Università degli Studi di Padova
Dipartimento di Matematica Pure ed Applicata
Facoltà di Scienze Matematiche, Fisiche e Naturali
Laurea Specialistica in Matematica
Relatore: Professore Luca Barbieri-Viale

Fourier-Mukai Transforms in Algebraic Geometry

Shane Kelly

June 2008

উৎসবকে, যে একাধারে আমার সহপাঠী, বন্ধু এবং ভাই – দুই বছর ধরে একসাথে ওরসে, প্যারিস, অমস্টারাদম, তুলুস, বোর্দো, দিল্লি, মানালি, মুম্বাই, গোয়া, চেন্নাই, কলকাতা, মিউনিখ আর প্যাডোভাতে হারিয়ে যাওয়ার জন্য। হাই ফাইভ!

Contents

Chapter 0

Preface

0.1 Introduction

Derived categories were originally introduced as a framework for derived functors, being used purely as a formal tool rather then being considered as interesting mathematical objects worthy of study in their own right. This has changed drastically over the last ten years. Amongst other positive attributes, it turns out that the derived category of a projective variety appears to be a reasonable invariant.

If a variety has ample canonical (or anti-canonical) bundle, then Bondal and Orlov have shown that the derived category uniquely determines the variety ([BO01] and Theorem 7.2.7 of this work). On the other hand, there are examples of nonisomorphic varieties with equivalent derived categories. The most prominant of these, and also the historical starting point for this theory, was observed by Mukai. He showed that the Poincaré bundle induces an equivalences between the derived category of an abelian variety A and its dual $\hat{A}$.

The way in which the Poincaré bundle induces this equivalence is easily generalized to a procedure which assigns to every object $E \in D^b(X \times Y)$ (called the kernel) an exact functor $\Phi_E : D^b(X) \to D^b(Y)$. Functors of this form are called *Fourier-Mukai transforms.* The main focus of this work is the proof of the following celebrated theorem of Orlov, which gives sufficient conditions for a functor to be a Fourier-Mukai transform:

Orlov's Theorem. *Let $F : D^b(X) \to D^b(Y)$ be an exact functor between the bounded derived categories of coherent sheaves on two smooth projective varieties X and Y over a field. Suppose F is full and faithful and has a right (and consequently, a left) adjoint functor.*

Then there exists an object $E \in D^b(X \times Y)$ such that F is isomorphic to the Fourier-Mukai transform Φ_E and this object is unique up to isomorphism.

It is suspected that every exact functor is a Fouier-Mukai transform although the question is still open (and has been open for some time). The equivalent statement in the world or DG-categories, however, has recently been proven to be true by Toën [Toë07].

There have been some notable improvements on this result in recent years. As a consequence of a result of Bondal and Van Den Bergh [BVdB02] saying that any cohomological functor of finite type is representable, the existence of an adjoint is automatically given. The first major generalization is due to Kawamata [Kaw02] who extended Orlov's Theorem to the case of smooth quotient stacks. More recently, Canonaco and Stellari [CS07] have generalized Orlov's Theorem to twisted coherent sheaves on smooth projective varieties. In addition, they have replaced the hypothesis that F is full and faithful to the hypothesis that $\hom(F(\mathcal{F}), F(\mathcal{G})[j]) = 0$ for all twisted coherent sheaves $\mathcal{F}, \mathcal{G}$ and $j < 0$. As any full functor satisfies this new hypothesis, this is a substantial improvement on Orlov's result. In this thesis however we restrict our attention to a study of Orlov's proof of his result as stated above.

Coming back to the view of derived categories as invariants, if we notice that the K-theory and cohomology functors factor through the derived category, we might be lead to ask whether there is any connection to the world of motives. This qustion is developed in [Orl05] where a procedure is discussed for associating a morphism of motives $M(X) \to M(Y)$ to every Fourier-Mukai transform $\Phi_E : D^b(X) \to D^b(Y)$.

In Chapter 1 we develop from scratch the background material that we need about derived categories. Proving results in the generality of triangulated categories is often a pleasant exercise in itself and we continue this in the next two chapters, discussing ample sequences in Chapter 2 and Postnikov systems in Chapter 3. Ample sequences are a formalism that packages basic properties of the tensor powers of an ample line bundle on a smooth projective variety. Postnikov systems are a kind of iterated cone (of morphisms of complexes), that allows us to convert a bounded complex of objects in an abstract triangulated category into a single object. In both of these chapters we develop only the material that is relevant for the proof of Orlov's Theorem.

In Chapter 4 we discuss the famous Beilinson resolution of the structure sheaf of the diagonal $\mathcal{O}_\Delta$ on $\mathbb{P}^n \times \mathbb{P}^n$, as well as the consequences of this that we need to prove Orlov's Theorem. In some sense this is the main tool, as it is how Orlov explicitely constructs a kernel for the functor F, whereas the theory of Postnikov systems and ample sequences are more technical formalisms that help us manage the conversion of a complex into a single object (the kernel), and construct the natural isomorphism between F and Φ_E. Their rôle should not be downplayed too much though, and the length of these chapters should give an idea as to how involved these procedures are.

Chapter 5 gives a very brief overview of Fourier-Mukai transforms before we get to Chapter 6, where we present the final arguments of the proof.

In Chapter 7 we move in quite a different direction and give an exposition of the ideas in [Orl05], which connect Fourier-Mukai transforms to the world of motives. In an attempt to better understand one of the conjectues of [Orl05], we consider the case of a Fourier-Mukai transform between the derived categories of two abelian varieties. This leads us to give a very condensed exposition of the ideas of [Orl02], which develops the theory of Fourier-Mukai transforms between abelian varieties, itself an interesting topic.

In this final chapter (Chapter 7) we present five conjectures of increasing specifity, all of which are still open. The first three of these are lifted directly from [Orl05] whereas Conjecture 4 and Conjecture 5 were communicated by Orlov to the author after a suggestion from Luca Barbieri-Viale and Paolo Stellari to consider the case of abelian varieties.

It is possibly worth mentioning that Corollary 7.2.9 and Example 7.2.5 are without reference. However, the former is a direct corollary of already known theorems and the latter a relatively simple extension to our context of examples discussed in [Huy06] and so they hardly qualify as “new”.

The chapters that deviate the most from the current literature are Chapter 3 and Chapter 6 which both have as their sole reference the original paper [Orl96] of Orlov. While not containing anything not found in [Orl96], the material has been restructured and embelished in an attempt to make this exposition clearer.

As for the recent improved proof and generalization of Orlov's Theorem of Canonaco and Stellari [CS07], as already mentioned we restrict our study to the original proof of the original statement contained in [Orl96]. The material on ample sequences and Postnikov systems developed here is still relevant and Canonaco and Stellari use these results in their proof, citing [Orl96]. Instead of using the Beilinson resolution of the diagonal, they use an arbitrary resolution of the diagonal of a similar form obtained from an ample sequence. Among other things, this means that they don't have to bother with the messy business of embedding the first variety X in projective space. Since the Beilinson resolution is such an important tool in the study of derived categories of smooth projective varieties, we don't feel that any harm is done by taking the older, more concrete path in this exposition.

Acknowledgements

Firstly, I would like to thank the ALGANT program for giving me the fantastic opportunity of studying for two years in Europe. It has helped me to improve my French and Italian, experience life in countries with histories that extend far past the age of Australia, enjoy a density of mathematics that is impossible in a country as sparsely spread as mine, and in particular study algebraic geometry, which again is very near to impossible in Australia. I am also extremely greatful for the generosity of the scholarship which helped ease the difficulties of living in countries with such different attitudes towards so many things.

I would like to thank those professors in Padova who I've had contact with: for the freedom I was given to choose study material that matched my frequently changing interests, and for the administrative work that made this possible. In particular, I would like to thank Frank Sullivan for suffering the student seminar on stacks and Luca Barbieri-Viale for offering me such interesting thesis topics and his support of the one I eventually chose.

For the thesis I would like to thank Luca Barbieri-Viale and Paolo Stellari for their reading and comments, Paolo Stellari for the help he gave me on the conjectures and for pointing out the improvements of Orlov's result, and Marco Maculan for the overabundance of help with the introduction.

Finally, I would like to thank my fellow students, flatmates, and of course Europe, for

offering me so many culturally enriching things to fill the time between mathematics.

0.2 Introduzione

Quando furono introdotte, le categorie derivate erano usate solo come struttura in cui poter lavorare coi funtori derivati e non erano studiate indipendentemente. Il punto di vista è cambiato molto durante gli ultimi dieci anni: tra i vari aspetti interessanti, pare che la categoria derivata di una varietà proiettiva sia un invariante ragionevole.

Se una varietà ha un fascio canonico (o anti-canonico) ampio, allora Bondal ed Orlov hanno mostrato che la categoria derivata identifica unicamente la varietà [BO01]. D'altra parte, ci sono degli esempi di varietà tra loro non isomorfe che hanno categorie derivate equivalenti. Fra queste, l'esempio prevalente – che è anche il punto di partenza di tale teoria – è stato osservato da Mukai. Lo stesso ha mostrato che il fascio di Poincaré induce un'equivalenza fra la categoria derivata di una varietà abeliana A e la sua duale $\hat{A}$.

Il modo in cui il fascio di Poincaré induce questa equivalenza si può generalizzare facilmente in un procedimento che dà, per ogni oggetto $E \in D^b(X \times Y)$ (chiamato il nucleo), un funtore esatto $\Phi_E : D^b(X) \to D^b(Y)$. I funtori formati in questo modo si chiamano *trasformate di Fourier-Mukai*. Il fulcro di questo lavoro è la dimostrazione del seguente celebrato teorema di Orlov:

Teorema di Orlov. *Sia $F : D^b(X) \to D^b(Y)$ un funtore esatto fra le categorie derivate di fasci coerenti di due varietà lisce proiettive X ed Y. Supponiamo che F sia pieno, fedele e dotato di aggiunto a destra (e quindi anche a sinistra).*

Allora, esiste un oggetto $E \in D^b(X \times Y)$ tale che F è isomorfo a Φ_E, ed è unico a meno di isomorfismi.

