the Krull topology. It is given by defining, for every $\sigma \in G(L|K)$, as a basis of neighbourhoods the cosets $\sigma G(L|M)$, where $M|K$ varies over the finite Galois subextensions of $L|K$.
$G(L|K)$ is thus turned into a compact, Hausdorff topological group. The main theorem of Galois theory then has to be modified in the infinite case by condition that the intermediate fields of $L|K$ correspond 1-1 to the closed subgroups of $G(L|K)$. Otherwise, everything goes through as in the finite case. So one tacitly restricts attention to closed subgroups, and accordingly to continious homomorphisms of $G(L|K)$.

So let $L|K$ be an arbitrary, finite or infinite, Galois extension with Galois group $G = G(L|K)$. If v is an (archimedean or nonarchimedean) valuation of K and ω an extension to L, then, for every $\sigma \in G$, $\omega \circ \sigma$ also extends v, so that the group G acts on the set of extensions $\omega|v$.

It can easily be seen that the group G acts transitively on the set of extensions $\omega|v$, i.e., every two extensions are conjugate.

Definition (Decomposition Group) 1.7.10. The *decomposition group* of an extension ω of v to L is defined by

$$G_\omega = G_\omega(L|K) = \{\sigma \in G(L|K) \mid \omega \circ \sigma = \omega\}.$$

If v is a nonarchimedean valuation, then the decomposition group contains two futher canonical subgroups

$$G_\omega \supseteq I_\omega \supseteq R_\omega$$

which are defined as follow. Let o, resp. O be the valuation ring, ρ,resp. β, the maximal ideal, and let $\kappa = o/\rho$, resp. $\lambda = O/\beta$, be the residue class field of v, resp. ω.

Definition (Inertia and Ramification Group) 1.7.11. The *inertia group* of $\omega|v$ is defined by

$$I_\omega = I_\omega(L|K) = \{\sigma \in G_\omega \mid \sigma x = x \; mod \; \beta \;\; for \; all \;\; x \in O\}$$

and the *ramification group* by

$$R_\omega = R_\omega(L|K) = \{\sigma \in G_\omega \mid \sigma x/x \equiv 1 \; mod \; \beta \;\; for \; all \;\; x \in L^*\}$$

Observe in this definition that, for $\sigma \in G_\omega$, the identity $\omega \circ \sigma = \omega$ implies that one always has $\sigma O = O$ and $\sigma x/x \in O$, for all $x \in L^*$.

The Groups G_ω, I_ω, R_ω of $G = G(L|K)$, are closed in the Krull topology.

Fact 1.7.12. For the extensions $K \subseteq M \subseteq L$, one has

$$G_\omega(L|M) = G_\omega(L|K) \cap G(L|M)$$

$$I_\omega(L|M) = I_\omega(L|K) \cap G(L|M)$$

$$R_\omega(L|M) = R_\omega(L|K) \cap G(L|M)$$

The inertia group I_ω is defined only if ω is a nonarchimedean valuation of L. It is the kernel of a canonical homomorphism of G_ω. For if O is the valuation ring of ω and β themaximal ideal, then, since $\sigma O = O$ and $\sigma\beta = \beta$, every $\sigma \in G_\omega$ induces a κ-automorphism

$$\bar{\sigma} : O/\beta \to O/\beta$$

$$x \;\; (mod \;\; \beta) \to \sigma x \;\; (mod \;\; \beta)$$

of the residue class field λ, and we obtain a homomorphism

$$G_\omega \to \mathrm{Aut}_\kappa(\lambda)$$

with kernel I_ω. That is, we have the following exact sequence of groups,

$$1 \to I_\omega \to G_\omega \to G(\lambda|\kappa) \to 1$$

Like the inertia group, the ramification group R_ω is the kernel of a canonical homomorphism

$$I_\omega \to \chi(L|K),$$

where

$$\chi(L|K) = \mathrm{Hom}(\Delta/\Gamma, \lambda^*),$$

where $\Delta = \omega(L^*)$, and $\Gamma = \upsilon(K^*)$. If $\sigma \in I_\omega$, then the associated homomorphism

$$\chi_\sigma : \Delta/\Gamma \to \lambda^*$$

is given as follow: for $\bar{\delta} = \delta \ \ mod \ \ \Gamma \in \Delta/\Gamma$, choose an $x \in L^*$ such that $\omega(x) = \delta$ and put

$$\chi_\sigma(\bar{\delta}) = \frac{\sigma x}{x} mod \ \ \beta.$$

Note that, this definition is independent of the choice of the representative $\delta \in \bar{\delta}$ and of $x \in L^*$. Because if $x' \in L^*$ is an element such that $\omega(x') \equiv \omega(x) \ \ mod \ \ \Gamma$, then $\omega(x') = \omega(xa)$, $a \in K^*$. Then $x' = xau, \ \ u \in O^*$, and since $\sigma u/u \equiv 1 \ \ mod \ \ \beta$ (because $\sigma \in I_\beta$), one gets $\sigma x'/x' \equiv \sigma x/x \ \ mod \ \ \beta$.

Fact 1.7.13. R_ω is the unique p-Sylow subgroup of I_ω and also we have the following exact sequence

$$1 \to R_\omega \to I_\omega \to \chi(L|K) \to 1.$$

Chapter 2

Elliptic Curve

2.1 Definition

Definition 2.1.1. An *elliptic curve* is a pair of (E, O), where E is a curve of genus 1 and $O \in E$. The elliptic curve E is defined over K, written E/K, if E is defined over K as a curve and $O \in E(K)$.
By using the Riemann-Roch theorem, it is easy to show that every elliptic curve can be written as a plane cubic; and conversely, every smooth Weierstrass plan cubic curve is an elliptic curve.

2.2 How can one make an elliptic curve into an abelian group with the identity O?

It can easily be seen that there is an isomophism

$$\kappa : E \xrightarrow{\sim} Pic^0(E)$$
$$P \to class\ \ (P) - (O)$$

Note that if E is given by a Weierstrass equation, then the "geometric group law" on E and the group law induced from $Pic^0(E)$ through the isomorphism above are the same.

Definition 2.2.1. Let E_1 and E_2 be elliptic curves. An *isogeny* between E_1 and E_2 is a morphism $\varphi : E_1 \to E_2$ satisfying $\varphi(O) = O$

Some Basic Facts about Isogenies

1. Let E_1 and E_2 be elliptic curves. Then the group of isogenies $Hom(E_1, E_2)$ is a torsion free $\mathbb{Z}$-module.

2. Let E be an elliptic curve. Then the endomorphism ring $End(E)$ is an integral domain of characteristic 0.

3. Let $\varphi : E_1 \to E_2$ be an isogeny. Then $\varphi(P+Q) = \varphi(P) + \varphi(Q)$ for all $P, Q \in E_1$.

4. Let E/K be an elliptic curve and let $m \in \mathbb{Z}$, $m \neq 0$. Then the multiplication by m map $[m] : E \to E$ is non-constant.

5. $Ker\varphi = \varphi^{-1}(O)$ is a finite subgroup.

Theorem 2.2.2. Let E be an elliptic curve and $m \in \mathbb{Z}$, $m \neq 0$.

1. $\deg[m] = m^2$.

2. If $char(K) = 0$ or if m is prime to $char(K)$, then $E[m] \cong (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})$.

3. If $char(K) = p$, then either

   $E[p^e] \cong \{0\}$ for all $e = 1, 2, 3, \ldots$; or
   $E[p^e] \cong \mathbb{Z}/p^e\mathbb{Z}$ for all $e = 1, 2, 3, \ldots$.

2.3 Tate Module

Motivation of the definition

Let E/K be an elliptic curve and $m \geq 2$ an integer (prime to $char(K)$ if $char(K) > 0$). By theorem above, $E[m] \cong (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})$.

The group $E[m]$ has more structures. Namely, each elements of the Galois group $G_{\bar{K}/K}$ acts on $E[m]$, since if $[m]P = O$ then $[m](P^\sigma) = ([m]P)^\sigma = O$. So we have

$$G_{\bar{K}/K} \to Aut(E[m]) \cong GL_2(\mathbb{Z}/m\mathbb{Z}),$$

where the later isomorphism is based on choosing a basis for $E[m]$.

As the matrices of representations above have coefficients in a ring not having characteristic 0, it is not usually easy to deal with. So the natural way of achieving this aim is to fit together all such representations when m varies.

Definition 2.3.1. Let E be an elliptic curve and $l \in \mathbb{Z}$ a prime. The (l-adic) *Tate module* of E is the group

$$T_l(E) = \varprojlim_n E[l^n]$$

Pay attention to the fact that each $E[l^n]$ is $\mathbb{Z}/l^n\mathbb{Z}$-module, so the Tate module is naturally a $\mathbb{Z}_l$-module and also, since multiplication-by-l maps are surjective, the *inverse limit topology* on $T_l(E)$ is equivalent to the *l-adic topology* given by being $\mathbb{Z}_l$-module.

Fact 2.3.2. As a $\mathbb{Z}_l$-module, the Tate module is,

1. $T_l(E) \cong \mathbb{Z}_l \times \mathbb{Z}_l$ if $l \neq char(K)$

2. $T_p(E) \cong \{0\}$ or $\mathbb{Z}_p$ if $p = char(K) > 0$.

2.3.1 Action of Galois Group on the Tate Module

In above, we saw that $G_{\bar{K}/K}$ acts on $E[l^n]$. This action commutes with multiplication-by-l maps, which was used to form the inverse limit. Therefore in a well-defined manner, $G_{\bar{K}/K}$ acts on $T_l(E)$ and also this action is continuous. (Because the pro-finite group $G_{\bar{K}/K}$ acts continuously on each finite (discrete) group $E[l^n]$).

2.3.2 Serre's Theorem

The l-adic representations (of $G_{\bar{K}/K}$ on E)

It is the map

$$\rho_l : G_{\bar{K}/K} \to Aut(T_l(E))$$

given by the action of $G_{\bar{K}/K}$ on $T_l(E)$, as we showed above.

Theorem(Serre) 2.3.3. Let K be a number field and E/K an elliptic curve without complex multiplication (namely, $End(E) \cong \mathbb{Z}$)

1. $\rho_l(G_{\bar{K}/K})$ is of finite index in $Aut(T_l(E))$ for all prime l.
2. $\rho_l(G_{\bar{K}/K}) = Aut(T_l(E))$ for all but finitely many primes l.

Chapter 3

Reduction of an Elliptic Curve

Notations: Suppose K is a local field, complete with respect to a discrete valuation v and R is its ring of integers (namely, $R = \{x \in K : v(x) \geq 0\}$)
We also put $M = \{x \in K : v(x) > 0\}$ to be the maximal ideal of R, π is a uniformizer for R (i.e. $M = \pi R$) and $k = R/M$ is the residue field of R.

3.1 Minimal Weierstrass Equation

Definition of Minimal Weierstrass Equation 3.1.1. Let E/K be an elliptic curve, and let

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

be a Weierstrass equation for E/K. Since replacing (x, y) by $(u^{-2}x, u^{-3}y)$ causes each a_i to become $u^i a_i$, if we choose u divisible by a large power of π, then we can find a Weierstrass equation with all coefficients $a_i \in R$. Then the discriminant Δ satisfies $v(\Delta) \geq 0$; and since we can look for an equation with $v(\Delta)$ as small as possible. Such an equation is called a *minimal (Weierstrass)* equation for E at v (means when $v(\Delta)$ is minimalized subject to the condition $a_1, a_2, a_3, a_4, a_6 \in R$). The $v(\Delta)$ is the valuation of the minimal discriminant of E at v.

Reduction modulo π

Let the operation of "reduction modulo π be denoted by a tilde ($\sim$). For example, the natural reduction map $R \to k = R/\pi R$ is denoted by $t \to \tilde{t}$.
If we choose a minimal Weierstrass equation for E/K, we can reduce its coefficients modulo π to obtain a (*possibly singular*) curve over k, namely

$$\tilde{E}: \quad y^2 + \tilde{a}_1xy + \tilde{a}_3y = x^3 + \tilde{a}_2x^2 + \tilde{a}_4x + \tilde{a}_6$$

The curve $\tilde{E}/K$ is called the reduction of E modulo π.
Let $P \in E(K)$. We can find homogeneous coordinates $P = [x_0, y_0, z_0]$ with $x_0, y_0, z_0 \in R$ and at least one of x_0, y_0, z_0 in R^*. Then the reduction point $\tilde{P} = [\tilde{x}_0, \tilde{y}_0, \tilde{z}_0]$ is in $\tilde{E}(k)$.

$$E(K) \to \tilde{E}(k)$$
$$P \to \tilde{P}$$

Now the curve$\tilde{E}/K$ may or may not be singular. In any case, we denote its set of non-singular points by $\tilde{E}_{ns}(k)$, which forms a group.

Two subsets of $E(K)$

1. $E_0(K) = \{P \in E(K) : P \in \tilde{E}_{ns}(k)\}$
2. $E_1(K) = \{P \in E(K) : tildeP = \tilde{O}\}$

(In fact, $E_0(K)$ is the set of points with non-singular reduction, and $E_1(K)$ is the kernel of reduction).

An exact sequence

There is an important exact sequence of abelian groups

$$0 \to E_1(K) \to E_0(K) \to \tilde{E}_{ns}(k) \to 0$$

where the right-hand map is reduction modulo π.

Proposition (Points of finite order) 3.1.2. It is an easy proposition, which provide a crucial ingredient in the proof of the *weak Mordell-Weil theorem* (The group $E(K)/mE(K)$ is a finite group when K is a number field and $m \geq 2$).

Let E/K be an elliptic curve and $m \geq 1$ and integer relatively prime to $char(k)$.

1. The subgroup $E_1(K)$ has no non-trivial points of order m.
2. If the reduced curve $\tilde{E}/K$ is non-singular, then the reduction map

$$E(K)[m] \to \tilde{E}(k)$$

is injective. ($E(K)[m]$ denotes the set of points of order m in $E(K)$).

Proof. From the exact sequence above, and our general knowledge on formal groups, we have, $E_1(K) \simeq \hat{E}(\mathscr{M})$ where $\hat{E}$ as mentioned, is the formal group associated to E. But $\hat{E}(\mathscr{M})$ has no non-trivial elements of order m.
This proves (1).

Now if $\tilde{E}$ is non-singular, then $E_0(K) = E(K)$ and $\tilde{E}_{ns}(k) = \tilde{E}(k)$, so the m-torsion in $E(K)$ injects into $\tilde{E}(k)$, which proves (2). □

Example 3.1.3. Let $E/\mathbb{Q}$ be the elliptic curve.

$$E: \quad y^2 + y = x^3 - x + 1$$

Its discriminant $\Delta = -643$ is prime, so $\tilde{E}$ (modulo 2) is non-singular. One easily can check that $\tilde{E}(F_2) = \{O\}$, hence from the proposition above, we conclude that $E(\mathbb{Q})$ (the set of rational points of the curve E) has no non-zero torsion points.

3.2 The Action of Inertia

We want to re-explain the injectivity of torsion (proposition above (b)) in terms of the action of Galois. First, we set the following notations:

K^{nr} the maximal unramified extension of K,
I_v the inertia subgroup of $G_{\bar{K}/K}$.

We know that the unramified extensions of K correspond to the extensions of the residue field k, $G_{\bar{K}/K}$ has a decomposition

$$\begin{array}{ccccccccc} 1 & \to & G_{\bar{K}/K^{nr}} & \to & G_{\bar{K}/K} & \to & G_{K^{nr}/K} & \to & 1 \\ & & \| & & & & \| & & \\ & & I_v & & & & G_{\bar{k}/k} & & \end{array}$$

In fact, the inertia group I_v is the set of all elements of $G_{\bar{K}/K}$ which act trivially on the residue field $\bar{k}$.

Definition 3.2.1. Let Σ be the set on which $G_{\bar{K}/K}$ acts. We say that Σ is unramified at v if the action of I_v on Σ is trivial.

Recall that if E/K is an elliptic curve, then we saw that $G_{\bar{K}/K}$ acts on torsion subgroup $E[m]$ and the Tate modules $T_l(E)$ of E.

Theorem 3.2.2. Let E/K be an elliptic curve, and suppose that reduced curve $\tilde{E}/k$ is non-singual.

1. Let $m \geq 1$ be an integer relatively prime to $char(k)$ (i.e. $v(m) = 0$). Then $E[m]$ is unramified at v.

2. Let $l \neq char(k)$ be a prime. Then $T_l(E)$ is unramified at v.

Proof. (1) We can choose a finite extension K'/K so that $E[m] \subset E(K')$, and let

R'= ring of integers of K'
$\mathcal{M}'$= maximal ideal of R'
k'= residue field of $R' = R'/\mathcal{M}'$
v'= valuation on K'.

By assumption, if we take a minimal Weierstrass equation for E at v, then its discriminant Δ satisfies $v(\Delta) = 0$ (since $\tilde{E}/k$ is non-singular.) But the restriction of v' on K is just a multiple of v, so $v'(\Delta) = 0$. Hence the Weierstrass equation is also minimal at v', and $\tilde{E}/k'$ is non-singular. Now proposition 2.4.2 (2) shows that the reduction map $E[m] \to \tilde{E}(k')$ is injective.

Let $\sigma \in I_v$ and $P \in E[m]$. We must show that $P^\sigma = P$. By the definition of inertia group, σ acts trivially on $\tilde{E}(k')$, so $\widetilde{P^\sigma - P} = \tilde{P}^\sigma - \tilde{P} = \tilde{O}$. But $P^\sigma - P$ is in $E[m]$, so from the injectivity we have $P^\sigma - P = O$.

(2) The second part of the theorem follows from the first part immediately and the definition $T_l(E) = \varprojlim E[l^n]$. □

The converse of the theorem above is known as the criterion of Neron-Ogg-Shafarevich.

3.3 Good and Bad Reduction

Definition 3.3.1. Let E/K be an elliptic curve, and let $\tilde{E}$ be the reduced curve for a minimal Weierstrass equation.

1. E has *good (or stable)* reduction over K if $\tilde{E}$ is non-singular.
2. E has *multiplicative (or semi-stable)* reduction over K of $\tilde{E}$ has a node.
3. E has *additive (or unstable)* reduction over K if $\tilde{E}$ has a cusp.

In the case (2) and (3), E is naturally said to have *bad* reduction.
If E has multiplicative reduction, then the reduction is said to be *split (respectively non-split)* if the slopes of the tangent lines at the node are in k (respectively, no in k.) Take a look at section 1.6.4.

Theorem (Recognizing the reduction type from a minimal Weierstrass equation) 3.3.2. Let E/K be an elliptic curve with minimal Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Let Δ be the discriminant of this equation and c_4 the usual combination of the a_i's.

1. E has good reduction if and only if $v(\Delta) = 0$ (i.e. $\Delta \in R^*$). In this case $\tilde{E}/k$ is an elliptic curve.
2. E has multiplicative reduction if and only if $v(\Delta) > 0$ and $v(c_4) = 0$ (i.e. $\Delta \in \mathscr{M}$ and $c_4 \in R^*$). In this case $\tilde{E}_{ns}$ is the multiplicative group,
$$\tilde{E}_{ns}(\bar{k}) \cong \bar{k}^*.$$
3. E has additive reduction if and only if $v(\Delta) > 0$ and $v(c_4) > 0$ (i.e. $\Delta, c_4 \in \mathscr{M}$). In this case $\tilde{E}_{ns}$ is the additive group,
$$\tilde{E}_{ns}(\bar{k}) \cong \bar{k}^+.$$

Proof. The theorem directly follows from 1.6.4. ◻

Example 3.3.3. Let $p \geq 5$ be a prime. Then the elliptic curve

$$E_1 : \quad y^2 = x^3 + px^2 + 1$$

has good reduction over $\mathbb{Q}_p$, while

$$E_2 : \quad y^2 = x^3 + x^2 + p$$

has (split) multiplicative reduction over $\mathbb{Q}_p$, and

$$E_3 : \quad y^2 = x^3 + p$$

has additive reduction over $\mathbb{Q}_p$.

Definition 3.3.4. Let E/K be an elliptic curve. E has *potential good reduction* over K if there is a finite extension K'/K so that E has good reduction over K'.

Theorem (semi-stable reduction theorem) 3.3.5. Let E/K be an elliptic curve.

1. Let K'/K be an unramified extension. Then the reduction type of E over K (i.e. good, multiplicative, or additive) is the same as the reduction type of E over K'.

2. Let K'/K be any finite extension. If E has either good or multiplicative reduction over K, then it has the same type of reduction over K'.

3. There exists a finite extension K'/K so that E has either good or (split) multiplicative reduction over K'. (That is the reason why we call them semi-stable and unstable reduction.)

Proof. (1) This follows from Tate's algorithm. We will assume that $char(k) \geq 5$, so E has a minimal Weierstrass equation over K of the form

$$E : \quad y^2 = x^3 + Ax + B$$

Let R' be the ring of integers in K', v' the valuation on K' extending v, and

$$x = (u')^2 x' \quad y = (u')^3 y'$$

a change of coordinates producing a minimal equation for E over K'. Since K'/K is unramified, we can find $u \in K$ with $(u/u') \in (R')^*$. Then the substitution

$$x = u^2 x' \quad y = u^3 y'$$

also gives a minimal equation for E/K', since

$$v'(u^{-12}\Delta) = v'((u')^{-12}\Delta)$$

But this new equation has coefficients in R, so by the minimality of the original equation over K, we have $v(u) = 0$. Hence the original equation is also minimal over K'.

