UNIVERSITY BORDEAUXI

HILBERT AND QUOT SCHEMES

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Introduction

Fix a base field k. Fix $n \in \mathbb{N}$ and a polynomial $P(t) \in \mathbb{Q}[t]$ of degree n such that $P(d) \in \mathbb{Z}$ for all integers $d \gg 0$. Suppose that we want to classify closed sub-schemes of a projective space over k with a given Hilbert polynomial $P(d)$, and moreover suppose that we want to show that the set of these classes can be endowed with the structure of an algebraic scheme. As we will describe, to each closed sub-scheme X one can associate a k-rational point $[X]_d$ of a Grassmannian scheme (see section 1.4). It is called *dth* Hilbert point of X. It can be shown that one also could recover X from $[X]_d$, for $d \geq d_0 := d_0(X)$. The main obstacle which one faces through this construction is actually to verify that if there exist a sufficiently large integer d_0 which uniformly works for all closed sub-schemes X, with given Hilbert polynomial $P(d)$.
To prevail over the problem, D.Mumford introduced the concept of m-regularity. We will present his results in second chapter, which leads to an important theorem, so called Uniform Vanishing Theorem (theorem 2.2.1). On the other hand by regarding Properness Theorem (theorem 3.1.7), it can be shown that it is in fact a proper scheme (see theorem 4.1.2). This almost finishes the story of the construction of the *Hilbert scheme* and even more, its generalization *Quot* scheme. However we should slightly modify the proofs in order to work in the relative case, i.e. over a general base scheme S, which makes some complications. We will close the chapter by giving some applications. For instance we prove a generalization of a result of Mori, and also other examples and applications will be given.

Since we are dealing with moduli spaces and in general one can not write the equations of Hilbert or Quot schemes explicitly, we shall give some criterions for testing smoothness, which are essentially based on the theory of infinitesimal deformations of algebraic schemes. This will be done in section 4.3, and will also make lots of explicit examples. For instance we produce an example of a singular reducible Hilbert scheme 4.3.5.

In the last chapter we study the Hilbert scheme of n points on a quasi-projective scheme X. We denote it by $X^{[n]}$. As a remarkable results of this chapter we discover a birational morphism $\rho : X^{[n]} \to X^{(n)}$, where $X^{(n)}$ is the symmetric *nth* power of X (see section 5.3.3). Furthermore we will prove that is in fact a resolution of singularities of $X^{(n)}$ when X is either a curve or a surface (see theorems 5.4.1 and 5.4.2). Finally by producing a counter example we will show that for sufficiently large n the irreducibility of $X^{[n]}$ fails when the dimension of X is greater than equal 3.

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Chapter 1

Preliminaries

1.1 Fine Moduli Spaces

In all what follows we are particularly interested in certain kinds of moduli spaces. Thus the functorial point of view is necessary. To this purpose let us first establish the following well-known assertion.

Proposition 1.1.1 *(**Yoneda's Lemma**)Let $\mathcal{C}$ be a category, Sets be the category of sets. Lets $\underline{Funct}(\mathcal{C}, \mathrm{Sets})$ denote the category of contravariant functors from $\mathcal{C}$ to sets. Then the following functor (of points)*

$$\mathcal{F} : \mathcal{C} \to \underline{Funct}(\mathcal{C}, \mathrm{Sets})$$

Which takes $X \in Ob(\mathcal{C})$ to $X(-) := Hom(-, X)$, and $\phi : T \to S$ to $X(\phi)$, is fully faithful. (i.e. for every pair of X, Y of objects of $\mathcal{C}$ the map on morphisms $h : Hom(X, Y) \to Hom(X(-), Y(-))$ is a bijection.)

Proof: To any natural transformation $\mathcal{H} : X(-) \to Y(-)$ we associate $\mathcal{H}(id_X) \in Hom(X, Y)$ and vice versa to a morphism $f \in Hom(X, Y)$ we associate $X(f) : X(-) \to Y(-)$. One can check easily that the compositions are identity.$\square$

Corollary 1.1.2 *the functors of points $X(-)$ and $Y(-)$ are isomorphic if and only if X is isomorphic to Y.*

Definition 1.1.3 *An S-scheme X is said to represent a functor $\mathcal{F}$ from the category of S-schemes to the category of Sets if $\mathcal{F}$ is isomorphic to $X(-)$. In this case X is called a fine moduli space for the moduli problem given by $\mathcal{F}$. Moreover if $\mathcal{F}$ is representable by the scheme X, then the object in $\mathcal{F}(X)$ associated to the $id_X \in X(X)$ is called the universal object.*

We now give a sort of moduli problems, which we will be particularly interested, in this thesis, to show that they are representable by some algebraic schemes.

Example 1.1.4 *Fix a Noetherian base scheme S. Let T be an S-scheme. Consider the following exact sequence of sheaves of $\mathcal{O}_T$-modules*

$$0 \to \mathcal{K} \to \mathcal{O}_T^n \to \mathcal{Q} \to 0 \tag{1.1}$$

where $\mathcal{K}$ and $\mathcal{Q}$ are locally free sheaves of rank m and $n-m$. we say that $0 \to \mathcal{K} \to \mathcal{O}_T^n \to \mathcal{Q} \to 0$ and $0 \to \mathcal{K}' \to \mathcal{O}_T^n \to \mathcal{Q} \to 0$ are equivalent if there exist isomorphisms $\alpha : \mathcal{K} \to \mathcal{K}'$ and $\beta : \mathcal{Q} \to \mathcal{Q}'$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{O}_T^n & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & \| & & \downarrow{\scriptstyle\beta} & & \\ 0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{O}_T^n & \longrightarrow & \mathcal{Q}' & \longrightarrow & 0, \end{array}$$

commutes.

the functor $\mathcal{G}r_{m,n}$ is defined as follows

$$\mathcal{G}r_{m,n}(T) := \{equivalence\ classes\ of\ exact\ sequences\ of\ type(1.1)\},$$

with $\mathcal{G}r_{m,n}(f) = f^*$.

We can even define the Grassmannian functor more generally, replacing $\mathcal{O}_s^n$ by any a locally free sheaf of rank n, $\mathcal{E}$. Then we denote the corresponding Grassmannian functor by $\mathcal{G}r_{m,\mathcal{E}}$

Example 1.1.5 *Let X be a projective S-scheme, let $\mathcal{E}$ be a coherent sheaf on X and a polynomial $P(d)$. Given a Noetherian S-scheme T, let $\mathcal{E}_T$ be the pull back of $\mathcal{E}$ to $X \times_S T$. Consider the coherent sheaf quotients $\mathcal{E}_T \to \mathcal{Q}$, modulo the relation $f \sim f'$ if and only if there exist*

$$\begin{array}{ccc} \mathcal{E}_T & \xrightarrow{f} & \mathcal{Q} \\ \| & & \downarrow \\ \mathcal{E}_T & \xrightarrow{f'} & \mathcal{Q}' \end{array}$$

Now we define a functor from Noetherian S-scheme to sets

$$\mathcal{Q}uot_{\mathcal{E},P(d)} := \{\mathcal{E}_T \to \mathcal{Q}; \mathcal{Q}\ is\ flat\ over\ T\ and\ each\ \mathcal{Q}_t\ has\ Hilbert\ polynomial\ P(d)\}/\sim$$

With $\mathcal{Q}uot_{\mathcal{E},P(d)}(f : U \to T) = (id \times f)^*$

In particular we define $\mathcal{H}ilb_{X,P(d)} := \mathcal{Q}uot_{\mathcal{O}_X,P(d)}$, alternatively, this functor can be defined as follows

$\mathcal{H}ilb_{X,P(d)} =$

$\{closed\ subschemes\ Z \subset X \times_S T\ that\ are\ flat\ over\ T\ with\ Hilbert\ polynomial\ P(d)\},$

via the identification of $i : Z \hookrightarrow X_T$ *with the quotient* $\mathcal{O}_{X_T} \to i_* \mathcal{O}_Z$.

1.2 Cohomology of quasi-coherent sheaves

In this section we give a list of Definitions and Theorems which will be of common use later on. We first state a well-known vanishing theorem of Grothendieck.

Theorem 1.2.1 *Let* X *be a Noetherian topological space of dimension* n. *Then* $H^i(X, \mathcal{F}) = 0$ *for all* $i > n$ *and all sheaves* $\mathcal{F}$ *of abelian groups on* X.

Proof: c.f. [4], Theorem III.2.7.

Theorem 1.2.2 *Let* A *be a Noetherian ring,* $B = A[x_0, \dots, x_n]$ *and* $X = Proj(B) = \mathbb{P}^n_A$ *with* $n \geq 1$. *Then:*

a) *The natural map* $B \to \Gamma_*(\mathcal{O}_{Proj(B)}) := \oplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n))$ *is an isomorphism of graded* B*-modules.*

b) $H^i(X, \mathcal{O}_X(r)) = 0$, $n > i > 0$ *and all* $r \in \mathbb{Z}$.

c) $H^n(X, \mathcal{O}_X(-n-1)) \cong A$.

d) *the natural map*

$H^0(X, \mathcal{O}_X(r)) \times H^n(X, \mathcal{O}_X(-n-r-1)) \to H^n(X, \mathcal{O}_X(-n-1)) \cong A,$

is a perfect paring of finitely generated A*-modules, for each* $r \in \mathbb{Z}$

Proof: [4], Theorem III.5.1.

Theorem 1.2.3 *Let* X *be a projective scheme over a noetherian ring* A, *and let* $\mathcal{O}_X(1)$ *be a very ample sheaf on* X *over* $Spec(A)$. *Let* $\mathcal{F}$ *be a coherent sheaf on* X. *Then*

a) *for each* $i \geq 0$, $H^i(X, \mathcal{F})$ *is finitely generated* A*-module.*

b) *there is an integer* n_0*, depending on* $\mathcal{F}$*, such that* $H^i(X, \mathcal{F}(n)) = 0$*, for each* $i \geq 1$ *and each* $n > n_0$*.*

Proof: c.f. [4] Theorem III.5.2.

Theorem 1.2.4 *If* X *is a projective scheme over a noetherian ring* A*, let* $\mathcal{O}_X(1)$ *denote a very ample invertible sheaf on* X*. Then there is a* $d_{\mathcal{F}}$ *for each coherent sheaf* $\mathcal{F}$ *on* X *so that* $\mathcal{F}(d) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ *is generated by its global sections, whenever* $d \geq d_{\mathcal{F}}$*.*

Proof:[4], Theorem II.5.17.

Definition 1.2.5 *Let* $(X, \mathcal{O}_X)$ *be a ringed space, and let* $\mathcal{F}$ *be an* $\mathcal{O}_X$*-module. We define the functors* $Ext^i(\mathcal{F}, .)$ *as the right derived functors of* $Hom(\mathcal{F}, .)$*, and* $\mathcal{E}xt^i(\mathcal{F}, .)$ *as the right derived functors of* $\mathcal{H}om(\mathcal{F}, .)$*.*

Consequently, according to the general properties of derived functors, we have $Ext^0 = Hom$.

proposition 1.2.6 *If* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is a short exact sequence in the category of sheaves of modules, then for any* $\mathcal{G}$ *we have a long exact sequence :*

$$0 \to Hom(\mathcal{F}'', \mathcal{G}) \to Hom(\mathcal{F}, \mathcal{G}) \to Hom(\mathcal{F}', \mathcal{G})$$
$$\to Ext^1(\mathcal{F}'', \mathcal{G}) \to Ext^1(\mathcal{F}, \mathcal{G}) \to ...,$$

and similarly for the $\mathcal{E}xt$ *sheaves.*

Proof: c.f. [4], Proposition III, 6.4.

For a locally free $\mathcal{O}_X$-module $\mathcal{L}$ of finite rank, we define the dual of $\mathcal{L}$, denoted $\check{\mathcal{L}}$, to be the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.

Lemma 1.2.7 *Let* $\mathcal{L}$ *be a locally free sheaf, and let* $\check{\mathcal{L}} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ *be its dual. Then for any* $\mathcal{F}$*,* $\mathcal{G}$ *in the category of sheaves of modules we have:*

$$Ext^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \simeq Ext^i(\mathcal{F}, \check{\mathcal{L}} \otimes \mathcal{G}),$$

and for the sheaf $\mathcal{E}xt$ *we have:*

$$\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \simeq \mathcal{E}xt^i(\mathcal{F}, \check{\mathcal{L}} \otimes \mathcal{G}) \simeq \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \check{\mathcal{L}}.$$

*Proof:*c.f. [4], Proposition III, 6.7.

1.2.1 Higher Direct Images of sheaves

We must remark that our intent to present this subsection is only to recall the reader some basic properties of the Higher Direct Image, and mainly to state the Base-Change Theorem which shall be of common use in this thesis. For the details and Proofs we refer to [4], section III.8.

The notion of the higher direct image provides strong tools to study the cohomology of a family of schemes $\pi : X \to Y$.

Definition 1.2.1.1 *Let $\pi : X \to Y$ be a continues map of topological spaces. Then we define the higher direct image functors $R^i\pi_*$ from the category of sheaves of abelian groups on X to the category of sheaves of abelian groups on Y to be the right derived functor of the functor π_*.*

Proposition 1.2.1.2 *For each $i \geq 0$ and each sheaf of abelian groups $\mathcal{F}$, $R^i\pi_*(\mathcal{F})$ is the sheaf associated to the presheaf*

$$V \mapsto H^i(\pi^{-1}(V), \mathcal{F}|_{\pi^{-1}(V)}).$$

Proof: c.f. [4], Proposition III.8.1.

Proposition 1.2.1.3 *Let X be a noetherian scheme over an affine $Y = Spec\ A$, $\pi : X \to Y$. Then for any quasi-coherent sheaf $\mathcal{F}$ on X, we have*

$$R^i\pi_*\mathcal{F} \cong \widetilde{H^i(X, \mathcal{F})}.$$

Proof: [4], Proposition III.8.5.

Proposition 1.2.1.4 *Let X be a noetherian scheme over Y, $\pi : X \to Y$. Then for any quasi-coherent sheaf $\mathcal{F}$ on X, the sheaves $R^i\pi_*\mathcal{F}$ are quasi-coherent on Y.*

The following theorem illustrate the relationship between higher direct image and cohomology of the fibers.

Theorem 1.2.1.5 *(Cohomology and Base Change). Let $\pi : X \to Y$ be a projective morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Then: a)if the natural map*

$$\varphi^i_y : R^i\pi_*(\mathcal{F}) \otimes k(y) \to H^i(X_y, \mathcal{F}_y)$$

is surjective, then it is an isomorphism, and the same is true for all y' in a suitable neighborhood of y;

b) Assume that $\varphi^i(y)$ is surjective. then the following conditions are equivalent:

i) φ^{i-1} is also surjevtive.

ii) $R^i\pi_*(\mathcal{F})$ is locally free in a neighborhood of y.

Proof: c.f. [4], Theorem III.12.11.

1.2.2 Hilbert polynomial and Euler characteristic

Let M be a finitely generated graded module over the polynomial ring $S := k[x_0, \ldots, x_n]$. The Hilbert function $\mathcal{H}_M$ of M is given by

$$\mathcal{H}_M(l) = dim_k(M_l)$$

for each $l \in \mathbb{Z}$. Then there is a unique polynomial $P_M(z) \in \mathbb{Q}[z]$ such that $\mathcal{H}_M(l) = P_M(l)$ for all $l \gg 0$. Furthermore $deg\ P_M = dim\ V_+(Ann\ M)$, for the proof we refer either to [4], Theorem I.7.5, or [6] section 5.13.