Si sospetta che ogni funtore esatto sia un trasformato di Fourier-Mukai sebbene la questione sia ancora aperta. Il problema equivalente nel mondo delle DG-categorie, invece, è stato risolto recentemente da Toën [Toë07].

Negli ultimi anni sono stati apportati sensibili miglioramenti: l'esistenza di un aggiunto segue automaticamente da un risultato di Bondal e Van Den Bergh [BVdB02], per il quale ogni funtore coomologico di tipo finito è rappresentabile. La prima importante generalizzazione si deve a Kawamata [Kaw02] che ha esteso il Teorema di Orlov al caso di uno stack quoziente liscio; recentemente, invece, Canonaco e Stellari [CS07] hanno generalizzato il Teorema di Orlov a fasci coerenti twistati su varietà proiettive lisce. Inoltre, l'ipotesi che F sia pieno e fedele è stata da loro rimpiazzata dalla richiesta che $\hom(F(\mathcal{F}), F(\mathcal{G})[j]) = 0$ per tutti i fasci coerenti twistati $\mathcal{F}, \mathcal{G}$ e $j < 0$: poiché ogni funtore pieno soddisfa quest'ultima condizione, si tratta di un sostanziale miglioramento del risultato. Nella tesi, comunque, ci focalizzeremo sul Teorema di Orlov – e della sua dimostrazione – come enunciato sopra.

Ritorniamo al punto di vista in cui le categorie derivate sono viste come degli invarianti. Visto che la K-teoria e la coomologia di una varietà provengono dalla sua categoria derivata, ci si può chiedere se ci sia o meno un rapporto col mondo di motivi. Questa è l'idea stata sviluppata in [Orl05], dove si esibisce un procedimento attraverso il quale, data una trasformata di Fourier-Mukai $\Phi_E : D^b(X) \to D^b(Y)$, si ottiene un morfismo di motivi $M(X) \to M(Y)$.

Nel Capitolo 1 svilupperemo da zero il materiale che ci servirà delle categorie derivate. Dimostrare i risultati nella generalità delle categorie triangolate può essere divertente e, infatti, proseguiremo con tale metodo: discuteremo le sequenze ampie nel Capitolo 2 e i sistemi di Postnikov nel Capitolo 3. Le sequenze ampie sono un formalismo astratto delle potenze tensoriali di un fascio ampio invertibile sopra una varietà proiettiva e liscia, mentre i sistemi di Postnikov sono un tipo di cono ripetuto nel senso delle categorie triangolate.

Nel Capitolo 4 discutiamo la famosa risoluzione di Beilinson del fascio strutturale della diagonale $\mathcal{O}_\Delta$ su $\mathbb{P}^n \times \mathbb{P}^n$ e delle sue conseguenze che ci serviranno per dimostrare il Teorema 1. In un certo senso questo è lo strumento principale, perché Orlov lo usa per costruire esplicitamente il nucleo del funtore F. I sistemi di Postnikov e le sequenze ampie sono formalismi piuttosto tecnici: ci aiuteranno, rispettivamente, a trasformare una versione alterata della risoluzione di Beilinson in un oggetto solo – che giocherà il ruolo del nucleo – e a costruire l'isomorfismo naturale fra F e Φ_E.

Il Capitolo 5 fornirà una breve visione d'insieme della trasformata di Fourier-Mukai prima di affrontare gli argomenti finali della dimostrazione, presentati nel Capitolo 6.

Nel Capitolo 7 ci muoveremo in una direzione diversa: daremo un'esposizione delle idee di [Orl05], collegamento tra le trasformate di Fourier-Mukai e il mondo dei motivi. Inoltre, considerando il caso di una transformata di Fourier-Mukai tra le categorie derivate di due varietà abeliane, tenteremo di capire meglio una delle congetture di [Orl05]. Questo ci spinge a presentare brevemente le idee di [Orl02], dove la teoria delle trasformate di Fourier-Mukai viene sviluppata fra varietà abeliane, un argomento di per sè molto interessante.

In questo capitolo finale (Capitolo 7) formuleremo cinque congetture di crescente specifità, tutte ancora aperte. Le prime tre sono prese direttamente da [Orl05], mentre Conjecture 4 e 5 sono state communicate da Orlov all'autore sequendo il suggerimento di Luca Barbieri-Viale e Paolo Stellari di considerare il caso di varietà abeliane.

Probabilmente è meglio rimarcare che Corollary 7.2.9 e l'Esempio 7.2.5 non hanno riferimanti bibliografici. Communque, il corollario seque direttamente da teoremi già conosciuti e gli esempi sono adattamenti, relativamente facili, al nostro contesto di esempi discussi in [Huy06] e difficilmente, quindi, qualificabili come "nuovi".

I capitoli che più si distanziano dalla letteratura corrente sono il Capitolo 3 e il Capitolo 6, avendo entrambi solamente l-articolo originale [Orl96] di Orlov come riferimento. Pur non contenendo niente non che non fosse già presente in [Orl96], si è cercato di organizzare e abbellire il materiale al fine di rendere più chiara l'esposizione.

Come detto sopra, per quanto riguarda il recente miglioramento della dimostrazione e la generalizzazione del teorema di Orlov da parte di Canonaco e Stellari [CS07], ci restringeremo alla dimostrazione dell'enunciato originale contenuto in [Orl96]. Il materiale presente nell'articolo concernente le sequenze ampie ed i sistemi di Postnikov è tutt'oggi rilevante: infatti, Canonaco e Stellari usano ancora tali risultati. Tuttavia nel loro lavoro viene usata, invece della risoluzione di Beilinson, una risoluzione della diagonale arbitraria, di forma simile, ottenuta da una sequenza ampia. Tra l'altro, ciò significa che non ci si deve preoccupare di immeregere la prima varietà X in uno spazio proiettivo. Comunque, visto che la risoluzione di Beilinson è uno strumento così importante nello studio delle categorie derivate di una varietà proiettiva liscia, non è male, in questa esposizione, seguire la strada vecchia e più concreta.

Chapter 1

Triangulated categories

In this chapter we present the basic theory behind derived categories. We begin with the basic definitions and properties of triangulated categories, then discuss triangulated categories that are obtained from abelian categories – derived categories. This material is available in a number of places (for example [Ver96], [Har66], [Wei94], [Huy06], and in particular [Noo07] which in the authors opinion, towers over the others pedagogically) and for this reason we don't give further references.

1.1 Definition and basic properties

Notation 1.1.1. Given a category $\mathcal{D}$ equipped with a autoequivalence $T : \mathcal{D} \to \mathcal{D}$ we use the notation $X[n]$ for $T^n X$ for an object X of $\mathcal{D}$ and $f[n]$ for $T^n f$ for a morphism f of $\mathcal{D}$. We only use the label T explicitely if there are more than one autoequivalences and confusion is likely to arise.

Definition 1.1.2. Let $\mathcal{D}$ be a category equipped with an autoequivalence. A *triangle* is a diagram of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

and a morphism of triangles is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

In this way, for any category with an autoequivalence we obtain a category of triangles. Using the usual conventions of category theory we can then talk about, in particular, isomorphisms of triangles.

Definition 1.1.3. A *triangulated category* is an additive category $\mathcal{D}$ equipped with an additive autoequivalence called the *shift* or *translation* functor and a set of *distinguished* triangles (sometimes called *exact* triangles) which satisfy the following axioms:

TR1 (a) Any triangle of the form

$$A \xrightarrow{id} A \longrightarrow 0 \longrightarrow A[1]$$

is distinguished.

(b) Any triangle isomorphic to a distinguished triangle is distinguished.

(c) Any morphism $f : A \to B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]$$

TR2 The triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a distinguished triangle if and only if the triangle

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is distinguished.

TR3 Suppose there exists a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

with the rows being distinguished triangles. Then there exists a morpism $h : C \to C'$ completing the diagram into a commutative diagram.

TR4 The octahedron axiom. This axiom is omitted as it is difficult to state and is not used in this present work.

Remark 1.1.4. In practice, most triangulated categories are built from certain kinds of complexs of objects in an abelian category. Heuristically, the triangules should correspond to short exact sequences, and this is where a lot of intuition about how the triangules should behave comes from (c.f. Proposition 1.3.3). In particular, we can make the following interpretations: TR1(c) says that any morphism has a cokernel, TR3 says that a morphism preserving a subobject passes to a morphism of cokernels, and TR4 (if it we here) can be interpretted in some sense as the Third Isomorphism Theorem ($(G/H)/(K/H) \cong G/K$).

Remark 1.1.5. We actually do use the octahedral axiom in the sense that it is used to prove the Nine Lemma (the triangulated category version) which we use at the end of the proof of Lemma 2.2.4. All of the octahedral axiom, the Nine Lemma, and its proof would add significant typesetting overhead compared to the small (but important) rôle it plays and so it was decided to ommit them.

Lemma 1.1.6. *Suppose we have a distinguished triangle*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

in a triangulated category $\mathcal{D}$. Then $g \circ f = 0$.