Since $\nu(\Delta) = \nu'(\Delta)$ and $\nu(c_4) = \nu'(c_4)$, using proposition above we deduce that E has the same reduction type over K and K'.

(2) Take a minimal Weierstrass equation for E over K, with corresponding quantities Δ and c_4. Let R' be the ring of integers in K', ν' the valuation on K' extending ν,

$$x = u^2x' + r \qquad y = u^3y' + su^2x' + t$$

by change of coordinates we get a minimal Weierstrass equation for E over K'. For this new equation the associated Δ' and $(c_4)'$ satisfy

$$0 \leq \nu'(u) \leq \min\{\frac{1}{2}\nu'(\Delta), \frac{1}{4}\nu'(c_4)\}$$

But for good (resp. multiplicative) reduction from proposition above (1) and (2), we have $\nu(\Delta) = 0$ (resp. $\nu(c_4) = 0$), so in both cases $\nu'(u) = 0$. Hence

$$\nu'(\Delta') = \nu'(\Delta) \quad and \quad \nu'(c_4') = \nu'(c_4),$$

again using the previous proposition, E has good (resp. multiplicative) reduction over K'.

(3) We assume $char(k) \neq 2$ and extend K so that E has a Weierstrass equation in Legendre normal form

$$E: \quad y^2 = x(x-1)(x-\lambda), \quad \lambda \neq 0, 1.$$

For this equation,

$$c_4 = 16(\lambda^2 - \lambda + 1) \quad and \quad \Delta = 16\lambda^2(\lambda - 1)^2.$$

We consider three cases.

Case 1. $\lambda \in R$, $\lambda \not\equiv 0, 1 (mod \mathscr{M})$. Then $\Delta \in R^*$, so the given equation has good reduction.

Case 2. $\lambda \in R$, $\lambda \equiv 0$ or 1 $(mod \mathscr{M})$. Then $\Delta \in \mathscr{M}$ and $c_4 \in R^*$, so the given equation has (split) multiplicative reduction.

Case 3. $\lambda \notin R$. Choose the integer $r \geq 1$ so that $\pi^r\lambda \in R^*$. Then the substitution $x = \pi^{-r}x'$, $y = \pi^{-3r/2}y'$ (where we replace K by $K(\pi^{1/2})$ if necessary) gives a Weierstrass equation

$$(y')^2 = x'(x' - \pi^r)(x' - \pi^r\lambda)$$

for E with integral coefficients, $\Delta' \in \mathscr{M}$, and $c_4' \in R^*$, so E has (split) multiplicative reduction. □

3.4 The Group E/E_0

Recall that the group $E_0(K)$ consists of those points of $E(K)$ whose reduction to $\tilde{E}(k)$ is not a singular point.

Theorem (Kodaira, Neron) 3.4.1. Let E/K be an elliptic curve. If E has split multiplicative reduction over K, then $E(K)/E_0(K)$ is a cyclic group of order $v(\Delta) = -v(j)$. In all other cases, $E(K)/E_0(K)$ is a finite group of order at most 4.

Proof. The finiteness of $E(K)/E_0(K)$ follows from the existence of the Neron model, which is a group scheme over $Spec(R)$ whose generic fiber is E/K. The specific description of $E(K)/E_0(K)$ comes from the complete classification of the possible special fibers of a Neron model. □

Corollary 3.4.2. The subgroup $E_0(K)$ is of finite index in $E(K)$.

Proof. It is clear from the theorem. □

3.5 The Criterion of Neron-Ogg-Shafarevich

If an elliptic curve E/K has a good reduction,and $m \geq 1$ is an integer prime to $char(k)$, we saw that the torsion subgroup $E[m]$ is unramified. Now its converse.

Theorem (Criterion of Neron-Ogg-Shafarevich) 3.5.1. Let E/K be an elliptic curve. The following are equivalent.

1. E has good reduction over K.
2. $E[m]$ is unramified at v for all integers $m \geq 1$ relatively prime to $char(k)$.
3. The Tate module $T_l(E)$ is unramified at v for some (all) primes l with $l \neq char(k)$.
4. $E[m]$ is unramified at v for infinitely many integer $m \geq 1$ relatively prime to $char(k)$.

Proof. We have already proven (1) ⇒ (2) in 2.5.5.
Also (2) ⇒ (3) ⇒ (4) clearly.

Pay attention to the fact that $T_l(E)$ is unramified if and only if $E[l^n]$ is unramified for all $n \geq 1$.) It remains to prove that (4) implies (1).
Assume (4) holds. Let K^{nr} be the maximal unramified extension of K.
Choose an integer m satisfying

1. m is relatively prime to $char(k)$;
2. $m > \#E(K^{nr})/E_0(K^{nr})$;
3. $E[m]$ is unramified at v.

Such an m exists, since we are assuming (4), and $E(K^{nr})/E_0(K^{nr})$ is finite from the previous theorem.

Now consider the two exact sequences

$$0 \to E_0(K^{nr}) \to E(K^{nr}) \to E(K^{nr})/E_0(K^{nr}) \to 0$$
$$0 \to E_1(K^{nr}) \to E_0(K^{nr}) \to \tilde{E}_{ns}(\bar{k}) \to 0$$

(Note that $\bar{k}$ is the residue field of the ring of integers in K^{nr}.) Since $E[m] \subset E(K^{nr})$, we see that $E(K^{nr})$ has a subgroup isomorphic to $(\mathbb{Z}/m\mathbb{Z})^2$. But from choice of m (2), $E(K^{nr})/E_0(K^{nr})$ has order strictly less than m. It follows from the first exact sequence that we can find a prime l dividing m so that $E_0(K^{nr})$ contains a subgroup $(\mathbb{Z}/l\mathbb{Z})^2$.

Now suppose that E has bad reduction over K^{nr}. If the reduction is multiplicative, then from theorem 2.6.2 (2),

$$\tilde{E}_{ns}(\bar{k}) = (\bar{k})^*;$$

but then the l-torsion in $\tilde{E}_{ns}(\bar{k})$ would be $\mathbb{Z}/l\mathbb{Z}$. Hence this type of reduction cannot occur. Similarly, if E has additive reduction over K^{nr}, then again from theorem 2.6.2 (3),

$$\tilde{E}_{ns}(\bar{k}) = \bar{k} \qquad (taken\ \ additively),$$

which has no l-torsion at all. This eliminates multiplicative and additive reduction as possibilities, so it remains the case that E has good reduction over K^{nr}. Finally, since K^{nr}/K is unramified, we conclude 2.6.4 (1) that E has good reduction over K and therefore we are done. □

Corollary 3.5.2. Let E_1, E_2/K be elliptic curves which are isogenous (namely, there is a non-constant isogenous between them) over K. Then either they both have good reduction over K or neither one does.

Proof. Let $\phi : E_1 \to E_2$ be a non-zero isogeny defined over K, and $m \geq 2$ be an integer relatively prime to both $char(k)$ and $deg\phi$. Then the induced map

$$\phi : E_1[m] \to E_2[m]$$

is an isomorphism of $G_{\bar{K}/K}$-modules, so in particular either both are unramified are v or neither one is. Now, the result deduces from the theorem ((1) $\Leftrightarrow$ (4)). □

Corollary 3.5.3. Let E/K be an elliptic curve. Then E has potential good reduction if and only if the inertia group I_v acts on the Tate module $T_l(E)$ through a finite quotient for some (all) prime(s) $l \neq char(k)$.

Proof. Suppose E has potential good reduction. Therefore there is a finite extension K'/K so that E has good reduction over K'. Extending K', we may assume K'/K is Galois. Let v' be the valuation on K' and I'_v the inertia group of K'. From the theorem, I'_v acts trivially on $T_l(E)$ for any $l \neq char(k)$. Hence the action of I_v on $T_l(E)$ factors through the finite quotient I_v/I'_v. This proves one implication.

Assume now that for some $l \neq char(k)$, I_v acts one $T_l(E)$ through a finite quotient, say I_v/J. Then the fixed field of J, which we denote $\bar{K}^J$, is a finite extension of $K^{nr} = \bar{K}^{I_v}$. Hence we can find a finite extension K'/K so that $\bar{K}^J$ is the compositum

$$\bar{K}^J = K'K^{nr}$$

Then the inertia group of K' is equal to J, and by assumption J acts trivially on $T_l(E)$. Now again from the theorem E has good reduction over K'. □

Chapter 4

Models and General Theory of Reduction

Basically, the main aim of defining models is the classification of nonsingular projective surfaces within a given birational equivalence class. In this case one knows that

- Every birational equivalence class of surfaces has a nonsingular projective surface in it.
- The set of nonsingular projective surfaces with a given function field K/k is a partially ordered set under the relation given by the existence of a birational morphism.
- Any birational morphism $f : X \to Y$ can be factored into a finite number of steps, each of which is a blowing-up of a point.
- Unless K is rational (i.e., $K = K(\mathbb{P}^2)$ or ruled (i.e., $K$is the function field of a product $\mathbb{P}^1 \times C$, where C is a curve), there is a unique *minimal model* of the function field K. (In fact, in the rational and ruled cases, there are infinitely many minimal elements, and their structure is also well-known.)

First of all, we define some basic notions required for introducing different types of models.

4.1 Basic Definitions

Definition 4.1.1. An integral domain A is called *normal* if it is integrally closed in $Frac(A)$, that is $\alpha \in Frac(A)$ integral over A implies that $\alpha \in A$.

Definition 4.1.2. Let X be a scheme. We say that X is *normal* at $x \in X$ or that x is *normal point* of X if $O_{X,x}$ is normal. We say that X is normal if it is irreducible and normal at all of its points.

Definition 4.1.3. In commutative algebra, a normal noetherian integral domain of dimension 1 is called a *Dedekind domain*. (Sometimes, in this definition we assume that the dimension of a Dedekind domain can also be zero so that we are able to make the class of Dedekind domains stable by localization.)

Definition 4.1.4. In scheme theory, we call a normal locally noetherian scheme of dimension 0 or 1 a *Dedekind scheme*.

Definition 4.1.5. Let (A, m) be a noetherian local ring. We say that A is *regular* if $\dim_k m/m^2 = \dim A$. By Nakayama's lemma, A is regular if and only if m is generated by $\dim A$ elements.

Definition 4.1.6. We say that a scheme is *regular* if its local rings are local regular.

Example 4.1.7. The spectrum of a Dedekind domain is a Dedekind scheme. (In fact, if X is a noetherian integral scheme, then X is a Dedekind scheme if and only if $O_X(U)$ is a Dedekind domain for every open subset U of X.

As you have already considered, we wish to study relative curves over a Dedekind scheme. We start by introducing fibered surface.

Definition of a fibered surface 4.1.8. Let S be a Dedekind scheme. We call an integral, projective, flat S-scheme $\pi : X \to S$ of dimension 2, a *fibered surface* over S.

We call X_η the *generic fiber* of X. A fiber X_s with $s \in S$ is called a *closed fiber*.
When $\dim S = 1$, X is also called *projective flat* S-curve.
(Note that the flatness of π is equivalent to the surjectivity of π.)
In other words, a fibered surface is a nonsingular projective surface S, a non singular curve C, and a surjective morphism $\pi : S \to C$. For any $t \in C$, the fiber of S lying over t is the curve $S_t = \pi^{-1}(t)$. Note that S_t will be a nonsingular curve for all but finitely many $t \in S$. We say that X is *normal* (respectively, *regular*) *fibered surface* if X is normal (respectively, regular).

It is quite well-known that a fibered surface is essentially determined by its points of codimension 1.

Fact 4.1.9. Let S be a Dedekind scheme of dimension 1, with generic point η. Let $X \to S$ be a fibered (normal fibered) surface. Then X_η is an integral (normal) curve over $K(S)$. For any $s \in S$, X_s is a projective curve over $k(s)$.

Definition 4.1.10. We call a regular fibered surface $X \to S$ over a Dedekind scheme S of dimension 1 an *arithmetic surface*.

4.2 Arithmetic Surface

Arithmetic surfaces are one of the most important objects in the study of surfaces. Let's take a look at them a little bit closer.
Let R be a Dedekind domain. An arithmetic surface over Spec(R) is the arithmetic

analogue of the fibered surfaces. Spec(R) plays the role of the base curve in here, and arithmetic surface is an R-scheme $\mathscr{C} \to$ Spec(R) whose fibers are curves. For example if R is a discrete valuation ring, then there will be two fibers. The generic fiber will be a curve over the fraction field of R and the special fiber will be a curve over the residue field of R.
Exactly similar to the case of fibered surfaces, an arithmetic surface $\mathscr{C}$ may be regular (nonsingular) even if it has singular fibers.

Theorem 4.2.1. Let $\pi : \mathscr{C} \to$ Spec(R) be a regular arithmetic surface over a Dedekind domain R, and let $p \in$ Spec(R).
Let $x \in \mathscr{C}_p \subset \mathscr{C}$ be a closed point on the fiber of $\mathscr{C}$ over p. Then

$$\mathscr{C}_p \text{ is nonsingular at } x \Leftrightarrow \pi^*(p) \not\subseteq \mathscr{M}^2_{\mathscr{C},x}.$$

The following important corollary says that the smooth part of a proper regular arithmetic surface is large enough to contain all of the rational points on the generic fiber. In turn, this also shows that the regularity condition is necessary for finding a "nice" model.
Recall that if $\pi : X \to S$ be an S-scheme, then a section of X was a morphism of S-scheme $\sigma : S \to X$. This amounts to saying that $\pi \circ \sigma = Id_S$. The set of sections of X is denoted by $X(S)$ (and also by $X(A)$ if $S =$ spec A.)
Now, let X be a scheme over a field k. Then we can identify $X(k)$ with the set of points $x \in X$ such that $k(x) = k$. Indeed, let $\sigma \in X(k)$, and let x be the image of the point of Spec(k). The homomorphism $\sigma^\#_x$ induces a field homomorphism $k(x) \to k$. As $k(x)$ is a k-algebra, this implies that $k(x) = k$.
Conversely, if $x \in X$ verifies $k(x) = k$, there exists a unique section Spec(k) $\to X$ whose image is x.

We call the points of $X(k)$, ($k-$) rational points of X. The notion of rational points is fundamental in arithmetic geometry.

Corollary 4.2.2. Let R be a Dedekind domain with fractional field K, let $\mathscr{C}/R$ be an arithmetic surface, and let C/K be the generic fiber of $\mathscr{C}$.

1. If $\mathscr{C}$ is proper pver R, then
$$C(K) = \mathscr{C}(R)$$
2. Suppose that the scheme $\mathscr{C}$ is regular, and let $\mathscr{C}^0 \subset \mathscr{C}$ be the largest subscheme of $\mathscr{C}$ such that the map $\mathscr{C}^0 \to$ Spec(R) is a smooth morphism. Then
$$\mathscr{C}(R) = \mathscr{C}^0(R)$$
3. In particular, if $\mathscr{C}$ is regular and proper over R, then
$$C(K) = \mathscr{C}(R) = \mathscr{C}^0(R)$$

.

Proof. The case (1) is really just a special case of the valuation criterion of properness. Any point in $\mathscr{C}(R)$ can be specialized to the generic fiber to give a point in $C(K)$, so there is a natural map $\mathscr{C}(R) \to C(K)$. This map is clearly one-to-one, since two morphisms $\text{Spec}(R) \to \mathscr{C}$ which agrees generically (i.e., one a dense open set) are the same. Thus $\mathscr{C}(R) \rightarrowtail C(K)$.
Let $P \in C(K)$ be a point. We are given that $\mathscr{C}$ is proper over R, so the valuation criterion says that there is a morphism $\sigma_P : \text{Spec}(R) \to \mathscr{C}$ making the following diagram commute:

$$\begin{array}{ccc} C = \mathscr{C} \times_R K & \to & \mathscr{C} \\ \uparrow P & & \uparrow \sigma_P \\ \text{Spec}(K) & \to & \text{Spec}(R). \end{array}$$

This proves that every point in $C(K)$ comes from a point in $\mathscr{C}(R)$, so $\mathscr{C}(R) = C(K)$.

(2) As theorem above says, every point in $\mathscr{C}(R)$ intersects each fiber at a nonsingular point of the fiber. But, by definition, $\mathscr{C}^0$, is the complement in $\mathscr{C}$ of the singular points on the fibers. Therefore the natural inclusion $\mathscr{C}^0 \to \mathscr{C}(R)$ is a bijection.

(3) is immediate from (1) and (2) □

At the end of this section, let us roughly explain the aim of finding several so-called nice models for an elliptic curve over a function field.
Like always, let R be a discrete valuation ring with maximal ideal p and fractional field K, and let E/K be an elliptic curve given by a Weierstrass equation

$$E: \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

, say with coefficients $a_1, a_2, a_3, a_4, a_6 \in R$. This equation can be used to define a closed subscheme $\mathscr{W} \subset \mathbb{P}^2_R$. An elementary property of closed subschemes of projective space says that every point of $E(K)$ extends to give a point of $\mathscr{W}(R)$, that is, a section $\text{Spec}(R) \to \mathscr{W}$.
An important property of the elleptic curve E is that it has the structure of a group variety, which means that there is a group law given by a morphism $E \times E \to E$. This group law will extend to a rational map $\mathscr{W} \times_R \mathscr{W} \to \mathscr{W}$, but in general it will not be a morphism, so $\mathscr{W}$ will not be a group scheme over R. However, if we discard all of the singular points on the special fiber of $\mathscr{W}$ (i.e., the singular points on the reduction of E modulo p) and call the resulting scheme $\mathscr{W}^0$, then we will prove that the group law on E does extend to a morphism $\mathscr{W}^0 \times_R \mathscr{W}^0 \to \mathscr{W}^0$. This makes $\mathscr{W}^0$ into a group scheme over R, but unfortunately, we may have lost the point extension property. In orther words, not every point of $E(K)$ will extend to give a point in $\mathscr{W}^0(R)$.

Here, a *Neron model* for E/K is a scheme $\mathscr{E}/R$ which has both of these desirable properties. Thus every point in $E(K)$ extends to a point in $\mathscr{E}(R)$, and further the group law on E extends to a morphism $\mathscr{E} \times_R \mathscr{E} \to \mathscr{E}$ which makes $\mathscr{E}$ into a group scheme over R however it is by no means clear that such a scheme exists.

4.3 Neron model for a curve

We begin this section with an important theorem showing the existence of a unique minimal proper regular model for an elliptic curve.

Theorem and Definition 4.3.1. Let R be a Dedekind domain with fraction field K, and let C/K be a nonsingular projective curve of genus g

1. (Resolution of Singularities for Arithmetic Surfaces, Abhyankar - Lipman) There exists a regular arithmetic surface $\mathscr{C}/R$, proper over R, whose generic fiber is isomorphic to C/K. We call $\mathscr{C}/R$ a *proper regular model for* C/K.

2. (Minimal Models Theorem, Lichtenbaum - Shafarevich) Assume that $g \geq 1$. Then there exists a proper regular model $\mathscr{C}^{min}/R$ for C/K with the following minimality property;

   Let $\mathscr{C}/R$ be any other proper regular model for C/K. Fix an isomorphism from the generic fiber of $\mathscr{C}$ to the generic fiber $\mathscr{C}^{min}$. Then the induced R-birational map
   $$\mathscr{C} \to \mathscr{C}^{min}$$
   is an R-isomorphism. We call $\mathscr{C}^{min}/R$ the *minimal proper regular model for* C/K. It is unique up to unique R-isomorphism.

Definition of a Neron model 4.3.2. Let R be a Dedekind domain with fraction field K, and let E/K be an elliptic curve. A *Neron model for* E/K is a (smooth) group scheme $\mathscr{E}/R$ whose generic fiber is E/K and which satisfies the following universal property, which is usually called *Neron Mapping Property*

Let $\mathscr{X}/R$ be a smooth R-scheme (i.e., $\mathscr{X}$ is smooth over R) with the generic fiber X/K, and let $\phi_K : X_{/K} \to E_{/K}$ be a rational map defined over K. Then there exists a unique R-morphism $\phi_R : \mathscr{X}_{/R} \to \mathscr{E}_{/R}$ extending ϕ_K.

The most important instance of the Neron mapping property is the case that $\mathscr{X} = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(K)$. Then the set of K-maps $X_{/K} \to E_{/K}$ is precisely the group of K-rational points $E(K)$, and the set of R-morphisms $\mathscr{X}_{/R} \to \mathscr{E}_{/R}$ is the group of sections $\mathscr{E}(R)$. So in this situation the Neron mapping property says that the natural inclusion
$$\mathscr{E}(R) \to E(K)$$
is a bijection. If R is a complete discrete valuation ring with algebraically closed residue field, then one can show that the equality $\mathscr{E}(R) = E(K)$ suffices to ensure that the group scheme $\mathscr{E}/R$ is a Neron model for E/K.

Theorem (Kodaira - Neron) 4.3.3. Let E/K be an elliptic curve over the fraction field K of a Dedekind domain R.

1. There is a regular projective two dimensional scheme $\mathscr{C}/\mathrm{Spec}(R)$ whose generic fiber $\mathscr{C} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$ is isomorphic (over K) to E/K. Suppose further that $\mathscr{C}$ is

minimal (i.e., the map $\mathscr{C} \to \mathrm{Spec}(R)$ cannot be factored as $\mathscr{C} \to \mathscr{C}' \to \mathrm{Spec}(R)$ in such a way that $\mathscr{C} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K) \to \mathscr{C}' \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$ is an isomorphism.) Then $\mathscr{C}$ is unique.