Let X be a projective scheme over a field k and $\mathcal{F}$ be a coherent sheaf on it. The Hilbert function of $\mathcal{F}$ is defined as follows:

$$\mathcal{H}_k(X, \mathcal{F}) : \mathbb{Z} \longrightarrow \mathbb{N}$$

$$d \mapsto dim_k\Gamma(X, \mathcal{F}(d))$$

We also define the Euler Characteristic of $\mathcal{F}(d)$:

$$\chi(X, \mathcal{F}(d)) = \sum_{i \geq 0} (-1)^i dim_k H^i(X, \mathcal{F}(d))$$

Remark 1.2.2.1 *Euler Characteristic is a finite integer by theorem 1.2.3 and the fact that $H^i(X, \mathcal{F}(d)) = 0$ if $i > dimX$ (Theorem 1.2.1). So we can rewrite this formula as follows:*

$$\chi(X, \mathcal{F}(d)) = \sum_{i=0}^{dimX} (-1)^i dim_k H^i(X, \mathcal{F}(d))$$

Example 1.2.2.2 *Let X be the projective space $\mathbb{P}_k^d$ over a field k. We get $\chi(X, \mathcal{O}_X(n)) = \binom{d+n}{d}$ if $n \geq 0$, in particular$\chi(X, \mathcal{O}_X) = 1$*

Remark 1.2.2.3 *Euler Characteristic is additive on exact sequences, i.e. if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of Coherent sheaves on X, then $\chi(X, \mathcal{F}) = \chi(X, \mathcal{F}') + \chi(X, \mathcal{F}'')$*

By theorem 1.2.1 there is a d_0 such that for every $d \geq d_0$, $\chi(X, \mathcal{F}(d)) = dim_k \Gamma(X, \mathcal{F}(d))$, therefore for sufficiently large d we have the following equality:

$$\chi(X, \mathcal{F}(d)) = \mathcal{H}_k(X, \mathcal{F})(d)$$

Lemma 1.2.2.4 *Hilbert function is a polynomial for sufficiently large d. We call it Hilbert polynomial.*

Proof: Let $f : X \to \mathbb{P}_k^n$ be a closed immersion, $\mathcal{F}$ a coherent sheaf over X and consider$\mathcal{E} = \mathcal{O}_{\mathbb{P}_k^n}(d)$ over $\mathbb{P}_k^n$. By projection formula (see [4] ex. II.5.1) we get the following isomorphism:

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{O}_{\mathbb{P}_k^n}(d)) \simeq f_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}_k^n}} \mathcal{O}_{\mathbb{P}_k^n}(d)$$

So:

$$dim_k \Gamma(X, \mathcal{F}(d)) = dim_k \Gamma(\mathbb{P}_k^n, f_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}_k^n}} \mathcal{O}_{\mathbb{P}_k^n}(d))$$

By the theorem 1.2.4, $\Gamma_* f_* \mathcal{F}$ is a finitely generated graded module over $k[x_0, ..., x_n]$, so we have reduced to the case of graded modules, hence we may conclude by the discussion given in the beginning of the subsection.

Note. One can prove an stronger result than the above lemma which asserts that Euler Characteristic can be given by a polynomial in d for all d.

Let $s \in \Gamma(X, \mathcal{O}_X(1))$ and $X_1, ..., X_r \subseteq X$ be the irreducible components of X with maximal dimension. We choose s such that it is nonzero on each of these components. Let$D \subseteq X$ be the zero scheme of s. We have $dim D = dim X - 1$. Consider the following exact sequence associated to the hyperplane intersection D:

$$0 \to i_*\mathcal{N} \longrightarrow \mathcal{F} \xrightarrow{.s} \mathcal{F}(1) \longrightarrow i_*\mathcal{Q} \to 0$$

Where $\mathcal{N}$ and $\mathcal{Q}$ are sheaves over D. We know that tensoring by $\mathcal{O}_X(d)$ is exact. Now to get the desired result, induction on dimension of X works.

1.3 Grassmannian

In this section we will introduce a scheme which represents Grassmannian functor, we call it naturally Grassmannian scheme. We will construct it over base scheme *Spec* $\mathbb{Z}$.

Suppose M is a $m \times n$ matrix and $I \subseteq \{1, ..., n\}$ a set with cardinality equal to m. Define the I-th minor M_I of M, to be the $m \times m$ minor of M whose columns are indexed by I.

Let X^I be a matrix such that X_I^I is identity matrix and the remaining entries are independent variables $x_{p,q}^I$ over $\mathbb{Z}$.

Suppose the affine pieces of Grassmannian scheme are $U^I = Spec\mathbb{Z}[X^I]$ where I varies over subsets of $\{1, ..., n\}$ with cardinality m, and $\mathbb{Z}[X^I]$ is the polynomial ring in variables $x_{p,q}^I$ over $\mathbb{Z}$.

Now we should glue these pieces to construct desired scheme. Let J be a subset of $\{1, ..., n\}$ with cardinality m, $P_J^I = det(X_J^I) \in \mathbb{Z}[X^I]$ and $U_J^I = Spec\mathbb{Z}[X^I, 1/P_J^I]$. In fact U_J^I is the open sub-scheme of U^I where P_J^I is invertible (i.e. X_J^I admits an inverse $(X_J^I)^{-1}$ on U_J^I). Suppose

$$\theta_{I,J} : \mathbb{Z}[X^J, 1/P_I^J] \to \mathbb{Z}[X^I, 1/P_J^I]$$

is a morphism such that the images of variables $x_{p,q}^J$ are given by the entries of the matrix formula $\theta_{I,J}(X^J) = (X_J^I)^{-1}X^I$. In particular $\theta_{I,J}(P_I^J) = 1/P_J^I$, so the map extends to $\mathbb{Z}[X^J, 1/P_I^J]$.

Note that $\theta_{I,I}$ is identity on $U_I^I = U^I$, moreover:

$$\theta_{K,I}\theta_{I,J}(X^J) = \theta_{K,I}((X_J^I)^{-1}, X^I) = (((X_I^K)^{-1}X^K)_J)^{-1}(X_I^K)^{-1}X^K$$

$$= ((X_I^K)^{-1}X_J^K)^{-1}(X_I^K)^{-1}X^K = (X_J^K)^{-1}(X_I^K)(X_I^K)^{-1}X^K$$

$$= (X_J^K)^{-1}X^K = \theta_{K,J}(X^J)$$

Therefore the schemes U^I, as I varies over all $\binom{n}{m}$ different subsets of $\{1, ..., n\}$ of cardinality m, can glued together by the cocycles $\theta_{I,J}$ to form a finite type scheme $Gr(m, n)$ over $\mathbb{Z}$. Each U^I is isomorphic to $\mathbb{A}_{\mathbb{Z}}^{m(n-m)}$, so $Gr(m, n) \to Spec(\mathbb{Z})$ is smooth of relative dimension $m(n - m)$.

Before going to prove the representibility of Grassmannian functor, by this scheme, we want to study more, about this scheme and its basic properties.

$Gr(m,n)$ is a separated scheme: We know $\theta_{I,J}(X^J) = (X^I_J)^{-1}X^I$, so we have glued X^J_I and $(X^I_J)^{-1}$ together, therefore the intersection of $Gr(m,n)$ with $U^I \times U^J$ can be seen as the closed subscheme $\Delta_{I,J} \subset U^I \times U^J$ defined by entries of the matrix formula $X^J_I X^I - X^J = 0$. So $Gr(m,n)$ is separated.

$Gr(m,n)$ is proper: Consider the morphism $\pi : Gr(m,n) \to Spec(\mathbb{Z})$. Let R be a discrete valuation ring, $\mathcal{K}$ its quotient field, and $\varphi : Spec(\mathcal{K}) \to Gr(m,n)$ a morphism of schemes.

We will show that the morphism φ extends to $\psi : Spec(R) \to Gr(m,n)$. φ gives us a morphism of rings:

$$f : \mathbb{Z}[X^I] \to \mathcal{K}$$

$P^I_I = 1$ so $\nu(f(P^I_I)) = 0$, where $\nu : \mathcal{K} \to \mathbb{Z} \cup \{\infty\}$ is the discrete valuation. Now choose J such that $\nu(f(P^I_J))$ is minimum. Obviously $\nu(f(P^I_J)) \leq 0$, so $f(P^I_J) \neq 0$ and so the matrix $f(X^I_J)$ lies in $GL_m(\mathcal{K})$.

Let $g : \mathbb{Z}[X^J] \to \mathcal{K}$ is defined by entries of the matrix formula:

$$g(X^J) = f((X^I_J)^{-1}X^I)$$

This morphism gives us a morphism of schemes from $Spec\mathcal{K}$ to $Gr(m,n)$ which is equal to φ. Moreover all $m \times m$ minors X^J_K satisfy $\nu(g(P^J_K)) \geq 0$. Now since X^J_J is identity, we get from the above that $\nu(g(x^J_{p,q})) \geq 0$ for all entries of X^J. Therefore $g : \mathbb{Z}[X^J] \to \mathcal{K}$ factors uniquely via $R \subset \mathcal{K}$. And the resulting morphism of schemes gives us the desired morphism ψ:

$$SpecR \to U^J \to Gr(m,n)$$

Now we conclude by using valuative criterion of properness for discrete valuation rings.

Let us now define the universal quotient. We define a rank m locally free sheaf $\mathcal{Q}$ on $Gr(m,n)$ together with a surjective homomorphism $\oplus\mathcal{O}^n_{Gr(m,n)} \to \mathcal{Q}$. On each U^I we define a surjective homomorphism $u^I : \oplus^n\mathcal{O}_{U^I} \to \oplus^m\mathcal{O}_{U^I}$ by the matrix X^I. Compatible with the co-cycle $(\theta_{I,J})$ for gluing the affine pieces U^I , we give gluing data $(g_{I,J})$ for gluing together the trivial bundles $\oplus^m\mathcal{O}_{U^I}$ by putting

$$g_{(I,J)} = (X^I_J)^{-1} \in GL_m(U^I_J).$$

This is compatible with the homomorphisms u^I, so we get a surjective homomorphism $u : \oplus^n\mathcal{O}_{Gr(m,n)} \to \mathcal{Q}$.

Theorem 1.3.1 *The Grassmannian $Gr(m, \mathcal{E})$ represents the following functor*

$\mathcal{G}r_{m,\mathcal{E}} := \{equivalence\ class\ of\ exact\ sequence\ of\ type(1.1)\}$.

Proof: Let us assume that $\mathcal{E} = \mathcal{O}_S^n$. Let us first define a transformation of functors $f : \mathcal{G}r_{m,n} \to Gr(m,n)(-)$. Let T be an S-scheme, and assume that a sequence $0 \to K \to \mathcal{O}_T^n \to Q \to 0$ is given. Choose a trivialization $T = \bigcup_i Spec\ A_i$ for the sequence, we may assume that T is affine, $T = Spec\ A$. Now $K|_T \cong \mathcal{O}_T^m \to \mathcal{O}_T^n$, which is given by $n \times m$ matrix, and since the cokernel of this map is also locally free of rank $n-m$, we see that is full rank. Clearly this gives a T-valued point of $Gr(m,n)$. On the other hand the universal quotient described above inverts f.

Now for $\mathcal{E}$ a locally free sheaf of rank n on S, again we take a trivialization for $\mathcal{E}$ over $\{Spec\ A_i\}$. Then the Grassmannian over the affines patch together to give a twisted Grassmannian $Gr(m, \mathcal{E})$ which represents $\mathcal{G}r(m, \mathcal{E})$.

□

Corollary 1.3.2 $G(m,n) \cong G(n-m,n)$

Remark 1.3.3 *if S is any scheme, take an affine cover $\{\mathrm{Spec}\, A_i\}$, then the Grasssmannian $G(m,n)$ over $\mathrm{Spec}\, A_i$ patches together and gives the Grassmannian over S. Even more for any locally free sheaf $\mathcal{E}$ of rank n on S we may take a trivialization of $\mathcal{E}$ over $\mathrm{Spec}\, A_i$, and then the Grassmannian over $\mathrm{Spec}\, A_i$ patches together to give it as a twisted Grassmannian $G(m, \mathcal{E})$ which represents the functor $\mathcal{G}r_{(m,\mathcal{E})} : Schs \to Sets$*

$\mathcal{G}r_{(m,\mathcal{E})}(T) :=\{sequences\ 0 \to \mathcal{K} \to \mathcal{E}_T \to \mathcal{Q} \to 0\ of\ loc.\ free\ sheaves\ over\ T\ with\ rk(\mathcal{K}) = m\ \}/\sim$,

where the equivalence relation is defined as the above case.

1.4 A rough description of the construction of Hilbert scheme

We work over a fixed base field k. Fix $n \in \mathbb{N}$ and $P(t) \in \mathbb{Q}[t]$ polynomial of degree n such that $P(d) \in \mathbb{Z}$ for integer $d \gg 0$.

Suppose that we want to classify projective varieties with a given Hilbert polynomial $P(d)$. According this paragraph we would like to give an sketchy description of the behind picture of the construction of Hilbert scheme.

Consider all closed $X \subset \mathbb{P}^n_k$, defined by a homogeneous ideal $J(X)$, with Hilbert polynomial $P(t)$. For this moment we denote the set of such subschemes by $\mathcal{H}ilb_p$, however we will endow $\mathcal{H}ilb_p$ with additional structure!

Choose $d_0(X)$ such that $P(d) = dim_k(k[x_0, \dots, x_n]/J(X))_d$ for each $d \geq d_0(X)$. For each $d \geq d_0(X)$, consider the linear subspace $J(X)_d \subset k[x_0, ..., x_n]_d$ which has codimension $P(d)$. The corresponding point in the Grassmannian

$$[X]_d \in Gr(\binom{n+d}{d} - P(d), k[x_0, ..., x_n]_d) \cong Gr(\binom{n+d}{d} - P(d), \binom{n+d}{d})$$

is called the dth Hilbert point of X.

Now suppose that $J(X)$ is generated by polynomials of degree $\leq d_0$. Then we claim that for each $d \geq d_0$, $[X]_d$ determines X uniquely. Indeed each subspace $\Lambda \subseteq k[x_0, ..., x_n]_d$ defines a homogeneous ideal $< \Lambda > \subseteq k[x_0, ..., x_n]$ and thus a projective scheme $V_+(< \Lambda >) \subseteq \mathbb{P}^n$. One can readily verify that $V_+(< J(X)_d >) = X$ and thus X is determined by its dth Hilbert point.

So the main obstacle which we face, through the construction of the Hilbert scheme, is that we need to show that there exist a sufficiently big integer d_0 which uniformly works for all schemes X, with given Hilbert polynomial $P(d)$. To prevail over the problem, Mumford introduced the concept of m-regularity. We shall present this result, in second chapter, so called Uniform Vanishing Theorem. On the other hand there is still another remarkable gap, that is to endow $\mathcal{H}ilb_p$ with the proper scheme structure. This gap will remove, by regarding a strong theorem which is called Properness Theorem, this will be proven in third chapter. This finishes the story of the construction. Furthermore since in general we do not have the equations of Hilbert scheme and its generalization Quot scheme, we shall give some criterion for testing smoothness, essentially based on the theory of infinitesimal deformations of algebraic schemes, and will make lots of examples.

Chapter 2

Castelnuovo-Mumford Regularity

2.1 Castelnuovo-Mumford Regularity

For an A-module M let $\widetilde{M}$ be the associated quasi coherent sheaf. This assignment gives an equivalence of categories between A-modules and quasi coherent sheaves over $Spec(A)$.

Now suppose M is a graded R-module, where $R = A[X_0, ..., X_n]$. There is natural sheaf $\widehat{M}$ of $\mathcal{O}_{\mathbb{P}^n_A}$-module, with the property that $\widehat{M}|_{D(f)} = \widetilde{(M_{(f)})}$, for a principal open affine $D(f)$. $\widehat{M}$ is coherent if A is noetherian and M is finitely generated. The module $M(d)$ is the sheafted graded R-module with $M(d)_e = M_{d+e}$. Set $\mathcal{O}_{\mathbb{P}^n_A}(d) := \widehat{R(d)}$. We then have $\mathcal{O}_{\mathbb{P}^n_A}(d)|_{D(X_i)} = X_i^d \mathcal{O}_{\mathbb{P}^n_A}$, therefore $\mathcal{O}_{\mathbb{P}^n_A}(d)$ is an invertible sheaf. We also mention that $\widehat{M(d)} = \widehat{M} \otimes_{\mathcal{O}_{\mathbb{P}^n_A}} \mathcal{O}_{\mathbb{P}^n_A}(d)$.

So from the above discussion we see that taking hat

$$\widehat{\quad} : M \mapsto \widehat{M}$$

gives a functor from the category of graded R-modules to the quasi coherent sheaves on $\mathbb{P}^n_A$. Notice that the above functor is not an equivalence of categories, in fact this functor takes the modules which become isomorphic after sufficiently high degree to the same sheaf. However the twisted global sections

$$\Gamma_* : \mathcal{F} \mapsto \bigoplus_{d \in \mathbb{Z}} H^0(\mathbb{P}^n_A, \mathcal{F}(d))$$

gives somehow a semi-inverse to this map, i.e. $\widehat{\Gamma_*(\mathcal{F})} \simeq \mathcal{F}$ and $\Gamma_*(\widehat{M})$ becomes isomorphic to M in sufficiently high degrees.