Proof. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "id"] \arrow[d, "id"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & A[1] \arrow[d, "f[1]"] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "h"] & A[1] \end{tikzcd}$$

By TR1 the upper row is distinguished and by assumption the lower row is distinguished. Hence, by TR3 there exists a morphism $0 \to C$ completing the diagram to a commutative diagram. Since in an additive category the only morphisms that have the zero object as target or source are trivial morphisms, this implies that the composition $A \to 0 \to C$ is trivial and therefore $g \circ f = 0$. $\square$

Lemma 1.1.7. *Suppose we have a distinguished triangle*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

in a triangulated category $\mathcal{D}$. Then for any object X the following two induced sequences are exact

$$\hom(X, A) \xrightarrow{f_*} \hom(X, B) \xrightarrow{g_*} \hom(X, C)$$

$$\hom(C, X) \xrightarrow{g^*} \hom(B, X) \xrightarrow{f^*} \hom(A, X)$$

Proof. We prove only that the first sequence is exact as the proof of the exactness of the second one is analogous. Suppose $b : X \to B$ is a morphism which when composed with $g : B \to C$ is the trivial morphism. Then we have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "b"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[r, "-id"] & X[1] \arrow[d, "b[1]"] \\ B \arrow[r, "g"] & C \arrow[r, "h"] & A[1] \arrow[r, "-f[1]"] & B[1] \end{tikzcd}$$

The rows are distinguished triangles by TR1 and TR2 and so by TR3 the morphism $b[1] : X[1] \to B[1]$ lifts to a morphism $X[1] \to A[1]$ and hence, $b : X \to B$ lifts to a morphism $X \to A$. So the Ker $g_* \subseteq$ Im f_*. It follows from Lemma 1.1.6 that Im $f_* \subseteq$ Ker g_*, hence, the sequence is exact. $\square$

Corollary 1.1.8. *Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ be a distinguished triangle in a triangulated category. Then for any object X we have two long exact sequences:*

$$\cdots \to \hom(X, A[i]) \xrightarrow{f_*} \hom(X, B[i]) \xrightarrow{g_*} \hom(X, C[i]) \xrightarrow{h_*} \hom(X, A[i+1]) \to \cdots$$

$$\cdots \to \hom(A[i+1], X) \xrightarrow{h^*} \hom(C[i], X) \xrightarrow{g^*} \hom(B[i], X) \xrightarrow{f^*} \hom(A[i], X) \to \cdots$$

Note: the morphism f_*, for example, is that defined by composition with $A[i] \xrightarrow{(-1)^i f[i]} B[i]$.

Proof. This follows directly from Lemma 1.1.7 and Axiom TR2. □

Lemma 1.1.9. *Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ be a distinguished triangle. Then $h = 0$ if and only if $B = A \oplus C$ with g the obvious projection.*

Proof. Applying $\hom(C, -)$ to the triangle we obtain a long exact sequence by Corollary 1.1.8. If $h = 0$ then $h_* = 0$ and so the identity morphism in $\hom(C, C)$ has a section $s \in \hom(C, B)$. Similarly, applying $\hom(-, A)$ to the triangle we have a long exact sequence and since $h^* = 0$ the identity morphism in $\hom(A, A)$ lifts to $t \in \hom(B, A)$. So now we can show that $A \oplus C \xrightarrow{f+s} B$ has inverse $B \xrightarrow{(t,g)} A \oplus C$.

Conversely, if $B = A \oplus C$ with f and g the obvious inclusion and projection then g has a section s and so $h = h \circ id = h \circ g \circ s = h_* g_*(s)$ where h_* and g_* are now morphisms in the long exact sequence coming from applying $\hom(C, -)$ to the triangle. Since the sequence is exact we find that $h = 0$. □

Lemma 1.1.10 (The Five Lemma.)**.** *Consider a morphism of distinguished triangles:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "b"] \arrow[d, "g"] & C \arrow[r, "h"] \arrow[d, "c"] & A[1] \arrow[d, "a[1]"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'[1]
\end{tikzcd}$$

If any two of a, b, c are isomorphisms then so is the third.

Proof. We prove the case where a and b are isomorphism, the other two cases follow from Axiom TR2. The lemma essentially follows from Yoneda's Lemma and the usual Five Lemma. Let X be an arbitrary object of the triangulated category and consider the diagram:

$$\begin{tikzcd}
\hom(X, A) \arrow[r] \arrow[d, "a_*"] & \hom(X, B) \arrow[r] \arrow[d, "b_*"] & \hom(X, C) \arrow[r] \arrow[d, "c_*"] & \hom(X, A[1]) \arrow[r] \arrow[d, "a[1]_*"] & \hom(X, B[1]) \arrow[d, "b[1]_*"] \\
\hom(X, A') \arrow[r] & \hom(X, B') \arrow[r] & \hom(X, C') \arrow[r] & \hom(X, A'[1]) \arrow[r] & \hom(X, B'[1])
\end{tikzcd}$$

The rows are exact sequences as a consequence of Corollary 1.1.8. Since a and b are isomorphisms all the morphisms $a_*, b_*, a[1]_*, b[1]_*$ are all isomorphisms and so it follows from the Five Lemma that c_* is also an isomorphism. Since X was arbitrary this means that c induces an isomorphism of functors $\hom(-, C) \cong \hom(-, C')$. Since the Yoneda embedding is fully faithful this means the original morphism c is an isomorphism. □

Lemma 1.1.11. *Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ be a distinguished triangle in a triangulated category. Then a is an isomorphism if and only if C is zero.*

Proof. First suppose that C is zero, let X be another object in the category and consider the exact sequence from Corollary 1.1.8:

$$\hom(X, C[-1]) \longrightarrow \hom(X, A) \xrightarrow{f_*} \hom(X, B) \longrightarrow \hom(X, C)$$

Since C is zero so is $C[-1]$ and so the first and last hom groups are zero showing that f_* is an isomorphism. Since X was arbitrary this means the natural transformation $f_* : \hom(-, A)\, \hom(-, B)$ is a natural isomorphism. Since the Yoneda embedding is full and faithful this means that the morphism f is an isomorphism.

Conversely, suppose that f is an isomorphism and consider the exact sequence from Corollary 1.1.8:

$$\hom(C, A) \xrightarrow{f_*} \hom(C, B) \longrightarrow \hom(C, C) \longrightarrow \hom(C, A[1]) \xrightarrow{f_*} \hom(C, B[1])$$

The morphisms f_* are isomorphisms and so $\hom(C, C) = 0$. In an additive category, any object with this property is isomorphic to the zero object. Hence, C is isomorphic to the zero object. $\square$

Lemma 1.1.12. *Let b be a morphism between two objects B and B' in two distinguished triangles*

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] & A[1] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'[1] \end{tikzcd}$$

If $g'bf = 0$ then there exist morphisms $a : A \to A'$ and $c : C \to C'$ such that the triple (a, b, c) is a morphism of triangles.

If in addition, $\hom(A[1], C') = 0$ then this triple is uniquely determined by the morphism b.

Proof. Since $g'bf = 0$ the morphism $A \xrightarrow{bf} B'$ is in the kernal of the induced morphism $g'_* : \hom(A, B') \to \hom(A, C')$. By Corollary 1.1.8 this means it is in the image of $a'_* : \hom(A, A') \to \hom(A, B')$ and so we obtain a morphism $A \xrightarrow{a} A'$ that makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] & A[1] \arrow[d, "a[1]"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'[1] \end{tikzcd}$$

We can then apply TR3 to find a morphism $c : C \to C'$ such that (a, b, c) is a morphism of triangles.

Now suppose that $\hom(A[1], C') = 0$. The translation functor is an autoequivalence so this implies that $\hom(A, C'[-1]) = 0$. Combining this with the exact sequence

$$\hom(A, C'[-1]) \xrightarrow{h'_*} \hom(A, A') \xrightarrow{f'_*} \hom(A, B')$$

of Corollary 1.1.8 we see that the morphism $f'_* : \hom(A, A') \to \hom(A, B')$ used to obtain a is injective and therefore there is a unique choice for a. Now suppose that there is a second morphism $C \xrightarrow{c'} C'$ such that (a, b, c') is also a morphism of triangles. Consider the exact sequence

$$\hom(A[1], C') \xrightarrow{h^*} \hom(C, C') \xrightarrow{g^*} \hom(B, C')$$

obtained again from Corollary 1.1.8. Since (a, b, c) and (a, b, c) are both morphisms of triangles we have $cg = g'b = c'g$ and so $c - c' \in \hom(C, C')$ is in the kernel of g^*. But by assumption $\hom(A[1], C')$ is zero and so g^* is injective. Hence, $c = c'$. $\square$

1.2 Exact functors

In this section we define exact functors between triangulated categories and prove that adjoints of exact functors are exact.

Definition 1.2.1. Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories and denote their shift functors by T and T' respectively. An additive functor $F : \mathcal{D} \to \mathcal{D}'$ is called *exact* if:

1. It commutes with the shift functors. That is, there is a fixed natural isomorphism
$$t_F : F \circ T \cong T' \circ F$$
2. It takes every distinguished triangle in $\mathcal{D}$ to a distinguished triangle of $\mathcal{D}'$ (using the isomorphism t_F we replace $F(TA)$ by $T'F(A)$).

Proposition 1.2.2 ([BK90])**.** *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories $\mathcal{D}, \mathcal{D}'$ and denote the shift functors by T and T' respectively. If F has a left (or right) adjoint $G : \mathcal{D}' \to \mathcal{D}$ then G is also exact.*

Proof. Since F is exact there are natural isomorphisms $F \circ T \cong T' \circ F$ and $T'^{-1} \circ F \cong F \circ T^{-1}$. So for any objects A in $\mathcal{D}$ and B in $\mathcal{D}'$ we have

$$\begin{aligned}
\hom_{\mathcal{D}}(A, GT'B) &\cong \hom_{\mathcal{D}'}(FA, T'B) \\
&\cong \hom_{\mathcal{D}'}(T'^{-1}FA, B) \\
&\cong \hom_{\mathcal{D}'}(FT'^{-1}A, B) \\
&\cong \hom_{\mathcal{D}}(T'^{-1}A, GB) \\
&\cong \hom_{\mathcal{D}}(A, T'GB)
\end{aligned}$$

All of these isomorphisms are natural in both objects and so we have an isomorphism of functors $\hom(-, GT'B) \cong \hom(-, T'GB)$ for any object B of $\mathcal{D}'$ and this isomorphism is natural in B. It follows from Yoneda's lemma then that $GT'B \cong T'GB$ and so we obtain a natural isomorphism $GT' \cong T'G$.