2. Let $\mathscr{E} \subset \mathscr{C}$ be the subscheme of $\mathscr{C}$ obtained by discarding all of the singular points of the special fiber $\tilde{\mathscr{C}} = \mathscr{C} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$. (i.e., we discard all multiple fibral components and all intersections of fibral components. Note that these are not singular points of $\mathscr{C}$ itself, which is regular.) Then $\mathscr{E}$ is a group scheme over $\mathrm{Spec}(R)$ whose generic fiber $\mathscr{E} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$ is isomorphic, as a group variety, to E/K. $\mathscr{E}$ is called *the Neron minimal model* of E/K.

3. The natural map $\mathscr{E}(R) \to E(K)$ is an isomorphism. (i.e., every section $\mathrm{Spec}(K) \to E$ on the generic fiber extends to a section $\mathrm{Spec}(R) \to \mathscr{E}$.)

4. Let $\tilde{\mathscr{E}} = \mathscr{E} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ be the special fiber of $\mathscr{E}$. Then $\tilde{\mathscr{E}}$ is an algebraic group over k, and we let $\tilde{\mathscr{E}}^0/k$ be its identity component (so $\tilde{\mathscr{E}}$ is an extension of $\tilde{\mathscr{E}}^0$ by a finite group.) Note that there is a reduction map $\mathscr{E}(R) \to \tilde{\mathscr{E}}(k)$. Then with the identification $\mathscr{E}(R) \cong E(K)$ from (3) above,

$$\tilde{\mathscr{E}}^0(k) \cong \tilde{E}_{ns}(k) \cong E_0(K)/E_1(K).$$

$$\tilde{\mathscr{E}}(k)/\tilde{\mathscr{E}}^0(k) \cong E(K)/E_0(K).$$

As the proof is not directly connected to our main goal of this thesis, we will not prove it here. Refer to "Quasi-fonctions et hauteurs sur les varietes abeliennes. Ann of Math. **82** (1965), 249-331" for the complete proof of this theorem.

4.4 A short note on Divisors

4.4.1 Weil Divisor

Definition 4.4.1. Let X be a noetherian scheme. A *prime cycle* on X is an irreducible closed subset of X. A *cycle* on X is an element of the direct sum $\mathbb{Z}^{(X)}$.
Thus any cycle Z can be written in a unique way as a finite sum

$$Z = \sum_{x \in X} n_x[x]$$

As we have a canonical bijection between X and the set of its irreducible closed subsets via the map $x \to \overline{\{x\}}$, we rather write Z as a finite sum

$$Z = \sum_{x \in X} n_x[\overline{\{x\}}].$$

The (finite) union of the $\overline{\{x\}}$ such that $n_x \neq 0$ is called the *support* of Z, and is denoted $\mathrm{Supp}(Z)$. (Note that $\mathrm{Supp}(Z)$ is closed subset of X. By convention, the support of 0 is the empty set.)

We say that a cycle is *of codimension 1* if the irreducible components of $\mathrm{Supp}(Z)$ are of codimension 1 in X. (Note that $\overline{\{x\}}$ is of codimension 1 if and only if $\dim O_{X,x} = 1$.)

The cycles of codimension 1 form a subgroup $Z^1(X)$ of the group of cycles on X.

Example 4.4.2. Let X be a curve over a field k. Then a cycle of codimension 1 on X is simply a finite sum $\sum_i n_i[x_i]$ with $n_i \in \mathbb{Z}$ and where the x_i are closed points of X.

Definition (Weil Divisor) 4.4.3. Let X be a noetherian integral scheme. A cycle of codimension 1 on X is called a *Weil* divisor on X.

4.4.2 Cartier Divisor

Let A be a ring and Frac(A) be the total ring of fraction of A, which is the localization of A, with respect to the multiplicatively closed subset of regular elements (those elements of A which are not zero divisor) of A and we know that it is a ring containing A as a subring, we denote the presheaf $\mathrm{Frac}(O_X(U))$ by K'_X (that is $K'_X(U) = \mathrm{Frac}(O_X(U))$.

Let the sheaf of algebra associated to the presheaf K'_X above be denoted by K_X, it is obvious that O_X is a subsheaf of K_X. We also denote the subsheaf of invertible elements of K_X by K_X^*.

Definition (Cartier Divisor) 4.4.4. Let X be a scheme. We denote the group $H^0(X, K_X^*/O_X^*)$ by $Div(X)$. The elements of $Div(X)$ are called *Cartier* divisors on X. We denote the group of all such, so that

$$\mathrm{Div}(X) = \Gamma(X, \mathscr{K}_X^*/O_X^*)$$

Conversely, then, a divisor $D \in \mathrm{Div}(X)$ is represented by data (U_i, f_i) consisting of an open covering U_i of X together with elements $f_i \in \Gamma(U_i, \mathscr{K}_X^*)$, having the property that $U_{ij} = U_i \cap U_j$ one can write

$$f_i = g_{ij} f_j \quad for\ some \quad g_{ij} \in \Gamma(U_{ij}, O_X^*).$$

f_i is called a *local equation* for D at any point $x \in U_i$. Two such collections determine the same Cartier divisor if there is a common refinement $\{V_k\}$ of the open coverings on which they are defined so that they are given by data $\{(V_k, f_k)\}$ and $\{(V_k, f'_k)\}$ with

$$f_k = h_k f'_k \ \ on \ \ V_k \ \ for \ \ some \ \ h_k \in \Gamma(V_k, O_X^*).$$

The group operation on Div(X) is always written additively: if $D, D' \in \mathrm{Div}(X)$ are represented respectively by data $\{(U_i, f_i)\}$ and $\{(U_i, f'_i)\}$, then $D + D'$ is given by by $\{(U_i, f_i f'_i)\}$.

The *support* of divisor $D = \{(U_i, f_i)\}$ is the set of points $x \in X$ at which a local equation of D at x is not a unit in $O_x X$. D is *effective* if $f_i \in \Gamma(U_i, O_X)$ is regular on U_i: this is written $D \geqslant 0$.

We fix a regular, Noetherian, connected scheme X of dimension 2. Let us recall that the Cartier divisor on X can be identified with the Weil divisor on X. For any Cartier divisor D, we let $O_X(D)$ denote the invertible sheaf associated to D. If D is effective, then $O_X(-D)$ is a sheaf of ideals of O_X. Consequently, D is naturally endowed with the closed subscheme structure $V(O_X(-D))$ of X.

Normal Crossing Divisor 4.4.5. Let Y be a regular Noetherian scheme, and let D be an effective Cartier divisor on Y. We say that D has *normal crossings* at a point $y \in Y$ if there exist a system of parameters $f_1, ..., f_n$ of Y at y, an integer $0 \leq m \leq n$, and integers $r_1, ..., r_m \geq 1$ such that $O_Y(-D)_y$ is generated by $f_1^{r_1}, ..., f_m^{r_m}$. We say that D has normal crossings if it has normal crossings at every point $y \in Y$.
We say that the prime divisors $D_1, ..., D_l$ *meet transversally* at $y \in Y$ if they are pairwise distinct and if the divisor $D_1 + ... + D_l$ has normal crossings at y.

The irreducible divisors on a fibered surface naturally divide into two differentsorts, those that lie in a single fiber and those that cover C.
More precisely, let $\Gamma \subset S$ be an irreducible curve lying on a fibered surface $\pi : S \to C$. Then π induces a map of curves $\pi : \Gamma \to C$ that is either constant or surjective.
If it is constant, say $\pi(\Gamma) = t$, then Γ lies entirely in the fiber S_t, and we call Γ *fibral* or *vertical*. If not, then $\pi : \Gamma \to C$ is a finite map of positive degree, and we call Γ *horizontal*. Now the formal definition;

Definition(Horizontal and Vertical Divisor) 4.4.6. Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme S. Let D be an irreducible Weil divisor. We say that D is *horizontal* if $\dim S = 1$ and if $\pi|_D : D \to S$ is surjective (hence finite). If $\pi(D)$ is reduced to a point, we say that D is *vertical*. An arbitrary Weil divisor is called horizontal (vertical) if its components are horizontal (vertical).

We will say that a Cartier divisor is horizontal (vertical) if its associated Weil divisor $[D]$ is horizontal (vertical). (Recall that $[D] = \sum_{x \in X, \dim O_{X,x}=1} mult_x(D)[\overline{\{x\}}] \in Z^1(X)$.)

Definition (Contraction) 4.4.7. Let $X \to S$ be a normal fibered surface. Let ϵ be a set of integral vertical curves on X. A normal fibered surface $Y \to S$ together with a projective birational morphism $f : X \to Y$ such that for every integral vertical curve E on X, the set $f(E)$ is a point if and only if $E \in \epsilon$ is called a *contraction* of the $E \in \epsilon$.

Definition (Exceptional Divisor) 4.4.8. Let $X \to S$ be a regular fibered surface. A prime divisor E on X (a closed integral subscheme of X which is of codimension one.) is called an *exceptional* divisor if there exists a regular fibered surface $Y \to S$ and a morphism $f : X \to Y$ of S-scheme such that $f(E)$ is reduced to a point, and $f : X \backslash E \to Y \backslash f(E)$ is an isomorphism.

In other words, an exceptional divisor is an integral curve that can be contracted to a regular point. Note that as $f(E)$ is a closed point, its image in S is also a closed point, hence E is vertical divisor.

4.5 Regular models in general

Definition (Relatively Minimal) 4.5.1. We say that a regular fibered $X \to S$ is *relatively minimal* if it does not contain any exceptional divisor.

By a theorem, this is equivalent to saying that every birational morphism of regular fibered surface $f : X \to Y$ is an isomorphism.

Definition (Minimal) 4.5.2. It is said that $X \to S$ is *minimal* if every birational map of regular fibered S-surfaces $Y \to X$ is a birational morphism. (Note that, every minimal surface is relatively minimal.)

Fact 4.5.3. Let $X \to S$ be an arithmetic surface. Then there exist only a finite number of fibers of $X \to S$ containing exceptional divisors.

Definition 4.5.4. Let S be a Dedekind scheme of dimension 1, with function field K. Let C be a normal, connected, projective curve over K. We call a normal fibered surface $\mathscr{C} \to S$ together with an isomorphism $f : \mathscr{C}_\eta \simeq C$ a *model* of C over S.

Definition 4.5.5. In general case, if $X \to S$ is a normal fibered surface, we will call a regular fibered surface $Y \to S$ together with a birational map $Y \to X$ a *(regular) model* of X over S. (Note that if $\dim S = 1$, then $Y_\eta \to X_\eta$ is a birational map of projective normal curves, therefore it is an isomorphism.)

Definition (Regular Model) 4.5.6. A *regular model* of C is a model $\mathscr{C}$ of C over S such that $\mathscr{C}$ is regular.

Definition (Weierstrass Model) 4.5.7. Let S =Spec A be an affine Dedekind scheme of dimension 1 and let E be an elliptic curve over $K = K(S)$, endowed with a privilege rational point $O \in E(K)$. By definition, E admits a homogeneous equation (Weierstrass equation)

$$v^2z + (a_1u + a_3z)vz = u^3 + a_2u^2z + a_4uz^2 + a_6z^3$$

with O corresponding to the point $(0, 1, 0)$, we then associate the S-scheme

$$W = \text{Proj} \quad A[u, v, z]/(v^2z + (a_1u + a_3z)vz - (u^3 + a_2u^2z + a_4uz^2 + a_6z^3))$$

to it. We call the surface $W \to S$ the *Weierstrass model* of E over S associated to it.

Note that, if we start with equation above, by a suitable change of coordinators we will obtain an integral equation for E. The Weierstrass model associated to this new equation admits a desingularization $\tilde{W}$ since E is smooth.

Fact 4.5.8. Let us S is affine. Let C be a smooth projective curve of genus (geometric genus) $g \geq 1$ over K. Then C admits a unique minimal regular model C_{min}.

Definition (Minimal Regular Model of an Elliptic Curve) 4.5.9. Let E be an elliptic curve over K. We call the minimal arithmetic surface $X \to S$ with generic fiber isomorphic to E the *minimal regular model* of E. (Such a model exists and it is unique by the definition of minimality.)

Example (Model of a curve) 4.5.10. If C is an elliptic curve over K, then the Weierstrass models of C over K are models of C over S.

4.6 Reduction of a normal projective curve

In this section, S will be again a Dedekind scheme of dimension 1, and we set $K = K(S)$.

Definition 4.6.1. Let C be a normal projective curve over K. Let us fix a closed point $s \in S$. We call the fiber $\mathscr{C}_s$ of a model $\mathscr{C}$ of C a reduction of C at s.

Naturally, if S is the spectrum of a Dedekind ring A, and if p is the maximal ideal of A corresponding to s, we also call C_s a reduction of C modulo p.

Definition (Good Reduction and Bad Reduction) 4.6.2. Let C be as above. We say that C has *good reduction* at $s \in S$ if it admits a smooth model over Spec $O_{S,s}$. We say that C has good reduction over S if it has good reduction at every $s \in S$. If C does not have good reduction at s, we will say that C has *bad reduction* at s.

Note if C has a non-smooth model over Spec $O_{S,s}$, this does not necessarily imply that C has bad reduction at s.

Example 4.6.3. Let p be a prime number $\neq 3$. Then the curve

$$C = \text{Proj } \mathbb{Q}[x, y, z]/(x^3 + y^3 + p^3 z^3)$$

admits a model C over $\mathbb{Z}$ by taking the same equation over $\mathbb{Z}$. The reduction C_p is a singular curve. Meanwhile, $C = \text{Proj } \mathbb{Q}[x, y, w]/(x^3 + y^3 + w^3)$, where $w = pz$ is also a model of C over $\mathbb{Z}$, but is smooth over p. Hence C has good reduction at p.

The following theorem is corresponded to the semi-stable reduction theorem (3.3.5)

Theorem 4.6.4. Let S be a Dedekind scheme of dimension 1, C a smooth projective curve over $K = K(S)$ of genus $g \geq 1$.

1. The curve C has good reduction at $s \in S$ except perhaps for a finite number of s.
2. Assume that S is affine. Then C has good reduction over S if and only if the minimal regular model $\mathscr{C}_{min}$ of C over S is smooth. Furthermore, this implies that $\mathscr{C}_{min}$ is a unique smooth model C over S.
3. (Etale base change) Let S' be a Dedekind scheme of dimension 1 that is etale over S, $S' \to S$. Let $s' \in S'$ and let s be its image in S. Then $C_{K'}$ has good reduction at s' if and only if C has good reduction at s.

Theorem 4.6.5. Let E be an elliptic curve over $K = K(S)$. Let $s \in S$, let W be the minimal Weierstrass model of E over Spec $O_{S,s}$ and Δ the discriminant of W. Then the following properties are equivalent:

1. E has good reduction at s;
2. W_s is smooth over $k(s)$;

3. $\Delta \in O^*_{S,s}$.

So far, we know that a curve that has bad reduction will have bad reduction after etale base change. However, if we admit ramified base change, the situation is different.

Definition 4.6.6. Let C be a smooth projective curve over $K = K(S)$. We say that C has *potential good reduction* at $s \in S$ if there exist a morphism $S' \to S$ from a Dedekind scheme S' to S and a point $s' \in S'$ lying above s such that $C_{K(S')}$ has good reduction at s'. If C has good reduction at s, then it has potential good reduction at s.

4.7 A Short Note on Algebraic Curve

By *Algebraic Curve*, we always mean an algebraic variety over a field k whose irreducible components are of dimension 1. Similarly, if all irreducible components are of dimension 2, we call the variety, an *algebraic surface* over k.

Let X be a reduced curve over a field k, and let $\pi : X' \to X$ be the normalization morphism. We know that π is a finite morphism. Take a look at the exact sequence of coherent sheaves on X,

$$0 \to O_X \to \pi_* O_{X'} \to S \to 0$$

The support of S is a closed set not containing any generic point of X; it is therefore a finite set, containing of the singular points of X.

For any $x \in X$, we have $S_x = O'_{X,x}/O_{X,x}$ where $O'_{X,x}$ is the integral closure of $O_{X,x}$ in $\mathrm{Frac}(O_{X,x})$, because normalization commutes with localization.

We set $\delta_x := length_{O_{X,x}} S_x$

Note that by the definition above, $\delta_x = 0$ if and only if x is a normal (and hence regular) point of X.

Definition (ordinary multiple and ordinary double point) 4.7.1. Again, let X be a reduced curve over an algebrically closed field k and $\pi : X' \to X$ be the normalization morphism. We say that a closed point $x \in X$ is an *ordinary multiple* if $\delta_x = m_x - 1$, where m_x is the number of $\pi^{-1}(x)$ (which we know it is finite.) If $m_x = 2$, we say that x is an *ordinary double point* or a *node*.

Definition (Euler-Poincare characteristic) 4.7.2. Let X be a projective variety over a field k. Let F be a coherent sheaf on X. We call the alternating sum

$$\chi_k(F) = \sum_{i \geqslant 0} (-1)^i \dim_k H^i(X, F)$$

The *Euler-Poincare characteristic* of F. It is finite integer (note that $H^i(X, F) = 0$ if $i > \dim X$.)

Definition (Arithmetic genus) 4.7.3. Let X be a projective curve over a field k. The *arithmetic genus* of X is defined to be the integer

$$p_a(X) := 1 - \chi_k(O_X)$$

If X is geometrically connected and geometrically reduced, so that we have $H^0(X, O_X) = k$, then $p_a(X) = \dim_k H^1(X, O_X)$.

In general, if X is a projective nonsingular curve, $H^1(X, O_X)$ and $H^0(X, \omega_X)$ are dual vector spaces. Hence the arithmetic genus $p_a = \dim H^1(X, O_X)$ and the geometric genus $p_g = \dim \Gamma(X, \omega_X)$ are equal.
If X is a projective nonsingular surface, then $H^0(X, \omega)$ is dual to $H^2(X, O_X)$, so $p_g = \dim H^2(X, O_X)$. One the other hand $p_a = \dim H^2(X, O_X) - \dim H^1(X, O_X)$. Thus $p_g \geq p_a$. The difference, $p_g - p_a = \dim H^1(X, O_X)$ is usually denoted by q, and is called the *irregularity* of X.

4.8 Stable Reduction of Algebraic Curve

Definition (Semi-stable and Stable) 4.8.1. Let C be an algebraic curve over an algebrically closed field k. We say that C is *semi-stable* if it is reduced and if its singular points are ordinary double points (node). We say that C is *stable* if moreover, the following conditions are varified:

1. C is connected and projective, of arithmetic genus $p_a(C) \geq 2$.
2. Let γ be an irreducible component of C that is isomorphism to $\mathbb{P}^1_k$. Then it intersects the other irreducible components at at least three points.

Naturally, we say that a curve C over a field k is *semi-stable* (respectively *stable*) if its extension $C_{\bar{k}}$ to the algebraic closure $\bar{k}$ of k is semi-stable (resp. stable) over $\bar{k}$.
As an example; a smooth curve over a field k is semi-stable.

Definition (Rational Point of a scheme) 4.8.2. Let X be a scheme over a field k. We call $x \in X$ a rational point of X if $k(x) = k$.

Definition (Split Ordinary Point) 4.8.3. Let C be a semi-stable curve over a field k, let $\pi : C' \to C$ be the normalization morphism, and $x \in C$ a singular point. We will say that x is a *split ordinary double point* (or simply that x is *split*) if the points of $\pi^{-1}(x)$ are all rational over k.

Definition 4.8.4. Let $f : X \to S$ be a morphism of finite type to a scheme S. We say that f is *semi-stable*, or that X is a semi-stable curve over S, if f is flat and if for any $s \in S$, the fiber X_s is a semi-stable curve over $k(s)$. We say that f is *stable* of genus $g \geq 2$, or that X is a stable curve over S, if f is proper, flat, with stable fibers of arithmetic genus g.

Now, let S be a Dedekind scheme of dimension 1. Let C be a smooth projective curve over $K(S)$. We say that C has semi-stable reduction (resp. stable reduction) at $s \in S$ if there exists a model $\mathscr{C}$ of C over Spec $O_{S,s}$ that is semi-stable (resp. stable) over Spec $O_{S,s}$. The special fiber $\mathscr{C}_s$ of a stable model over Spec $O_{S,s}$ is called the stable reduction of C at s. (It can be seen that stable model -and therefore the stable reduction- is unique.)

We also say that C has semi-stable (resp. stable) reduction over S if the property is true for every $s \in S$. A model $\mathscr{C}$ of C over S is called a stable model if $\mathscr{C} \to S$ is a stable curve. For instance, if C has good reduction at s, then it has stable (and a fortiori semi-stable) reduction.

Here we explain two theorems about semi-stable reduction of an algebraic curve over a Dedekind scheme of dimension 1.

Theorem 4.8.5. Let S be an affine Dedekind scheme of dimension 1. Let C be a smooth projective scheme over $K = K(S)$ of genus $g \geq 1$. Let us suppose that C has semi-stable reduction over S, then

1. The minimal regular model $\mathscr{C}_{min}$ of C over S is semi-stable over S.
2. The curve C admits a semi-stable model over S. If C has stable reduction over S, then it admits a stable model over S.