To construct the *Quot* scheme we essentially need to know that when $\widehat{\Gamma_*(\mathcal{F})}$ can be generated by elements of degree less than g. This is the basic idea behind the strange definition of Castelnuovo-Mummford regularity. In fact the idea is to give a cohomological criteria for this property of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$. So we start with a hyperplane $H \subset \mathbb{P}^n$, and we assume that the intersection with hyperplane H is sharp, i.e. H does not contain any associated point of $\mathcal{F}$, then we have the short exact sequence

$$0 \to \mathcal{F}(m-i-1) \xrightarrow{h} \mathcal{F}(m-i) \longrightarrow \mathcal{F}_H(m-i) \to 0$$

where h is given by multiplication with a defining equation of H. This gives rise to the exact sequence

$$\ldots \to H^i(\mathbb{P}^n, \mathcal{F}(m-i)) \to H^i(\mathbb{P}^n, \mathcal{F}_H(m-i)) \to H^{i+1}(\mathbb{P}^n, \mathcal{F}(m-i-1)) \to \ldots$$

If for an integer m, $H^i(\mathbb{P}^n_k, \mathcal{F}(m-i))$ vanishes for each $i \geq 1$ then so is its restriction to $H \cong \mathbb{P}^{n-1}$, $\mathcal{F}_H$. Due to this motivation we give the following definition.

Definition 2.1.1 *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$ is called m-regular if*

$$H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) = 0,$$

for all $i > 0$.

As we have already seen, we have restricted ourselves to the coherent sheaves, in fact this implies that the set of associated points of $\mathcal{F}$ is finite which guarantees the existence of such a hyperplane H, when k is infinite. Notice that when k is finite the m-regularity of $\mathcal{F}$ would preserve, when we base-change with an infinite extension.

We Claim that $\mathcal{F}$ is m-regular then it is $m+1$-regular. We prove by induction. The claim is obvious for $n = 0$. Consider the exact sequence

$$H^i(\mathbb{P}^n, \mathcal{F}(m-i)) \to H^i(\mathbb{P}^n, \mathcal{F}(m+1-i)) \to H^i(H, \mathcal{F}_H(m+1-i))$$

The first term vanishes by our hypothesis, on the other hand the last term also vanishes because $\mathcal{F}_H$ is m-regular and by the inductive hypothesis also $m+1$-regular. Therefore $H^i(\mathbb{P}^n, \mathcal{F}(m+1-i))$ vanishes.

Suppose $\mathcal{F}$ is m-regular. We are now going to prove that the following natural morphism

$$H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n_k, \mathcal{F}(m)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(m+1)),$$

is surjective, again by induction on n. We come back to the following exact sequence

$$0 \to H^0(\mathbb{P}^n, \mathcal{F}(m-1)) \to H^0(\mathbb{P}^n, \mathcal{F}(m)) \to H^0(H, \mathcal{F}_H(m)) \\ \to H^1(\mathbb{P}^n, \mathcal{F}(m-1))$$

Since $\mathcal{F}$ is m-regular, therefore the last term vanishes i.e.

$$H^0(\mathbb{P}^n, \mathcal{F}(m-1)) \to H^0(H, \mathcal{F}_H(m-1))$$

is surjective.

Notice that the canonical morphism $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(H, \mathcal{O}_H(1))$ is surjective, therefore $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(H, \mathcal{O}_H(d))$ is surjective. Thus the top map in the following commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{F}(r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) & \xrightarrow{\alpha} & H^0(H, \mathcal{F}_H(r)) \otimes H^0(H, \mathcal{O}_H(1)) \\ {\scriptstyle\mu}\Big\downarrow & & {\scriptstyle\gamma}\Big\downarrow \\ H^0(\mathbb{P}^n, \mathcal{F}(r+1)) & \xrightarrow{\beta} & H^0(H, \mathcal{F}_H(r+1)) \end{array}$$

is surjective. The vertical map γ is surjective by induction hypothesis. Therefore $\gamma \circ \alpha$ is surjective, so the composition $\beta \circ \mu$ is surjective too. Hence α, β and γ are surjective, and since $im(h : H^0(\mathbb{P}^n, \mathcal{F}(r)) \to H^0(\mathbb{P}^n, \mathcal{F}(r+1)))$ which is given by multiplication with a certain section $h \in \mathcal{O}_{\mathbb{P}^n}(1)$ is inside $im(\mu)$, thus we conclude that μ is surjective.

Let us summarize these results in the following theorem.

Theorem 2.1.2 *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$. If $\mathcal{F}$ is m-regular then:*

i) $\mathcal{F}$ is m'-regular for all $m' \geq m$.

ii) the multiplication map:

$$H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n_k, \mathcal{F}(r)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(r+1))$$

is surjective for $r \geq m$.

Corollary 2.1.3 *Let $\mathcal{F}$ be a coherent sheaf. If $\mathcal{F}$ is m-regular then:*

i) $H^i(\mathbb{P}^n_k, \mathcal{F}(d)) = 0$ for all $i > 0$and $d \geq m-1$,

ii) $\mathcal{F}(d)$ is generated by the global sections for all $d \geq m$.

Proof:

i) Part a) of the above theorem implies that $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ for $i \geq 1$ when $r \geq m$.

ii) By the above theorem we know that

$$H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(p)) \otimes H^0(\mathbb{P}^n_k, \mathcal{F}(r)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(r+p)),$$

is surjective, for $r \geq m$ and $p \geq 0$. We know from theorem 1.2.4 that for sufficiently large p, $\mathcal{F}(r+p)$ is generated by its global sections. Hence $\mathcal{F}(r)$ is also generated by its global sections for $r \geq m$.

□

Remark 2.1.4 *Note that 2.1.1 .ii) tells us that if $\mathcal{F}$ is m-regular then the graded $k[X_0, \ldots, X_n]$-module $\Gamma_*\mathcal{F}$ can be generated by elements of degree less $\leq m$. In particular, if an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}_k{}^n}$ is m-regular then the homogeneous ideal*

$$I := \Gamma_*(\mathcal{I}) \subset k[X_0, \ldots, X_n]$$

is generated by elements of degree $\leq m$.

2.2 Uniform Vanishing Theorem

Recall the description we gave in section 1.4. Due to the inspiration we got from the description, we will try to attack to the problem, in the general case, i.e. when our projective scheme X is over a general base scheme S.

Let $X = \mathbb{P}^n_S$, take an element $Z \in \mathcal{Q}uot_{X/S}(T)$, assume that is correspond to the following short exact sequence of sheaves

$$Z: \quad 0 \to \mathcal{K} \to \oplus^p \mathcal{O}_{\mathbb{P}^n_T} \to \mathcal{F} \to 0.$$

From the Theorem 1.2.4 for a sufficiently big number d_0 we get the following short exact sequence

$$0 \to \pi_*\mathcal{K}(d_0) \to \pi_* \oplus^p \mathcal{O}_{\mathbb{P}^n_T}(d_0) \to \pi_*\mathcal{F}(d_0) \to 0.$$

where $\pi : \mathbb{P}^n_T \to T$ is the porjection morphism. So the above short exact sequence gives an element $[Z]_{d_0}$ in the relative Grassmannian scheme, as we described in 1.4 . Therefore we would be very glad if we could find an integer d_0 which works for each coherent sub-sheaf $\mathcal{K}$ of $\mathcal{O}_{\mathbb{P}^n_T}$ with fixed Hilbert polynomial $P(d)$, because this fact guarantees the existence of a morphism from the *Quot* scheme to the Grassmannian! To this purpose we establish the following theorem. We will actually follow Mumford's proof [5].

Theorem 2.2.1 *Let k be any field. There exist a polynomial $F_{p,n}$ in $n+1$ variables with integral coefficients, such that any coherent sub-sheaf $\mathcal{K}$ of $\oplus^p\mathcal{O}_{\mathbb{P}^n}$, with Hilbert polynomial*

$$\chi(\mathcal{K}(d)) = \sum_{i=0}^{n} a_i \binom{d}{i}$$

is $F_{p,n}(a_0, \ldots, a_n)$-regular.

Proof: We prove by induction on n. The statement is clear for $n = 0$. Let $n \geq 1$. Note that we may assume that k is infinite, thus there exist a hyperplane $H \subset \mathbb{P}^n$ which does not contain any associated point of $\oplus^p\mathcal{O}_{\mathbb{P}^n}/\mathcal{K}$.

Tensoring the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{h} \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_H \to 0$$

with $\mathcal{K}(d)$ we get

$$0 \to \mathcal{K}(d-1) \xrightarrow{h} \mathcal{K}(d) \longrightarrow \mathcal{K}_H(d) \to 0 \qquad (\star)$$

Thus we get

$$\chi(\mathcal{K}_H(d)) = \chi(\mathcal{K}(d)) - \chi(\mathcal{K}(d-1)) = \sum_{i=0}^{i=n} a_i \binom{d}{i} - \sum_{i=0}^{i=n} a_i \binom{d-1}{i-1}$$
$$= \sum_{i=0}^{n} a_i \binom{d-1}{i-1}.$$

Note that the torsion sheaf $Tor_1^{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_H, \oplus^p\mathcal{O}_{\mathbb{P}^n}/\mathcal{K})$ vanishes therefore the short exact sequence

$$0 \to \mathcal{K} \to \oplus^p\mathcal{O}_{\mathbb{P}^n} \to \oplus^p\mathcal{O}_{\mathbb{P}^n}/\mathcal{K} \to 0,$$

restricts to a short exact sequence

$$0 \to \mathcal{K}_H \to \oplus^p\mathcal{O}_H \to \oplus^p\mathcal{O}_H/\mathcal{K} \to 0,$$

i.e. $\mathcal{K}_H$ can be identified with a sub-sheaf of $\mathcal{O}_{\mathbb{P}^{n-1}}$, with Hilbert polynomial $\chi(\mathcal{K}_H(d)) = \sum_{j=0}^{n-1} b_j \binom{d}{j}$

So by the inductive hypothesis, there exist a polynomial $F_{p,n-1}(x_0, \ldots, x_{n-1})$ such that $\mathcal{K}_H$ is $m_0 := F_{p,n-1}(b_0, \ldots b_{n-1})$-regular. Substituting $b_j = g_j(a_0, \ldots, a_n)$ we get $m_0 = G(a_0, \ldots, a_n)$, where G is a polynomial with integral coefficients independent of the field k and the sheaf $\mathcal{K}$. Thus from $(\star)$ we get a long exact cohomology sequence for $m \geq m_0 - 1$

$$\begin{aligned} 0 \to H^0(\mathcal{K}(m-1)) \to H^0(\mathcal{K}(m)) \to H^0(\mathcal{K}_H(m)) \\ \to H^1(\mathcal{K}(m-1)) \to H^1(\mathcal{K}(m)) \to 0 \to \ldots \\ \to 0 \to H^i(\mathcal{K}(m-1)) \xrightarrow{\sim} H^i(\mathcal{K}(m)) \to 0 \to \ldots \end{aligned}$$

Notice that by theorem 1.2.3 $H^i(\mathcal{K}(m)) = 0$ for $m \gg 0$, hence

$$H^i(\mathcal{K}(m)) = 0 \quad for\ all\ i \geq 2\ and\ m \geq m_0 - 2$$

The surjection $H^1(\mathcal{K}(m-1)) \to H^1(\mathcal{K}(m))$ shows that $h^1(\mathcal{K}(d)) \leq h^1(\mathcal{K}(d-1))$, where $h^1(\mathcal{K}(d))$ is the function $h^1(\mathcal{K}(d)) := dim H^1(\mathcal{K}(d))$. Now notice that the equality holds for $d \geq d_0$ if and only if the restriction map $r_H : H^0(\mathcal{K}(d)) \to H^0(\mathcal{K}_H(d))$ is surjective. As $\mathcal{K}_H$ is m-regular, it follows from Theorem 2.1.2 the restriction morphism $r : H^0(\mathcal{K}(j)) \to H^0(\mathcal{K}_H(j))$ is surjective for all $j \geq d_0$, so $h^1(\mathcal{K}(j-1)) = h^1(\mathcal{K}(j))$ for all $j \geq d_0$. As $h^1(\mathcal{K}(j))$ vanishes for $j \gg 0$, thus the function $h^1(\mathcal{K}(d))$ is strictly decreasing for $d \geq d_0$ until its value reaches zero. Therefore

$$H^1(\mathcal{K}(d)) = 0 \quad for\ \ d \geq d_0 + h^1(\mathcal{K}(d_0)) \qquad (\star\star)$$

We are now going to find a bound on $h^1(\mathcal{K}(d_0))$. As $\mathcal{K} \subset \oplus_p \mathcal{O}_{\mathbb{P}^n}$ we have

$$h^0(\mathcal{K}(r)) \leq p h^0(\mathcal{O}_{\mathbb{P}^n}(r)) = p\binom{n}{n+r}.$$

So we get

$$h^1(\mathcal{K}(d_0)) = h^0(\mathcal{K}(d_0)) - \chi(\mathcal{K}(d_0)) \leq p\binom{n+d_0}{n} - \sum_{i=0}^{n} a_i \binom{d_0}{i} = P(a_0, \ldots, a_n)$$

where $P(a_0, \ldots a_n)$ is a polynomial expression in $a_0, \ldots a_n$, obtained by substituting $d_0 = G(a_0, \ldots, a_n)$ in the last term of the above inequality.

Thus we see that the coefficient of the polynomial $P(x_0, \ldots, x_n)$ are again independent of the field k and the sheaf $\mathcal{K}$. Hence from $(\star\star)$ we get

$$H^1(\mathcal{K}(d)) = 0 \quad for \;\; d \geq G(a_0, \ldots, a_n) + P(a_0, \ldots a_n)$$

Note that $P(a_0, \ldots a_n) \geq h^1(\mathcal{K}(d_0)) \geq 0$. Let $F_{p,n}(x_0, \ldots, x_n) := G(x_0, \ldots, x_n) + P(x_0, \ldots x_n)$, then $\mathcal{K}$ is $F_{p,n}(a_0, \ldots, a_n)$-regular.

$\square$

Chapter 3

Flatness and Stratification

3.1 Constancy of Hilbert polynomial

Let X be an S-scheme via $\pi : X \to S$ and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is flat over S at $x \in X$ if the stalk $\mathcal{F}_x$ be a flat $\mathcal{O}_{s,\pi(x)}$-module. If $\mathcal{F}$ is flat over S at each point of X, we say simply $\mathcal{F}$ is flat over S. A morphism $g : X \to Y$ of schemes is called flat if $\mathcal{O}_X$ be flat over Y.

Example 3.1.1 *Classical smooth families are flat, i.e. Let $f : X \to Y$ is a morphism of nonsingular varieties over an algebraically closed field k such that $dimX - dimY = n$ and $\Omega_{X/Y}$ is locally free of rank n, then f is flat.*

Lemma 3.1.2 *The following statements are true:*
i.Open immersions are flat morphisms.
ii.Flat morphisms are stable under base change.
iii.Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms, then $g \circ f$ is a flat morphism
iv.The ring homomorphism $A \to B$ is flat if and only if the morphism of schemes $SpecB \to SpecA$ is flat.

Proof: It is an easy consequence of general properties of flat modules.

Our first goal in this chapter is to prove the Properness theorem.
Let X be an S-scheme with associated morphism $f : X \to S$. $\{X_s := X \times_S Spec(k(s)) | s \in S\}$ is the family of fibers of X which parametrized by S. Here $Spec(k(s)) \to S$ is the inclusion of the point $s \in S$. In a similar way we can get a family of sheaves from a given sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules.In fact this family is: $\{\mathcal{F}_s := \mathcal{F}|_{X_s} | s \in S\}$.
Now for the closed immersion $X \subseteq \mathbb{P}^n_S$, we have $X_s \subseteq \mathbb{P}^n_{k(s)}$. In this case we

can compute the Hilbert polynomial of $\mathcal{F}_s$, but it depends upon s. We will see that under suitable conditions for a flat sheaf $\mathcal{F}$ these Hilbert polynomials are independent of point s !
Let $\mathcal{F}$ be a coherent sheaf on $X := \mathbb{P}^n_A$ where A is a local noetherian domain. Let us write the Cech Complex $C^{\cdot}(\mathcal{U}, \mathcal{F}(m))$, for standard open affine cover $\mathcal{U}$ and integer number m:

$$0 \to H^0(X, \mathcal{F}(m)) \to C^0(\mathcal{U}, \mathcal{F}(m)) \to C^1(\mathcal{U}, \mathcal{F}(m)) \to ... \to C^n(\mathcal{U}, \mathcal{F}(m)) \to 0$$

But for sufficiently large m, $H^i(X, \mathcal{F}(m)) = 0$ (for $i > 0$), and therefore the complex is a resolution for $H^0(X, \mathcal{F}(m))$. Now if $\mathcal{F}$ be flat then $C^i(\mathcal{U}, \mathcal{F}(m))$ is flat for every i, so we we see that $H^0(X, \mathcal{F}(m))$ is flat. On the other hand by 1.2.3 it is finitely generated. Thus $H^0(X, \mathcal{F}(m))$ is flat and finitely generated, and so free of finite rank. Therefore we have proved if $\mathcal{F}$ is flat then $H^0(X, \mathcal{F}(m))$ is free of finite rank for sufficiently large m. Now

we want to verify that whether the inverse implication also holds or not ! Suppose $H^0(X, \mathcal{F}(m))$ is free of finite rank for $m \geq m_0$. Let us define the

graded $A[x_0, ..., x_n]$-module M as follow:

$$M = \bigoplus_{m \geq m_0} H^0(X, \mathcal{F}(m))$$

Clearly M is free and hence flat. M and $\Gamma_* \mathcal{F}$ have the same terms in degrees $m \geq m_0$, so $\widehat{M} = \widehat{\Gamma_* \mathcal{F}}$, but $\widehat{\Gamma_* \mathcal{F}} = \mathcal{F}$, hence $\mathcal{F}$ is flat.
Therefore $\mathcal{F}$ is flat if and only if $H^0(X, \mathcal{F}(m))$ is free of finite rank.
Let $\mathcal{F}$ be flat over $SpecA$. We will show that the Hilbert polynomial of $\mathcal{F}_s$is independent of the chosen point $s \in SpecA$. Set $P_t$be the Hilbert polynomial of $\mathcal{F}_t$ for the point $t \in SpecA$. It is enough to show

$$P_t(m) = rank_A H^0(X, \mathcal{F}(m))$$

for $m >> 0$.