Now we show that distinguished triangles get taken to distinguished triangles. Suppose that

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a distinguished triangle in $\mathcal{D}'$ and consider the morphism $Gf : GA \to GB$. Using T1 we complete this to a distinguished triangle

$$GA \xrightarrow{Gf} GB \xrightarrow{g'} C' \xrightarrow{h'} TGA \tag{1.1}$$

in $\mathcal{D}$. We push this triangle through F and use the adjunction natural transformation $FG \to id$ to obtain a diagram that can be completed to a morphism of triangles using TR3:

$$\begin{tikzcd} FGA \arrow[r, "FGf"] \arrow[d] & FGB \arrow[r, "Fg'"] \arrow[d] & FC' \arrow[r, "Fh'"] \arrow[d, dotted] & TFGA \arrow[d] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "h"] & TA \end{tikzcd} \tag{1.2}$$

We then push everything back through G and use the other adjunction natural transformation $id \to GF$ to obtain a morphism between Diagram (1.1) and Diagram (1.2):

$$\begin{tikzcd} GA \arrow[r, "Gf"] \arrow[d] & GB \arrow[r, "g'"] \arrow[d] & C' \arrow[r, "h'"] \arrow[d] & TGA \arrow[d] \\ GFGA \arrow[r, "GFGf"] \arrow[d] & GFGB \arrow[r, "GFg'"] \arrow[d] & GFC' \arrow[r, "GFh'"] \arrow[d, dotted] & TGFGA \arrow[d] \\ GA \arrow[r, "Gf"] & GB \arrow[r, "Gg"] & GC \arrow[r, "Gh"] & TGA \end{tikzcd}$$

It is a formal consequence of the definition of adjoint that the composition $GA \to (GF)GA = G(FG)A \to GA$ is the identity and so we thus obtain the following morphism of triangles

$$\begin{tikzcd} GA \arrow[r, "Gf"] \arrow[d, equal] & GB \arrow[r, "g'"] \arrow[d, equal] & C' \arrow[r, "h'"] \arrow[d] & TGA \arrow[d, equal] \\ GA \arrow[r, "Gf"] & GB \arrow[r, "Gg"] & GC \arrow[r, "Gh"] & TGA \end{tikzcd}$$

It now follows from the Five Lemma 1.1.10 that $C' \to GC$ is an isomorphism. Hence, $GA \to GB \to GC \to TGA$ is isomorphic to a distingusihed triangle and is therefore itself distinguished by TR1(b). $\square$

1.3 Derived categories of abelian categories

We now arrive at an important class of triangulated categories – derived categories of abelian categories. In this section we prove some results that we need in Section 2. As it turns out, most of these results also highlight the strong relationship between an abelian category and its derived category (c.f. Proposition 1.3.3, Proposition 1.3.4, Proposition 1.3.6). We ommit the construction of the derived categories due to space restrictions. As already mentioned, this material (and the construction details) is available in [Ver96], [Har66], [Wei94], and [Noo07], where the latter gives the most accessible account.

Remark 1.3.1. A word of motivation: Derived categories were the impetus for Verdier [Ver96] to develop the machinery of triangulated categories (although there are important triangulated categories in topology see [Wei94, Section 10.9] for a brief account). The "problem" is that when using abelian categories (for example, categories of sheaves or modules) it is an all-too-common procedure to replace an object by a complex, and work with the complex instead. Examples of this are using projective (injective) resolutions to define (co)homology, or taking a free resolution of a finitely generated module. This suggests that complexes are more fundamental objects, but in the category of complexes, an object and its resolution are not always isomorphic. The derived category is an alteration of the category of complexes which fixes this problem (c.f. Proposition 1.3.6 and Proposition 1.3.4). Once the work is done to build the derived category, using language of derived categories and derived functors makes the statement of many results a much cleaner.

Notation 1.3.2. Let $\mathcal{A}$ be an abelian category. We use the following notation:

$Ch(\mathcal{A})$ The category of chain complexes.

$K(\mathcal{A})$ The homotopy category of chain complexes. This has the same objects as $Ch(\mathcal{A})$ but the hom groups are homotopy equivalence classes of morphisms.

$D(\mathcal{A})$ The derived category of $\mathcal{A}$ this is $S^{-1}K(\mathcal{A})$ where S is the multiplicative system of quasi–inverses in $K(\mathcal{A})$.

We append one of $+, -$ or b as a superscript to indicate same categories, but built using complexes A such that $A^i = 0$ for $i \ll 0, i \gg 0$ or $|i| \gg 0$ respectively, for example $D^b(A)$.

Proposition 1.3.3. *Suppose $A \xrightarrow{a} B \xrightarrow{b} C$ is a diagram in an abelian category $\mathcal{A}$. The sequence $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ is exact sequence if and only if $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1]$ is a distinguished triangle for some morphism $C \xrightarrow{c} A[1]$.*

Proof. Suppose $0 \to A \to B \to C \to 0$ is exact in $\mathcal{A}$. By definition, the cone of a is the complex $C' = \{A \xrightarrow{a} B\}$ concentrated in degrees -1 and 0, and the triangle $A \to B \to C' \to A[1]$ is distinguished also by definition. It is now a simple matter to show that the exactness of $0 \to A \to B \to C \to 0$ implies that C' and C are isomorphic, and that this isomorphism $C \xrightarrow{f} C'$ fits into an isomorphism of triangles (id_A, id_B, f).

Now suppose that $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1]$ is a distinguished triangle for some morphism c. Lemma 1.1.6 gives exactness at B. Let $C' = Cone(A \xrightarrow{a} B)$ and apply Axiom TR3 to

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, equal] & C \arrow[r] & A[1] \arrow[d, equal] \\ A \arrow[r] & B \arrow[r] & C' \arrow[r] & A[1] \end{tikzcd}$$

to obtain a morphism $C \to C'$ which is an isomorphism by the Five Lemma 1.1.10. It follows directly from the definition of the cone of a morphism of complexes (applied to $A \xrightarrow{a} B$) that $H^{-1}(C') = \ker(A \xrightarrow{a} B)$ and $H^0(C') = \mathrm{coker}(A \xrightarrow{a} B)$. Since isomorphisms in $D(\mathcal{A})$ induce isomorphisms of cohomology, we see that $\ker(A \to B) = H^{-1}(C) = 0$ and $\mathrm{coker}(A \to B) = H^0(C) = C$. Therefore $0 \to A \to B \to C \to 0$ is a short exact sequence. □

Proposition 1.3.4. *Let $A^\bullet$ be a complex with $H^i(A^\bullet) = 0$ for $i > m$ (resp. $i < m$, resp. $i \neq m$). Then $A^\bullet$ is quasi–isomorphic to a complex with $A^i = 0$ for $i > m$ (resp. $i < m$, resp. $i \neq m$).*

Proof. Consider the two morphisms of complexes:

$$\begin{tikzcd} \cdots \arrow[r] & A^{m-2} \arrow[r] \arrow[d] & A^{m-1} \arrow[r] \arrow[d] & ker(d^m) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & A^{m-2} \arrow[r, "d^{m-2}"] \arrow[d] & A^{m-1} \arrow[r, "d^{m-1}"] \arrow[d] & A^m \arrow[r, "d^m"] \arrow[d] & A^{m+1} \arrow[r, "d^{m+1}"] \arrow[d] & A^{m+2} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & coker(d^m) \arrow[r] & A^{m+1} \arrow[r] & A^{m+2} \arrow[r] & \cdots \end{tikzcd}$$

It can be seen that they are both quasi–isomorphisms. The case $H^i(A^\bullet) = 0$ for $i \neq m$ is a combination of the other two cases. □

Lemma 1.3.5. *Let A be a complex with $H^i(A) = 0$ for $i > 0$. Then there exists a morphism $\phi : A \to H^0(A)$ such that the induced morphism $H^0(\phi)$ is the identity.*

Proof. By Proposition 1.3.4 we can assume that $A^i = 0$ for $i > 0$, and so $H^0(A) = \mathrm{coker}\, d{-}1$. We then have the obvious morphism ϕ of complexes

$$\begin{tikzcd} \cdots \arrow[r] & A^{-2} \arrow[r] \arrow[d] & A^{-1} \arrow[r, "d^{-1}"] \arrow[d] & A^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H^0(A) \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

which clearly satisfies the condition that $H^0(\phi)$ is the identity. □

Proposition 1.3.6. *The composition of the functors:*

$$\mathcal{A} \to Ch(\mathcal{A}) \to K(\mathcal{A}) \to D(\mathcal{A})$$

is fully faithful.

Remark 1.3.7. Recall that the functor $\mathcal{A} \to Ch(\mathcal{A})$ sends an object A to the complex with A in degree zero and the zero object in all other degrees and the functor $K(\mathcal{A}) \to D(\mathcal{A})$ sends a $K(\mathcal{A})$–morphism $A \to B$ represented by $f \in \hom_{Ch(\mathcal{A})}(A, B)$ to the $D(\mathcal{A})$–morphism represented by $A \xleftarrow{id} A \xrightarrow{f} B$.

Proof. Let A and B be objects of $\mathcal{A}$. That $\mathcal{A} \to Ch(\mathcal{A})$ is fully faithful is clear. By definition $\hom_{Ch(\mathcal{A})}(A, B) \to \hom_{K(\mathcal{A})}(A, B)$ is surjective so consider a morphism $f \in \hom_{Ch(\mathcal{A})}(A, B)$ homotopically equivalent to zero (with A and B still in the image of $\mathcal{A} \to Ch(\mathcal{A})$). That means there is a morphism $s \in \hom_{Ch(\mathcal{A})}(A[1], B)$ such that $ds - sd = f$. But all differentials in the complexes A and B are zero and so $f = 0$. Hence, $\hom_{Ch(\mathcal{A})}(A, B) \cong \hom_{K(\mathcal{A})}(A, B)$.