Example 4.8.6. Let E be an elliptic curve over $K(S)$. If E has semi-stable reduction at a point $s \in S$ if and only if it has good reduction at s or multiplicative reduction at s.

Theorem 4.8.7. Let S be a Dedekind scheme of dimension 1, C a smooth projective curve over $K(S)$ with $p_a(C) \geq 1$, and S' a Dedekind scheme of dimension 1 that dominates S (roughly, means S is dense in S'.). Let $K' = K(S')$.

1. If C has semi-stable (resp. stable) reduction over S, then $C_{K'}$ has semi-stable (resp. stable) reduction over S'. If $\mathscr{C}$ is a semi-stable (resp. stable) model of C over S, then $\mathscr{C} \times_S S'$ is a semi-stable (resp. stable) model of $C_{K'}$ over S'.
2. Let $\mathscr{C}_s$ be the stable reduction (of which we suppose the existence) of C at a point $s \in S$. Let $s' \in S'$ lie above s. Then the stable reduction of $C_{K'}$ at s' is isomorphic to $\mathscr{C}_s \times_{\mathrm{Spec}\, k(s)} \mathrm{Spec}\, k(s')$.
3. Let us moreover suppose that $S' \to S$ is etale surjective. If $C_{K'}$ has semi-stable (resp. stable) reduction over S', then C has semi-stable (resp. stable) reduction over S.

Chapter 5

Stable Reduction Theorem

First of all, we note that if C is a smooth projective curve over a discrete valuation field K, it does not always have semi-stable reduction over O_K. This naturally motivates the definition below.

Definition 5.0.1. Let C be a smooth, projective, geometrically connected curve over $K(S)$, where S is a Dedekind scheme of dimension 1. Let L be a finite extension of $K(S)$ and S' the normalization of S in L. This is a Dedekind scheme of dimension one. We say that C *has semi-stable* (resp. *stable*) *reduction over* S' if C_L has semi-stable (resp. stable) reduction over S'.

Let's take a look at some special cases right here. Suppose $g(C) = 0$ and $L|K$ be a finite separable extension such that $C(L) \neq \emptyset$. (We know that in general if X is a geometrically reduced algebraic variety over a field k, and if k^s is the separable closure of k, then $X(k^s) \neq \emptyset$.). Hence $C_L \simeq \mathbb{P}^1_L$, which, has good reduction over O_L. Hence C has semi-stable reduction over Spec(O_L). If C is of genus 1, then after separable extension, we can suppose that $C(K) \neq \emptyset$, and hence C is an elliptic curve. We have already seen that there exists a finite extension $L|K$ such that C_L has semi-stable reduction over Spec(O_L).

There are several equivalent versions of the stable reduction theorem of Deligne - Mumford. Here we would like to state some of them without showing their coincidence.

5.1 1

Stable Reduction Theorem (Deligne – Mumford) 5.1.1. Let S be a Dedekind scheme of dimension 1, C a smooth, projective, geometrically connected curve of genus $g \geq 2$ over $K(S)$. Then there exists a Dedekind scheme S' that is finite flat over S such that $C_{K(S')}$ has a unique stable model over S'. Moreover, we can take $K(S')$ separable over $K(S)$.

5.2 2

Stable Reduction Theorem (Deligne - Mumford) 5.2.1. Let R be a discrete valuation ring, and let X be a smooth, projective, geometrically connected curve of genus $g(X) \geq 2$ over $K = \mathrm{Frac}(R)$. Then there exists a finite, separable extension $K \subset L$ such that $X \otimes_K L$ has stable reduction over R_L, the integral closure of R in L. that is, there exists a stable curve χ_L over $\mathrm{Spec}(R_L)$ such that the generic fiber is isomorphic to $X \otimes_K L$.

The following version of the theorem is more generalized.

5.3 3

Stable Reduction Theorem (Deligne - Mumford) 5.3.1. Let R be a discrete valuation ring with quotient field K. Let A be an abelian variety over K. Then there exists a finite algebraic extension L of K such that, if R_L= integral closure of R in L, and if $\mathscr{A}_L$ is the Neron model of $A \times_K L$ over R_L, then the closed fiber $A_{L,s}$ of $\mathscr{A}_L$ has no unipotent radical.

This theorem was first proven by Deligne and Mumford using the theorem of Grothendieck on the semi-abelian reduction of abelian varieties, and a theorem of Raynaud that links the reduction of a regular model of C over S to that of the Neron model of $\mathrm{Jac}(C)$ over S.

Mumford had previously given a proof in characteristic $\neq 2$ using the theta function. Afterwards, there were different proofs: Artin - Winters, Bosch - Lutkebohmert and van der Put using rigid analytic geometry, Saito with the theory of vanishing cycle, which we are going to present in this thesis. Saito's proof gives, moreover a characterization of the case when $S' \to S$ is wildly ramified (in which S is local).

As we mentioned above, Deligne and Mumford proved this fact using Picard schemes. However, Saito's proof is purely cohomological one.
More precisely, Saito's new proof of the stable reduction theorem uses l-adic cohomology and theory of vanishing cycles, relating the monodromy action of the Galois group $\mathrm{Gal}(K_{sep}/K)$ on $H^1(X_{K_{sep}}, \mathbb{Q}_l)$ with the geometry of certain normal crossing models for X over R.

5.4 4

Stable Reduction Theorem (Deligne - Mumford) 5.4.1. Assume X is a flat and separated S-scheme of finite type purely of relative dimention 1, (S is the spectrum of a strictly local discrete valuation ring with algebrically closed residue field of $ch = p \geq 0$) μ (resp. s) is the generic point (resp. the closed point) of S. Suppose also X_μ is a proper smooth geometrically connected curve over μ of genus ≥ 2, and X is its minimal regular model. Then the following conditions are equivalent.

1. The action of I on $H^1(X_{\bar{\mu}}, \Lambda)$ is unipotent.

2. X_s is a normal crossing divisor in X.

Also the more general version of the theorem above holds which is as follows,

Stable Reduction Theorem 5.4.2. Suppose X is a normal S-curve, x is a closed point of X_s such that $X - \{x\}$ is smooth over S (i.e., x is an isolated singularity of $X \to S$) and Y is a minimal regular model of X. Then the following conditions are equivalent.

1. The action of I on $R^1\psi\Lambda_x$ (the sheaf of vanishing cycles) is unipotent.

2. Y_s is a normal crossing divisor in Y.

In the next chapter, we explain the important notions of Etale cohomology and also briefly construct the theory of vanishing cycles, which are the main tools in Saiot's new approach to the stable reduction theorem.

Chapter 6

Étale cohomology and the theory of vanishing cycles

For a variety X over the complex numbers, $X(\mathbb{C})$ acquires a topology from that on $\mathbb{C}$, and so one can apply the machinery of algebraic topology to its study. For example, one can define the Betti numbers $\beta^r(X)$ of X to be the dimensions of the vector spaces $H^r(X(\mathbb{C}), \mathbb{Q})$, and such theorems as the Lefschetz fixed point formula are available.
For a variety X over an arbitrary algebraically closed field k, there is only the Zariski topology, which is too coarse (i.e., has too few open subsets) or the methods of algebraic topology to be useful. For example, if X is irreducible, then the groups $H^r(X, \mathbb{Z})$, computed using the Zariski topology, are zero for all $r > 0$.

In the 1940s, Weil observed that some of his results on the numbers of points on certain varieties (curves, abelian varieties, diagonal hypersurfaces ...) over finite fields would be explained by the existence of a cohomology theory giving vector spaces over a field of characteristic zero for which a Lefschetz fixed point formula holds. His results predicted a formula for the Betti numbers of a diagonal hypersurface in $\mathbb{P}^{d+1}$ over $\mathbb{C}$ which was later verified by Dolbeault.

About 1958, Grothendieck defined the étale "topology" of a scheme, and the theory of étale cohomology was worked out by him with the assistance of M. Artin and J.-L. Verdier. The whole theory is closely modelled on the usual theory of sheaves and their derived functor cohomology on a topological space. For a variety X over $\mathbb{C}$, the étale cohomology groups $H^r(X_{et}, \Lambda)$ coincide with the complex groups $H^r(X(\mathbb{C}), \Lambda)$ when Λ is finite, the ring of l-adic integers $\mathbb{Z}_l$, or the field $\mathbb{Q}_l$ of l-adic numbers (but not for $\Lambda = \mathbb{Z}$). When X is the spectrum of a field K, the étale cohomology theory for X coincides with the Galois cohomology theory of K. Thus étale cohomology bridges the gap between the first case, which is purely geometric, and the second case, which is purely arithmetic.

Let us briefly review the origins of the theory on which etale cohomology is modelled.

Algebraic topology had its origins in the late 19th century with the work of Riemann, Betti, and Poincaré on "homology numbers". After an observation of Emmy Noether, the focus shifted to "homology groups".

By the 1950s there were several different methods of attaching (co)homology groups to a topological space, for example, there were the singular homology groups of Veblen, Alexander, and Lefschetz, the relative homology groups of Lefschetz, the Vietoris homology groups, the Cech homology groups, and the Alexander cohomology groups.

The situation was greatly clarified by Eilenberg and Steenrod 1953, which showed that for any "admissible" category of pairs of topological spaces, there is exactly one cohomology theory satisfying a certain short list of axioms. Consider, for example, the category whose objects are the pairs (X, Z) with X a locally compact topological space and Z a closed subset of X, and whose morphisms are the continuous maps of pairs. A cohomology theory on this category is a contravariant functor attaching to each pair a sequence of abelian groups and maps

$$\cdots \to H^{r-1}(U) \to H^r_Z(X) \to H^r(X) \to H^r(U) \to \ldots, \quad U = X \backslash Z$$

satisfying the following axioms:

1. (exactness axiom) the above sequence is exact;
2. (homotopy axiom) the map f^* depends only on the homotopy class of f;
3. (excision) if V is open in X and its closure is disjoint from Z, then the inclusion map $(X \backslash V, Z) \to (X, Z)$ induces an isomorphisms $H^r_Z(X) \to H^r_Z(X \backslash V)$;
4. (dimension axiom) if X consists of a single point, then $H^r(P) = 0$ for $r \neq 0$.

The topologists usually write $H^r(X, U)$ for the group $H^r_Z(X)$. The axioms for a homology theory are similar to the above except that the directions of all the arrows are reversed. If $(X, Z) \to H^r_Z(X)$ is a cohomology theory such that the $H^r_Z(X)$ are locally compact abelian groups (e.g., discrete or compact), then $(X, Z) \to H^r_Z(X)^\vee$ (Pontryagin dual) is a homology theory. In this approach there is implicitly a single coefficient group.

In the 1940s, Leray attempted to understand the relation between the cohomology groups of two spaces X and Y for which a continuous map $Y \to X$ is given. This led him to the introduction of sheaves (local systems of coefficient groups), sheaf cohomology, and spectral sequences (about the same time as Roger Lyndon, who was trying to understand the relations between the cohomologies of a group G, a normal subgroup N, and the quotient group G/N).

Derived functors were used systematically in Cartan and Eilenberg 1956, and in his 1955 thesis a student of Eilenberg, Buchsbaum, defined the notion of an abelian category and extended the Cartan-Eilenberg theory of derived functors to such categories. (The name "abelian category" is due to Grothendieck).

Finally Grothendieck, showed that the category of sheaves of abelian groups on a topological space is an abelian category with enough injectives, and so one can define the cohomology groups of the sheaves on a space X as the right derived functors of the functor taking a sheaf to its abelian group of global sections. One recovers the cohomology of a fixed coefficient group Λ as the cohomology of the constant sheaf it defines. This is now the accepted definition of the cohomology groups, and it is the approach we follow to define the étale cohomology groups. Instead of fixing the coefficient group and having to consider all (admissible) pairs of topological spaces in order to characterize the cohomology groups, we fix the topological space but consider all sheaves on the space.

6.1 Review of sheaf cohomology

Let X be a topological space. We make the open subsets of X into a category with the inclusions as the only morphisms, and define a presheaf to be a contravariant functor from this category to the category Ab of abelian groups. Thus, such a presheaf $\mathscr{F}$ attaches to every open subset U of X an abelian group $\mathscr{F}(U)$ and to every inclusion $V \subset U$ a restriction map $\rho_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ in such way that $\rho_V^U = id_{\mathscr{F}(U)}$ and, whenever $W \subset V \subset U$,

$$\rho_W^U = \rho_W^V \circ \rho_V^U.$$

For historical reasons, the elements of $\mathscr{F}(U)$ are called the sections of $\mathscr{F}$ over U, and the elements of $\mathscr{F}(X)$ the global sections of $\mathscr{F}$. Also, one sometimes writes $\Gamma(U, \mathscr{F})$ for $\mathscr{F}(U)$ and $s|V$ for $\rho_V^U(s)$.
A presheaf $\mathscr{F}$ is said to be a sheaf if

1. a section $f \in \mathscr{F}(U)$ is determined by its restrictions $\rho_U^{U_i}(f)$ to the sets of an open covering $(U_i)_{i \in I}$ of U;

2. a family of sections $f_i \in \mathscr{F}(U_i)$ for $(U_i)_{i \in I}$ an open covering of U arises by restriction from a section $f \in \mathscr{F}(U)$ if $f_i|U_i \cap U_j = f_j|U_i \cap U_j$ for all i and j.

In other words, $\mathscr{F}$ is a sheaf if, for every open covering $(U_i)_{i \in I}$ of an open subset U of X, the sequence

$$\mathscr{F}(U) \to \prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} \mathscr{F}(U_i \cap U_j)$$

is exact by definition, this means that the first arrow maps $\mathscr{F}(U)$ injectively onto the subset of $\prod \mathscr{F}(U_i)$ on which the next two arrows agree. The first arrow sends $f \in \mathscr{F}(U)$ to the family $(f|U_i)_{i \in I}$, and the next two arrows send $(f_i)_{i \in I}$ to the families $f_i|U_i \cap U_{j(i,j) \in I \times I}$ and $(f_j|U_i \cap U_j)_{(i,j) \in I \times I}$ respectively. Since we are considering only (pre)sheaves of abelian groups, we can restate the condition as: the sequence

$$0 \to \mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i) \to \prod_{(i,j)\in I\times I} \mathscr{F}(U_i \cap U_j) \to 0$$

$$f \to (f|U_i)$$
$$(f_i) \to (f_j|U_i \cap U_j - f_i|U_i \cap U_j)$$

is exact. When applied to the empty covering of the empty set, the condition implies that $\mathscr{F}(\emptyset) = 0$.

For example, if Λ is a topological abelian group (e.g., $\mathbb{R}$ or $\mathbb{C}$), then we can define a sheaf on any topological space X by setting $\mathscr{F}(U)$ equal to the set of continuous maps $U \to \Lambda$ and taking the restriction maps to be the usual restriction of functions. When Λ has the discrete topology, every continuous map $f : U \to \Lambda$ is constant on each connected component of U, and hence factors through $\pi_0(U_1)$, the space of connected components of U. When this last space is discrete, $\mathscr{F}(U)$ is the set of all maps $\pi_0(U) \to \Lambda$, i.e., $\mathscr{F}(U) = \Lambda^{\pi_0(U)}$. In this case, we call $\mathscr{F}$ the constant sheaf defined by the abelian group Λ.

Grothendieck showed that, with the natural structures, the sheaves on X form an abelian category. Thus, we have the notion of an injective sheaf: it is a sheaf $\mathscr{I}$ such that for any subsheaf $\mathscr{F}'$ of a sheaf $\mathscr{F}$, every homomorphism $\mathscr{F}' \to \mathscr{I}$ extends to a homomorphism $\mathscr{F} \to \mathscr{I}$. Grothendieck showed that every sheaf can be embedded into an injective sheaf. The functor $\mathscr{F} \mapsto \mathscr{F}(X)$ from the category of sheaves on X to the category of abelian groups is left exact but not (in general) right exact. We define $H^r(X, .)$ to be its rth right derived functor. Thus, given a sheaf $\mathscr{F}$, we choose an exact sequence

$$0 \to \mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \mathscr{I}^2 \to \dots$$

with each $\mathscr{I}^r$ injective, and we set $H^r(X, \mathscr{F})$ equal to the rth cohomology group of the complex of abelian groups

$$\mathscr{I}^0(X) \to \mathscr{I}^1(X) \to \mathscr{I}^2(X) \to \dots$$

While injective resolutions are useful for defining the cohomology groups, they are not convenient for computing it. Instead, one defines a sheaf $\mathscr{F}$ to be *flabby* (or *flasque*) if the restriction maps $\mathscr{F}(U) \to \mathscr{F}(V)$ are surjective for all open $U \supset V$, and shows that $H^r(X, \mathscr{F}) = 0$ if $\mathscr{F}$ is flabby. Thus, resolutions by flabby sheaves can be used to compute cohomology.

6.2 Why is the Zariski topology inadequate?

As we noted above, for many purposes, the Zariski topology has too few open subsets. We list some situations where this is evident.

The cohomology groups are zero Recall that a topological space X is said to be irreducible if any two nonempty open subsets of X have nonempty intersection, and that a variety (or scheme) is said to be irreducible if it is irreducible as a(Zariski) topological space.
Theorem (GROTHENDIECKS Theorem) If X is an irreducible topological space, then $H^r(X, \mathscr{F}) = 0$ for all constant sheaves and all $r > 0$.
PROOF. Clearly, any open subset U of an irreducible topological space is connected. Hence, if $\mathscr{F}$ is the constant sheaf defined by the group Λ, then $\mathscr{F}(U) = \Lambda$ for every nonempty open U. In particular, $\mathscr{F}$ is flabby, and so $H^r(X, \mathscr{F}) = 0$ for $r > 0$.

REMARK The Cech cohomology groups are also zero. Let $\mathscr{U} = (U_i)_{i \in I}$ be an open covering of X. Then the Cech cohomology groups of the covering $\mathscr{U}$ are the cohomology groups of a complex whose rth group is

$$\prod_{(i_0,\ldots,i_r)\in I^{r+1} \quad U_{i_0}\cap\ldots\cap U_{i_r}\neq\emptyset} \Lambda$$

with the obvious maps. For an irreducible space X, this complex is independent of the space X; in fact, it depends only on the cardinality of I (assuming the U_i are nonempty). It is easy to construct a contracting homotopy for the complex, and so deduce that the complex is exact.

The inverse mapping theorem fails A C^∞ map $\phi : N \to M$ of differentiable manifolds is said to be étale at $n \in N$ if the map on tangent spaces $d\phi : Tgt_n(N) \to Tgt_{\phi(n)}(M)$ is an isomorphism.

Theorem (Inverse Mapping Theorem) A C^∞ map of differentiable manifolds is a local isomorphism at any point at which it is étale, i.e., if '$\phi : M \to N$ is étale at $n \in N$, then there exist open neighbourhoods V and U of n and $\phi(n)$ respectively such that ϕ restricts to an isomorphism $V \to U$.

Let X and Y be nonsingular algebraic varieties over an algebraically closed field k. A regular map $\phi : Y \to X$ is said to étale at $y \in Y$ if $d\phi : Tgt_y(Y) \to Tgt_{\phi(y)}(X)$ is an isomorphism.

For example, as we shall see shortly, $x \to x^n : \mathbb{A}^1_k \to \mathbb{A}^1_k$ is étale except at the origin (provided n is not divisible by the characteristic of k). However, if $n > 1$ this map is not a local isomorphism at any point; in fact, there do not exist nonempty open subsets V and U of $\mathbb{A}^1_k$ such that map $x \to x^n$ sends V isomorphically onto U. To see this, note that $x \to x^n$ corresponds to the homomorphism of k-algebras $T \to T^n : k[T] \to k[T]$. If $x \to x^n$ sends V into U, then the corresponding map $k(U) \to k(V)$ on the function fields is $T \to T^n : k(T) \to k(T)$ If $V \to U$ were an isomorphism, then so would be the map on the function fields, but it is not.

6.3 Étale topology

Let X and Y be smooth varieties over an algebraically closed field k. A regular map $\phi : Y \to X$ is said to be étale if it is étale at all points $y \in Y$. An étale map is quasifinite (its fibres are finite) and open.
The étale "topology" on X is that for which the "open sets" are the étale morphisms $U \to X$. A family of étale morphisms $(U_i \to U)_{I \in I}$ over X is a covering of U if $U = \cup \phi_i(U_i)$.
An étale neighbourhood of a point $x \in X$ is an étale map $U \to X$ together with a point $u \in U$ mapping to x.
Define Et/X to be the category whose objects are the étale maps $U \to X$ and whose arrows are the commutative diagrams

$$\begin{array}{ccc} V & \rightarrow & U \\ & \searrow \; \swarrow & \\ & X & \end{array}$$

with the maps $V \to X$ and $U \to X$ étale (then $V \to U$ is also automatically étale).
A presheaf for the étale topology on X is a contravariant functor $\mathscr{F} : Et/X :\to Ab$. It is a sheaf if the sequence

$$\mathscr{F}(U) \to \prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} \mathscr{F}(U_i \times_U U_j)$$

is exact for all étale coverings $U_i \to U$. One shows, much as in the classical case, that the category of sheaves is abelian, with enough injectives. Hence one can define étale cohomology groups $H^r(X_{et}, \mathscr{F})$ exactly as in the classical case, by using the derived functors of $\mathscr{F} \to \mathscr{F}(X)$.