Fix $t \in T := SpecA$ and let $T' := SpecA_p$ where p is the prime ideal corresponding to t. Each Cech complex on X, localizes to a Cech complex on X', where $X' := X \times_T T'$.
Moreover localizingis exact, therefore we get the following isomorphism:

$$H^i(X, \mathcal{F}) \otimes_A A_p \xrightarrow{\simeq} H^i(X', \mathcal{F}')$$

Here $\mathcal{F}'$ is the pull back of $\mathcal{F}$ to X'. So we may reduced to the case where t is a closed point.
Take a presentation of $k(t)$ over A:

$$A^q \to A \to k := k(t) \to 0 \qquad (*)$$

$\mathcal{F}$ is flat, therefore we get the following exact sequence:

$$\mathcal{F}^q \to \mathcal{F} \to \mathcal{F}_t \to 0$$

Thus by 1.2.3 we get another exact sequence:

$$H^0(X, \mathcal{F}(m)^q) \to H^0(X, \mathcal{F}(m)) \to H^0(X_t, \mathcal{F}_t(m)) \to 0$$

On the other hand if we tensor the sequence $(*)$ with $H^0(X, \mathcal{F}(m))$, we will have the exact sequence:

$$H^0(X, \mathcal{F}(m)) \otimes A^q \to H^0(X, \mathcal{F}(m)) \otimes A \to H^0(X, \mathcal{F}(m)) \otimes k \to 0$$

$\mathcal{F}$ is flat so $H^0(X, \mathcal{F}(m))$ is free of finite rank for enough large m, Thus:

$$H^0(X, \mathcal{F}(m)) \otimes_A A^q \simeq H^0(X, \mathcal{F}(m)^q)$$

Moreover

$$H^0(X, \mathcal{F}(m)) \otimes_A A \simeq H^0(X, \mathcal{F}(m))$$

Therefore by comparing these two exact sequences, we see that:

$$H^0(X_t, \mathcal{F}_t(m)) \simeq H^0(X, \mathcal{F}(m)) \otimes_A k(t)$$

for all $m \geq 0$
So we see that under our assumptions, If $\mathcal{F}$ is flat then P_t does not depend upon t. It is interesting to know that whether the inverse implication is true or not.

We may assume in the above discussion that T is an integral noetherian scheme, because we have the following fiber product:

$$\begin{array}{ccc} X_p & \longrightarrow & X \\ \downarrow & & \downarrow \\ Spec(A_p) & \longrightarrow & Spec(A) \end{array}$$

And moreover flatness is a local property.Now let $X \subseteq \mathbb{P}^n_T$ be an arbitrary closed subscheme.
Considering the inclusion $i : X \to \mathbb{P}^n_T$, we can push forward the sheaf $\mathcal{F}$ on X to a sheaf $f_*\mathcal{F}$ on $\mathbb{P}^n_T$ and use the fact that $\mathcal{F}$ is flat if and only if $f_*\mathcal{F}$ is flat.
So let us summerize the above discussion:

Theorem 3.1.3 *(Constancy of Hilbert polynomial) Let T be an integral noetherian scheme and $X \subseteq \mathbb{P}^n_T$ be a closed subscheme. Let $\mathcal{F}$ be a coherent sheaf on X. Consider the Hilbert polynomial $P_t \in Q[z]$ of the fiber X_t considered as a closed subscheme of $\mathbb{P}^n_{k(t)}$, then the following are equivalent:*

i) $\mathcal{F}$ is flat over $SpecA$.
ii) $H^0(X, \mathcal{F}(m))$ is locally free, for all $m >> 0$.
iii) The Hilbert polynomial $\chi(X_t, \mathcal{F}_t(m))$ is independent of $t \in T$.

Using the fact that any two points on a connected scheme can be joined by a sequence of integral subschemes, we get the following corollary:

Corollary 3.1.4 *Let S be a noetherian connected scheme and $X \subseteq \mathbb{P}^n_S$ be a projective S-scheme, flat over S. Then the following holds:*

i)The fiber dimensions, $dim(X_s)$ are constant.
ii)The Euler Characterisric, $\chi(X_s, \mathcal{O}_{X_s})$ is constant.
iii)The degree of $X_s \in \mathbb{P}^n_s$ is constant.

Let $f : X \to S$ be a morphism of schemes, where S is irreducible and regular of dimension one. Set $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.
For a moment suppose $\mathcal{F}$ is flat and x is a point of X with closed image $s = f(x) \in S$. Here $\mathcal{O}_{S,s}$ is DVR and m_s, the maximal ideal of $\mathcal{O}_{S,s}$ has a uniformizing parameter t. The local parameter t is not a zero divisor so its image $f^\sharp t$ can not be a zero divisor in the flat module $\mathcal{F}_x$. So x is not an associated point. Therefore if f is flat then every associated point of $\mathcal{F}$ go to the generic point via f.

Vice versa, let f take every associated point of $\mathcal{F}$ to the generic point of S. Let $x \in X$ be such that $f(x) = s$ is generic point of $\mathcal{O}_{S,s}$. $\mathcal{O}_{S,s}$ is a field, so $\mathcal{F}_x$ is a vector field over $\mathcal{O}_{S,s}$ and in particular a flat $\mathcal{O}_{S,s}$-module. So $\mathcal{F}$ is flat at each point which map to the generic point of S. If $f(x) = s$ is closed and $\overline{\{x\}}$ does not contain an associated point of $\mathcal{F}$, then multiplication by the uniformizing parameter $f^\sharp(t)$ is injective on $\mathcal{F}_x$, and since $< t >= m_s$, it follows that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$.

Let us summarize this discussion in the following theorem:

Theorem 3.1.5 *If $f : X \to S$ and S is irreducible and regular of dimension one, then a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is flat over S if and only if f maps each associated point of $\mathcal{F}$ to the generic point of S.*

Let S and X be as above, $s \in S$ be a closed point and $U = S - \{s_0\}$. Consider a coherent sheaf $\mathcal{E}$ on X and define $\mathcal{E}_U := \mathcal{E}|X_U$, where $X_U = f^{-1}(U)$. Suppose that $\mathcal{E}_U \to \mathcal{Q}$ is a surjection, where $\mathcal{Q}$ a coherent sheaf flat over U. We want to extend this flat surjection to X. To get this goal, in the first step we will show that there is a universal extension for $\mathcal{E}_U \to \mathcal{Q}$ (i.e. if $\mathcal{E}_U \to \mathcal{Q}'$ is another surjection extending $\mathcal{E}_U \to \mathcal{Q}$, then there is a unique factorization $\mathcal{E} \to \mathcal{Q}' \to \overline{\mathcal{Q}}$). Afterward we will prove that this extension is necessarily flat over S, and moreover such an extension is unique.

Clearly it is sufficient to prove for affine base scheme $S = SpecB$. Moreover it is enough to show the existence of the extension for the case $U = SpecB_g$. Here g is an element of B. Notice that we may let $X = SpecA$, because if we could get the desired result in this case, then extensions over the affines will patch by the universal property. Therefore we have $X_U = SpecA_g$. We can set $\mathcal{E} = \widetilde{M}$ and $\mathcal{Q} = \widetilde{N}$ (since $\mathcal{E}$ and $\mathcal{Q}$ are coherent).
Consider the A-module map $M \to M_g \to N$ and set $\overline{N}$ to be the image of this map. now $\widetilde{\overline{N}}$ is the universal extension. Let $\mathcal{E} \to \mathcal{Q}'$ be another extension of $\mathcal{E}_U \to \mathcal{Q}$ and $\mathcal{Q}' = \widetilde{K}$ over $SpecA$ for some A-module K. We have the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & K \\ \downarrow & & \downarrow g \\ M \otimes A_g & \longrightarrow & K \otimes A_g \\ \cong\downarrow & & \downarrow h \\ M_g & \longrightarrow & N \end{array}$$

Thus we get the desired morphism α:

$$\alpha = h \circ g.$$

The commutativity of the large square and surjectivity of f, shows that the image of α is equal to $\overline{N}$.

We will show that this universal extension is the unique flat extension of $\mathcal{E}_U \to \mathcal{Q}$.

If $\overline{\mathcal{Q}}$ is not flat, then there is an associated point of $\overline{\mathcal{Q}}$ like x, which goes to a closed point like $s_1 \in S$ via f. (i.e. $\exists s \in S$ s.t. $\overline{\{x\}} = Supp(s)$, moreover $\exists s_1 \in S$ s.t. $f(x) = s_1$ and s_1 is closed)

We claim $s_1 = s_0$, otherwise $x \in X_U$ and the closure of $\{x\}$ in X_U is equal to $Supp(s \otimes 1)$, where $s \otimes 1$ is the section of $\mathcal{Q}$, induced by s. (Note that $\overline{\mathcal{Q}} \otimes A_g = \mathcal{Q}$)
So x is an associated point of $\mathcal{Q}$ which goes to a closed point, and this is in contradiction with flatness of $\mathcal{Q}$ (see **??**)
So f takes x to s_0. Now the qoutient of $\overline{\mathcal{Q}}$ by the subsheaf generated by sections, is obviously an extension of $\mathcal{E}_U \to \mathcal{Q}$ which violate universality of $\overline{\mathcal{Q}}$. So the assumption that tell $\overline{\mathcal{Q}}$ is not flat, is false ! And $\overline{\mathcal{Q}}$ is flat over S.

Let $\mathcal{Q}'$ be an extension of $\mathcal{E}_U \to \mathcal{Q}$, by universality of $\overline{\mathcal{Q}}$ we get a morphism of sheaves:

$$f : \mathcal{Q}' \to \overline{\mathcal{Q}}.$$

Its kernel is nontrivial, so set of associated points of $kerf$ is nonempty. We have:

$$x \in X_U \Rightarrow (Kerf)_x = 0 \Leftrightarrow x \notin \bigcup_{s \in kerf} Supp(s)$$

Now let x be an associated point of $kerf$, we conclude x is not in X_U. So associated points of $kerf$ are over s_0. But the set of these points is nonempty (since $kerf$ is not trivial) and moreover:

$$Ass(kerf) \subseteq Ass(\mathcal{Q}'),$$

which means that $\mathcal{Q}'$ can not be flat. So we have proved the following theorem:

Theorem 3.1.6 *If S is irreducible and regular of dimension one and $U \subseteq S$ is the complement of a closed point $s_0 \in S$, then flat quotients uniquely extend across s_0. That is, given $f : X \to S$ and a coherent sheaf $\mathcal{E}$ on X, let $X_U = f^{-1}(U)$ and $\mathcal{E}_U = \mathcal{E}|_{X_U}$. Then a surjection $\mathcal{E}_U \to \mathcal{Q}$ to a coherent sheaf $\mathcal{Q}$ that is flat over U extends uniquely to a surjection $\mathcal{E} \to \overline{\mathcal{Q}}$ where $\overline{\mathcal{Q}}$ is coherent and flat over S.*

□

Now by using the above theorems, we can prove an important theorem which will play an essential role in the rest of story.

Theorem 3.1.7 *(Properness) Let S be a nonsingular curve, and $U := S - \{s_0\} \subset S$ for a point s_0 of S. Every family of projective schemes $X_U \subseteq \mathbb{P}^n_U$ with constant Hilbert polynomial $P(d)$, extends uniquely to a family $X_S \subseteq \mathbb{P}^n_S$ such that the Hilbert polynomial of the limit $X_{s_0} \subseteq \mathbb{P}^n_{k(s_0)}$ is also $P(d)$.*

Proof: Consider the morphism $\mathbb{P}^n_S \to S$ and the coherent sheaf $\mathcal{O}_{\mathbb{P}^n_S}$ on $\mathbb{P}^n_S$. Let $\mathcal{Q}$ be the sheaf $i_*\mathcal{O}_{X_U}$ on $\mathbb{P}^n_U$ induced by the closed immersion $i : X_U \hookrightarrow \mathbb{P}^n_U$.
We have the surjection $\mathcal{O}_{\mathbb{P}^n_S}|_{\mathbb{P}^n_U} \to \mathcal{Q}$ and moreover $\mathcal{Q}$ is flat over $\mathcal{U}$. By flatness over nonsingular curves, there is a unique extension $\mathcal{O}_{\mathbb{P}^n_S} \to \overline{Q}$ such that $\overline{Q}$ is flat over S. We get the result.

Theorem 3.1.8 *(Generic Flatness Theorem)Let S be a reduced noetherian scheme and $f : X \to S$ a morphism of finite type. Consider a coherent sheaf $\mathcal{F}$ on X. There is a nonempty open subset $U \subset S$ such that $\mathcal{F}_U$ is flat over U. Here $\mathcal{F}_U$ is $\mathcal{F}|_{X_U}$ and X_U is defined to be $f^{-1}(U) = X \times_S U$.*

To prove this theorem we need the following lemma:

Lemma 3.1.9 *A and B are integral domain, A is noetherian and B is a finitely generated A-algebra. There is an element $f \in A$ such that B_f is a free A_f-algebra.*

Proof: We will prove this lemma by induction on n, the transcendence degree of B over A. Let k be the field of fraction of A. Suppose $b_1, ..., b_r$ are the generators of B as an A-algebra.

$B \otimes_A k$ is a finitely generated k-algebra, so by noether normalization $B \otimes_A k$ is integral over $k[f_1, ..., f_n]$ with $f_1, ..., f_n \in B$. Let f denote the product of denominators appearing in the minimal polynomials of b_i 's. Then B_f is integral over $A_f[f_1, ..., f_n]$ and therefore it is of finite rank as an $A_f[f_1, ..., f_n]$-module. Let $rank(B_f) = m$. Consider the following exact sequence of $A_f[f_1, ..., f_n]$-modules:

$$0 \to A_f[f_1, ..., f_n]^m \to B_f \to Q \to 0.$$

Here Q is a torsion module. There is a filtration for Q:

$$0 \subset Q_1 \subset Q_2 \subset ... \subset Q_n = Q, \qquad \textit{where} \qquad Q_{i+1}/Q_i \simeq A_f[f_1, ..., f_n]/P_i$$

Let us turn to the case where B' is an integral domain and A-algebra of smaller transcendence degree. By induction the result follows.

We are now going to prove theorem 3.1.8.

We may assume f is dominant, because otherwise $U = S - f(X)$ is nonempty and this is the desired open subset of S. We can replace the base

scheme by an open subset of this scheme. So we can suppose $S = SpecA$ where A is an integral domain.
Now cover X by a finite number of open affines, $SpecB$. Write $\mathcal{F}|_{SpecB} = \widetilde{M}$. It is enough to prove the claim for those $SpecB$ that dominates $SpecA$. Now by existence of a filtration for M it is enough to prove the statement for $M = B/p$. We can finish the proof by using the lemma.

3.2 Flattening Stratification

Keep the setting of the theorem 3.1.8. By induction and using Generic Flatness (3.1.8), we can find, reduced and locally closed subschemes V_i of S such that $\mathcal{F}_{V_i}$ is flat and S is the distinct union of these V_i:

$$S = \coprod V_i.$$

This is called week stratification. Moreover the Hilbert polynomials of the $\mathcal{F}_s$ for $s \in S$ vary in a finite set, if $f : X \to S$ be a projective morphism with closed immersion $X \subseteq \mathbb{P}^n_S$.