Now consider the morphism $\hom_{\mathcal{A}}(A, B) \to \hom_{D(\mathcal{A})}(A, B)$.

Surjectivity. Let $A \xleftarrow{s} C \xrightarrow{f} B$ represent a morphism in $\hom_{D(\mathcal{A})}(A, B)$. So s is a quasi–isomorphism. Then

$$H^i(C) = \begin{cases} 0 & i \neq 0 \\ A & i = 0 \end{cases}$$

and so by Proposition 1.3.4 there exists a quasi–isomorphism $A \xrightarrow{t} C$. This means the composition $A \to C \to A$ is also a quasi–isomorphism and since A is concentrated in degree zero $st = id_A$. So the diagram

$$\begin{tikzcd}
 & C \arrow[dl, "s"'] \arrow[dr, "f"] & \\
A \arrow[r, equal] \arrow[dr, equal] & A \arrow[u, "t"'] \arrow[r, "ft"] \arrow[d, equal] & B \\
 & A \arrow[ur, "ft"'] &
\end{tikzcd}$$

commutes and therefore, $A \xleftarrow{s} C \xrightarrow{t} B$ is equivalent to $A \xleftarrow{id} A \xrightarrow{ft} B$ which is in the image of $\hom_{K(\mathcal{A})}(A, B) \xrightarrow{\sim} \hom_{Ch(\mathcal{A})}(A, B) \to \hom_{D(\mathcal{A})}(A, B)$.

Injectivity. A morphism $f \in \hom_{\mathcal{A}}(A, B)$ gets sent to zero if and only if there is a commutative diagram:

$$\begin{tikzcd}
 & A \arrow[dr, "f"] & \\
A \arrow[ur, equal] \arrow[dr, equal] & C \arrow[l] \arrow[u, "g"] \arrow[r] \arrow[d] & B \\
 & A \arrow[ur, "0"'] &
\end{tikzcd}$$

where the inside morphisms are quasi–equivalences. That is, if and only if there is a complex C and a quasi–equivalence $g : C \to A$ such that $fg = 0$ in $K(\mathcal{A})$. But this means that the morphism $H^0C \cong A \to B$ induced on zeroth homology groups is also zero. Hence $A \to B$ is zero. □

We finish this section with the statement of some propositions that we will have cause to refer to, but not not the space to prove. The first is immediate from the definitions of cone, and the triangulated structure $K(\mathcal{A})$ and $D(\mathcal{A})$.

Proposition 1.3.8 ([Huy06, Exercise 2.28]). *Let $A \to B \to C \to A[1]$ be a distinguished triangle in $K(\mathcal{A})$ or $D(\mathcal{A})$. Then there is a long exact sequence*

$$\cdots \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to H^{i+1}(B) \to \dots$$

Proposition 1.3.9 ([Huy06, Example 2.70]). *Let $A, B \in D(\mathcal{A})$ with B bounded below and suppose that $\mathcal{A}$ has enough injectives. Then there exists a spectral sequence:*

$$E_2^{p,q} = \hom_{D(\mathcal{A})}(A, H^q(B)[p]) \implies \hom_{D(\mathcal{A})}(A, B[p+q])$$

If, instead, $\mathcal{A}$ contains enough projectives and A is bounded above, then we have a spectral sequences:

$$E_2^{p,q} = \hom_{D(\mathcal{A})}(H^{-q}(A), B[p]) \implies \hom_{D(\mathcal{A})}(A, B[p+q])$$

Proposition 1.3.10 ([Huy06, Remark 2.67]). *Let $F : K^+(\mathcal{A}) \to K(\mathcal{B})$ be an exact functor for two abelian categories $\mathcal{A}, \mathcal{B}$ and let $A \in D^+(\mathcal{A})$. Then there exists a spectral sequence:*

$$E_1^{p,q} = R^qF(A^p) \implies R^{p+q}F(A)$$

1.4 Derived categories of smooth projective varieties.

The derived categories that we are most interested in are the derived categories associated to the abelian category of coherent sheaves on a smooth projective variety X over a field k. These are denoted $D^b(X)$. To finish this section we state some results about these derived categories that we will have cause to use later on.

Remark 1.4.1. As we only ever use derived categories, derived functors between them (direct image Rf_*, inverse image Lf^*, tensor product $\otimes^L$, ...) will always be denoted simply f_*, f^*, $\otimes$, ... with the implication that they are derived.

Theorem 1.4.2 (Serre duality. [Huy06, Theorem 3.12]). *Let X be a smooth projective variety over a field k. Then for complexes $E, F \in D^b(X)$ there is a functorial isomorphism*

$$\hom_{D^b(X)}(E, F) \cong \hom_{D^b(X)}(F, E \otimes \omega_X[n])^\vee$$

A consequence of Serre Duality is the following.

Proposition 1.4.3 ([Huy06, Remark 1.31]). *Let X and Y be smooth projective varieties over a field. Let $F : D^b(X) \to D^b(Y)$ be an exact functor between their derived categories and suppose that F has a left (resp. right) adjoint F^* (resp. $F^!$). Then F has a right (resp. left) adjoint $F^!$ (resp. F^*) and*

$$F^! \circ S_Y = S_X \circ F^*$$

where S_Y and S_X are the functors $(-) \otimes \omega_Y[\dim Y]$ and $(-) \otimes \omega_X[\dim X]$ respectively.

Proposition 1.4.4 (Flat base change [Har77, Proposition III.9.3]). *Consider a fibre product diagram*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "v"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ X \arrow[r, "u"] & Z \end{tikzcd}$$

with X, Y, Z smoth projective varieties. Then there is a functorial isomorhpism

$$u^* f_* F \cong g_* v^* F$$

for any object $F \in D(Qcoh(Y))$ in the unbounded derived category of quasicoherent sheaves on Y.

Proposition 1.4.5 (Projection formula [Har66, Proposition II.5.6]). *Let $f : X \to Y$ be a proper morphism of projective schemes over a field k. For any $F \in D^b(X)$, $E \in D^b(Y)$ there exists a natural isomorphism*

$$(f_* F) \otimes E \cong f_*(F \otimes f^* E)$$

Chapter 2

Ample sequences

One of the key pieces in Orlov's proof that every derived equivalence is Fourier-Mukai is the result that a natural isomorphism between an exact autoequivalence and the identity on an ample sequence can be extended to a natural transformation on the whole category (Proposition 2.2.1). This is the main goal of this section.

We begin by defining ample sequences and developing some basic properties, then we state and prove Proposition 2.2.1.

Remark 2.0.6. We will fix an abelian category $\mathcal{A}$ and the assumption that $\mathcal{A}$ is k-linear, and has finite dimensional hom vector spaces.

2.1 Definition and basic properties.

Definition 2.1.1. Let $\{L_i\}_{i\in\mathbb{Z}}$ be a sequence of objects from $\mathcal{A}$. This sequence is called *ample* if for every object A of $\mathcal{A}$ there exists $N \in \mathbb{Z}$ such that for all $i < N$ the following conditions hold:

AS1 the canonical morphism $\hom(L_i, A) \otimes L_i \to A$ is surjective,

AS2 $\operatorname{Ext}^j(L_i, A) = 0$ for any $j \neq 0$,

AS3 $\hom(A, L_i) = 0$

Remark 2.1.2. By $\hom(L_i, A) \otimes L_i \to A$ we mean the morphism $\sum_{i=1}^n f_i : L_i^{\oplus n} \to A$ where $n = \dim \hom(L_i, A)$ and $\{f_1, \ldots, f_n\}$ is a basis for $\hom(L_i, A)$.

The choice of basis does not matter since given any other basis the change of basis matrix defines an isomorphism $L_i^{\oplus n} \xrightarrow{\sim} L_i^{\oplus n}$ which commutes with the morphisms $L_i^{\oplus n} \to A$.

Example 2.1.3. The relevant example is when $\mathcal{A}$ is the abelian category of coherent sheaves for some projective variety X over a field. In this case the sequence $\{\mathcal{L}^{\otimes i}\}_{i\in\mathbb{Z}}$ is ample in $D^b(X)$ where $\mathcal{L}$ is an ample invertible sheaf on the variety [BO01].

Remark 2.1.4. For the rest of this section we assume that $\mathcal{A}$ has an ample sequence and denote it by $\{L_i\}$.

Lemma 2.1.5. *For an object $A \in D^b(\mathcal{A})$ we have*

$$\hom_{D^b(\mathcal{A})}(L_i, H^q(A)) = \hom_{D^b(\mathcal{A})}(L_i, A[q])$$

for all q and all $i \ll 0$.

Proof. We use the spectral sequence 1.3.9:

$$E_2^{p,q} = \hom_{D^b(\mathcal{A})}(L_i, H^q(A)[p]) \implies \hom_{D^b(\mathcal{A})}(L_i, A[p+q])$$

Since $A \in D^b(\mathcal{A})$ there are only finitely many q for which $H^q(A) \neq 0$. For each of these finitely many $H^q(A)$, due to condition AS2 there is some i_q such that $\hom_{D^b(\mathcal{A})}(L_i, H^q(A)[p]) = 0$ for $i < i_q$ and $p \neq 0$. Fix one i' that is smaller than each i_q. Then for $i < i'$ the spectral sequence is concentrated on the vertical axis and so $\hom(L_i, H^q(A)) = \hom(L_i, A[q])$ for all q and all $i < i'$. □

Lemma 2.1.6. *If A is an object in $D^b(\mathcal{A})$ such that* $\hom(L_i, A[j]) = 0$ *for all j and all $i \ll 0$, then A is the zero object.*

Proof. Using Proposition 1.3.6 and Lemma 2.1.5 we have

$$\hom_{\mathcal{A}}(L_i, H^j(A)) \cong \hom_{D^b(\mathcal{A})}(L_i, H^j(A)) \cong \hom_{D^b(\mathcal{A})}(L_i, A[j])$$

for all j, and for $i \ll 0$. By assumption, the rightmost group is zero and therefore the leftmost group is also zero. Property AS1 says that $\hom(L_i, H^j(A)) \otimes L_i \to H^j(A)$ is surjective and therefore, we see that $H^j(A) = 0$ for all j. Hence, A is isomorphic to the zero object in $D^b(\mathcal{A})$. □

Definition 2.1.7. An abelian category $\mathcal{A}$ is said to be of *finite homological dimension* if there exists an integer ℓ such that $\hom_{\mathcal{D}(A)}(A, B[i]) = 0$ for all objects A and B of $\mathcal{A}$ and all $i > \ell$.