6.4 The Étale Fundamental Group

The étale fundamental group classifies the finite étale coverings of a variety (or scheme) in the same way that the usual fundamental group classifies the covering spaces of a topological space. We begin by reviewing the classical theory from a functorial point of view.

The topological fundamental group Let X be a connected topological space. In order to have a good theory, we assume that X is pathwise connected and semi-locally simply connected (i.e., every $P \in X$ has a neighbourhood U such that every loop in U based at P can be shrunk in X to P).
Fix an $x \in X$. The fundamental group $\pi_1(X, x)$ is defined to be the group of homotopy classes of loops in X based at x.
A continuous map $\pi : Y \to X$ is a covering space of X if every $P \in X$ has an open

neighbourhood U such that $\pi^{-1}(U)$ is a disjoint union of open sets U_i each of which is mapped homeomorphically onto U by π. A map of covering spaces $(Y, \pi) \to (Y', \pi')$ is a continuous map $\alpha : Y \to Y'$ such that $\pi' \circ \alpha = \pi$. Under our hypotheses, there exists a simply connected covering space $\tilde{\pi} : \tilde{X} \to X$. Fix an $\tilde{x} \in \tilde{X}$ mapping to $x \in X$. Then $(\tilde{X}, \tilde{x})$ has the following universal property: for any covering space $Y \to X$ and $y \in Y$ mapping to $x \in X$, there is a unique covering space map $\tilde{X} \to Y$ sending $\tilde{x}$ to y. In particular, the pair $(\tilde{X}, \tilde{x})$ is unique up to a unique isomorphism; it is called *the universal covering space* of (X, x).

Let $\mathrm{Aut}_X(\tilde{X})$ denote the group of covering space maps $\tilde{X} \to \tilde{X}$, and let $\alpha \in \mathrm{Aut}_X(\tilde{X})$. Because α is a covering space map, $\alpha\tilde{x}$ also maps to $x \in X$. Therefore, a path from $\tilde{x}$ to $\alpha\tilde{x}$ is mapped by $\tilde{\pi}$ to a loop in X based at x. Because $\tilde{X}$ is simply connected, the homotopy class of the loop does not depend on the choice of the path, and so, in this way, we obtain a map $\mathrm{Aut}_X(\tilde{X}) \to \pi_1(X, x)$. It can be shown that this map is an isomorphism.

Let $Cov(X)$ be the category of covering spaces of X with only finitely many connected components the morphisms are the covering space maps. We define $F : Cov(X) \to Sets$ to be the functor sending a covering space $\pi : Y \to X$ to the set $\pi^{-1}(x)$. This functor is representable by $\tilde{X}$, i.e.,

$$F(Y) \simeq \mathrm{Hom}_X(\tilde{X}, Y) \qquad functorially\ \ in\ \ Y$$

Indeed, we noted above that to give a covering space map $\tilde{X} \to Y$ is the same as to give a point $y \in \pi^{-1}(x)$.
If we let $\mathrm{Aut}_X(\tilde{X})$ act on $\tilde{X}$ on the right, then it acts on $\mathrm{Hom}_X(\tilde{X}, Y)$ on the left:

$$\alpha f \stackrel{def}{=} f \circ \alpha \quad \alpha \in \mathrm{Aut}_X(\tilde{X}), \qquad f : \tilde{X} \to Y$$

Thus, we see that F can be regarded as a functor from $Cov(X)$ to the category of $\mathrm{Aut}_X(\tilde{X})$ (or $\pi_1(X, x)$) sets.
That $\pi_1(X, x)$ classifies the covering spaces of X is beautifully summarized by the following statement:

The functor F defines an equivalence from $Cov(X)$ to the category of $\pi_1(X, x)$-sets with only finitely many orbits.

Now, we would like to construct the étale fundamental group.
Let X be a connected variety (or scheme). We choose a geometric point $\bar{x} \to X$, i.e., a point of X with coordinates in a separably algebraically closed field. When X is a variety over an algebraically closed field k, we can take $\bar{x}$ to be an element of $X(k)$. For a scheme X, choosing $\bar{x}$ amounts to choosing a point $x \in X$ together with a separably algebraically closed field containing the residue field $k(x)$ at x.

Recall that a finite étale map $\pi : Y \to X$ is open (because it is étale) and closed (because it is finite) and so it is surjective (provided $Y \neq \emptyset$). If X is a variety over an algebraically closed field and $\pi : Y \to X$ is finite and étale, then each fibre of π has exactly the same number of points. Moreover, each $x \in X$ has an étale neighbourhood $(U, u) \to (X, x)$ such that $Y \times_X U$ is a disjoint union of open subvarieties (or subschemes) U_i each of which is mapped isomorphically onto U by $\pi \times 1$. Thus, a finite étale map is the natural analogue of a finite covering space.

We define FEt/X to be the category whose objects are the finite étale maps $\pi : Y \to X$ (sometimes referred to as finite étale coverings of X) and whose arrows are the X-morphisms.

Define $F : FEt/X \to Sets$ to be the functor sending (Y, π) to the set of $\bar{x}$-valued points of Y lying over x, so $F(Y) = \mathrm{Hom}_X(\bar{x}, Y)$. If X is a variety over an algebraically closed field and $\bar{x} \in X(k)$, then $F(Y) = \pi^{-1}(\bar{x})$.
We would like to define the universal covering space of X to be the object representing F , but unfortunately, there is (usually) no such object. For example, let $\mathbb{A}^1$ be the affine line over an algebraically closed field k of characteristic zero. Then the finite étale coverings of $\mathbb{A}^1\backslash\{0\}$ are the maps

$$t \mapsto t^n : \mathbb{A}^1 \to \mathbb{A}^1\backslash\{0\}$$

Among these coverings, there is no "biggest" one in the topological case, with $k = \mathbb{C}$, the universal covering is

$$\mathbb{C} \xrightarrow{\exp} \mathbb{C}\backslash\{0\}.$$

which has no algebraic counterpart.

However, the functor F is pro-representable. This means that there is a projective system $\tilde{X} = (X_i)_{i\in I}$ of finite étale coverings of X indexed by a directed set I such that

$$F(Y) = \mathrm{Hom}(\tilde{X}, Y) \overset{def}{=} \varprojlim_{i\in I} \mathrm{Hom}(X_i, Y) \qquad functorially\ \ in\ \ Y$$

In the example considered in the last paragraph, $\tilde{X}$ is the family $t \mapsto t^n : \mathbb{A}^1\backslash\{0\} \to \mathbb{A}^1\backslash\{0\}$ indexed by the positive integers partially ordered by division.

We call $\tilde{X}$ *"the" universal covering space of* X. It is possible to choose $\tilde{X}$ so that each X_i is Galois over X, i.e., has degree over X equal to the order of $\mathrm{Aut}_X(X_i)$. A map $X_j \to X_i$, $i \leq j$, induces a homomorphism $\mathrm{Aut}_X(X_j) \to \mathrm{Aut}_X(X_i)$, and we define

$$\pi_1(X, \bar{x}) = \mathrm{Aut}_X(\tilde{X}) \overset{def}{=} \varprojlim_{i} \mathrm{Aut}_X(X_i),$$

endowed with its natural topology as a projective limit of finite discrete groups. If $X_n \to \mathbb{A}^1\backslash\{0\}$ denotes the covering in the last paragraph, then

$$\mathrm{Aut}_X(X_n) = \mu_n(k) \ \ (group\ \ of\ \ nth\ \ roots\ \ of\ \ 1\ \ in\ \ k$$

with$\zeta \in \mu_n(k)$ acting by $x \mapsto \zeta x$. Thus

$$\pi_1(\mathbb{A}^1\backslash 0, \bar{x}) = \varprojlim n\mu_n(k) \approx \hat{\mathbb{Z}}.$$

Here $\hat{Z} \simeq \prod_l \mathbb{Z}_l$ is is the completion of $\mathbb{Z}$ for the topology defined by the subgroups of finite index. The isomorphism is defined by choosing a compatible system of isomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mu_n(k)$, or, equivalently, by choosing primitive nth roots ζ_n of 1 for each

n such that $\zeta_{mn}^m = \zeta_n$ for all $m, n > 0$.
The action of $\pi_1(X, x)$ on $\tilde{X}$ (on the right) defines a left action of $\pi_1(X, x)$ on $F(Y)$ for each finite étale covering Y of X. This action is continuous when $F(Y)$ is given the discrete topology this simply means that it factors through a finite quotient $\mathrm{Aut}_X(X_i)$ for some $i \in I$.

Theorem The functor $Y \mapsto F(Y)$ defines an equivalence from the category of finite étale coverings of X to the category of finite discrete $\pi_1(X, \bar{x})$-sets.

Thus $\pi_1(X, \bar{x})$ classifies the finite étale coverings of X in the same way that the topological fundamental group classifies the covering spaces of a topological space.

Remark

1. If $\bar{\bar{x}}$ is a second geometric point of X, then there is an isomorphism $\pi_1(X, \bar{x}) \to \pi_1(\bar{\bar{x}})$, well-defined up to conjugation.
2. The fundamental group $\pi_1(X, \bar{x})$ is a covariant functor of $(X, \bar{x})$.

Important Example

The spectrum of a field For $X = \mathrm{Spec}(k)$, k a field, the étale morphisms $Y \to X$ are the spectra of étale k-algebras A, and each is finite. Thus, rather than working with FEt/X, we work with the opposite category Et/k of étale k-algebras.
The choice of a geometric point for X amounts to the choice of a separably algebraically closed field Ω containing k. Define $F : Et/k \to Sets$ by

$$F(A) = \mathrm{Hom}_k(A, \Omega)$$

Let $\tilde{k} = (k_i)_{i \in I}$ be the projective system system consisting of all finite Galois extensions of k contained in Ω. Then $\tilde{k}$ ind-represents F , i.e.,

$$F(A) \simeq \mathrm{Hom}_k(A, \tilde{k}) \overset{def}{=} \varinjlim_{i \in I} \mathrm{Hom}_k(A, k_i) \qquad functorality \;\; in \;\; A$$

- this is obvious. Define

$$\mathrm{Aut}_k(\tilde{k}) = \varprojlim_{i \in I} \mathrm{Aut}_{k-alg}(k_i).$$

Thus

$$\mathrm{Aut}_k(\tilde{k}) = \varprojlim_{i} \mathrm{Gal}(k_i/k) = \mathrm{Gal}(k^{sep}/k)$$

where k^{sep} is the separable algebraic closure of k in Ω. Moreover, F defines an equivalence of categories from Et/k to the category of finite discrete $\mathrm{Gal}(k^{sep}/k)$-sets. This statement summarizes, and is easily deduced from, the usual Galois theory of fields.

Normal Varieties (or Schemes) For a connected normal variety (or scheme) X, it is most natural to take the geometric point $\bar{x}$ to lie over the generic point x of X. Of course, strictly speaking, we cannot do this if X is a variety because varieties do not have generic points, but what it amounts to is choosing a separably algebraically closed field Ω containing the field $k(X)$ of rational functions on X. We let L be the union of all the finite separable field extensions K of $k(X)$ in Ω such that the normalization of X in K is étale over X; then

$$\pi_1(X, \bar{x}) = \mathrm{Gal}(L/k(X))$$

with the Krull topology.

The Case of Variety Let X be a varity over an algebrically closed field k. Recall that an étale neighbourhood of a point $x \in X$ is an étale map $U \to X$ together with a point $u \in U$ mapping to x. A morphism (or map) of étale neighbourhoods $(V, v) \to (U, u)$ is a regular map $V \to U$ over X sending v to u. It can be shown that there is at most one map from a connected étale neighbourhood to a second étale neighbourhood. The connected affine étale neighbourhoods form a directed set with the definition,

$$(U, u) \leq (U', u') \quad if\ there\ exists\ a\ map\ (U', u') \to (U, u),$$

and we define *the local ring at x for étale topology* to be

$$O_{X,\bar{x}} = \varinjlim_{(U,u)} \Gamma(U, O_U).$$

Since every open Zariski neighbourhood of x is also an étale neighbourhood of x, we get a homomorphism $O_{X,x} \to O_{X,\bar{x}}$. Similarly, we get a homomorphism $O_{U,u} \to O_{X,\bar{x}}$ for any étale neighbourhood (U, u) of x, and clearly

$$O_{X,\bar{x}} = \varinjlim_{(U,u)} O_{U,u}.$$

The transition maps $O_{U,u} \to O_{V,v}$ in the direct system are all flat (hence injective) unramified local homomorphisms of local rings with Krull dimension $\dim X$.

Fact. The ring $O_{X,\bar{x}}$ is a local Noetherian ring with Krull dimension $\dim X$.

Proof. The direct limit of a system of local homomorphisms of local rings is local (with maximal ideal the limit of the maximal ideals); hence $O_{X,\bar{x}}$ is local. The maps on the completions $\hat{O}_{U,u} = \hat{O}_{X,x}$ are all isomorphisms and it follows that $\hat{O}_{X,\bar{x}} = \hat{O}_{X,x}$. An argument of Nagata now shows that $O_{X,\bar{x}}$ is Noetherian, and hence has Krull dimension $\dim X$.

It is quite well-known that every complete local ring is Henselian. Now in the case of variety we have;
Theorem For any point x in X, $O_{X,\bar{x}}$ is Henselian.

The Case of Scheme A *geometric point* of a scheme X is a morphism $\bar{x} : \mathrm{Spec}\,\Omega \to X$ with Ω a separably closed field. An *étale neighbourhood* of such a point $\bar{x}$ is an étale map $U \to X$ together with a geometric point $\bar{u} : \mathrm{Spec}\,\Omega \to U$ lying over $\bar{x}$. The *local* *ring at $\bar{x}$ for the étale topology* is

$$O_{X,\bar{x}} \stackrel{def}{=} \varinjlim_{(U,\bar{u})} \Gamma(U, O_U)$$

where the limit is over the connected affine étale neighbourhoods $(U, \bar{u})$ of $\bar{x}$.
When X is a variety and $x = \bar{x}$ is a closed point of X, this agrees with the previous definition.
Most of the results for varieties over algebraically closed fields extend to schemes. For example, $O_{X,\bar{x}}$ is a strict Henselization of $O_{X,x}$.

In the remainder of these notes, the local ring for the étale topology at a geometric point $\bar{x}$ of a scheme X (or variety) will be called the strictly local ring at $\bar{x}$, $O_{X,\bar{x}}$.

6.5 Étale Cohomology Groups

In applications to algebraic geometry over a finite field $\mathbb{F}_q$, the main objective was to find a replacement for the singular cohomology groups with integer (or rational) coefficients, which are not available in the same way as for geometry of an algebraic variety over the complex number field. Étale cohomology works fine for coefficients $\mathbb{Z}/n\mathbb{Z}$ for n coprime to the characteristic, but gives unsatisfactory results for non-torsion coefficients. To get cohomology groups without torsion from étale cohomology one has to take an inverse limit of étale cohomology groups with certain torsion coefficients; this is called l-adic cohomology. Here "l" stands for any prime number different from p, where p is the characteristic of $\mathbb{F}_q$. One considers, for schemes V, the cohomology groups

$$H^i(V, \mathbb{Z}, l^k\mathbb{Z})$$

and defines the l-adic cohomology group

$$H^i(V, \mathbb{Z}_l) = \varprojlim H^i(V, \mathbb{Z}/l^k\mathbb{Z})$$

as their inverse limit. Here $\mathbb{Z}_l$ denotes the l-adic integers, but the definition is by means of the system of "constant" sheaves with the finite coefficients $\mathbb{Z}/l_k\mathbb{Z}$. (There is a trap here: cohomology does not commute with taking inverse limits, and the l-adic cohomology group, defined as an inverse limit, is not the cohomology with coefficients in the étale sheaf $\mathbb{Z}_l$; In fact, the latter cohomology group exists but gives the "wrong" cohomology groups.)

More generally, if $\mathscr{F}$ is an inverse system of étale sheaves $\mathscr{F}_i$, then the cohomology of $\mathscr{F}$ is defined to be the inverse limit of the cohomology of the sheaves $\mathscr{F}_i$,

$$H^q(X, \mathscr{F}) = \varprojlim H^q(X, \mathscr{F}_i)$$

and though there is a natural map

$$H^q(X, \varprojlim \mathscr{F}_i) \to \varprojlim H^q(X, \mathscr{F}_i)$$

this is "not" usually an isomorphism. An l-adic sheaf is a special sort of inverse system of étale sheaves $\mathscr{F}_i$, where i runs through positive integers, and $\mathscr{F}_i$ is a module over $\mathbb{Z}/l_i\mathbb{Z}$ and the map from $\mathscr{F}_{i+1}$ to $\mathscr{F}_i$ is just reduction mod $\mathbb{Z}/l_i\mathbb{Z}$.

In the case that V is a non-singular algebraic curve and $i = 1$, H^1 is a free $\mathbb{Z}_l$-module of rank $2g$, dual to the Tate module of the Jacobian variety of V, where g is the genus of V. Since the first Betti number of a Riemann surface of genus g is $2g$, this is isomorphic to the usual singular cohomology with $\mathbb{Z}_l$ coefficients for complex algebraic curves. It also shows one reason why the condition $l \neq p$ is required: when $l = p$ the rank of the Tate module is at most g.

To remove any torsion subgroup from the l-adic cohomology groups and get cohomology groups that are vector spaces over fields of characteristic 0 one defines

$$H^i(V, \mathbb{Q}_l) = H^i(V, \mathbb{Z}_l) \otimes \mathbb{Q}_l$$

(though this notation is misleading: $\mathbb{Q}_l$ is neither an étale sheaf nor an l-adic sheaf).

Examples of Étale Cohomology Groups

1. If X is the spectrum of a field K with absolute Galois group G, then étale sheaves over X correspond to continuous sets (or abelian groups) acted on by the (profinite) group G, and étale cohomology of the sheaf is the same as the group cohomology of G, i.e. the Galois cohomology of K.
2. If X is a complex variety, then étale cohomology with finite coefficients is isomorphic to singular cohomology with finite coefficients. (This does not hold for integer coefficients.) More generally the cohomology with coefficients in any constructible sheaf is the same.
3. If $\mathscr{F}$ is a coherent sheaf then the étale cohomology of $\mathscr{F}$ is the same as Serre's coherent sheaf cohomology calculated with the Zariski topology (and if X is a complex variety this is the same as the sheaf cohomology calculated with the usual complex topology).
4. For abelian varieties and curves there is an elementary description of l-adic cohomology. For abelian varieties the first l-adic cohomology group is the dual of the Tate module, and the higher cohomology groups are given by its exterior powers. For curves the first cohomology group is the first cohomology group of its Jacobian. This explains why Weil was able to give a more elementary proof of the Weil conjectures in these two cases: in general one expects to find an elementary proof whenever there is an elementary description of the l-adic cohomology.

6.6 Vanishing Cycle

First of all, let us recall the definition of *monodromy* in the sense of algebraic topology.

Definition of Monodromy 6.6.1. Let X be a connected and locally connected based topological space with base point x, and let $p : \tilde{X} \to X$ be a covering with fiber $F = p^{-1}(x)$. For a loop $\gamma : [0, 1] \to X$ based at x, denote a lift under the covering map (starting at a point $\tilde{x} \in F$) by . Finally, we denote by $\tilde{x}.\gamma$ the endpoint $\tilde{\gamma}(1)$, which is generally different from $\tilde{x}$. There are theorems which state that this construction gives a well-defined group action of the fundamental group $\pi_1(X, x)$ on F, and that the stabilizer of $\tilde{x}$ is exactly , that is, an element $[\gamma]$ fixes a point in F if and only if it is represented by the image of a loop in $\tilde{X}$ based at $\tilde{x}$. This action is called the *monodromy action* and the corresponding homomorphism $\pi_1(X, x) \to \mathrm{Sym}(F)$ into the symmetric group on F is the monodromy. The image of this homomorphism is *the monodromy group*.

Historically, a classical result is the Picard-Lefschetz formula, detailing how the monodromy round the singular fiber acts on the vanishing cycles, by a shear mapping (transvection).

The classical, geometric theory of Lefschetz was recast in purely algebraic terms, in SGA7. This was for the requirements of its application in the context of l-adic cohomology; and eventual application to the Weil conjectures. There the definition uses derived categories, and looks very different. It involves a functor, the nearby cycle functor, with a definition by means of the higher direct image and pullbacks.
The vanishing cycle functor then sits in a distinguished triangle with the nearby cycle functor and a more elementary functor. This formulation has been of continuing influence, in particular in D-module theory.

Here, we are going to explain briefly the theory of vanishing cycle in the sense of complex analysis and the algebraic counterpart is derived accordingly.

We begin with some definitions, Let R be a regular noetherian ring with finite Krull dimension (e.g., $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$) A complex $(\mathbf{A}^\bullet, \mathbf{d}^\bullet)$

$$\dots \mathbf{A}^{-1} \xrightarrow{d^{-1}} \mathbf{A}^0 \xrightarrow{d^0} \mathbf{A}^1 \xrightarrow{d^1} \mathbf{A}^2 \xrightarrow{d^2} \dots$$

of sheaves of R-modules on a complex analytic space (or a scheme), X, is bounded if $\mathbf{A}_p = 0$ for $|p|$ large.

The cohomology sheaves $\mathbf{H}^p(\mathbf{A}^\bullet)$ arise by taking the (sheaf-theoretic) cohomology of the complex.