Theorem 3.2.1 ***(Flattening Stratification)*** *Given a projective morphism $f : X \to S$ over a noetherian scheme S and a coherent sheaf $\mathcal{F}$ on X, there is a (unique) stratification:*

$$S = \coprod S_i$$

of S by locally closed subschemes S_i such that $\mathcal{F}_{S_i}$ is flat over each S_i and more generally, given a morphism $g : T \to S$ and hence $\widetilde{g} : X \times_S T \to X$, then $\widetilde{g}^*\mathcal{F}$ is flat over T if and only if g factors through:

$$g : T \to \coprod S_i \to S$$

We may assume $S = SpecA$, because we can glue the flattening stratifications for an open cover by their universal property. Also we may assume $T = SpecB$. For a given morphism $g : T \to S$ we have the following diagram:

$$\begin{array}{ccc} X_B & \xrightarrow{\widetilde{g}} & X_A \\ {\scriptstyle \widetilde{f}}\downarrow & & {\scriptstyle f}\downarrow \\ SpecB & \xrightarrow{g} & SpecA \end{array}$$

For a sheaf $\mathcal{E}$ on X_A, there are always maps:

$$H^i(X_A, \mathcal{E}_A) \otimes_A B \to H^i(X_B, \mathcal{E}_B) \qquad (*)$$

because:
First, Any Cech complex of A-modules like :

$$0 \to H^0(X_A, \mathcal{E}_A) \to C^0(\mathcal{U}, \mathcal{E}_A) \to ... \to C^n(\mathcal{U}, \mathcal{E}_A) \to 0$$

becomes a Cech complex for $\mathcal{E}_A := \widetilde{g}^* \mathcal{E}$, by tensoring with B.
Second, tensor product is right exact.
Morphisms $(*)$ are isomorphism if B be flat over A.

Consider a weak stratification $S = \coprod V_i$ of S by previous corollary of Generic Flatness. Let $SpecA_i$ be an affine open subset of V_i, so by theorem 1.2.3 we get the following Cech complex which is an exact sequence of flat A_i modules for large d ($for\ d \geq d_0$):

$$0 \to H^0(X_{A_i}, \mathcal{F}_{A_i}(d)) \to C^0(\mathcal{U}, \mathcal{F}_{A_i}(d)) \to ... \to C^n(\mathcal{U}, \mathcal{F}_{A_i}(d)) \to 0$$

Tensoring by $k(s)$ gives us a Cech complex for $\mathcal{F}_s(d)$ which is exact because of flatness. Note that here $s \in SpecA_i$. We can choose d_0 large enough to work for every A_i. Therefore we have:

(i) $H^i(X_s, \mathcal{F}_s(d)) = 0$ for all $s \in S$ and $d \geq d_0$

We can increase d_0 if it is necessary to get the following result (By use of Stable base change result):

$$H^0(X_A, \mathcal{F}_A(d)) \xrightarrow{\sim} H^0(X_{A_i}, \mathcal{F}_{A_i}(d)) \qquad for\ all\ A_i\ and\ d \geq d_0$$

Moreover by a little bit work on the above exact sequence, we get the following isomorphism:

(ii) $\quad H^0(X_A, \mathcal{F}_A(d)) \otimes_A k(s) \xrightarrow{\sim} H^0(X_s, \mathcal{F}_s(d)) \qquad for\ all\ s \in S\ and\ d \geq d_0$

Let $\mathcal{E}$ be a coherent sheaf on S such that its fiber $\mathcal{E}(s)$ at $s \in S$ has rank e . Consider the following presentation of $\mathcal{E}|_U$:

$$\mathcal{O}_U^f \xrightarrow{\psi_{ij}} \mathcal{O}_U^e \longrightarrow \mathcal{E}|_U \to 0$$

Where U is a neighborhood of s. Note that this presentation exists, by the Nakayama lemma.

Consider the locally closed subset V_e of S, where $\mathcal{E}$has constant rank e. we can put a scheme structure on it, by patching via the equations $\psi_{ij} = 0$. It is not difficult to prove the following:

$$g^*\mathcal{E} \; is \; locally \; free \; of \; rank \; e$$

$$\Updownarrow$$

$$g(T) \subset V_e \; (as \; sets) \; and \; g^*\psi_{ij} = 0 \; for \; all \; \psi_{ij}$$

$$\Updownarrow$$

$$g : T \to S \quad factors \; through \; S_e$$

When e varies over local ranks of $\mathcal{E}$, we get a locally free stratification $S = \coprod S_e$ associated to $\mathcal{E}$.
Now choose $n+1$ to exceed the degree of the Hilbert polynomial of each $\mathcal{F}_s$ in the family. So by (i) the ranks of $H^0(X_s, \mathcal{F}_s(d_0)), H^0(X_s, \mathcal{F}_s(d_0 + 1)), ..., H^0(X_s, \mathcal{F}_s(d_0 + n))$ give us the value of the Hilbert polynomial of $\mathcal{F}$ in $n+1$ points: $d_0, d_0+1, ..., d_0+n$. Thus if we have these ranks we can then compute the corresponded Hilbert polynomial. Since $f_*\mathcal{F}(d_0), f_*\mathcal{F}(d_0 + 1), ..., f_*\mathcal{F}(d_0 + n)$ are the sheafifications of

$$H^0(X_A, \mathcal{F}_A(d_0)), H^0(X_A, \mathcal{F}_A(d_0 + 1)), ..., H^0(X_A, \mathcal{F}_A(d_0 + n))$$

, by (ii) we see that $H^0(X_s, \mathcal{F}_s(d_0)), H^0(X_s, \mathcal{F}_s(d_0 + 1)), ..., H^0(X_s, \mathcal{F}_s(d_0 + n))$ are the fibers of $f_*\mathcal{F}(d_0), f_*\mathcal{F}(d_0 + 1), ..., f_*\mathcal{F}(d_0 + n)$.

Now take a locally free stratification for each of the sheaves $f_*\mathcal{F}(d_0), f_*\mathcal{F}(d_0 + 1), ..., f_*\mathcal{F}(d_0+n)$, and intersect the locally closed sub-schemes that we have gotten above. This gives us a locally free stratification indexed by (distinct) Hilbert polynomials, with the property that

$$g^*f_*\mathcal{F}(d_0), g^*f_*\mathcal{F}(d_0 + 1), ..., g^*f_*\mathcal{F}(d_0 + n)$$

are simultaneously locally free if and only if $g : T \to S$ factors through $\coprod S_{P(d)}$. Further intersecting with the schemes obtained from the locally free stratifications of the rest of the sheaves $f_*\mathcal{F}(d_0 + n + 1)$, ... only serves to shrink the scheme structure on $S_{P(d)}$ leaving the same underlying reduced scheme $V_{P(d)}$. Since S is noetherian, it follows that after finitely many such intersections, we obtain the limit scheme structure with the property that:

$$g : T \to \; S \; factors \; through \; \coprod S_{P(d)}$$
$$\Updownarrow$$
$$g^* f_* \mathcal{F}(d) \; is \; locally \; free \; for \; all \; d \geq d_0$$

Suppose $g^* f_* \mathcal{F}(d)$ is locally free for all $d \geq d_0$. by stable base change result the following maps are isomorphism for sufficiently large d:

$$g^* f_* \mathcal{F}(d) = H^0(X, \mathcal{F}(d)) \otimes_A B \to H^0(X_B, \mathcal{F}_B(d)) = \widetilde{f}_* \widetilde{g}^* \mathcal{F}(d)$$

Thus $\widetilde{f}_* \widetilde{g}^* \mathcal{F}(d)$ is locally free for enough large d. Therefore by constancy of the Hilbert polynomial, $\widetilde{g}^* \mathcal{F}(d)$ is flat over T.

Conversely Consider $\widetilde{g}^* \mathcal{F}(d)$ is flat over T. Let $t \in Spec(B)$ maps to $s \in Spec(A)$, so $Spec(k(t))$ is flat over $Spec(k(s))$ therefore by (i) we have:

$$0 = H^i(X_s, \mathcal{F}_s(d)) \otimes_{k(s)} k(t) \xrightarrow{\sim} H^i(X_t, \mathcal{F}_t(d))$$

for $d \geq d_0$ and all $i > 0$. Now by flatness of $\widetilde{g}^* \mathcal{F}(d)$, $\widetilde{f}_* \widetilde{g}^* \mathcal{F}(d)$ is locally free for all $d \geq d_0$. The map $g^* f_* \mathcal{F}(d) \to \widetilde{f}_* \widetilde{g}^* \mathcal{F}(d)$ induces the following isomorphisms fibers:

$$H^0(X_s, \mathcal{F}_s(d)) \otimes_{k(s)} k(t) \xrightarrow{\sim} H^0(X_t, \mathcal{F}_t(d))$$

using (ii) and cohomology and base change. By Nakayama every map from a coherent sheaf to a locally free sheaf, which is isomorphism on fibers is itself an isomorphism. So we conclude.

Chapter 4

Construction of Hilbert and Quot Schemes

4.1 Constructions

Let us indicate that in the subsection 1.3 of the first chapter we have discussed about the Grassmannian scheme and we have shown that this scheme is indeed a fine moduli space for the moduli problem $\mathcal{G}_{(m,\mathcal{E})}$. In this chapter we will focus on a more sophisticated cases.

In order to define the Quot functor, let us fix a base scheme S, a projective S-scheme X, a coherent sheaf $\mathcal{E}$ on X and a polynomial $P(d)$. Let us denote the coherent quotient $q : \mathcal{E}_T \to \mathcal{Q} \to 0$ by the pair $< \mathcal{E}_T, q >$. We say that two such pairs $< \mathcal{E}_T, q >$ and $< \mathcal{E}_T, q' >$ are equivalent if there is a commuting diagram:

$$\begin{array}{ccccc} \mathcal{E}_T & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\ \| & & \downarrow & & \\ \mathcal{E}_T & \longrightarrow & \mathcal{Q}' & \longrightarrow & 0, \end{array}$$

Definition 4.1.1 *The quot functor is defined as follows $\mathcal{Q}uot_{\mathcal{E},P(d)}$:*

$\mathcal{Q}uot_{\mathcal{E},P(d)}(T) = \{< \mathcal{E}, q >$; $\mathcal{Q}$ is flat over T with Hilbert polynomial $P(d)\}/\sim$

$\mathcal{Q}uot_{\mathcal{E},P(d)}(f : W \to T) = f_X^*$

Theorem 4.1.2 *We keep the above notation. The functor $\mathcal{Q}uot_{\mathcal{E},P(d)}$ is always represented by a projective scheme $Quot(\mathcal{E}, P(d))$.*

Proof: Let us first prove the theorem for $X = \mathbb{P}^m_A$ and $\mathcal{E} = \mathcal{O}^n_{\mathbb{P}^m_A}(l)$, for some integer l and m. We proceed in 4 steps

Step1: We plan first to show that for sufficiently large d_0 there is a transformation of functors $\mathcal{Q}uot_{\mathcal{E},P(d)} \to G(\cdot)$, where

$$G := Gr(n\binom{m+l+d_0}{l+d_0} - P(d), H^0(\mathbb{P}^m_A, \mathcal{E}(d_0)))$$

and $G(\cdot)$ is its functor of points.

Let $q : \mathcal{E}_T \to \mathcal{Q} \in \mathcal{Q}uot_{\mathcal{E},P(d)}(T)$, and let $\mathcal{K}$ be its kernel. Notice that $\mathcal{E}_T$ and $\mathcal{Q}$ are flat over T, hence $\mathcal{K}$ is flat over T, with constant Hilbert polynomial $P'(d) := n\binom{m+l+d}{l+d} - P(d)$ on the fibers.

Now for each such $\mathcal{K}$ we have $\mathcal{K}_t \hookrightarrow \mathcal{E}_t$. Since they are subsheaves of a fixed locally free sheaf for each t, we may apply the uniform vanishing theorem, to deduce that there is d_0 independent of t and the chosen element q such that

$$H^i(\mathbb{P}^m_{k(t)}, \mathcal{K}_t(d)) = 0 forall\ t \in T,\ i > 0,\ and\ d \geq d_0$$

By flatness the sequences

$$0 \to \mathcal{K}_t \to \mathcal{E}_t \to \mathcal{Q}_t \to 0$$

are all exact, and uniform vanishing for $\mathcal{K}_t$ and $\mathcal{E}_t$ implies uniform vanishing for $\mathcal{Q}_t$. These vanishing together with cohomology and base-change theorem (Theorem 1.2.1.5) tells us that the sequence

$$0 \to \pi_*\mathcal{K}(d_0) \to \pi_*\mathcal{E}_T(d_0) \to \pi_*\mathcal{Q}(d_0) \to 0$$

is exact, where $\pi : \mathbb{P}^n_T \to T$ is the projection. Hence this sequence gives a T-valued point of G. Notice that $\pi_*\mathcal{Q}(d_0)$ and $\pi_*\mathcal{K}(d_0)$ of rank $P(d_0)$ and $P'(d_0)$ respectively. This completes the proof of the first step.

Step2 We have already constructed the map to the Grassmannian. As the second step we shall discover the image.
Let K be the universal subbundle on the Grassmannian. Let $Quot(\mathcal{E}, P(d))$ be the term in the flattening stratification of G with respect to $\mathcal{F}$ over which $\mathcal{F}$ is flat with Hilbert polynomial $P(d_0 + d)$.

Consider the cokernel

$$\pi^*K \to \mathcal{E}_G(d_0) \to \mathcal{F} \to 0$$

where the first map is the composition $\pi^*K \to \pi^*\pi_*\mathcal{E}_G(d_0) \to \mathcal{E}_G(d_0)$.

From the uniform vanishing we know that $\mathcal{K}_t(d_0)$ is generated by its global sections, therefore by the cohomology and base change the natural map $\psi : \pi^*\pi_*\mathcal{K}(d_0) \to \mathcal{K}(d_0)$ is surjective. Thus if $T \to G$ is a morphism associated to the quotient $\mathcal{E}_{Quot(\mathcal{E},P(d))} \to \mathcal{F}(-d_0)|_{\mathbb{P}^m_{Quot}}$, then $(*)$ pulls back to

$$\pi^*\pi_*\mathcal{K}(d_0) \to \mathcal{E}_T(d_0) \to \mathcal{Q}(d_0) \to 0$$

where the first map factors through ψ. This proves that the transformation factors through $Quot(.)$ and moreover the universal quotient inverse the transformation.

Step3 By the prior two steps $Quot(\mathcal{E}, P(d))$ is constructed as a quasi-projective sub-scheme of G. Now we want to show that is in fact projective. But actually this is the content of the Theorem 3.1.6 together with the Valuative Criterion.

Step4 In this step we complete the proof for the general case. Let $\mathcal{E}$ be a coherent sheaf on a projective scheme $i : X \subset \mathbb{P}^m_A$. Let T be an S-scheme. Flat quotients of $\mathcal{E}_T$ push forward under i_T to flat quotients of $\mathcal{O}^n_{\mathbb{P}^m_T}$ which determines a T-valued point of $Quot(\mathcal{O}^n_{\mathbb{P}^m_T}, P(d))$. Let $\mathcal{G}$ be the kernel on

$$0 \to \mathcal{G} \to \mathcal{O}^n_{\mathbb{P}^m_{Quot}}(l) \to i_{Quot_*}\mathcal{E}_{Quot} \to 0$$

and choose $d_1 \geq d_0$ so that $\pi^*\pi_*\mathcal{G}(d_1) \to \mathcal{G}(d_1)$ is surjective. We have already seen that $\pi_*\mathcal{Q}(d_1)$ is locally free on Quot by cohomology and base change. It follows that the zero locus of $\pi_*\mathcal{G}(d_1) \to \pi_*\mathcal{Q}(d_1)$ is a closed subscheme $Z \subset Quot(\mathcal{O}^n_{\mathbb{P}^m_A}(l), P(d))$. Over this subscheme,the universal quotient lifts to a quotient: $i_{Z*}\mathcal{E}_Z \to \mathcal{Q}|_{\mathbb{P}^m_Z}$ and it follows that $\mathcal{Q}|_{\mathbb{P}^m_Z}$ is the push-forward of a sheaf on X_Z which is the universal quotient for $\mathcal{E}_Z$. In other words, $Quot(\mathcal{E}, P(d)) := Z$ represents $\mathcal{Q}uot(\mathcal{E}, P(d))$.

$\square$

Definition 4.1.3 *The Hilbert functor is defined as follows*

$\mathcal{H}ilb_{X,P(d)}(T) =\{$*closed sub-schemes* $Z \subseteq X \times_S T$ *which are flat over* T *with Hilbert polynomial* $P(d)\}$

the correspondence of morphisms is clear.

Notice that via the identification $i : Z \hookrightarrow X_T$ with the quotient $\mathcal{O}_{X_T} \to i_*\mathcal{O}_Z$ we have $\mathcal{H}ilb_{X,P(d)} = \mathcal{Q}uot_{\mathcal{O}_X,P(d)}$, therefore we in particular deduce the following corollary.