Example 2.1.8. Consider the category $Coh(X)$ of coherent sheaves on a smooth projective variety X over a field. It holds generally that $\hom_{D^b(X)}(F, G[i]) = 0$ for $i < 0$. Now using Serre Duality 1.4.2 we have $\hom_{D^b(X)}(F, G[i]) \cong \hom_{D^b(X)}(G, F \otimes \omega_X[n-i])^\vee$ (where $n = \dim X$) and as we just mentioned, this vanishes for $n - i < 0$. So $\hom_{D(X)}(F, G[i]) = 0$ for $i \notin [0, \dots, \dim X]$ and therefore $Coh(X)$ has finite homological dimension.

Lemma 2.1.9. *Assume that $\mathcal{A}$ is of finite homological dimension. If A is an object in $D^b(\mathcal{A})$ such that* $\hom(A, L_i[j]) = 0$ *for all j and all $i \ll 0$ then A is the zero object.*

Proof. If A has a nonzero cohomology object we will construct objects $B_1, B_2, \ldots$ and nonzero morphisms $A \to B_m[m]$ which will provide a contradiction with the assumption that $\mathcal{A}$ has finite homological dimension.

If all the cohomology groups are zero then A is quasi–isomorphic to zero as a complex and therefore isomorphic to zero in $D(\mathcal{A})$. Assume there are nonzero cohomology groups, and in fact, assume that $H^0(A) \neq 0$ and $H^i(A) = 0$ for $i > 0$ (this assumption is so we can invoke Lemma 1.3.5 later). Choose a surjective morphism $L_{i_1}^{\oplus k_1} = \hom(L_{i_1}, H^0(A)) \otimes L_{i_1} \to H^0(A)$ (from Property AS1) and let B_1 denote the kernel. From Proposition 1.3.3 since $0 \to B_1 \to L_{i_1}^{\oplus k_1} \to H^0(A) \to 0$ is exact in $\mathcal{A}$ we have a distinguished triangle $B_1 \to L_{i_1}^{\oplus k_1} \to H^0(A) \to B_1[1]$ in $D(\mathcal{A})$ and therefore, by Corollary 1.1.8 an exact sequence:

$$\hom(A, L_{i_1}^{\oplus k_1}) \to \hom(A, H^0(A)) \to \hom(A, B_1[1])$$

Since $\hom(A, L_{i_1}) = 0$ the first group in this sequence is zero and so the second morphism is injective. Hence, the image ϕ_1 of the nonzero morphism $\phi_0 : A \to H^0(A)$ from Lemma 1.3.5 is nonzero. Now take another surjective morphism $L_{i_2}^{\oplus k_2} = \hom(L_{i_2}, B_1) \otimes L_{i_2} \to B_1$ and denote its kernel by B_2. Using the same reasoning we have an exact sequence

$$\hom(A, L_{i_2}^{\oplus k_2}[1]) \to \hom(A, B_1[1]) \to \hom(A, B_2[2])$$

where, again, the first group is zero by assumption and so the image of ϕ_1 in $\hom(A, B_1[1])$ is a nonzero morphism ϕ_2 in $\hom(A, B_2[2])$. Iterating this procedure we eventually find an object Y_m of $\mathcal{A}$ with $m > \ell$ for ℓ in the definition of finite homological dimension such that $\hom(A, B_m[m])$ is nonzero, which is a contradiction. So the original assumption that A has nonzero cohomology groups was false and A is isomorphic to the zero object in $D(\mathcal{A})$. □

Lemma 2.1.10. *Let $\mathcal{B}$ be another abelian category, suppose that $\mathcal{A}$ has finite homological dimension. Suppose $F : D^b(\mathcal{A}) \to D^b(\mathcal{B})$ is an exact functor such that it has left and right adjoints F^* and $F^!$ respectively. If the maps*

$$\hom_{D^b(A)}(L_i, L_j[k]) \to \hom_{D^b(B)}(F(L_i), F(L_j)[k])$$

are isomorphisms for $i < j$ and all k then F is full and faithful.

Remark 2.1.11. Throughout the proof of this lemma we make heavy use of the functorality of the adjunction isomorphisms $\phi_{A,B} : \hom(GA, B) \cong \hom(A, HB)$ (for adjoint functors, say, G and H) as well as the adjunction natural transformations $id \to HG$ and $GH \to id$. The naturality says

$$\phi(h \circ g \circ Gf) = Hh \circ \phi g \circ f$$

and in particular, this means that $\phi g = \phi(id \circ g) = Hid \circ \phi$

Proof. First we show that the adjunction natural transformation $id \to F^! F$ restricted to the full subcategory with objects $\{L_i\}$ is a natural isomorphisms. Then we use this to show that the adjunction natural transformation $F^*F \to id$ is a natural isomorphisms (on all of $\mathcal{A}$). Since $\hom(A,B) \to \hom(FA,FB)$ is the composition of the two morphisms

$$\hom(F^*FA,B) \to \hom(A,B) \qquad \hom(F^*FA,B) \xrightarrow{\sim} \hom(FA,FB)$$

(see the remark above) this shows that F is full and faithful.

Step 1: $id|_{L_i} \cong F^!F|_{L_i}$. Consider the adjunction morphisms $f_j : L_j \to F^!F(L_j)$ and let C_j be the cone of f_j for each j. The morphism $f_{j*} : \hom(L_i, L_j[k]) \to \hom(L_i, F^!F(L_j)[k])$ can be written as the composition

$$\hom(L_i, L_j[k]) \to \hom(F(L_i), F(L_j)[k]) \cong \hom(L_i, F^!F(L_j)[k])$$

where the first morphism is application of F and the second is the adjunction isomorphism. If we choose $i < j$ then the first morphism is an isomorphism (by assumption) and hence the composition is an isomorphism. So in the exact sequence from Corollary 1.1.8

$$\begin{aligned}\hom(L_i, L_j)[k] \to \hom(L_i, F^!F(L_j)[k]) \to \hom(L_i, C_j[k])&\\ \to \hom(L_i, L_j[k+1]) \to \hom(L_i, F^!F(L_j)[k+1])&\end{aligned}$$

the first and last morphisms are isomorphisms, whence it follows that $\hom(L_i, C_j[k]) = 0$ for all $i < j$ and all k. We apply Lemma 2.1.6 and see that $C_j = 0$ and therefore by Proposition 1.1.11 each f_j is an isomorphism.

Step 2: $F^*F \cong id$. Now consider the adjunction morphism $g_A : F^*F(A) \to A$ where A is an arbitrary object of $D^b(\mathcal{A})$. The morphism $(g_A)^* : \hom(A, L_i[k]) \to \hom(F^*F(A), L_i[k])$ can be written as the composition

$$\hom(A, L_i[k]) \xrightarrow{(f_i)_*} \hom(A, F^!F(L_i)[k]) \xrightarrow{\sim} \hom(F(A), F(L_i)[k]) \xrightarrow{\sim} \hom(F^*F(A), L_i[k])$$

where the second two morphisms are the adjunction isomorphisms. We have just seen that f_i is an isomorphism and so the composition $(g_A)^*$ of the three morphisms above is an isomorphism. We denote the cone of g_A by C_A and again use the exact sequence of Corollary 1.1.8

$$\begin{aligned}\hom(A, L_i[k]) \overset{g_A^*}{\cong} \hom(F^*F(A), L_i[k]) \to \hom(C_A, L_i[k])&\\ \to \hom(A, L_i[k-1]) \overset{g_A^*}{\cong} \hom(F^*F(A), L_i[k-1])&\end{aligned}$$

to see that $\hom(C_A, L_i[k]) = 0$ for all i and k. It now follows from Lemma 2.1.9 that C_A is zero and so by Proposition 1.1.11 the natural transformation $g : F^*F \to id$ is an isomorphism.

As mentioned at the beginning of the proof, this is enough to conclude that F is full and faithful. $\square$

2.2 Extending natural isomorphisms on ample sequences.

We now come to the main result using ample sequences.

Proposition 2.2.1. *Let $F : D^b(\mathcal{A}) \to D^b(\mathcal{A})$ be an exact autoequivalence. Suppose*

$$f : id_{\{L_i\}} \xrightarrow{\sim} F|_{\{L_i\}}$$

is an isomorphism of functors on the full subcategory $\{L_i\}$ given by an ample sequence $\{L_i\}$ in $\mathcal{A}$. Then there exists a unique extension to an isomorphism

$$\tilde{f} : id \xrightarrow{\sim} F$$

The proof is quite lengthy and so we follow Huybrechts [Huy06] (who follows Orlov [Orl96]) and break it into steps.

Lemma 2.2.2. *An object $A \in D^b(\mathcal{A})$ is isomorphic to an object in the image of $\mathcal{A} \to D^b(\mathcal{A})$ if and only if*

$$\hom^j_{D^b(\mathcal{A})}(L_i, A) = 0$$

for all $j \neq 0$ and $i \ll 0$.