The stalk of $\mathbf{H}^p(\mathbf{A}^\bullet)$ at a point x is written $\mathbf{H}^p(\mathbf{A}^\bullet)_x$ and is isomorphic to what one gets by first taking stalks and then taking cohomology, i.e., $\mathbf{H}^p(\mathbf{A}^\bullet_x)$.

A single sheaf $\mathbf{A}$ on X is considered a complex, $\mathbf{A}^\bullet$, on X by letting $\mathbf{A}^0 = \mathbf{A}$ and $\mathbf{A}^i = 0$ for $i \neq 0$; thus, $\mathbf{R}^\bullet_X$ denotes the constant sheaf on X.

The shifted complex $\mathbf{A}^\bullet[n]$ is defined by $(\mathbf{A}^\bullet[n])^k = \mathbf{A}^(n+k)$ and differential $d^k_{[n]} = (-1)^n d^{k+n}$.

A map of complexes is a graded collection of sheaf maps $\Phi^\bullet : \mathbf{A}^\bullet \to \mathbf{B}^\bullet$ which commute with the differentials. The shifted sheaf map $\Phi^\bullet_{[n]} : \mathbf{A}^\bullet[n] \to \mathbf{B}^\bullet[n]$ is defined by $\Phi^k_{[n]} := \Phi^{k+n}$ (note the lack of a $(-1)^n$). A map of complexes is a *quasi-isomorphism* provided that the induced maps

$$\mathbf{H}^p(\Phi^\bullet) : \mathbf{H}^p(\mathbf{A}^\bullet) \to \mathbf{H}^p(\mathbf{B}^\bullet)$$

are isomorphisms for all p.

If $\Phi^\bullet : \mathbf{A}^\bullet \to \mathbf{I}^\bullet$ is a quasi-isomorphism and each $\mathbf{I}^p$ is injective, then $\mathbf{I}^\bullet$ is called an injective resolution of $\mathbf{A}^\bullet$. Injective resolutions always exist (in our setting), and are unique up to chain homotopy. However, it is sometimes important to associate one particular resolution to a complex, so it is important that there is a canonical injective resolution which can be associated to any complex (we shall not describe the canonical resolution here).

If $\mathbf{A}^\bullet$ is a complex on X, then *the hypercohomology module*, $\mathbb{H}^p(X, \mathbf{A}^\bullet)$, is defined to be the p-th cohomology of the global section functor applied to the canonical injective resolution of $\mathbf{A}^\bullet$.

Note that if $\mathbf{A}$ is a single sheaf on X and we form $\mathbf{A}^\bullet$, then $\mathbb{H}^p(X, \mathbf{A}^\bullet) = H^p(X, \mathbf{A})$ = ordinary sheaf cohomology. In particular, $\mathbb{H}^p(X, \mathbf{R}^\bullet_X) = H^p(X, R)$.

Note also that if $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are quasi-isomorphic, then $\mathbb{H}^*(X, \mathbf{A}^\bullet) \cong \mathbb{H}^*(X, \mathbf{B}^\bullet)$.

The usual Mayer-Vietoris sequence is valid for hypercohomology; that is, if U and V form an open cover of X, then there is an exact sequence

$$\ldots \mathbb{H}^i(X, \mathbf{A}^\bullet) \to \mathbb{H}^i(U, \mathbf{A}^\bullet) \oplus \mathbb{H}^i(V, \mathbf{A}^\bullet) \to \mathbb{H}^i(U \cap V, \mathbf{A}^\bullet) \to \mathbb{H}^{i+1}(X, \mathbf{A}^\bullet) \to \ldots.$$

Note that hypercohomology is not a homology invariant.

For construction of the theory, we assume our objects are in the category $\mathbf{D}^b_c(X)$;
The complex $\mathbf{A}^\bullet$ is *constructible* with respect to a complex stratification, $S = \{S_\alpha\}$, of X provided that, for all α and i, the cohomology sheaves $\mathbf{H}^i(\mathbf{A}^\bullet{}_{|S_\alpha})$ are locally constant and have finitely generated stalks (the fact which is correspondingly met in our setting on schemes); we write $\mathbf{A}^\bullet \in \mathbf{D}_S(X)$. Moreover if $\mathbf{A}^\bullet \in \mathbf{D}_S(X)$ and $\mathbf{A}^\bullet$ is bounded, we write $\mathbf{A}^\bullet \in \mathbf{D}^b_S(X)$.

Notations;

$\Gamma(X,.)$ (global sections);
$\Gamma_c(X,.)$ (global sections with compact support);
f_* (direct image);
$f_!$ (direct image with proper supports); and
f^* (pull-back or inverse image),
where $f : X \to Y$ is a continuous map. If the functor T is an exact functor from sheaves to sheaves, then $RT(\mathbf{A}^\bullet) \cong T(\mathbf{A}^\bullet)$; in this case, we normally suppress the R. Hence, if $f : X \to Y$, $\mathbf{A}^\bullet \in \mathbf{D}^b(X)$, and $\mathbf{B}^\bullet \in \mathbf{D}^b(Y)$, we write:

$f^*\mathbf{B}^\bullet$;
$f_!\mathbf{A}^\bullet$, if f is the inclusion of a subspace and, hence, $f_!$ is extension by zero;
$f_*\mathbf{A}^\bullet$, if f is the inclusion of a closed subspace.
Note that hypercohomology is just the cohomology of the derived global section functor, i.e., $\mathbb{H}^*(X,.) = H^* \circ R\Gamma(X,.)$. The cohomology of the derived functor of global sections with compact support is the *compactly supported hypercohomology* and is denoted $\mathbb{H}^*_c(X,\mathbf{A}^\bullet)$.

If $f : X \to Y$ is the inclusion of a subset and $\mathbf{B}^\bullet \in D^b(Y)$, then the *restriction* of $B^\bullet$ to X is defined to be $f^*(\mathbf{B}^\bullet)$, and is usually denoted by $\mathbf{B}^\bullet_{|X}$.

If $f : X \to Y$ is continuous and $\mathbf{A}^\bullet \in D^b_c(X)$, there is a canonical map

$$Rf_!\mathbf{A}^\bullet \to Rf_*\mathbf{A}^\bullet.$$

For $f : X \to Y$ continuous, there are canonical isomorphisms

$$R\Gamma(X,\mathbf{A}^\bullet) \cong R\Gamma(Y,Rf_*\mathbf{A}^\bullet) \ \ \textit{and} \ \ R\Gamma_c(X,\mathbf{A}^\bullet) \cong R\Gamma_c(Y,Rf_!\mathbf{A}^\bullet)$$

which lead to canonical isomorphisms

$$\mathbb{H}^*(X,\mathbf{A}^\bullet) \cong \mathbb{H}^*(Y,Rf_*\mathbf{A}^\bullet) \ \ \textit{and} \ \ \mathbb{H}^*_c(X,\mathbf{A}^\bullet) \cong \mathbb{H}^*_c(Y,Rf_!\mathbf{A}^\bullet)$$

for all $\mathbf{A}^\bullet$ in $D^b_c(X)$.

If $f : X \to Y$ is continuous, $\mathbf{A}^\bullet \in D^b_c(X)$, and $\mathbf{B}^\bullet \in D^b_c(Y)$, there are natural maps induced by restriction of sections

$$\mathbf{B}^\bullet \to Rf_*f^*\mathbf{B}^\bullet \ \ \textit{and} \ \ f^*Rf_*\mathbf{A}^\bullet \to \mathbf{A}^\bullet.$$

Fix a complex $\mathbf{B}^\bullet$ on X. There are two covariant functors which we wish to consider: the functor $\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, *)$ from the category of complexes of sheaves to complexes of sheaves and the functor $Hom^\bullet(\mathbf{B}^\bullet, *)$ from the category of complexes of sheaves to the category of complexes of R-modules. These functors are given by

$$(\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet))^n = \prod_{p\in\mathbb{Z}} \mathbf{Hom}(\mathbf{B}^p, \mathbf{A}^{n+p})$$

and

$$(Hom^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet))^n = \prod_{p\in\mathbb{Z}} Hom(\mathbf{B}^p, \mathbf{A}^{n+p})$$

with differential given by

$$[\partial^n f]^p = \partial^{n+p} f^p + (-1)^{n+1} f^{p+1}\partial^p$$

The associated derived functors are $R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, *)$ and $RHom^\bullet(\mathbf{B}^\bullet, *)$, respectively.

If $\mathbf{P}^\bullet \to \mathbf{B}^\bullet$ is a projective resolution of $\mathbf{B}^\bullet$, then, in $D_c^b(X)$, $R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet)$ is isomorphic to $\mathbf{Hom}^\bullet(\mathbf{P}^\bullet, \mathbf{A}^\bullet)$. For all k, $R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet[k]) = R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet)[k]$.

The functor $RHom^\bullet(\mathbf{B}^\bullet, *)$ is naturally isomorphic to the derived global sections functor applied to $R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, *)$, i.e., for any $\mathbf{A}^\bullet \in D_c^b(X)$,

$$RHom^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet) \cong R\Gamma(X, R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{A}^\bullet)).$$

We wish now to describe an analogous adjoint for $Rf_!$

Let $\mathbf{I}^\bullet$ be a complex of injective sheaves on Y. Then, $f^!(\mathbf{I}^\bullet)$ is defined to be the sheaf associated to the presheaf given by

$$\Gamma(U, f^!\mathbf{I}^\bullet) = Hom^\bullet(f_!\mathbf{K}_U^\bullet, \mathbf{I}^\bullet),$$

for any open $U \subseteq X$, where $\mathbf{K}_U^\bullet$ denotes the canonical injective resolution of the constant sheaf $\mathbf{R}_U^\bullet$. For any $\mathbf{A}^\bullet \in D_c^b(X)$, define $f^!\mathbf{A}^\bullet$ to be $f^!\mathbf{I}^\bullet$, where $\mathbf{I}^\bullet$ is the canonical injective resolution of $\mathbf{A}^\bullet$.

Now that we have this definition, we may state:

(Verdier Duality) If $f : X \to Y$, $\mathbf{A}^\bullet \in D_c^b(X)$, and $\mathbf{B}^\bullet \in D_c^b(Y)$, then there is a canonical isomorphism in $D_c^b(Y)$:

$$Rf_* R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, f^!\mathbf{B}^\bullet) \cong R\mathbf{Hom}^\bullet(Rf_!\mathbf{A}^\bullet, \mathbf{B}^\bullet)$$

and so

$$Hom_{D_c^b(X)}(\mathbf{A}^\bullet, f^!\mathbf{B}^\bullet) \cong Hom_{D_c^b(Y)}(Rf_!\mathbf{A}^\bullet, \mathbf{B}^\bullet).$$

If $\mathbf{B}^\bullet$ and $\mathbf{C}^\bullet$ are in $D_c^b(Y)$, then we have an isomorphism

$$f!R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{C}^\bullet) \cong R\mathbf{Hom}^\bullet(f^*\mathbf{B}^\bullet, f^!\mathbf{C}^\bullet).$$

Nearby and Vanishing Cycles

Historically, there has been some confusion surrounding the terminology *nearby (or neighboring) cycles* and *vanishing cycles*; now, however, the terminology seems to have stabilized. In the past, the term "vanishing cycles"was sometimes used to describe what are now called the "nearby cycles"

The point is: one should be very careful when reading works on nearby and vanishing cycles.

Let $S = \{S_\alpha\}$ be a Whitney stratification of X and suppose $\mathbf{F}^\bullet \in D^b_S(X)$. Given an analytic map $f : X \to \mathbb{C}$, define a (stratified) critical point of f (with respect to S) to be a point $x \in S_\alpha \subseteq X$ such that $f_{|S_\alpha}$ has a critical point at x (in our setting, it corresponds to a singular point of the scheme); we denote the set of such critical points by $\Sigma_S f$.

We wish to investigate how the cohomology of the level sets of f with coefficients in $\mathbf{F}^\bullet$ changes at a critical point (which we normally assume lies in $f^{-1}(0)$).

Consider the diagram

$$\begin{array}{ccccc}
 & & E & \rightarrow & \widetilde{\mathbb{C}^*} \\
 & & \downarrow \hat{\pi} & & \downarrow \pi \\
 & & X - f^{-1}(0) & \xrightarrow{\hat{f}} & \mathbb{C}^* \\
 & & \downarrow i & & \\
f^{-1}(0) & \xrightarrow{j} & X & &
\end{array}$$

where:

$j : f^{-1}(0) \to X$ is inclusion;

$i : X - f^{-1}(0) \to X$ is inclusion;

$\hat{f}$ = restriction of f;

$\widetilde{\mathbb{C}^*}$ = cyclic (universal) cover of $\mathbb{C}^*$;

and E denotes the pull-back.

The *nearby (or neighboring) cycles* of $\mathbf{F}^\bullet$ along f are defined to be

$$\psi_f \mathbf{F}^\bullet := j^* R(i \circ \hat{\pi})_*(i \circ \hat{\pi})^* \mathbf{F}^\bullet.$$

Note that this is a sheaf on $f^{-1}(0)$.

As $\psi_f(\mathbf{F}^\bullet[k]) = (\psi_f \mathbf{F}^\bullet)[k]$, we may write $\psi_f \mathbf{F}^\bullet[k]$ unambiguously. In fact, it is frequently useful to consider the functor where one first shifts the complex by k and then takes the nearby cycles; thus, we introduce the notation $\psi_f[k]$ to be the functor such that $\psi_f[k]\mathbf{F}^\bullet = \psi_f \mathbf{F}^\bullet[k]$ (and which has the corresponding action on morphisms). The functor ψ_f takes distinguished triangles to distinguished triangles.

If $\mathbf{P}^\bullet$ is a perverse sheaf on X, then $\psi_f[-1]\mathbf{P}^\bullet$ is perverse on $f^{-1}(0)$. (Actually, to conclude that $\psi_f[-1]\mathbf{P}^\bullet$ is perverse, we only need to assume that $\mathbf{P}^\bullet_{|X - f^{-1}(0)}$ is perverse.)

Because $\psi_f[-1]$ takes perverse sheaves to perverse sheaves, it is useful to include the shift by -1 in many statements about ψ_f. Consequently, we also want to shift $j^*\mathbf{F}^\bullet$ by -1 in many statements, and so we write $j^*[-1]$ for the functor which first shifts by -1 and then pulls-back by j.

As there is a canonical map $\mathbf{F}^\bullet \to Rg_* g^* \mathbf{F}^\bullet$ for any map $g : Z \to X$, there is a map

$$\mathbf{F}^\bullet \to R(i \circ \hat{\pi})_*(i \circ \hat{\pi})^* \mathbf{F}^\bullet$$

and, hence, a canonical map, called the *comparison map*:

$$j^*[-1]\mathbf{F}^\bullet \xrightarrow{c} j^*[-1]R(i \circ \hat{\pi})_*(i \circ \hat{\pi})^* \mathbf{F}^\bullet = \psi_f[-1]\mathbf{F}^\bullet$$

If you wish to look at the analytic root of the theory, note that for $x \in f^{-1}(0)$, the stalk cohomology of $\psi_f \mathbf{F}^\bullet$ at x is the cohomology of the "Milnor fibre" of f at x with coefficients in $\mathbf{F}^\bullet$, i.e., for all $\epsilon > 0$ small and all $\xi \in \mathbb{C}^*$ with $|\xi| << \epsilon$,

$$\mathbf{H}^i(\psi_f \mathbf{F}^\bullet)_x \cong \mathbb{H}^i(\overset{o}{B}_\epsilon(x) \cap X \cap f^{-1}(\xi), \mathbf{F}^\bullet),$$

where the open ball $\overset{o}{B}_\epsilon(x)$ is taken inside any local embedding of (X, x) in affine space. The sheaf $\psi_f \mathbf{F}^\bullet$ only depends on f and $\mathbf{F}^\bullet_{|X - f^{-1}(0)}$.

While the above definition of the nearby cycles treats all angular directions equally, it is perhaps more illuminating to fix an angle θ and describe the nearby cycles in terms of moving out slightly along the ray $e^{i\theta}[0, \infty)$. Consider the three inclusions

$$k_\theta : f^{-1}(e^{i\theta}(0,\infty)) \to f^{-1}(e^{i\theta}[0,\infty));$$
$$m_\theta : f^{-1}(0) \to f^{-1}(e^{i\theta}[0,\infty));$$
$$l_\theta : f^{-1}(e^{i\theta}[0,\infty)) \to X.$$

Then, one can define *the nearby cycles at angle* θ to be $\psi_f^\theta \mathbf{F}^\bullet := m_\theta^* R k_{\theta *} k_\theta^* l_\theta^* \mathbf{F}^\bullet$.

For each θ there is a canonical isomorphism $\psi_f \mathbf{F}^\bullet \cong \psi_f^\theta \mathbf{F}^\bullet$. By letting θ travel around a full circle, we obtain isomorphisms $\psi_f^\theta \mathbf{F}^\bullet \cong \psi_f^{\theta+2\pi} \mathbf{F}^\bullet$. These isomorphisms correspond to the *monodromy* automorphism $T_f : \psi_f[-1]\mathbf{F}^\bullet \to \psi_f[-1]\mathbf{F}^\bullet$, which comes from the deck transformation obtained in our definition of $\psi_f \mathbf{F}^\bullet$. (and, hence, $\psi_f[-1]\mathbf{F}^\bullet$) by traveling once around the origin in $\mathbb{C}$. Actually, T_f is a natural automorphism from the functor $\psi_f[-1]$ to itself; thus, strictly speaking, when we write $T_f : \psi_f[-1]\mathbf{F}^\bullet \to \psi_f[-1]\mathbf{F}^\bullet$, we should include $\mathbf{F}^\bullet$ in the notation for T_f.

There is a natural distinguished triangle

$$\begin{array}{ccc} j^*[-1]Ri_*i^*\mathbf{F}^\bullet & \to & \psi_f[-1]\mathbf{F}^\bullet \\ [1] \nwarrow & & \swarrow\ T_f - id \\ & \psi_f[-1]\mathbf{F}^\bullet & \end{array}$$

Since we have a map $c[1] : j^*\mathbf{F}^\bullet \to \psi_f \mathbf{F}^\bullet$, the third vertex of a distinguished triangle is defined up to quasi-isomorphism. We define *the sheaf of vanishing cycles*, $\phi_f \mathbf{F}^\bullet$, of $\mathbf{F}^\bullet$ along f to be this third vertex, i.e., there is a distinguished triangle

$$\begin{array}{ccc} j^*\mathbf{F}^\bullet & \to & \psi_f \mathbf{F}^\bullet \\ [1] \nwarrow & & \swarrow \\ & \phi_f \mathbf{F}^\bullet & \end{array}$$

Letting $\phi_f[-1]$ denote the functor which first shifts by -1 and then applies ϕ_f, we can write the triangle above as

$$\begin{array}{ccc} j^*[-1]\mathbf{F}^\bullet & \xrightarrow{c} & \psi_f[-1]\mathbf{F}^\bullet \\ [1] \nwarrow & & \swarrow \\ & \phi_f[-1]\mathbf{F}^\bullet & \end{array}$$

Note that this is a triangle of sheaves on $f^{-1}(0)$. Note also that, by replacing $\mathbf{F}^\bullet$ with $i_! i^! \mathbf{F}^\bullet$, we conclude that there is a natural isomorphism $\psi_f[-1]\mathbf{F}^\bullet \cong \phi_f[-1](i_! i^! \mathbf{F}^\bullet)$. There is another natural isomorphism $\psi_f[-1]\mathbf{F}^\bullet \cong \phi_f[-1](Ri_* i^* \mathbf{F}^\bullet)$.

By now, we have constructed the nearby cycles and the sheaf of vanishing cycles for a complex $\mathbf{F}^\bullet$. Among the constant sheaves, which can be thought as a complex, the skyscraper sheaves are of the most importance in our discussion. Here we recall that, in general, a sheaf $\mathscr{F}$ is said to be *skyscraper sheaf* if $\mathscr{F}_{\bar{x}} = 0$ except for a finite number of x. (Recall that $\bar{x}$ denotes a geometric point of X with image $x \in X$). We shall need some special skyscraper sheaves. Let X be a topological space (or scheme), and let $x \in X$. Let Λ be an abelian group. Define $\Lambda^x(U) = \Lambda$ if $x \in U$ and 0 otherwise. Then Λ^x is a sheaf on X. Obviously the stalk of Λ^x at $y \neq x$ is 0, and at x it is Λ.

In the case of having a scheme, we choose x in X to be its generic point.

Chapter 7

Geometry of curves over a discrete valuation ring

As we already pointed out, the stable reduction theorem gives an equivalence geometrical condition for the action of the inertia group to be unipotent in the global situation, where "global" means that the curve in consideration is proper.

Here, we generalize the stable reduction theorem in two directions.

1. To the local situation with the isolated singularity.
2. To consider the geometrical condition for the action of the wild ramification group to be trivial

In this way, we give a purely cohomological proof of the stable reduction theorem. Throughout this chapter, we treat a strictly local discrete valuation ring (a ring satisfying the Hensel's lemma and also has a separably closed residue field) with algebrically closed residue field, however the results are true in the case we have a discrete valuation ring with perfect residue field since they are étale local. (Recall that a D.V.R R is called Henselian if it satisfies Hensel's lemma; that is, R is Henselian if for any monic polynomial $f(x) \in R[x]$ and any element $a \in R$ satisfying

$$f(a) \equiv 0 \pmod{p} \quad and \quad f'(a) \not\equiv 0 \pmod{p}$$

there exists a unique element $\alpha \in R$ satisfying

$$\alpha \equiv a \pmod{p} \quad and \quad f(\alpha) = 0.$$

First of all, let's set up our notations;

- S : the spectrum of a strictly local discrete valuation ring with algebrically closed residue field of ch $= p \geq 0$.