Corollary 4.1.4 *The functor $\mathcal{H}ilb_{X,P(d)}$ is representable by a projective scheme $Hilb(X, P(d))$.*

4.2 Some examples and applications

Example-Theorem 4.2.1 *Let A be a Dedekind domain and $S = Spec\ A$. Let X be a projective scheme over S. Assume that for infinitely many $s \in S$, the fiber X_s contains a closed variety with Hilbert polynomial $P(d)$ (we say a solution of type $P(d)$). Then the fiber over the generic point η also contains a solution of the same type over a finite field extension of $k(\eta)$.*

*Proof:*Consider the Hilbert scheme $\mathcal{H} := Hilb_{X,P(d)}$. By the theorem 4.1.2, $\mathcal{H}$ is projective. Therefore it consists only finitely many components, this together with our assumption implies that it has a component whose image contains infinitely many primes. On the other hand by properness of $\mathcal{H}$ the image must be closed, thus the component dominate S.

Now any L-rational point in the fiber of the component over S gives the desired solution, where L is a finite field extension of $k(\eta)$. Notice that $\mathcal{H}_\eta(L)$ is non-empty for some finite extension L (c.f. [3], Proposition 3.2.20).
□

Example 4.2.2 *Suppose X is a projective k-scheme for a field k and let $P(d)$ be the Hilbert polynomial of a locally free sheaf $\mathcal{Q}$ of rank $n - m$ which is a quotient of a trivial bundle via $g : \mathcal{O}_X^n \to \mathcal{Q}$. The morphism g gives us a k-rational point $q \in Quot(\mathcal{O}_X^n, P(d))$ and a morphism $f_q : X \to Gr(m, n)$. Note that the universal quotient $\mathcal{O}_{X\times Quot} \to \mathcal{Q}$ is locally free in a neighborhood of $X \times \{q\}$. By properness theorem there is an open neighborhood $q \in U$ in the Quot scheme such that $\mathcal{Q}|_{X\times Quot}$ is locally free of rank $n - m$. Therefore all the points in U parametrize morphisms $X \to Gr(m, n)$. Thus components of the Quot scheme containing U could be seen as a compactification of a space of maps from X to the Grassmannian.*

Example 4.2.3 *Let $P(d)$ be the Hilbert polynomil of a linear subspace $V \subset \mathbb{P}_k^{n-1}$ of dimension $m - 1$. We have an inclusion of projective bundles $P(K) \subset \mathbb{P}_k^{n-1} \times G$, which is flat over G and we have gotten it from the universal subbundle $\mathcal{K} \hookrightarrow \mathcal{O}_G^n$. This gives us a morphism $Gr(n, m) \to H_{\mathbb{P}_k^{n-1},P(d)}$. Moreover one can show that the only projective subschemes of $\mathbb{P}_k^{n-1}$ with Hilbert polynomial $P(d)$ are linear subspaces. Therefore we have an isomorphism $Gr(n, m) \simeq H_{\mathbb{P}_k^{n-1},P(d)}$.*

4.3 Infinitesimal deformations of the Quot scheme

In this section we study local properties of Quot and therefore in particular Hilb scheme by studying infinitesimal deformations. We will derive to a criterion for smoothness of Quot scheme in terms of the vanishing of Ext. Our discussions also leads to a dimension estimate for the Quot scheme.

In this chapter by X we mean a nonsingular projective variety over a field k. Consider the projective scheme $Y := Quot(\mathcal{O}_X^n, P(d))$. Let $q : \mathcal{O}_X^n \to \mathcal{Q}$ be a k-rational point of the Quot scheme, and let $\mathcal{K}$ be its kernel. Giving a vector v in Zariski tangent space to Y at y is equivalent to giving a section $\bar{v}$ which makes the diagram

$$\begin{tikzcd} Spec(k) \arrow[r] \arrow[d] & Y \arrow[d, equal] \\ Spec(k[\epsilon]) \arrow[r, "\bar{v}"] & Y \end{tikzcd}$$

commutative, i.e. giving v is equivalent to giving a $k[\epsilon]$-valued point $\bar{v} : \mathcal{O}^n_{X_{k[\epsilon]}} \to \overline{\mathcal{Q}}$ which extends the quotient $q : \mathcal{O}_X^n \to \mathcal{Q}$. As an example of such a flat extension, we have the trivial one $q \otimes 1$.

Lemma 4.3.1 *Let B be a finitely generated k-algebra, and A is a Noetherian local k-algebra with residue field k. Now suppose that $B_A^n \to \overline{Q}$ is a quotient of B_A-modules with kernel $\overline{\mathcal{K}}$. Then $\overline{Q}$ is flat over A if and only if $\mathfrak{m}_A\overline{\mathcal{K}} = \overline{\mathcal{K}} \cap \mathfrak{m}_A B_A^n$*

Proof: Consider the following commutative diagram

$$\begin{tikzcd} \mathfrak{m}_A\overline{\mathcal{K}} \arrow[r] \arrow[d] & \mathfrak{m}_A B_A^n \arrow[r, "\bar{v}"] \arrow[d] & \mathfrak{m}_A\overline{\mathcal{Q}} \arrow[d] \\ \overline{\mathcal{K}} \arrow[r] \arrow[d] & B_A^n \arrow[r, "q"] \arrow[d] & \overline{\mathcal{Q}} \arrow[d] \\ \mathcal{K} \arrow[r, "\beta"] & B^n \arrow[r] & \mathcal{Q}, \end{tikzcd}$$

where the lower two rows and all the columns are exact. Now $\overline{\mathcal{Q}}$ is flat if and only if β is injective, if and only if $\mathfrak{m}_A\overline{\mathcal{K}} = \overline{\mathcal{K}} \cap \mathfrak{m}_A B_A^n$. Note that $k = A/\mathfrak{m}$ is a test module for flatness (i.e. a finitely generated module M over B_A is flat over A if and only if $Tor_1^A(k, M)$ vanishes).

Proposition 4.3.2 *Let $q \in Quot(\mathcal{O}_X^n, P(d))$, corresponds to a quotient $\mathcal{O}_X^n \to \mathcal{Q}$ with kernel $\mathcal{K}$. The tangent space to $Quot(\mathcal{O}_X^n, P(d))$ at q is isomorphic to $Hom_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q})$ as a vector space over k.*

Proof: First we prove the Theorem for the affine case. So let $B := k[X]$ and suppose that $(A, \mathfrak{m})$ is a local ring with residue field k. Now $\bar{v} : B_A^n \to \overline{Q}$ is a flat extension of $q : B^n \to Q$ (i.e v reduces to q mod $\mathfrak{m}_A$).

Consider the short exact sequence $K \to B^n \to Q$ and let us write a presentation of K inside B^n

$$F' \xrightarrow{r} F \xrightarrow{g} B^n$$

Where F and F' are free k-modules. Since $\bar{v} : B_A^n \to \overline{Q}$ is an extension of $q : B^n \to Q$, the Nakayama's Lemma implies that the above presentation extends to the following presentation of $\overline{K}$

$$F'_A \xrightarrow{\overline{r}} F_A \xrightarrow{\overline{g}} B_A^n$$

For a given lifts $\overline{r}$ and $\overline{g}$, since $\overline{r} \circ \overline{g}$ is zero modulo ϵ, the following composition of morphisms

$$F'_A \xrightarrow{\overline{r}} F_A \xrightarrow{\overline{g}} B_A^n \xrightarrow{q \otimes 1} Q_A$$

descends to a morphism $F' \to \epsilon Q$. Since $q \otimes 1 \circ \overline{g}$ is already zero mod ϵ therefore the morphism $F' \to \epsilon Q$ does not depend upon the lift $\overline{r}$, and thus we can simply denote it by $\varphi_{\overline{g}}$.

Our strategy to work out the theorem, is simply to breaks down the correspondence to the following one to one correspondences

$$T_q Quot(X, P(d))$$

$$\Updownarrow$$

$$\{\overline{v} : B_A^n \to \overline{Q}; \quad \overline{v} \; is \; a \; flat \; extension \; of \; q : B^n \to Q\}$$

$$\Updownarrow$$

$$\{\overline{g} : F_A \to B_A^n; \; \overline{g} \; is \; a \; lift \; of \; g \; s.t. \; \varphi_{\overline{g}} = 0\}/\sim$$

$$(\overline{g} \sim \overline{g}' \Leftrightarrow Im(\overline{g}) = Im(\overline{g}'))$$

$$\Updownarrow$$

$$Hom(K, \epsilon Q),$$

here $A = k[\epsilon]$.

The top correspondence has already described. The second one from up to down, as we mentioned earlier is given by taking a presentation for $\overline{K}$. Vise versa, to an arbitrary lift $\overline{g}$, we associate its $coker(\overline{g}) : B_A^n \to B_A^n/K$, where $K := Im(\overline{g})$. We need also to verify that $coker(\overline{g})$ is flat. For each $\overline{r}$, the image of $\overline{g} \circ \overline{r}$ is in $\overline{K} \cap \epsilon B_A^n$ on the other hand since $\varphi_{\overline{g}} = 0$ thus it is also in $\epsilon\overline{K} = \epsilon K$. Take $x \in \overline{K} \cap \epsilon B_A^n$. Choose $y \in F_A$ such that $\overline{g}(y) = x$. Then the image of y in F is in the image of F', so we can find a $z \in F'_A$ with the same image in F, therefore $\overline{r}(z) - y \in \epsilon F$, and hence $\overline{g} \circ \overline{r}(z) - x \in \epsilon K$. Therefore $\epsilon K = \overline{K} \cap \epsilon B_A^n$, so that from the previous lemma we deduce that $coker(\overline{g})$ is flat.

To prove the last correspondence assume that a lift $\overline{g}$ with $\varphi_{\overline{g}} = 0$ is given, then $(q \otimes 1) \circ \overline{g}$ descends to a morphism $F \to \epsilon Q$ as above, and further to

$$\psi_{\overline{g}} : K \to \epsilon Q$$

We should point out that the above correspondence only depends to the image of $\overline{g}$, and if $\varphi_{\overline{g}} = \varphi_{\overline{g}'}$ then $Im(\overline{g} - \overline{g}') \subseteq \epsilon K$, so they have the same image and thus define the same class.

Finally, any morphism $\psi : K \to \epsilon Q$ gives rise to a morphism $F_A \to Q_A$ which factors through a $\overline{g}$ such that $\psi_{\overline{g}} = \psi$. This completes the proof for the local case. To globalize the result choose an affine cover $\{U_i\}$ where $U_i = Spec(B_i)$. Over each U_i we know that $\overline{v} : \mathcal{O}_{X_A}^n \to \overline{Q}$ gives $\varphi_i \in Hom_{B_i}(K_i, Q_i)$ from the above local version.Since these local data are agree on the overlaps they patch together to define an element in $Hom_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q})$. Vise versa an element of $Hom_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q})$ gives local quotients, which patch to a global extention of q.

$\square$

Using the above theorem and more advance techniques of deformation theorey one has proved the following theorem.

Theorem 4.3.3 *Let $X \to S$ be a projective morphism of algebraic schemes, $\mathcal{F}$ a coherent sheaf, flat over S, and $\pi : Q := Quot(\mathcal{F}, X) \to S$ the associated Quot scheme over S. Let $s \in S$ be a k-rational point and $q \in Q_s$ corresponding to a coherent quotient $f : \mathcal{F} \to \mathcal{Q}$ with kernel $\mathcal{K}$. Let*

$$f_s : \mathcal{F}_s \to \mathcal{Q}_s$$

be the restriction of f to the fiber X_s, whose kernel is $\mathcal{K}_s = \mathcal{K} \otimes \mathcal{O}_{X_s}$(by the flatness of $\mathcal{F}$). Then there is an exact sequence

$$0 \to Hom(\mathcal{K}_s, \mathcal{Q}_s) \to T_qQ \xrightarrow{d\pi_q} T_sS \to Ext^1_{\mathcal{O}_{X_s}}(\mathcal{K}_s, \mathcal{Q}_s)$$

Moreover π is smooth at q if $Ext^1_{\mathcal{O}_{X_s}}(\mathcal{K}_s, \mathcal{Q}_s) = 0$.

Proof: c.f. [9], Proposition 4.4.4.

Now we have enough tools to treat a wide domain of various applications. So let us back to the earth, and give a sort of examples which will give us of course this comprehension that theorems would have something to do with reality. The first part of the following example will in particular illustrate some local properties of the Hilbert scheme of points on a smooth curve, the Hilbert scheme of points will be studied in full generality in the next chapter.

Example 4.3.4 *Let C be a nonsingular projective curve, then the associated Hilbert polynomial $P(d)$ is either linear or constant. So we will first consider the Quot scheme $Quot(\mathcal{O}_C^n, P(d))$ with constant Hilbert polynomial $P(d)$.*

i) Let $P(d) = a$, and q be a k-valued point of $Quot(\mathcal{O}_C^n, a)$, $q : \mathcal{O}_C^n \to \mathcal{Q}_Z$, where $\mathcal{Q}_Z$ is supported on a zero dimensional sub-scheme $Z \subset C$ and the kernel $\mathcal{K}$ is locally free of rank n. By theorem 4.3.3, the tangent space to $Quot(\mathcal{O}_C^n, a)$ at q is:

$$dim_k T_q Quot(\mathcal{O}_C^n, a) = Hom(\mathcal{K}, \mathcal{Q}_Z) = dim_k Hom(\mathcal{O}_C, \check{\mathcal{K}} \otimes \mathcal{Q}_Z)$$

$$= dim_k \Gamma(X, \check{\mathcal{K}} \otimes \mathcal{Q}_Z) = na,$$

on the other hand by the lemma1.2.7

$$Ext^1(\mathcal{K}, \mathcal{Q}) = H^1(X, \check{\mathcal{K}} \otimes \mathcal{Q}_Z)$$

but since $\check{\mathcal{K}} \otimes \mathcal{Q}_Z$ is supported on a zero dimensional locus the last term vanishes, and therefore $Quot(\mathcal{O}_C^n, a)$ is non singular by the above theorem.

ii) Let $P(d) = (n - m)d + b$, where $n \geq m \geq 0$. Consider the following short exact sequence correspond to the point $q \in Quot(\mathcal{O}_C^n, P(d))$

$$0 \to \mathcal{K} \longrightarrow \mathcal{O}_C^n \xrightarrow{q} \mathcal{Q} \to 0$$

Tensoring with $\check{\mathcal{K}}$ and taking long exact cohomology sequence

$$H^1(C, \check{\mathcal{K}} \otimes \mathcal{O}_C^n) \to H^1(C, \check{\mathcal{K}} \otimes \mathcal{Q}) \to 0$$

But the right hand side is isomorphic to $Ext^1(\mathcal{K},\mathcal{Q})$, *by the lemma 1.2.7. Therefore the vanishing of* $H^1(C,\check{\mathcal{K}})$ *implies the vanishing of* $Ext^1(\mathcal{K},\mathcal{Q})$ *and thus the Theorem 4.3.3 implies the smoothness of* $Quot(\mathcal{O}_C^n, P(d))$ *at* q. *The Quot scheme is not in general connected, but this can be proven for the lower dimensional case, namely for curves and surfaces (for instance see [2], section 7.2).*

Let us make a more concrete example.

Example 4.3.5 *Let* $\underline{X} = (X_0, ..., X_3)$ *be the homogeneous coordinates of* $\mathbb{P}^3$ *consider the following family of curves over* $\mathbb{A}^1$ *with parameter* u

$$C_u = Proj(\frac{k[\underline{X}]}{(X_2, X_3)}) \cup Proj(\frac{K[\underline{X}]}{(X_1, X_3 - uX_0)})$$

If $u \neq 0$ *then* C_u *consists of two disjoint lines, while* C_0 *is a degenerated conic. A simple computation shows that*

$$P_u(d) = \begin{cases} 2d+2 & u \neq 0 \\ 2d+1 & u = 0, \end{cases}$$

So from Theorem 3.1.3 we see that $\{C_u\}$ *can not be a the set of fibers of a flat family of closed subschemes of* $\mathbb{P}^3$.

We now try to construct a morphism whose fibers are the C_u*'s. Consider the closed subscheme* $\chi := Proj(\frac{k[u][\underline{X}]}{J}) \subset \mathbb{P}^3 \times \mathbb{A}^1$ *where*

$$J = (X_2, X_3) \cap (X_1, X_3 - uX_0) = (X_1X_2, X_1X_3, X_2(X_3 - uX_0), X_3(X_3 - uX_0)).$$

Since locally on $\frac{k[u][\underline{X}]}{J}$, *any non constant polynomial* $g(u)$ *is not a zero divisor, therefore by the Theorem 3.1.5 the above scheme is flat over* $\mathbb{A}^1$. *The fibers of* χ *are*

$$\chi_u = \begin{cases} C_u & u \neq 0 \\ C_0 \cup Proj(\frac{k[\underline{X}]}{(X_1, X_2, X_3^2)}) & u = 0, \end{cases}$$

Thus we see that χ_0 *has obtained from* C_0 *by adjoining an embedded point in* $(1,0,0,0)$. *Notice that Theorem 3.1.7 ensures us that* χ *is uniquely determined by the fibers over* $\mathbb{A}^1 \backslash \{0\}$.