Proof. If A is an object of $\mathcal{A}$ then by AS2 we have $\hom^j_{D^b(\mathcal{A})}(L_i, A) = 0$ for all $j \neq 0$ and $i \ll 0$. For the other direction, recall that from Lemma 2.1.5 we have $\hom(L_i, H^j(A)) = \hom(L_i, A[j])$ for all j and $i \ll 0$. So the assumption says that $\hom(L_i, H^j(A)) = 0$ for $j \neq 0$ and $i \ll 0$. It then follows from Property AS1 that $H^j(A) = 0$ for $j \neq 0$ and so by Proposition 1.3.4 A is in the image of $\mathcal{A} \to D^b(\mathcal{A})$. □

Corollary 2.2.3. *Under the assumptions of Proposition 2.2.1 the image of any object of $\mathcal{A}$ is isomorphic to an object of $\mathcal{A}$.*

Proof. We have isomorphisms

$$\hom(L_i, F(A)[j]) \cong \hom(F(L_i), F(A)[j]) \cong \hom(L_i, A[j])$$

where the first comes from the isomorphism $id_{\{L_i\}} \cong F|_{\{L_i\}}$ and the second comes from the assumption that F is an exact autoequivalence. It follows from Lemma 2.2.2 that the last group is zero for $j \neq 0$ and $i \ll 0$ and hence, so is the first. So it follows, still from Lemma 2.2.2, that $F(A)$ is isomorphic to an object in $\mathcal{A}$. □

Lemma 2.2.4. *Under the assumptions of Proposition 2.2.1 the natural transformation f extends to a natural transformation*

$$f^{\mathcal{A}} : id_{\mathcal{A}} \xrightarrow{\sim} F|_{\mathcal{A}}$$

This subpart of the proof of Proposition 2.2.1 is long in itself and so we break it up into steps as well.

Proof. Step a. Construction of the morphisms $f_A^{\mathcal{A}}$ for each object A in $\mathcal{A}$. We use the first property of ample sequences to construct an exact sequence in $\mathcal{A}$ of the form

$$0 \to B \to L_i^{\oplus k} \to A \to 0$$

with $i \ll 0$ (recall that we are taking $\hom(L_i, A) \otimes L_i$ to be $L_i^{\oplus k}$ where $k = \dim \hom(L_i, A)$). This exact sequence of $\mathcal{A}$ gives a distinguished triangle in $D^b(\mathcal{A})$ (Proposition 1.3.3) which gets taken to a distinguished triangle by F (since F is exact). By Corollary 2.2.3 each of the objects $F(B), F(L_i^{\oplus k})$ and $F(A)$ are again in $\mathcal{A}$ and since the embedding $\mathcal{A} \to D^b(\mathcal{A})$ is fully faithful (Proposition 1.3.6) we can use Proposition 1.3.3 again to see that the distinguished triangle $F(B) \to F(L_i^{\oplus k}) \to F(A) \to F(B)[1]$ corresponds to an exact sequence $0 \to F(B) \to F(L_i^{\oplus k}) \to F(A) \to 0$ in $\mathcal{A}$. So we have a diagram in $\mathcal{A}$:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "b"] & L_i^{\oplus k} \arrow[r, "a"] \arrow[d, "f_{L_i}^{\oplus k}"] & A \arrow[r] & 0 \\
0 \arrow[r] & F(B) \arrow[r, "F(b)"] & F(L_i^{\oplus k}) \arrow[r, "F(a)"] & F(A) \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact and the vertical morphism is an isomorphism. We will find a unique morphism $A \to F(A)$ making the diagram commute by showing that $F(a) \circ f_{L_i}^{\oplus k} \circ b = 0$.

Choose a surjection $L_j^{\oplus \ell} \xrightarrow{b'} B$ for $j \ll 0$ using Property AS1. Then we have a commutative diagram:

$$\begin{tikzcd}
L_j^{\oplus \ell} \arrow[d, "f_{L_j}^{\oplus \ell}"'] \arrow[dr, "b'"'] \arrow[drr, "p"] & & & & \\
F(L_j)^{\oplus \ell} \arrow[drr, "F(p)"] & B \arrow[r, "b"'] & L_i^{\oplus k} \arrow[r, "a"'] \arrow[d, "f_{L_i}^{\oplus k}"] & A \arrow[r] & 0 \\
& F(B) \arrow[r, "F(b)"'] & F(L_i)^{\oplus k} \arrow[r, "F(a)"'] & F(A) \arrow[r] & 0
\end{tikzcd}$$

Since the composition $B \to L_i^{\oplus k} \to A$ is zero, so is the composition $L_j^{\oplus \ell} \to L_i^{\oplus k} \to A$ and therefore, also its image $F(L_j)^{\oplus \ell} \to F(L_i)^{\oplus k} \to F(A)$ under F. So the composition $L_j^{\oplus \ell} \xrightarrow{p} B \xrightarrow{b} L_i^{\oplus k} \xrightarrow{f_{L_i}^{\oplus k}} F(L_i)^{\oplus k} \xrightarrow{F(a)} F(A)$ is zero and since p is a surjection, this impies that $F(a) \circ f_{L_i}^{\oplus k} \circ b = 0$.

This means that $F(a) \circ f_{L_i}^{\oplus k}$ factors through A giving a morphism $A \to F(A)$ which makes the diagram commutative. Furthermore, it is unique since a being an epimorphism implies that $a^* : \hom(A, F(A)) \to \hom(L_i^{\oplus k}, F(A))$ is injective.

Step b. The morphisms $f_A^{\mathcal{A}}$ are independent of the choice of surjection $L_i^{\oplus k} \to A$. Since for any two surjections $L_i^{\oplus k} \to A$, $L_{i'}^{\oplus k'} \to A$ of this form we have a third surjection

$L_{i''}^{\oplus k''} \to A$ forming a commutate diagram

$$\begin{tikzcd} L_{i''}^{\oplus k''} \arrow[r] \arrow[d] \arrow[dr] & L_{i'}^{\oplus k'} \arrow[d] \\ L_i^{\oplus k} \arrow[r] & A \end{tikzcd}$$

we can restrict to the case where we have diagram of the form

$$\begin{tikzcd} L_j^{\oplus \ell} \arrow[r] \arrow[d, "f_{L_j}^{\oplus \ell}"] & L_i^{\oplus k} \arrow[r] \arrow[d, "f_{L_i}^{\oplus k}"] & A \arrow[d, "f_A^{\mathcal{A}}"] \\ F(L_j)^{\oplus \ell} \arrow[r] & F(L_i)^{\oplus k} \arrow[r] & F(A) \end{tikzcd}$$

where $f_A^{\mathcal{A}}$ is induced by the first surjection and the two squares commute. In this case the outside rectangle commutes as well and since our choice for $f_A^{\mathcal{A}}$ was unique the morphism $f_A^{\mathcal{A}}$ nust be the same as the morphism induced by the surjecion $L_j^{\oplus \ell} \to A$.

Step c. The morphisms $f_A^{\mathcal{A}}$ are functorial in A. Consider a morphism $\phi : A_1 \to A_2$ and the corresponding diagram

$$\begin{tikzcd} A_1 \arrow[r, "f_{A_1}^{\mathcal{A}}"] \arrow[d, "\phi"] & F(A_1) \arrow[d, "F(\phi)"] \\ A_2 \arrow[r, "f_{A_2}^{\mathcal{A}}"] & F(A_2) \end{tikzcd}$$

We wish for this diagram to be commutative. We obtain the morphisms $f_{A_i}^{\mathcal{A}}$ by choosing surjections onto the A_i:

$$\begin{tikzcd} L_i^{\oplus k} \arrow[r] & A_1 \arrow[d] \\ L_j^{\oplus \ell} \arrow[r] & A_2 \end{tikzcd} \tag{2.1}$$

We want to add a morphism $L_i^{\oplus k} \to L_j^{\oplus \ell}$ to make the diagram commutative. Let B_2 be the kernel of $L_j^{\oplus \ell} \to A_2$ and then consider the distinguished triangle $B_2 \to L_j^{\oplus \ell} \to A_2 \to B_2[1]$ of $D^b(\mathcal{A})$ associated to the exact sequence $0 \to B_2 \to L_j^{\oplus \ell} \to A_2 \to 0$ (Proposition 1.3.3). Putting this through the functor $\hom_{D^b(\mathcal{A})}(L_i, -)$ gives an exact sequence

$$\hom(L_i, L_j^{\ell}) \to \hom(L_i, A_2) \to \hom(L_i, B_2[1])$$

by Corollary 1.1.8. But $\hom(L_i, B_2[1]) = 0$ from AS2 (at least for $i \ll 0$) and so $\hom(L_i, L_j^{\ell}) \to \hom(L_i, A_2)$ is surjective and we can find a moprhism $L_i^{\oplus k} \to L_j^{\oplus \ell}$ making Diagram 2.1 commute.

Now inserting the morphisms $f^{\mathcal{A}}$ we obtain

$$
\begin{tikzcd}
F(L_i)^{\oplus k} \arrow[rrr] \arrow[ddd] & & & F(A_1) \arrow[ddd] \\
& L_i^{\oplus k} \arrow[ul] \arrow[r] \arrow[d] \arrow[u, phantom, "\circlearrowleft"] \arrow[l, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & A_1 \arrow[ur] \arrow[d] \arrow[dr, phantom, "?"] & \\
& L_j^{\oplus \ell} \arrow[dl] \arrow[r] \arrow[d, phantom, "\circlearrowleft"] & A_2 \arrow[dr] & \\
F(L_j)^{\oplus \ell} \arrow[rrr] & & & F(A_2)
\end{tikzcd}
$$

where we know that all quadrangles marked with $\circlearrowleft$ are commutative as well as the outside square and we wish to show the commutativity of the quadrangle marked with ?. Using the known commutativities we see that the composition of the two morphisms $A_1 \to F(A_2)$ with $L_i^{\oplus k} \to A_1$ gives the same morphism. Since $L_i^{\oplus k} \to A_1$ is surjective, this is enough to show that the quadrangle marked with ? is commutative. Hence, the morphisms $f^{\mathcal{A}}$ form a natural transformation.