- s : the closed point S.
- η (resp. $\bar{\eta}$): the (resp. geometric) generic point of S.
- I : the inertia group of S.
- P : the wild ramification group of S.
- $\Lambda := \mathbb{Q}_l$, where l is a prime number different from p.
- S-curve or curve over S: flat and separated S-scheme of finite type purely of relative dimension 1.
- n.c.d: a normal crossing divisor which is a closed subscheme in a regular scheme defined étale locally by an ideal $(\prod_i f_i)$ where $(f_i)_i$ forms a part of a regular system of parameters.
- $R\psi\Lambda$: The sheaf of vanishing cycles.
- $\tilde{X}_{\bar{x}}$: the strict localization of X at a geometric point $\bar{x}$ of X.
- X_i : the set of all closed integral subschemes of X of dim i.

All we want to do can be summarized into four main theorems. Two of them are involved with the inertia group I and the rest of them are involved with the wild ramification group P. (the P-versions)

Now let's take a look at the main theorems;

The Main Theorem 1 (Deligne - Mumford, Grothendieck) Suppose X_η is a proper smooth geometrically connected curve over η of genus ≥ 2, and X is its minimal regular model. Then the following conditions are equivalent.

1. The action of I on $H^1(X_{\bar{\eta}}, \Lambda)$ is unipotent.
2. X_s is a normal crossing divisor in X.

(Deligne - Mumford proved the theorem above using the result of Raynaud on Picard schemes.)

The Main Theorem 2 (The local analogue of the stable reduction theorem) Suppose X is a normal S-curve, x is a closed point of X_s such that $X - \{x\}$ is smooth over S (i.e., x is an isolated singularity of $X \to S$) and Y is a minimal regular model of X. Then the following conditions are equivalent.

1. The action of I on $R^1\psi\Lambda_x$ is unipotent.
2. Y_s is a normal crossing divisor in Y.

For the P-versions of the theorems above, we will need the notion of relatively minimal regular n.c.d. model of X_η.

P – version of the main theorem 1 Suppose X_η is a proper smooth geometrically connected curve over η of genus $\neq 1$, and X is a relative minimal regular n.c.d. model of X_η. Then the following conditions are equivalent.

1. The action of P on $H^1(X_{\bar{\eta}}, \Lambda)$ is trivial.
2. Every irreducible component C of X_s whose multiplicity in X_s is divisible by p satisfies the following condition (*).

(*) C is isomorphic to $\mathbb{P}^1_s$ and intersects with other components of X_s at exactly two points and these components have prime-to-P multiplicities in X_s.

P – version of the main theorem 2 Suppose X is a normal S-curve, x is a closed point of X_s such that $X - \{x\}$ is smooth over S (i.e., x is an isolated singularity of $X \to S$) and Y is a minimal regular n.c.d. model of X. Then the following conditions are equivalent.

1. The action of P on $R^1\psi\Lambda_x$ is trivial.
2. Every irreducible component C of Y_s whose multiplicity in Y_s is divisible by p satisfies the following condition (*).

(*) C is isomorphic to $\mathbb{P}^1_s$ and intersects with other components of Y_s at exactly two points and those components have prime-to-p multiplicities in Y_s.

Let's talk about the sketch of proof. For Λ-vector space V of finite dimension with a continuous and quasi-unipotent action I, we put

$\dim_s(V) := \dim_\Lambda((\text{semi-simplification of } V)^I)$.

$\dim_t(V) := \dim_\Lambda(V^P)$.

Of course, we have $\dim_s(V) \leq \dim_t(V) \leq \dim_\Lambda(V)$.

It is obvious that the action of I (resp. P) on V is unipotent (resp. trivial) if and only if $\dim_s(V) = \dim_\Lambda(V)$ (resp. $\dim_t(V) = \dim_\Lambda(V)$). We define the following quantities as well;

$$h^1(X_{\bar{\eta}}) := \dim_\Lambda H^1(X_{\bar{\eta}}, \Lambda), \quad h^1_t(X_{\bar{\eta}}) := \dim_t H^1(X_{\bar{\eta}}, \Lambda),$$
$$h^1_s(X_{\bar{\eta}}) := \dim_s H^1(X_{\bar{\eta}}, \Lambda).$$

(resp. $r^1_x := \dim_\Lambda R^1\psi\Lambda_x$, $r^1_{t,x} := \dim_t R^1\psi\Lambda_x$, and $r^1_{s,x} := \dim_s R^1\psi\Lambda_x$)

The key point in reaching to a proof of our main theorems is to represent the numbers $h^1_t(X_{\bar{\eta}}),\ h^1_s(X_{\bar{\eta}}),\ r^1_{t,x}$ and $r^1_{s,x}$ using intersection theory. In fact, we use some linear algebra over $\mathbb{Z}$ to make a correspondence between the geometrical properties stated in our main theorems above and the equalities between these numbers.

Note that $\dim_t$ and $\dim_s$ are also defined for an object of $D^b_c(\eta, \Lambda)$ such that the action of I on each cohomology group is quasi-unipotent.

7.1 sheaves $R^1\psi_t\Lambda$.

As we said, in this section we are going to represent numbers $h^1_t(X_{\bar{\eta}}),\ h^1_s(X_{\bar{\eta}}),\ r^1_{t,x}$ and $r^1_{s,x}$ by intersection theory. We recall Riemann - Roch theorem formula which is $\chi(O_E) = -(E, E + K)/2$.

Keep these nitations in this section;

- Y : a regular S-curve such that $Y_{s,red}$ is a normal cossing divisor (an n.c.d.) in Y.
- K : the relative canonical divisor of Y over S.

For $C \in (Y_s)_1$.

- r_C : the multiplicity of C in Y_s,
- m_C : the prime-to-p part of r_C (resp. $m_c := r_c$) if $p \neq 0$ (resp. $p = 0$.)

Theorem 7.1.1. Suppose Z is a reduced subscheme of Y_s which is a proper s-curve. Let D_0 be the effective divisor $Y_{s,red}$ and put

$$E := \Sigma r_C C, \qquad E_1 := \Sigma m_C C, \qquad E_0 := \Sigma C$$

where C runs over Z_1,

$$\chi_t(Z, R\psi\Lambda) := \dim_t R\Gamma(Z, R\psi\Lambda) \ \ \textit{and}$$
$$\chi_s(Z, R\psi\Lambda) := \dim_s R\Gamma(Z, R\psi\Lambda).$$

(we know that the action of I on $R\Gamma(Z, R\psi\Lambda)$ is quasi-unipotent.) Then we have,

$$\chi_t(Z, R\psi\Lambda) = -(E_1, K + D_0),$$
$$\chi_s(Z, R\psi\Lambda) = -(E_0, K + D_0).$$

Furthermore, if we assume that $Y - Z$ is smooth over S, then

$$\chi_t(Z, R\psi\Lambda) = -(E_1, K + E_0 - E) \ \ \textit{and}$$

$$\chi_s(Z, R\psi\Lambda) = -(E_0, K + E_0 - E).$$

The following corollary is crucial for our aim. In the corollary, we assume that an S-curve X satisfies either of the following conditions.

- Global X is proper over S and regular, $X_{s,red}$ is an n.c.d. in X and X_η is geometrically connected over η.
- Local There exists a closed point x in X_s such that $X - \{x\}$ and $X_s - \{x\}$ are regular and X is normal.

Defiition 7.1.2. If X satisfies the local condition above, *a regular n.c.d. model of* X is a regular S-curve Y with a proper birational S-morphism $Y \to X$ which is an isomorphism over $X - \{x\}$ and whose reduced special fiber $Y_{s,red}$ is an n.c.d. in Y. If X satisfies the global (resp. local) condition, we put

$$h_t^1(X_{\bar{\eta}}) := \dim_t H^1(X_{\bar{\eta}}, \Lambda) \quad \textit{and} \quad h_s^1(X_{\bar{\eta}}) := \dim_s H^1(X_{\bar{\eta}}, \Lambda)$$

(resp. $r_{t,x}^1 := \dim_t R^1\psi\Lambda_x$, and $r_{s,x}^1 := \dim_s R^1\psi\Lambda_x$).

Now the corollary,

Corollary 7.1.3. Suppose that X satisfies the global condition and put $Y = X$ and $Z = X_{s,red}$. Then we have

$$h_t^1(X_{\bar{\eta}}) = 2 + (E_1, E_0 + K) \quad \textit{and} \quad h_s^1(X_{\bar{\eta}}) = 2 + (E_0, E_0 + K).$$

Now, suppose that X satisfies the local condition and that Y is a regular n.c.d. model of X and Z is $Y_{s,red}$. Then,

$$r_{t,x}^1 = 1 + (E_1, K + E_0 - E) \quad \textit{and} \quad r_{s,x}^1 = 1 + (E_0, K + E_0 - E)$$

if $Y \neq X$.
$r_{t,x}^1 = r_{s,x}^1 = 1$ if $Y = X$ and X is not smooth over S at x.
$r_{t,x}^1 = r_{s,x}^1 = o$ if $Y = X$ and X is smooth over S at x.

Proof. By using the theorem (7.1.1) and Theorem 3.3 of [D-I], it suffices to remark that if the assumption of the first part of this theorem (resp. the assumption of the second part) is satisfied, then $H^0(X_{\bar{\eta}}, \Lambda) = \Lambda$ and $H^2(X_{\bar{\eta}}, \Lambda) = \Lambda(-1)$ (resp. $R^0\psi\Lambda_x = \Lambda$). Now the deduction is clear since if it is necessary we can replace X by the connected component of x in X and we may suppose X_η is geometrically integral over η. □

Now, it is time for explaining the action of the inertia goup I in the case that the special fiber is reduced. Assume that X is a regular S-curve such that the special fiber X_s is an n.c.d. in X. If x is a closed point of X_s, B_x denotes the set of the branches of X_s at x (i.e., $(\tilde{X_s})_x)_1$) and $\Lambda(x)$ denotes the cokernel of the diagonal morphoism $\Lambda \to \Lambda^{B_x}$.

We mention the following theorem (Formule de Picard-Lefschetz of Deligne) that you can find it in [D-XV] Sections (3.3) and (3.4).

Theorem (Deligne) 7.1.4. Let X be as above. Then

$$R^0\psi\Lambda = \Lambda \quad and \quad R^1\psi\Lambda_{|X_{s,red}} = 0.$$

Assume $x \in X_{s,sing}$. Then,

$$R^1\psi\Lambda_x = \Lambda(x)(-1),$$

$$H^i_x(X_s, R\psi\Lambda) \simeq \begin{cases} \Lambda(x) & (i = 1) \\ \Lambda(-1) & (i = 2) \\ 0 & (otherwise), \end{cases}$$

where the isomorphism $H^1_x(X_s, R\psi\Lambda) \simeq \Lambda(x)$ is such that the following diagram is commutative,

$$\begin{array}{ccc} H^0((\tilde{X}_s)_x - \{x\}, R\psi\Lambda) & \to & H^1_x(X_s, R\psi\Lambda) \\ \downarrow & & \downarrow \\ \Lambda^{B_x} & \to & \Lambda(x) \end{array}$$

where the vertical arrows are isomorphisms and all morphisms are canonical.

Poincaré duality The cup product

$$R^1\psi\Lambda_x \times H^1_x(X_s, R\psi\Lambda) \to H^2_x(X_s, R\psi\Lambda) \simeq \Lambda(-1)$$

The cup product gives a dual of the above diagram, the following diagram is also commutative,

$$\begin{array}{ccc} R^1\psi\Lambda_x & \to & H^2_x(X_s, \Lambda) \\ \downarrow & & \downarrow \\ \Lambda(x)(-1) & \to & \Lambda^{B_x}(-1) \end{array}$$

where again the vertical arrows are isomorphisms and all the morphisms are the dual of those canonical ones.

We have the variation morphism

$$\begin{array}{ccc} Var(\sigma)_x : R^1\psi\Lambda_x & \to & H^1_x(X_s, R\psi\Lambda) \\ \downarrow & & \downarrow \\ \Lambda(x)(-1) & \to & \Lambda(x) \end{array}$$

(the vertical arrows are isomorphisms)

The variation morphism is given by multiplication $-t_l(\sigma)$ where t_l is the canonical surjection, $t_l : I \to \mathbb{Z}_l(1)$.

The composite morphism

$$R^1\psi\Lambda_x \xrightarrow{Var(\sigma)} H^1_x(X_s, R\psi\Lambda) \to H^1(X_s, R\psi\Lambda) \to R^1\psi\Lambda_x$$

is equal to 0.

In the following theorem we use the notation of the theorem (7.1.1) and also assume that the special fiber Y_s is reduced.

Theorem 7.1.5. Let N be the logarithm of monodromy in $H^1(Z, R\psi\Lambda)$ and $W := Z_{sing}$. Then

$$\mathrm{Im}(N : H^1(Z, R\psi\Lambda)(1) \to H^1(Z, R\psi\Lambda)) = \mathrm{Im}(\underset{y\in W}{\oplus} H^1_y(Z, R\psi\Lambda) \to H^1(Z, R\psi\Lambda)),$$

and rank $N = \mathrm{Card}\ W -\ \mathrm{Card}\ Z_1 + \dim_\Lambda H^0(Z, \Lambda)$.

Corollary 7.1.6. Suppose X, Y and Z satisfy the condition of Corollary (7.1.3 (the first part)) and Y_s is reduced. Let N be the logarithm of monodromy on $H^1(X_{\bar{\eta}}, \Lambda)$. Then we have

$$\text{rank } N = \mathrm{Card}(X_s)_{sing} - \mathrm{Card}(X_s)_1 + 1.$$

Corollary 7.1.7. Suppose X, Y and Z satisfy the condition of Corollary (7.1 (the second part)) and Y_s is reduced. Let N be the logarithm of monodromy on $R^l\psi\Lambda_x$. Then we have

$$\text{rank } N = \mathrm{Card}(Y_x)_{sing} - \mathrm{Card}(Y_x)_1 + 1 \quad if \quad Y \neq X$$

$$\text{rank } N = 0 \quad if \quad Y = X$$

Proof. By knowing the fact that $H^0(X_{\bar{\eta}}, \Lambda) = \Lambda$ and that $R^0\psi\Lambda_x = \Lambda$ the results deduce from the theorem directly. □

7.2 Lemmas

The aim of this section is to prove some lemmas which make a link between the geometric properties stated in the main theorems and the quantities defined in the previous section as dimension of some special vector spaces.

Definition of forms 7.2.1. A form is a free $\mathbb{Z}$-module M of finite rank with the following objects on M, which satisfies the properties 1 to 3 below.

- a basis B
- a symmetric bilinear form (,).
- a linear form (, K).
- an element $E = \Sigma_{C\in B} r_C C$, $r_C \geq 1$ for all $C \in B$.

(Property 1) $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ is negative and satisfies either of the following conditions;

- (1-1) The kernel is generated by E.

- (1-2) $M_{\mathbb{Q}}$ is nondegenerate and $(C, -E) \geq 0$ for all $C \in B$.

(If M satisfies (1-1) (resp. (1-2)), M is called *global.* (resp. *local*))

(Property 2) $(C, C') \geq 0$, for all $C, C' \in B$, $C \neq C'$, and B is connected i.e., there is no nontrivial partition of the basis B, $B = B_1 \sqcup B_2$, such that $C, C') = 0$ for all $C \in B_1$ $C' \in B_2$.

(Property 3) $(C, C + K)/2 + 1 \in \mathbb{N}$ for all $C \in B$

We put $E_0 := \Sigma_{C \in B} C$, and $F := (\quad , K - E + E_0)$.
We define the conditions (**RM**) and (**RMN**) on forms as follows.

- (**RM**) There exists no $C \in B$ such that $(C, C) = (C, K) = -1$.
- (**RMN**) There exists no $C \in B$ such that $F(C) \leq 0$ and $-(C, C) = 1$.

Since $F(C) \leq 0$ and $(C, C) = -1$ imply $(C, C + K) = -2$, the condition (RM) implies (RMN).

Definition of exceptional forms 7.2.2. A form is exceptional if it is isomorphic to one of the following types, where always $r \in \mathbb{N}, \ \geq 1$.

- $({}_{\mathbf{r}}\mathbf{I_0})$

  rank $M = 1$, $B = \{C\}$, $(C, C) = 0$, $(C, K) = 0$ and $E = rC$.

- $({}_{\mathbf{r}}\mathbf{I_k})$

  $(k \in \mathbb{N}, \ \geq 2)$, rank $M = k$, $B = \{C_i; \ i \in \mathbb{Z}/k\mathbb{Z}\}$,

  If $k = 2$ (,) is

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

  and

$$\begin{pmatrix} -2 & 1 & & & & & & 1 \\ 1 & -2 & 1 & & & & & \\ & 1 & : & \vdots & & & & \\ & & & & \vdots & \vdots & : & 1 \\ & & & & & 1 & -2 & 1 \\ 1 & & & & & & 1 & -2 \end{pmatrix}$$

  otherwise,

  $(C, K) = 0$ for all $C \in B$ and $E = rE_0$.

- **($_r$II)**

  rank $M = 4$, $B = \{C_{0,1}, C_{0,2}, C_{0,3}, C\}$,

  (,) is

$$\begin{pmatrix} -6 & 0 & 0 & 1 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

  (, K) is

$$\begin{pmatrix} 4 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

  , and $E = r(C_{0,1} + 2C_{0,2}, 3C_{0,3}, 6C)$.

- **($_r$III)**

  rank $M = 4$, $B = \{C_{0,1}, C_{0,2}, C_{0,3}, C\}$, (,) is

$$\begin{pmatrix} -4 & 0 & 0 & 1 \\ 0 & -4 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

  (, K) is

$$\begin{pmatrix} 2 \\ 2 \\ 0 \\ -1 \end{pmatrix}$$

  $E = r(C_{0,1} + C_{0,2} + 2C_{0,3} + 4C)$

- **($_r$IV)**

  rank $M = 4$, $B = \{C_{0,1}, C_{0,2}, C_{0,3}, C\}$,

  (,) is

$$\begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

(, K) is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

, and $E = r(C_{0,1} + C_{0,2} + C_{0,3} + 3C)$

- $({}_{\mathbf{r}}\mathbf{I}^*_{\mathbf{k}})$ $(k \in \mathbb{N})$,

  rank $M = k + 5$, $B = \{C_{0,1}, C_{0,2}, C_0, C_1, \ldots, C_k, C_{k,3}, C_{k,4}\}$,

  (,) is

$$\begin{pmatrix}
-2 & 0 & 1 & & & & & & \\
0 & -2 & 1 & & & & & & \\
1 & 1 & -2 & 1 & & & & & \\
 & & 1 & -2 & . & & & & \\
 & & & . & . & . & & & \\
 & & & & . & . & 1 & & \\
 & & & & & 1 & -2 & 1 & 1 \\
 & & & & & & 1 & -2 & 0 \\
 & & & & & & 1 & 0 & -2
\end{pmatrix}$$

  ,

  (, K) = 0, and $E = r(C_{0,1} + C_{0,2} + \sum_{i=0}^{k} 2C_i + C_{k,3} + C_{k,4})$.

- $({}_{\mathbf{r}}\mathbf{II}^*)$

  $B = \{C_{0,1}, C_{1,1}, C_{2,1}, C_{3,1}, C_{4,1}, C_{0,2}, C_{1,2}, C_{0,3}, C\}$

  (,) is

$$\begin{pmatrix}
-2 & 1 & & & & & & & \\
1 & -2 & 1 & & & & & & \\
 & 1 & -2 & 1 & & & & & \\
 & & 1 & -2 & 1 & & & & \\
 & & & 1 & -2 & & & & 1 \\
 & & & & & -2 & 1 & & \\
 & & & & & 1 & -2 & & 1 \\
 & & & & & & & -2 & 1 \\
 & & & & 1 & & 1 & 1 & -2
\end{pmatrix}$$

,

$(\ , K) = 0$, and $E = r(C_{0,1}+2C_{1,1}+3C_{2,1}+4C_{3,1}+5C_{4,1}+2C_{0,2}+4C_{1,2}+3C_{0,3}+6C)$.

- ($_{\mathbf{r}}$**III***)

  rank $M = 8$, $B = \{C_{0,1}, C_{1,1}, C_{2,1}, C_{0,2}, C_{1,2}, C_{2,2}, C_{0,3}, C\}$

  (,) is

$$\begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & & & & & 1 \\ & & & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & \\ & & & & 1 & -2 & & 1 \\ & & & & & & -2 & 1 \\ & & 1 & & & 1 & 1 & -2 \end{pmatrix}$$

  ,

  $(\ , K) = 0$ and $E = r(C_{0,1} + 2C_{1,1} + 3C_{2,1} + C_{0,2} + 2C_{1,2} + 3C_{2,2} + 2C_{0,2} + 4C)$.