We use this family to show that $Hilb_{\mathbb{P}^3, 2(d+1)}$ *is singular at* $[X]$, *where* $X := \chi_0$.

If $u \neq 0$ *then* $\chi_u = C_u$ *is a pair of disjoint lines. Thus we have*

$$dim\ Hom_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{I}_{\chi_u}, \mathcal{O}_{\chi_u}) = dim\ Hom_{\mathcal{O}_{\chi_u}}(\mathcal{I}_{\chi_u}/\mathcal{I}^2_{\chi_u}, \mathcal{O}_{\chi_u}) = h^0(\chi_u, \mathcal{N}_{\chi_u}) = 8$$

On the other hand $Ext^1(\mathcal{I}_{\chi_u}, \mathcal{O}_{\chi_u}) = H^1(\chi_u, \mathcal{N}_{\chi_u}) = 0$*, therefore by the Theorem 4.3.3* $c(u)$ *is a nonsingular point and the above computation shows that the tangent space has dimension 8.*

Notice that the family χ *over* $\mathbb{A}^1$ *defines the classifying map*

$$c : \mathbb{A}^1 \to Hilb_{\mathbb{P}^3,2(d+1)}.$$

This shows that $c(0)$ *and* $c(u)$ *belong to the same irreducible component of* $Hilb_{\mathbb{P}^3,2(d+1)}$*. Thus to prove that* $[X]$ *is singular, it is sufficient to show that* $h^0(X, \mathcal{N}_X) > 8$*.*

X *is defined by* $J_0 := (X_1X_2, X_1X_3, X_2X_3, X_3^2)$*. We have the following surjective morphism of graded* $k[X_0, \ldots X_3]$*-modules*

$$k[X_0, \ldots, X_3](-2)^{\oplus 4} \xrightarrow{e} J_o \to 0,$$

the left hand side is shifted in order to get a morphism of degree zero.

Let us extend e *to a resolution of* J_0*. Let* $e_1, \ldots, e_4$ *be the canonical basis for* $k[X_0, \ldots, X_3](-2)^{\oplus 4}$*. The kernel consists of four relations*

$$ker(e) :=< X_3e_1 - X_2e_2, X_3e_2 - X_1e_3, X_3e_2 - X_1e_4, X_3e_3 - X_2e_4 > .$$

So we get

$$r : k[\underline{X}](-3)^{\oplus 4} \to ker(e) \to 0$$

by sending $e'_1 \mapsto X_3e_1 - X_2e_2$*,* $e'_2 \mapsto X_3e_2 - X_1e_3$*,* $e'_3 \mapsto X_3e_2 - X_1e_4$ *and* $e'_4 \mapsto X_3e_3 - X_2e_4$*.*

The kernel of the above morphism is generated only by one element $X_3e_1 - X_3e_2 + X_2e_3 - X_1e_4$*.*

So we get the following resolution for J_0

$$0 \to k[\underline{X}](-4) \xrightarrow{B} k[\underline{X}](-3)^{\oplus 4} \xrightarrow{A} k[\underline{X}](-2)^{\oplus 4} \xrightarrow{e} J_0 \to 0 \qquad (\star)$$

Where A *and* B *are given by the following matrices*

$$A = \begin{bmatrix} X_3 & X_3 & 0 & 0 \\ -X_2 & 0 & X_3 & 0 \\ 0 & -X_1 & 0 & X_3 \\ 0 & 0 & -X_1 & -X_2 \end{bmatrix}$$

$$B = \begin{bmatrix} X_3 \\ -X_3 \\ X_2 \\ -X_1 \end{bmatrix}$$

Taking hat from the above resolution $(\star)$, *we get*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \xrightarrow{B} \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 4} \xrightarrow{A} \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 4} \xrightarrow{e} \mathcal{I}_X \to 0$$

By taking $Hom(-, \mathcal{O}_X)$ *we obtain the following exact sequence*

$$0 \to \mathcal{N}_X \longrightarrow \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 4} \xrightarrow{A^T} \mathcal{O}_{\mathbb{P}^3}(3)^{\oplus 4}$$

Since the global section functor is left exact we get

$$0 \to H^0(X, \mathcal{N}_X) \longrightarrow H^0(X, \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 4}) \xrightarrow{A^T} H^0(X, \mathcal{O}_{\mathbb{P}^3}(3)^{\oplus 4}).$$

Therefore $H^0(X, \mathcal{N}_X)$ *can be identified with*

$$ker[A^T : H^0(X, \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 4}) \to H^0(X, \mathcal{O}_{\mathbb{P}^3}(3)^{\oplus 4})].$$

So we see that the the column vectors of the following matrix

$$\begin{bmatrix} X_1^2 & X_1X_0 & X_2^2 & X_2X_0 & X_3X_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_1^2 & X_1X_0 & X_3X_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_2^2 & X_2X_0 & X_3X_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_3X_0 \end{bmatrix}$$

give a basis for $H^0(X, \mathcal{N}_X)$, *and therefore* $h^0(X, \mathcal{N}_X) = 12 > 8$. *So we have shown that* $[X]$ *is a singular.*

Now, we construct another flat family $\mathcal{Y}$ *over* $\mathbb{A}^1$.

$$\mathcal{Y} := Proj(k[v][X_0, X_1, X_2, X_3]/I)$$

where

$$I = (X_1X_2, X_1X_3 + vX_1X_0, X_2X_3 + vX_2X_0, X_3^2 - v^2X_0^2)$$

$$= (X_1, X_2, X_3 - vX_0) \cap (X_3 + vX_0, X_1X_2).$$

Thus we see that for $v \neq 0$, $\mathcal{Y}_v$ is the disjoint union of a conic and a point. Notice that the Theorem 3.1.6 guarantees the flatness of the family $\mathcal{Y}$ over $\mathbb{A}^1$.

As we mentioned for all $v \neq 0$, $\mathcal{Y}_v = Q_v \cup \{p_v\}$, where Q is a conic and p_v is a point. So we have

$$h^0(\mathcal{Y}_v, \mathcal{N}_{\mathcal{Y}_v}) = h^0(Q_v, \mathcal{N}_{Q_v}) + h^0(Q_{p_v}, \mathcal{N}_{p_v}) = 8 + 3 = 11$$

and $h^1(\mathcal{Y}_v, \mathcal{N}_{\mathcal{Y}_v}) = 0$. Hence for all $v \neq 0$, $\mathcal{Y}_v$ is a nonsingular point of a component of dimension 11 of $Hilb(\mathbb{P}^3, 2(d+1))$. So we realize that $[X]$ belongs to two irreducible components of $Hilb(\mathbb{P}^3, 2(d+1))$ of dimensions 8 and 11. Let us indicate that this example illustrates that the Hilbert schemes are not necessarily irreducible and even equidimensional.

Chapter 5

Hilbert scheme of points

5.1 Introduction

In this chapter we will actually try to study the Example 4.3.4, in more general case, namely when X is a quasi-projective scheme.

Fix an algebraically closed field k. Let X be a quasi-projective scheme over k with an ample line bundle $\mathcal{O}(1)$. The Hilbert scheme $Hilb(X, P(d))$ of X parametrizes all closed, proper sub-schemes of X, with Hilbert polynomial $P(d)$. Set $P(d) = n$, where n is an integer, and set $X^{[n]} := Hilb(X, P(d))$. As the degree of Hilbert polynomial is 0, thus $X^{[n]}$ parametrizes the zero dimentional sub-schemes of length n of X, i.e.

$$dim H^0(Z, \mathcal{O}_Z) = \sum_{p \in Supp(Z)} dim_k(\mathcal{O}_{Z,p}) = n$$

Recall that we have already seen in Example for the simplest case, when $X := C$ is a nonsingular curve, $X^{[n]}$ is also nonsingular. In order to get some impression let us have a look to the simplest case, namely when $X = Spec\ k[x]$ or more generally $X = Spec\ k[x]_S$, where S is a multiplicatively closed subset of $k[x]$.

Example 5.1.1 *Let $t_1, \ldots, t_n$ be independent variables over a commutative ring A. Let $s_1, \ldots, s_n$ denote the elementary symmetric functions in $t_1, \ldots, t_n$. Denote the polynomial ring of symmetric functions as $Sym^n_A = A[s_1, \ldots, s_n]$. Let S denote the multiplicatively closed subset of $A[x]$. It can be shown in elementary fashion that the Hilbert functor of n-points on $Spec\ k[x]_S$ is represented by the spectrum of fraction ring $H := (Sym^n_A)_{S(n)}$, where $S(n) = \{f(t_1) \ldots f(t_n); f \in S\}$ and the universal family is the ideal generated by $\Delta_n(x) \in H \otimes_A A[x]_S$, where $\Delta_n(x) = \prod_{i=1}^n (x - t_i)$ (for the* *details we refer to [11]). As an application we consider the following cases:*

i) Let $S = \{f^i\}$ where $f \in k[x]$. Then Spec$(k[x]_S) = D(f)$ is a basic open subscheme of $\mathbb{A}^1_k$, the affine line over k. The Hilbert scheme of n-points on $D(f)$ is then the spectrum of $(Sym^n_k)_{S(n)}$, where $S(n) = \{(f(t_1)\dots f(t_n))^m\}$. Hence the Hilbert functor of n- points on a basic open subscheme $D(f)$ of the line is represented by a basic open subscheme $D(f(t_1)\dots f(t_n))$ of the Hilbert scheme of n-points on the line.

ii) We produce a Hilbert scheme without rational points. Assume that the base ring A is an integral domain and k its field of fractions. Let $S = A[x]^\times$ be the set of non-zero polynomials. We then have $A[X]_S = k(X)$ is the function field of the line. Clearly, since $k(X)$ is a field, the Hilbert scheme of n-points on Spec$(k(X))$ has no k-valued points. The coordinate ring of the Hilbert scheme is the fraction ring of the symmetric functions Sym^n_k with respect to the set $S(n)$ of products $f(t_1)\dots f(t_n)$, for any non-zero $f \in k[x]$.

The symmetric nth power of X quotiented by the action of $\mathcal{S}_n$, given by the permuting the factors $X^{(n)}$, parametrizes effective 0-cocycles of degree n on X. As a remarkable results of this chapter we will discover a morphism $\rho : X^{[n]} \to X^{(n)}$ and moreover we will prove that is in fact the resolution of singularities of $X^{(n)}$ when X is either a curve or a surface.

5.2 Preliminaries

Definition 5.2.1 *Suppose X is a quasi-projective variety over k and G is a group acting on X (by automorphism).A quotient of X by G is a variety Y together with a surjective morphism $\pi : X \to Y$ which satisfy the following conditions:*
(a) The fibers of π are the orbits of G.
(b) Any G-invariant morphism $\varphi : X \to Z$ to a scheme Z factors through π.

If the quotient exists, is unique up to isomorphism. We denote it by X/G. If G is finite and X is quasi-projective, there exists the quotient of X by G:

Theorem 5.2.2 *If X is a quasi-projective variety with an action of a finite group G then the quotient X/G exists as a variety.*

Proof: We just explain the sketch of the proof. First we consider that X is affine. Denote the affine coordinate ring of X by $k[X]$ Condition (b) tells us that if X/G exists then $k[X/G]$ should be the ring of invariants $k[X]^G \subset k[X]$. As $k[X]^G$ is finitely generated k-algebra, it is natural to define:

$$X/G := Spec(k[X]^G)$$

Now $k[X]^G \hookrightarrow k[X]$ induces a morphism from X to the affine variety $Spec\ k[X]^G$:

$$\pi : X \to X/G$$

One can prove easily that π is surjective and its fibers are the orbits of the action G on X. So we have the result for affine X.

If (U_i) is an affine cover, then we can make another affine cover such that every orbit is contained in one of the U_i. Make W_i in this way:

$$W_i = \bigcap_{g \in G} g(U_i)$$

Now (W_i) is the desired cover which is G-invariant. This is really an open affine cover, because X is quasi-projective and therefore separated, so the intersection of affines is affine. It is not difficult to show that W_i/G glue to give the quotient X/G.

In general case it is not easy to say if there is a quotient of X by G or not.

As a special case of the above theorem, if S_n is the symmetric group which acts on a quasi-projective variety X^n, by permutation of the factors, then the symmetric power $X^{(n)} := X^n/S_n$ exists.

Example 5.2.3 (a) *By using fundamental theorem on symmetric function* $k[x_1, ..., x_n]^{S_n} = k[s_1, ..., s_n]$ *where s_i are the elementary symmetric functions in the x_i. Therefore $(\mathbb{A}^1)^{(n)} = \mathbb{A}^n$. Similarly $(\mathbb{P}^1)^{(n)} = \mathbb{P}^n$.*

(b) *We want to calculate $(\mathbb{A}^2)^{(2)}$. Let x_1, y_1 and x_2, y_2 be the coordinates on the two factors. Now put $x := x_1 - x_2,\ y := y_1 - y_2,\ x' = x_1 + x_2,\ y' = y_1 + y_2$. Obviously $x,\ y,\ x',\ y'$ are also coordinates on $(\mathbb{A}^2)^{(2)}$. But transposition τ of S_2 acts on $(\mathbb{A}^2)^2$ as follows:*

$$\tau(x) = -x,\ \tau(y) = -y,\ \tau(x') = x',\ \tau(y') = y'$$

Therefore $k[x, y, x', y']^\tau = k[x, y]^\tau [x', y']$. Now we have: $\tau(x^i Y^j) = (-1)^{i+j} x^i y^j$ (i.e. every monomial in x, y is an eigenvector for τ). So $k[x, y]$ has a basis of *eigenvectors for τ. Therefore $k[x,y]^{\tau}$ has the basis of all eigenvectors with eigenvalue 1 for τ, i.e. the eigenvectors x^iy^j with $i+j$ even. Put:*

$$u = x^2,\ v = xy,\ w = y^2$$

Therefore $k[x,\ y,\ x',\ y']^{S_2}$ is the subalgebra generated by $u,\ v,\ w,\ x',\ y'$, which is isomorphic to $k[u,v,w,x',y']/(uw-v^2)$. Moreover we see that the singular locus of $(\mathbb{A}^2)^{(2)}$ is the image of the diagonal in $(\mathbb{A}^2)^2$.

5.3 Hilbert-Chow morphism

5.3.1 A Rough Description of The Idea

In this sub-section we would like to evolve the ideas behind the construction of Hilbert-chow morphism. In this description we avoid to work over general base scheme, so fix an algebraically closed field k. Let $H \subset \mathbb{P}^d \times \check{\mathbb{P}}^d$ be the incidence correspondence, i.e. $\{(x,l); x \in l\}$. H is a fiber bundle over $\mathbb{P}^d$ with fiber $\mathbb{P}^{d-1}$. Let Z be a closed sub-scheme, flat, of degree n. Consider the following diagram

$$\begin{array}{ccc} Z^* := p^{-1}Z \hookrightarrow H & \xrightarrow{\check{p}} & \check{\mathbb{P}}^d \\ \quad\quad\quad\quad\ \ {\scriptstyle p}\downarrow & & \\ Z \quad \hookrightarrow \mathbb{P}^d, & & \end{array}$$

where p and $\check{p}$ are the projections. The main idea is to transmit a zero dimensional scheme Z of degree n inside $\mathbb{P}^n$ to a divisor in $\check{\mathbb{P}}^n$ via the above projections, more precisely $H_q := \check{p}(p^{-1}(q))$ is a divisor in $Div^1(\check{\mathbb{P}}^d)$, sending $[Z] \mapsto \sum_{q \in supp(Z)} len(\mathcal{O}_{Z,q})H_q$, defines a morphism

$$\rho : (\mathbb{P}^d)^{[n]} \to Div^n(\check{\mathbb{P}}^d).$$

On the other hand sending $(q_1, \ldots, q_n)$ to $\sum_{q_i} H_{q_i}$ defines a morphism

$$ch : (\mathbb{P}^d)^{(n)} \to Div^n(\check{\mathbb{P}}^d),$$

which is called chern morphism. Now clearly ρ factors through ch

$$\begin{array}{ccc} & (\mathbb{P}^d)^{[n]} & \\ \swarrow & \downarrow & \\ (\mathbb{P}^d)^{(n)} & \to & Div^n(\check{\mathbb{P}}^d). \end{array}$$

Over a general base scheme S, this situation is much harder. Our next task is to plow this ahead!

To this goal we first introduce a construction of Mumford which associates to a coherent sheaf $\mathcal{F}$ on a scheme Y an effective Cartier divisor $div\mathcal{F}$ on Y.

5.3.2 Some homological stuff

Lemma 5.3.2.1 *If $(A, \mathfrak{m})$ is a regular local ring, then:*

a) for every M, $pd(M) \le dimA$.

b) if $k = A/\mathfrak{m}$, then $pd(k) = dimA$.

Proof: c.f. [6], p. 131.□

Lemma 5.3.2.2 *Let A be a regular local ring of dimension n and let M be a finitely generated A-module. Then we have*

$$pd(M) + depth(M) = n$$

Proof: c.f. [6], p. 113.□

Lemma 5.3.2.3 *Let A be a regular local ring and M an A-module that admits a resolution*

$$0 \to A^n \xrightarrow{r} A^n \xrightarrow{g} M \to 0.$$

Then the class of $det(r) \in A/A^\times$ depends only on M and not on the resolution chosen, moreover $det\ r \in A/A^\times$ is not a zero divisor.

Proof: Fix a surjective map $g : A^n \to M$. The possible r' are obtained from r by composing with an automorphism α of A^n. Thus $det(r') = det(\alpha)det(r)$, with $det(\alpha) \in A^\times$.

Let $g : A^n \to M$ be given by the choice of a set $\{g_1, \dots, g_n\}$ of generators of M. We obtain any other choice of generators by successively adding removing generators. Assume that $g' : A^{n+1} \to M$ be given by $g_1, \dots, g_n, x \in M$. Write $x = \sum a_i g_i$. Thus from a given resolution

$$0 \to A^n \xrightarrow{r} A^n \xrightarrow{g} M \to 0.$$

we get a resolution

$$0 \to A^{n+1} \xrightarrow{r'} A^{n+1} \xrightarrow{g'} M \to 0.$$

where $r' = \begin{bmatrix} [r] & 0 \\ a_1 \dots a_n & 1 \end{bmatrix}$, thus $det(r') = det(r)$. □

Lemma 5.3.2.4 *Let $0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \ldots \to \mathcal{E}_0 \to 0$ be an exact sequence of locally free sheaves on a scheme Y. then there is a canonical isomorphism $\bigotimes (det\ \mathcal{E}_i)^{-1^i} \to \mathcal{O}_Y$.*

Proof: c.f. [2], Lemma 7.1.12.

Lemma 5.3.2.5 *Let X be a smooth projective variety of dimension n. let $\mathcal{F}$ be a coherent sheaf on $X \times S$ which is flat over S. Then $\mathcal{F}$ admits a locally free resolution of length n.*

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \ldots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

Proof: c.f. [2], Lemma 7.1.7.

5.3.3 Construction of $div(\mathcal{F})$

Let X be a smooth connected variety over k and Let $\mathcal{F}$ be a coherent sheaf on X with $Supp(\mathcal{F}) \neq X$. For an irreducible hypersurface $V \subset X$ let $[V]$ be its generic point. Then $\mathcal{F}_{[V]}$ is an $A := \mathcal{O}_{X,[V]}$-module. Since A is a DVR, by Lemma 5.3.2.1 and Lemma 5.3.2.2 the projective dimension of $\mathcal{F}_{[V]}$ is equal to one. Thus there exist a free resolution

$$0 \to A^n \xrightarrow{\ r\ } A^n \xrightarrow{\ g\ } M \to 0.$$

Let m_V be the order of vanishing of $det(r)$. Note that the Lemma 5.3.2.3 guarantees that this order m_V is not depend upon the choice of the resolution. So we may define

$$div(\mathcal{F}) = \sum_V m_V[V].$$

Note that the sum is finite because m_V can only be nonzero if $V \subset Supp(\mathcal{F})$.

Our goal is to construct such a divisor in a relative situation. So assume that $\mathcal{F}$ is a coherent sheaf on $X \times S$, flat over S. This is not as good as the above case, but at least we know from 5.3.2.5 that $\mathcal{F}$ admits a locally free resolution of length n

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \ldots \to \mathcal{E}_0 \to \mathcal{F} \to 0.$$

Now let $U \subset X \times S$ be an open subset such that all $\mathcal{E}_i$s are free on U, and $V := U \setminus Supp(\mathcal{F})$. Note that by our assumption V is not empty. We restrict the above resolution to V then the Lemma tells that we have an isomorphism

$$\mathcal{O}_{X\times S}|_V \cong \bigotimes_{i=0}^{n} det(\mathcal{E}_i)^{(-1)^i}|_V$$

On the other hand $det(\mathcal{E}_i)$ is isomorphic to $\mathcal{O}_{X\times S}$ over U; the isomorphism is unique up to a unit. So we have the following isomorphism

$$\bigotimes_{i=0}^{n} det(\mathcal{E}_i)^{(-1)^i}|_U \cong \mathcal{O}_{X\times S}|_U$$

which is unique up to a unit. The composition of the above isomorphism gives a section $f \in \mathcal{O}_{X\times S}(V)^*$ unique up to a section of $\mathcal{O}_{X\times S}(U)^*$. Since V is an open dense subset of U, any section of $\mathcal{O}_{X\times S}(V)^*$ defines a section of $\mathcal{K}^*(U)$. Therefore we may associate to our choice of the resolution and of the open subset U a Cartier divisor on U. It can be shown that the Cartier divisor we have just constructed locally does not depend upon the choice of resolution, and thus glue to give an effective Cartier divisor $div(\mathcal{F})$ on $X \times S$.

Notice that this construction of $div(\mathcal{F})$ is compatible with base change. Indeed for a given morphism $g : T \to S$ the pullback of a resolution of $\mathcal{F}$ $0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \ldots \to \mathcal{E}_0 \to \mathcal{F} \to 0.$ on $X \times S$

$$0 \to (\mathcal{E}_n)_T \to (\mathcal{E}_{n-1})_T \to \ldots \to (\mathcal{E}_0)_T \to \mathcal{F}_T \to 0.$$

is a resolution of $\mathcal{F}_T$. Hence the pullback of $div(\mathcal{F})$ is $div(\mathcal{F}_T)$.

So finally we may counstract the Hilbert-Chow morphism $\rho : X^{[n]} \to X^{(n)}$.Just we have only to modify the situation in subsection 5.2.1 for the relative case. Let H be the incidence correspondence as before. Let $Z \subset \mathbb{P}^n_S$ be a closed subscheme, flat of degree n over S. Let $\mathcal{F} := (\check{p}_S)_*(\mathcal{O}_{Z^*})$. Then $\mathcal{F}$ is a coherent sheaf on $\check{\mathbb{P}}^n_S$, flat over S. Clearly $Supp(\mathcal{F}_s) \neq \check{\mathbb{P}}^n$. Thus $div(\mathcal{F})$ is a relative Cartier divisor on $\check{\mathbb{P}}^n_S$, and we have constructed $\rho : X^{[n]} \to Div^n(\check{\mathbb{P}}^d)$. Since $div(\mathcal{F}_s) = \sum_{q\in supp(Z_s)} len(\mathcal{O}_{Z,q})H_q$, thus we see that the support of the image of $X^{[n]}$ is $X^{(n)}$, so if we give $X^{[n]}$ the reduced structure, the morphism factors through $X^{(n)}$. Thus we have proven the following Theorem

Theorem 5.3.3.1 *Let X be a smooth projective variety. There is a surjective morphism $\rho : X^{[n]}_{red} \to X^{(n)}$, given on the level of points by $Z \mapsto \sum_{p\in Supp(Z)} leg(\mathcal{O}_{Z,p})[p]$.*

5.4 Curves and Surfaces

Let X be a nonsingular quasi-projective scheme of dimension d. Let $X_0^n \subseteq X^n$ be the open set of $(p_0, ..., p_n)$ with the (p_i) distinct. The open subscheme $X_0^n \subseteq X^n$ is clearly dense. Let $X_0^{(n)}$ is given by passing to the quotient via the action of S_n. Since S_n acts freely on X_0^n, thus $X_0^{(n)}$ is nonsingular of dimension nd.
For a given point $[Z]$ in the pre-image of $(p_0, ..., p_n)$, $p_i \neq p_j$ via $\rho : X^{[n]} \to X^{(n)}$ we have $leg(\mathcal{O}_{Z,p_i}) = 1$ and therefore $hom(\mathcal{I}_Z, \mathcal{O}_Z) = nd$. Thus by the theorem 4.3.3 we see that the dimension of the tangent space in the $X_0^{[n]} := \rho^{-1}(X_0^{(n)})$ is also nd.
Moreover $\rho|_{X_0^{[n]}}$ is an isomorphism. Hence $X^{[n]}$ contains a nonsingular open subset, which is isomorphic to an open subset of $X^{(n)}$.

The Hilbert scheme of points is almost singular, but there are two remarkable exceptions, namely the case of curves and surfaces.

Theorem 5.4.1 *(smoothness of Hilb (X, n) for curves) Let C be an irreducible nonsingular quasi projective curve and $n \geq 0$. Then $C^{[n]}$ is nonsingular and irreducible of dimension n.*

Proof: We have already shown (see example 4.3.4) that $C^{[n]}$ is smooth. Now by the connectedness (see 4.3.4), $C^{[n]}$ can not be reducible (otherwise the irreducible components would have to intersect, which violate the smoothness).

Theorem 5.4.2 *(smoothness of $Hilb(X, n)$ for surfaces) Let S be an irreducible nonsingular quasi-projective surface and $n \geq 0$. Then $S^{[n]}$ is nonsingular and irreducible of dimension $2n$.*

Proof: By the previous discussion we know that $S^{[n]}$ is connected and contains an open subset of dimension $2n$. We will prove that the dimension of the tangent space $T_{[Z]}X^{[n]}$ is $2n$ for all $[Z] \in X^{[n]}$. This implies that $\overline{X_0^{[n]}}$ is nonsingular, moreover the connectedness tells us that there is no more irreducible components.
Applying $\mathrm{Hom}(-, \mathcal{O}_Z)$ to $0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$ we get:

$$H^0(Z, \mathcal{O}_Z) \xrightarrow{\sim} H^0(Z, \mathcal{O}_Z) \longrightarrow Hom_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) \longrightarrow Ext^1_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_Z)$$

Thus we see that the tangent space $T_{[Z]}X = Hom_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z)$ sits inside $Ext^1_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_Z)$. So it suffices to prove that $ext^1(\mathcal{O}_Z, \mathcal{O}_Z) \leq 2n$. By the Serre duality we have:

$$Ext^2(\mathcal{O}_Z, \mathcal{O}_Z) = H^0(\mathcal{O}_Z \otimes \mathcal{K}_s)\check{} = k^n$$

Now write a locally free resolution of $\mathcal{O}_Z$ on S:

$$0 \to \mathcal{E}_l \to ... \to \mathcal{E}_0 \to \mathcal{O}_Z \to 0$$

Now we have:

$$\chi(\mathcal{O}_Z, \mathcal{O}_Z) = \sum_{i=0}^{l} (-1)^i \ \chi(\mathcal{E}_i, \mathcal{O}_Z) = \ n \sum_{i=0}^{l} (-1)^i \ rk(\mathcal{E}_i)$$

But since $0 \to \mathcal{E}_l \to ... \to \mathcal{E}_0 \to \mathcal{O}_Z \to 0$ is a locally free resolution, the last term vanishes, and hence:

$$0 = \chi(\mathcal{O}_Z, \mathcal{O}_Z) = ext^0(\mathcal{O}_Z, \mathcal{O}_Z) - ext^1(\mathcal{O}_Z, \mathcal{O}_Z) + ext^2(\mathcal{O}_Z, \mathcal{O}_Z)$$

$$= n - ext^1(\mathcal{O}_Z, \mathcal{O}_Z) + n,$$

i.e. $ext^1(\mathcal{O}_Z, \mathcal{O}_Z) = 2n$. □

Example 5.4.3 *Let X be a nonsingular variety of dimension 3. Let $[Z] \in X^{[4]}$ corresponds to the quotient $\mathcal{O}_Z = \mathcal{O}_p/m^2$, then:*

$$T_{[Z]}X^{[4]} = Hom_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) = Hom_k(m^2/m^3, m/m^2) \simeq k^{18}$$

Therefore the dimension of the tangent space is $> dn = 12$ hence $X^{[4]}$ is singular.

Theorem 5.4.4 *Let C be a nonsingular quasi projective curve, then $\rho : C^{[n]} \to C^{(n)}$ is an isomorphism.*

Proof: As the stalks of $\mathcal{O}_C$ at a closed point p is a discrete valuation ring, all ideals in $\mathcal{O}_{C,p}$ are powers of maximal ideal m_p. Hence for all $[Z] \in C^{[n]}$ we have:

$$\mathcal{O}_Z = \bigoplus_i \mathcal{O}_{C,p_i}/m_{p_i}^{n_i} \quad , \qquad \sum n_i = n$$

and then ρ sends $Z \mapsto \sum n_i[p_i]$.
Then ρ is bijective, since ρ is also birational. The theorem [3], 4.4.6 (Zariski's Main Theorem) implies that ρ is an isomorphism.

□

Let us finish the chapter by giving an example which shows that the above theorems are no longer true, when the dimension of X becomes higher than three, in fact we will show that the irreduciblity may fail when the dimension growths.

Example 5.4.5 *Let X be a nonsingular variety with dimension $d \geq 3$. Let $p \in X$ be a closed point and $\mathfrak{m}_p$ the maximal ideal of $\mathcal{O}_{X,p}$. Let $\varphi_r : \mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/\mathfrak{m}^{r+1}$ be the quotient morphism. Take a sub-vector space V in $\mathfrak{m}_p^r/\mathfrak{m}_p^{r+1}$ of codimension l. We now produce an element $Z \in \rho^{-1}(n[p])$, which is defined by the ideal $I = \varphi_r^{-1}(V)$. Notice that since X is nonsingular thus $\mathfrak{m}_p$ can be generated by a regular sequence of length d. Let us compute the length of $\mathcal{O}_{Z,p}$.*

$$leg(\mathcal{O}_{Z,p}) = l + \sum_{i=0}^{r-1} \binom{i+d-1}{d-1} = l + \binom{d}{d} + \sum_{i=1}^{r-1} \binom{i+d-1}{d-1}$$

$$= l + \binom{d+1}{d} + \sum_{i=2}^{r-1} \binom{i+d-1}{d-1} = ... = l + \binom{r+d-2}{d-1} + \sum_{i=r-1}^{r-1} \binom{r+d-1}{d-1}$$

$$= l + \binom{r+d-1}{d}.$$

Set $n := l + \binom{r+d-1}{d}$. Thus we get a closed sub-scheme of $\rho^{-1}(n[p])$, which is isomorphic to $Gr(l, \binom{n+d-1}{d-1})$. It has the dimension $l.(s-l)$, where $s := \binom{r+d-1}{d-1}$. So if $d \geq 3$ and r is sufficiently large and l is near $s/2$ then it is easy to see that $dim(\rho^{-1}(n[P])) > l(s-l) \geq nd$. On the other hand $X_0^{[n]}$ has dimension nd, thus its closure is an irreducible component of dimension nd which can not contain $(\rho^{-1}(n[P])$. Therefore $X^{[n]}$ is reducible.

For instance if dim $X = 4$ and $r = 8$ we have $s = 165$ and $n = 1683$. Set $l = 78$ then $l.(s-l) = 78.78 = 6786 > 1683.4 = 6732$. Thus $X^{[1683]}$ is reducible.

Bibliography

[1] A. Grothendieck, *Fondements de la Géométrie Algébrique, Extraits du Seminarie Bourbaki (1957-62)*. Lecture Notes in Math. vol. 224, Springer, 1971.

[2] B. Fantechi, *Fundamental Algebraic Geometry, Grothendieck's FGA Explaind (2005).*

[3] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford University Press, 2002.

[4] R. Hartshorne, Algebraic geometry. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[5] D. Mumford, Lectures on curves on an algebraic surface. With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59. Princeton University Press, Princeton, N.J., 1966.

[6] H. Matsumura, Commutative Algebra, W. A. Benjamin Co., New York, 1970

[7] L. Gotische, Hilbert Scheme of Zero-Dimensional Subschemes of Projective Varieties, Lecture Notes in Mathematics(1572), 1991.

[8] D. Eisenbud, Commutative Algebra with view Toward to Algebraic Geometry, Graduate text in Mathematics(150),

[9] E. Sernesi, Deformation of Algebraic Schemes, Springer, A Series of Comprehensives Studies in Math. Volume 334, 2006.

[10] A. Bertram, Lecture notes on Construction of the Hilbert Scheme ,http://www.math.utah.edu/ bertram/courses/hilbert/, 1999.

[11] R. Skjelnes, Resultants and the Hilbert scheme of points on the line , Arkiv fr Matematik, Springer Netherlands, 2006.

[12] J. Harris and I. Morrison, Moduli of Curves, Graduate Texts in Math. 187, Springer, 1991.