- ($_{\mathbf{r}}$**IV***)

  rank $M = 7$, $B = \{C_{0,1}, C_{1,1}, C_{0,2}, C_{1,2}, C_{0,3}, C_{1,3}, C\}$,

  (,) is

$$\begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & & & & & 1 \\ & & -2 & 1 & & & \\ & & 1 & -2 & & & 1 \\ & & & & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & 1 & & 1 & & 1 & -2 \end{pmatrix}$$

  , $(\ , K) = 0$ and $E = r(C_{0,1} + 2C_{1,1} + C_{0,2} + 2C_{1,2} + C_{0,3} + 2C_{1,3} + 3C)$.

- $_{\mathbf{r}}$**O**

  rank $M = 1$, $B = \{C\}$, $(C, C) = 0$, $(C, K) = -2$ and $E = rC$.

  (Here is the end of definition of exceptional forms.)

Note that a form is exceptional of type I or I* - IV* (resp. I - IV or I* - IV*) if and only if it is global and satisfies (RM) (resp. (RMN)) and $F(E) = 0$.

We again emphasize that p denotes a prime number and, for $r \in \mathbb{N}$ and ≥ 1, m denotes the prime-to-p part of r and $E_l := \Sigma_{C \in B} m_c C$.

Lemma (G-I) 7.2.3. If M is a global form satisfying (RMN), the following conditions are equivalent.

- $F(E) = F(E_0)$.
- $E = E_0$, or M is of type I.

Lemma (L-I) 7.2.4. If M is a local form satisfying (RMN), the following conditions are equivalent.

- $F(E) = F(E_0)$.
- $E = E_0$.

Lemma (G-P) 7.2.5. If M is a global form satisfying (RMN), the following conditions are equivalent.

- $F(E) = F(E_1)$.
- M satisfies either of the following conditions (i) and (ii).

(i) Every $C \in B$ such that $p|r_c$ satisfies the following property (*).

(*) $F(C) = 0$, $(C, C + K) = -2$ and, for all C' such that $(C', C) \neq 0$ and $C \neq C'$, $(p, r_{c'}) = 1$,

(ii) M is of one of the following exceptional types.

- type I - IV or I* - IV*, (if $p \neq 2, 3$).
- type I, III, I* or III*, (if $p = 3$).
- type I, IV or IV*. (if $p = 2$).

Lemma (L-P) 7.2.6. If M is a local form satisfying (RMN), the following conditions are equivalent.

- $F(E) = F(E_1)$.
- Every $C \in B$ such that $p|r_c$ satisfies the following property (*).

(*) $F(C) = 0$, $(C, C + K) = -2$ and, for all C' such that $(C', C) \neq 0$ and $C \neq C'$, $(p, r_{c'}) = 1$.

Theorem 7.2.7. Assume that M satisfies the condition (RMN). If there is a $C \in B$ such that $F(C) - ch(C) = -2$, the following conditions are equivalent.

- $F(E) = F(E_1)$ (resp. $F(E) = F(E_0)$)
- M is one of the following types,

– type ${}_rO$ such that $(p, r) = 1$,
– type ${}_rII$ - ${}_rIV$ or ${}_rI^*$ - ${}_rIV^*$ such that $p|r$ ($p \neq 2, 3$),
– type ${}_rIII$, ${}_rI^*$ or ${}_rIII^*$ such that $p|r$ ($p = 3$),
– type ${}_rIV$ or ${}_rIV^*$ such that $p|r$ ($p = 2$).

(resp. type ${}_10$.)

Theorem 7.2.8. Here we assume that $p \neq 2$. Let M (as always) be a form. Then the following conditions are equivalent pairwise.

- M is of type ${}_rI_k^*$ such that $p|r$ and $k \geq 2$.
- There exists a $C \in B$ such that $F(C) = 0$ and $ch_A(C) = 2$.
- M is of type ${}_rI_k^*$ such that $p|r$ and $k \geq 1$.
- There exists a $C \in B$ such that $F(C) = 1$, $ch_A(C) = 1$, $ch_\alpha(C) = 2$ and $r_C = 2r_{0,\lambda}$ for all $\lambda \in ch_\alpha(C)$.
- M is of type ${}_rI_0^*$ such that $p|r$.
- There exists a $C \in B$ such that $F(C) = 2$, $ch_A(C) = 0$, $ch_\alpha(C) = 4$ and $r_C = 2r_{0,\lambda}$ for all $\lambda \in ch_\alpha(C)$.
- M is of type ${}_rII$ (resp. ${}_rIII$, ${}_rIV$) such that $p|r$.
- There exists a $C \in B$ such that $F(C) = 1$, $ch_A(C) = 0$, $ch_\alpha(C) = 3$, $r_C = r_{0,\lambda} + r_{n_\lambda,\lambda}$ for all $\lambda \in ch_\alpha(C)$ and $(r_C/r_{0,\lambda}) = (2, 3, 6)$ (resp. $(2, 4, 4)$, $(3, 3, 3)$) up to order.
- M is of type ${}_rII^*$ (resp. ${}_rIII^*$, ${}_rIV^*$) such that $p|r$.
- There exists a $C \in B$ such that $F(C) = 1$, $ch_A(C) = 0$, $ch_\alpha(C) = 3$, $r_{0,\lambda} = r_{n_\lambda,\lambda}$ for all $\lambda \in ch_\alpha(C)$ and $(r_C/r_{0,\lambda}) = (2, 3, 6)$ (resp. $(2, 4, 4)$, $(3, 3, 3)$) up to order.

And in this case, M satisfies $F(E) = F(E_1)$.

7.3 Global Case

Definition 7.3.1. Assume that X_η is a regular proper η-curve. A proper S-curve Y with an η-isomorphism $X_\eta \simeq Y_\eta$ is *a regular (resp. regular n.c.d.) model* (abbreviated R-model (resp. N-model)) of X, if Y is regular (resp. regular and $Y_{s,red}$ is an n.c.d. in Y).

An S-morphism $f : X' \to X''$ where X', X'' are R-models (resp. N-models) of X, is a morphism of R-models (resp. N-models) of X_η if it induces the canonical isomorphism $X'_\eta \simeq X''_\eta$.

An R-model (resp. N-model) X of X_η is *a relatively minimal regular (resp. regular n.c.d.) model* (abbr. RM-model (resp. RMN-model)) if every morphism $X \to X'$ of

R-model (resp. N-model) of X_η is an isomorphism.

An R-model (resp. N-model) of X, is *the minimal regular (resp. regular n.c.d.) model* (abbr. M-model (resp. MN-model)) if it is the final object of the category of R-models (resp. N-models) of X_η.

Suppose X is a normal S-curve. An S-curve Y with a proper birational S-morphism $Y \to X$ is a regular (resp. reg. n.c.d.) model (abbr. R-model (resp. N-model)) of X if Y is regular (resp. regular and $Y_{s,red}$ is an n.c.d. in Y).

We can define RM-models and RMN-models, etc. of X similarly as above.

The natural question here is to ask if these models always exist. If not, under which assumptions can we find such models?

If a proper smooth curve X_η, over η is given, we know that there exists an R-model of X_η. From this it is easily seen that an N-model, an RM-model and an RMN-model of X_η exist. If further X_η is geometrically integral over η, except the case where there exists an R-model X of X_η such that $X_s \simeq \mathbb{P}^1_s$, the M-model of X_η exists. Therefore the MN-model does.

In the exceptional case, every RM-model X Of X_η satisfies $X_s \simeq \mathbb{P}^1_s$, in particular, X is smooth over S.

In the following theorem we will see when the generic fiber X_η satisfies the conditions of the stable reduction theorem, then we can find a model among each of those models we defined above such that it has reduced special fiber.

Theorem 7.3.2. Consider the following conditions,

a Every RMN-model has reduced special fiber.

a' Every RM-model is an N-model and has reduced special fiber.

b There exists an RMN-model with reduced special fiber.

b' There exists an N-model with reduced special fiber.

If X_η is a proper smooth geometrically integral curve over η or X is a normal S-curve such that an R-model exists, the above conditions (a) - (b') on X_η, or X are all equivalent to one another.

Definition 7.3.3. Under the assumption of the theorem above X_η, (resp. X) is *semi-stable* if it satisfies the equivalent conditions (a) - (b') of the theorem.

Note to the fact that If X_η (resp. X) is semi-stable, so is it after any finite extension of traits $S' \to S$. This is easily seen by calculation of the blowing up.

Definition 7.3.4. Suppose X is a regular S-curve and Z is a connected closed subscheme of X_s which is a proper curve over s. The form of (X, Z) is the free Z-module M of basis Z_1, with the following objects,

- $B := Z_1$,
- $(\, , \,)$: the intersection product on M,
- $(\, , K)$: the intersection with the relative canonical divisor of X over S.
- $E := \Sigma_{C \in B} r_C C$, where r_C is the multiplicity of C in X_s.

We can check that the form of (X, Z) is a form in the sense of Definition (7.2.1) above. It is global if $Z = X_s$, and it is local otherwise.

(X, Z) is said to be *exceptional of type N* if so is the form of (X, Z) in the sense of Definition (7.2.2). If X is an R-model (resp. N-model) of a proper regular geometrically connected η-curve X_η, X is *relatively minimal* if and only if the form of (X, X_s) satisfies the condition (RM) (resp. (RMN)).

Suppose X is a normal S-curve, x is a closed point of X_s such that $X - \{x\}$ is regular and is an R-model (resp. N-model) of X such that $Y - Y_x \simeq X - \{x\}$. Then Y is relatively minimal if and only if the form of (Y, Y_x) satisfies the condition (RM) (resp. (RMN)).

Now, we are ready to give a new proof of the stable reduction theorem.

Theorem (Deligne - Mumford, Grothendieck) 7.3.5. Suppose X is an RMN-model of a proper smooth geometrically connected curve X, over η. Then the following conditions are equivalent.

1. The action of I on $H^1(X_{\bar{\eta}}, \Lambda)$ is unipotent.
2. X has reduced special fiber (i.e., X_η, is semi-stable), or (X, X_s) is of type ${}_rI_k$ ($r, k \in \mathbb{N}$ and $r \geq 1$) (in the latter case, the genus of X_η is 1).

Proof. As we pointed out above, the form M of (X, X_s) satisfies the assumption of the lemma (G-I).

On the other hand, the condition (2) of the theorem is equivalent to that $E = E_0$ or M is of type ${}_rI_k$. Therefore it suffices to show that the condition (1) is equivalent to $F(E) = F(E_0)$.

It is equivalent because we have $h^1_s(X_{\bar{\eta}}) = 2 + F(E_0)$ by Corollary (7.1.3) and the fact that $(D_1, D) = 0$. But, we have

$$h^1(X_{\bar{\eta}}) := \dim H^1(X_{\bar{\eta}}, \Lambda) = 2 - \chi(X_{\bar{\eta}}) = 2 - 2\chi(O_{X_\eta}) = 2 - 2\chi(O_{X_s})$$

(by the invariance of Euler-Poincare characteristic (EGA III)) it is equal to

$$= 2 + (E, E + K) = 2 + F(E).$$

So it is the condition (1), which is $h^1(X_{\bar{\eta}}) = h^1_s(X_{\bar{\eta}})$, is equivalent to $F(E) = F(E_0)$. □

Corollary 7.3.6. Suppose X_η is a proper smooth geometrically connected curve over η. Then, there exists a finite separable extension η' of η such that X'_η over η' is semi-stable.

Corollary 7.3.7. Let the assumption be the same as in the theorem except that if the genus of $X_\eta = 1$, we further assume that $r_C = 1$ for a component C of X_s (i.e., X has a section). Then the following conditions are equivalent.

1. The action of I on $H^1(X_{\bar{\eta}}, \Lambda)$ is trivial.
2. X_s is reduced and $\mathrm{Card}(X_s)_1 = \mathrm{Card}(X_s)_{sing} + 1$.

Proof. Again, this is immediately deduced from the theorem and the corollary (7.1.6) □

Now, the P-version of the theorem above,

Theorem 7.3.8. Let the assumption be the same as in the theorem (7.2.13). Then the following conditions are equivalent.

1. The action of P on $H^1(X_{\bar{\eta}}, \Lambda)$ is trivial.
2. X satisfies either of the following conditions (a), (b).

- Every component C of X_s whose multiplicity in X_s is divisible by p satisfies the following condition (*).

  (*) C is isomorphic to $\mathbb{P}^1_s$ and intersects with other components of X_s at exactly two points and the multiplicities of those components which intersect with C are prime to p.

- (X, X_s) is of following exceptional type.

  (in this case, the genus of X_η is 1.)

  (X, X_s) is of type (I - IV) or (I* - IV*) (if $p \neq 2, 3$). (i.e., X_η is an arbitrary curve of genus 1.

  (X, X_s) is of type (I), (III), (I*), or (III*) (if $p = 3$).

  (X, X_s) is of type (I), (IV), or (IV*) (if $p = 2$).

Proof. The proof of this theorem is analogue to the proof of theorem (7.2.13). The only difference is that we use here the lemma (G-P) and the fact that $h^1_t(X_{\bar{\eta}}) = 2 + F(E_1)$. □

Corollary 7.3.9. Suppose X_η is a proper smooth geometrically connected η-curve of genus = 1 (which may not have section) J_η is the Jacobian of X_η Let X (resp. J) be the MN-model of X_η, (resp. J_η). Then,

(X, X_s) is of type (I) if and only if (J, J_s) is of type (I).

If $p = 2$ (resp. 3),

(X, X_s) is of type (IV) or (IV*) (resp. (III), (I*) or (III*) if and only if so is (J, J_s).

Proof. The proof is deduced from the theorems above and the theory of degeneration of elliptic curves. Theorem (7.2.13) and the classification show that the following conditions are equivalent.

1. The action of I on $H^1(X_{\bar{\eta}}, \Lambda)$ is unipotent.

2. (X, X_s) is of type (I).

The assertion (1) follows from the isomorphism $H^1(X_{\bar{\eta}}, \Lambda) \simeq H^1(J_{\bar{\eta}}, \Lambda)$. The second assertion is deduced by the same argument.

If $p = 2$ (resp. $= 3$), Theorem (7.2.16) and the classification show that the following conditions are equivalent.

- The action of P on $H^1(X_{\bar{\eta}}, \Lambda)$ is trivial.

- (X, X_s) is of type (I), (IV) or (IV*) (resp. (I), (I*), (III) or (III*)).

The equivalence above shows that the second assertion is true. □

7.4 Local Case

In this section we take a look at the local case which is the case when we face with an isolated singularity. First, we begin with defining some quantities.

Definition 7.4.1. Let X be a reduced s-curve where s is the spectrum of a field k, and x is a closed point of X.

$$\delta_{X,s}(x)(= \delta_s(x)) := \dim_k(O_{X,x}^{normal}/O_{X,x}),$$

$$p_{X,s}(= p_s(x)) := \mathrm{Card}((\tilde{X}_x)_1).$$

Now, suppose X is an S-curve such that X_s is reduced and x is a closed point of X_s.

$$\delta_{X,s}(= \delta_s(x)) := \delta_{X_s,s},$$

$$p_{X,s}(x)(= p_s(x)) := p_{X_s,s}(x).$$

Suppose that X is an S-curve and x is a closed point of X_s.

$$r_x^1 := \dim_\Lambda R^1\psi\Lambda_x.$$

The following theorem is important for resolving the local results in this section. We are not going to proof the theorem here. You can find it as Propsition (5.9) of [K] and also Proposition (4.2) [T] with another proof.

Theorem 7.4.2. Suppose X is a normal S-curve and x is a closed point of X_s such that $X - \{x\}$ is smooth over S. Then,

$$r_x^1 = 2\delta_s(x) - p_s(x) + 1.$$

Definition 7.4.3. Suppose X is a normal S-curve and x is a closed point of X_s such that $X - \{x\}$ is smooth over S and $\phi : Y \to X$ is a regular model of X such that $Y - Y_x \simeq X - \{x\}$. Then the proper transform $X{*}_s$ of X_s in Y is a reduced divisor and the exceptional divisor $E = E_x$ is the divisor $Y_s - X_s^*$ in Y.

Theorem 7.4.4. We keep the notation of the definition above, we have

$$\delta_s(x) = \Sigma_y \delta_{s,X_s^*}(y) - (E, E) - \chi(O_E)$$

where y runs over inverse image of x in X_s^*.

Proof. By the Zariski's main theorem, the restriction map $\phi_{|X_s^*} : X_s^* \to X_s$ is finite and we have

$$\delta_s(x) - \Sigma_y \delta_{s,X_s^*}(y) = \dim_k(\phi_* O_{X_s^*,x} / O_{X_s,x})$$

Because we have

$$\delta_s(x) = \dim_k(O_{X_s,x}^{normal} / O_{X_s,s}) \quad \textit{and}$$

$$\Sigma_y \delta_{s,X_s^*}(y) = \dim_k(O_{X_s,x}^{normal} / \phi_* O_{X_s^*,x})$$

But we have

$$-(E, E) = (E, (\pi) - E) = (E, X_s^*) = \chi(O_E \otimes_{O_Y} O_{X_s^*}).$$

One the other hand we have the following exact sequence

$$0 \to O_{Y_s} \to O_{X_s^*} \times O_E \to O_E \otimes_{O_Y} O_{X_s^*} \to 0,$$

Hence, it is enough to show that

$$\chi_s(O_{X_s,x} \to R\phi_* O_{Y_s,x}) := \dim_k(Mapping\ \ cone\ (O_{X_s,x} \to R\phi_* O_{Y_s,x})) = 0.$$

To show this, we take a proper S-curve $\bar{X}$ with an open immersion $X \to \bar{X}$ and put $\bar{Y}$ the gluing of Y and $\bar{X}$ by the isomorphism $Y - E \simeq X - \{x\}$.

Then we will have

$$\chi_s(O_{X_s,x} \to R\phi_* O_{Y_s,x}) = \chi(O_{\bar{Y}_s}) - \chi(O_{\bar{X}_s}) = \chi(O_{\bar{Y}_\eta}) - \chi(O_{\bar{X}_\eta}).$$

But it is equal to zero by the invariance of Euler-Poincaré characteristic) □

Corollary 7.4.5. Under the assumption of the theorem (7.4.2), suppose that Y is an N-model of X and $E_0 := E_{red}$. Then

$$r_x^1 = \begin{cases} 1 + (E, K + E_0 - E) & if \quad Y \neq X, \\ 1 \quad if \quad Y = X \ and \ X \ is \ not \ smooth \ over \ S, \\ 0 \quad if \quad Y = X \ and \ X \ is \ smooth \ over \ S. \end{cases}$$

Proof. Only It is obvious that only the first case needs to be proved. By theorem (7.4.2), it is enough to show

$$2\delta_s(x) = (E, K - E) \quad and \quad p_s(x) = -(E, E_0).$$

For the first equality, we have $\delta_{X_s^*}(y) = 0$ for ally $y \longmapsto x$, because Y_x, is connected and Y is an N-model.

On the other hand, we have $-2\chi(O_E) = (E, E + K)$. So it suffices to apply theorem (7.4.4).

For the second, it is sufficient to remark that $(-E, E_0) = \mathrm{Card}(X_s^* \times_{X_s} x)$ (it follows from the fact that Y_x, is connected and that Y is an N-model.) □

Now, the local version of our main theorem.

Theorem 7.4.6. Suppose x is a normal S-curve, x is a closed point of X_s such that $X - \{x\}$ is smooth over S and Y is the MN-model of X. Then the following conditions are equivalent.

1. The action of I on $R^1\psi\Lambda_x$ is unipotent.
2. Y has reduced special fiber (i.e., X is semi-stable).

Proof. The assertion is clear if $Y = X$, we assume $Y \neq X$. By the explanations given after the definition (7.3.4), the form of (Y, Y_x) satisfies the assumption of the lemma (L-I).

The above condition (2) is equivalent to $E = E_0$. On the other hand, the condition (1) is equivalent to $r_x^1 = r_{s,x}^1$. Since $r_x^1 = 1 + F(E)$ by Corollary above and $r_{s,x}^1 = 1 + F(Eo)$ by Corollary (7.1.3 (the second part)) is also equivalent to $F(E) = F(E_0)$. Thus it is enough to apply the lemma (L-I) and we are done. □

Corollary 7.4.7. Let the assumption be the same as in Theorem (7.4.6) above. Then there exists a finite separable extension η' of η such that X'_S, over S' is semi-stable where S' is the integral closure of S in η'.

Proof. The assertion is directly deduced. □

Corollary 7.4.8. Again, let the assumption be the same as in Theorem (7.4.6). Then the following conditions are equivalent.

1. The action of I on $R^1\psi\Lambda_x$ is trivial.

2. The MN-model Y of X has a reduced special fiber and

$$\mathrm{Card}(Y_x)_1 = \mathrm{Card}(Y_x)_{sing} + 1,$$

or $Y = X$.

Proof. The proof of this corollary follows immediately from the theorem (7.4.6) and the Corollary (7.1.7). □

And at the end, the P-version of our main theorem in the local case.

Theorem 7.4.9. Let the assumption be the same as in Theorem (7.4.6). Then the following conditions are equivalent.

1. The action of P on $R^1\psi\Lambda_x$ is trivial.
2. Every component C of Y_x whose multiplicity in Y_s is divisible by p satisfies the following condition (*).

   (*) C is isomorphic to $\mathbb{P}^1_s$ and intersects with other components of Y_s at exactly two points and the multiplicities of those components which intersect with C are prime to p.

Proof. The proof is analogue to the proof of theorem (7.4.6) but here we use the lemma (L-P) instead. □

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