Step d. The morphisms $f_A^{\mathcal{A}}$ are isomorphisms. Recall the diagram that was used to construct $f_A^{\mathcal{A}}$ for some object A in $\mathcal{A}$:

$$
\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "f_B^{\mathcal{A}}"] & L_i^{\oplus k} \arrow[r] \arrow[d, "f_{L_i}^{\oplus k}"] & A \arrow[r] \arrow[d, "f_A^{\mathcal{A}}"] & 0 \\
0 \arrow[r] & F(B) \arrow[r] & F(L_i)^{\oplus k} \arrow[r] & F(A) \arrow[r] & 0
\end{tikzcd}
\tag{2.2}
$$

where we now include the morphisms $f^{\mathcal{A}}$. It follows directly from the diagram (using that f_{L_i} is an isomorphism) that $f_A^{\mathcal{A}}$ is surjective and $f_B^{\mathcal{A}}$ is injective. Moving into the derived category and converting the exact sequences in $\mathcal{A}$ to distinguished triangles in $D^b(\mathcal{A})$ (using Proposition 1.3.3) we obtain the commutative diagram

$$
\begin{tikzcd}
B \arrow[r] \arrow[d] & L_i^{\oplus k} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B[1] \\
F(B) \arrow[r] \arrow[d] & F(L_i)^{\oplus k} \arrow[r] \arrow[d] & F(A) \arrow[r] \arrow[d] & F(B)[1] \\
\operatorname{coker}(f_B^{\mathcal{A}}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ker(f_A^{\mathcal{A}})[1] \arrow[r] \arrow[d] & \operatorname{coker}(f_B^{\mathcal{A}})[1] \\
B[1] & L_i^{\oplus k}[1] & A[1] &
\end{tikzcd}
$$

where the top two rows and all the columns are distinguished triangles. It then follows from the Nine Lemma ([May01]) that the bottom row is also distinguished and therefore from Lemma 1.1.11 that $\ker(f_A^{\mathcal{A}}) \cong \operatorname{coker}(f_B^{\mathcal{A}})$. Since there is another diagram of the form of Diagram (2.2) but with B in the place of A (and some other i, k, and some other object, say C, in place of B), we see from the same reasoning as above that $f_B^{\mathcal{A}}$ is surjective. Hence, $\operatorname{coker}(f_B^{\mathcal{A}}) = 0$, $\ker(f_A^{\mathcal{A}}) = 0$, and finally $f_A^{\mathcal{A}} : A \to F(A)$ is an isomorphism. $\square$

We now (finally) prove Proposition 2.2.1. This proof follows a similar pattern to the proof of Lemma 2.2.4. We need the following natural definition. It is placed here rather than in Section 1 since this proof is the only place it is used.

Definition 2.2.5. The length of an object $A \in D^b(\mathcal{A})$ is

$$\operatorname{length}(A) = \max\{q_1 - q_2 : H^{q_1}(A) \neq 0 \neq H^{q_2}(A)\} + 1$$

Proof of Proposition 2.2.1. Step a. Construction of the (unique) isomorphisms $\tilde{f}_A$ for each object A in $D^b(\mathcal{A})$. We do this by induction on the length of the complex A. The case for complexes of length 1 is the content of Lemma 2.2.4 thanks to Proposition 1.3.6.

Suppose we have defined $\tilde{f}_A$ for every complex A of length $< N$ and suppose we have a complex A of length N. By Proposition 1.3.4 we can assume that A is of the form

$$\cdots \to A^{m-2} \to A^{m-1} \to A^m \to 0 \to 0 \to \ldots$$

for some integer m.

Using AS1, choose a surjection $L_i^{\oplus k} \to A^m$, consider this surjection as a morphism $L_i^{\oplus k}[-m] \to A$, and complete it to a distinguished triangle

$$L_i^{\oplus k}[-m] \longrightarrow A \longrightarrow B \longrightarrow L_i^{\oplus k}[1-m]$$

The section

$$H^j(L_i^{\oplus k}[-m]) \to H^j(A) \to H^j(B) \to H^{j+1}(L_i^{\oplus k}[-m])$$

of the long exact cohomology sequence (Proposition 1.3.8) for this distinguished triangle shows that $H^j(B) \cong H^j(A)$ for $j \neq m-1, m$. Similarly, the section

$$H^m(L_i^{\oplus k}[-m]) \to H^m(A) \to H^m(B) \to H^{m+1}(L_i^{\oplus k}[-m])$$

together with the facts that $H^m(L_i^{\oplus k}[-m]) \to A \to H^m(A)$ is surjective (since it is the composition $L_i^{\oplus k} \to A^m \to \operatorname{coker}(A^{m-1} \to A^m)$) and $H^{m+1}(L_i^{\oplus k}[-m]) = 0$ show that $H^m(B) = 0$. Hence, $\operatorname{length}(B) < N$. So we can use the induction hypothesis to form a diagram

$$\begin{tikzcd}
L_i^{\oplus k}[-m] \arrow[r] \arrow[d, "f_{L_i}^{\oplus k}"] & A \arrow[r] & B \arrow[r] \arrow[d, "\tilde{f}_B"] \arrow[r, phantom, shift right=6ex, "\circlearrowleft"] & L_i^{\oplus k}[1-m] \arrow[d, "f_{L_i}^{\oplus k}"] \\
F(L_i^{\oplus k})[-m] \arrow[r] & F(A) \arrow[r] & F(B) \arrow[r] & F(L_i^{\oplus k})[1-m]
\end{tikzcd} \tag{2.3}$$

The rows of this diagram are distinguished triangles and so using TR3 there is a morphism $A \xrightarrow{\tilde{f}_A} F(A)$ such that the diagram commutes. Furthermore, since $f_{L_i}^{\oplus k}$ and $\tilde{f}_B$ are isomorphisms it follows from the Five Lemma 1.1.10 that so is $\tilde{f}_A$.

We also show that $\tilde{f}_A$ is unique. Pushing the lower distinguished triangle through $\hom(A, -)$ gives us an exact sequence (Corollary 1.1.8):

$$\hom(A, F(L_i^{\oplus k})[-m]) \to \hom(A, F(A)) \to \hom(A, F(B)) \tag{2.4}$$

The first term in this sequence is zero since

$$\hom(A, F(L_i^{\oplus k})[-m]) \cong \hom(A, L_i^{\oplus k}[-m]) \cong \hom(H^m(A), L_i^{\oplus k}) = 0$$

where the first isomorphisms is induced by the isomorphism $f_{L_i}^{\oplus k}$, the equality is from AS3, and the second isomorphism comes from the spectral sequence 1.3.9 (we have $\hom(H^{-q}(A), L_i^{\oplus k}[p]) = 0$ for $p < 0$ and by assumption $H^{-q}(A) = 0$ for $q < -m$ so the only nonzero entry on the diagonal $p + q = -m$ is $\hom(H^m(A), L_i^{\oplus k})$). So the second morphism in sequence (2.4) is injective. This means that if there were two morphisms $A \to F(A)$ making the diagram 2.3 commute then they would be sent to the same morphism $A \to F(B)$. But we have just seen that $\hom(A, F(A)) \to \hom(A, F(B))$ is injective and so any morphism $A \to F(A)$ making 2.3 commute is unique.

Step b. The isomorphisms $\tilde{f}_A$ are independent of the choices. In this proof steps b and c are actually intertwined in the sense that we use the inductive hypothesis: for any complex of length less than N the isomorphisms $\tilde{f}_A$ are independent of the choices AND for any morphism $\phi : A \to C$ with length$(A)+$ length $(C) < N$ the isomorphisms $\tilde{f}_A$ and $\tilde{f}_C$ are functorial with respect to ϕ.

As in Step b. of the proof of Lemma 2.2.4, to prove that the morphism $\tilde{f}_A$ is independent of the choice of surjection $L_i^{\oplus k} \to A^m$ we need only consider the case $L_j^{\oplus \ell} \to L_i^{\oplus k} \to A^m$. In this situation we have the following commutative diagram:

$$\begin{tikzcd}
L_j^{\oplus \ell}[-m] \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & B_1 \arrow[r] & L_j^{\oplus \ell}[1-m] \arrow[d] \\
L_i^{\oplus k}[-m] \arrow[r] & A \arrow[r] & B_2 \arrow[r] & L_i^{\oplus k}[1-m]
\end{tikzcd}$$

where B_1 and B_2 are the cones of $L_j^{\oplus \ell}[-m] \to A$ and $L_i^{\oplus k}[-m] \to A$ respectively. We use TR3 to complete this diagram to a morphism of distinguished triangles giving a morphism

$B_1 \to B_2$. We now have

$$\begin{tikzcd}
F(A) \arrow[rrr] \arrow[ddd, equal] & & & F(B_1) \arrow[ddd] \\
& A \arrow[r] \arrow[d, equal] \arrow[ul, "a"'] \arrow[ur, phantom, "\circlearrowleft"] & B_1 \arrow[d] \arrow[ur] \arrow[dr, phantom, "\circlearrowleft"] & \\
{} \arrow[r, phantom, "?"] & A \arrow[r] \arrow[dl, "b"] \arrow[dr, phantom, "\circlearrowleft"] \arrow[ur, phantom, "\circlearrowleft"] & B_2 \arrow[dr] & {} \\
F(A) \arrow[rrr] & & {} & F(B_2)
\end{tikzcd}$$

where all the quadrangles marked with $\circlearrowleft$ commute as well as the outside square and we are trying to show that a and b are the same morphism. it can be seen from the commutativities of this diagram that the two compositions $A \xrightarrow{a} F(A) \to F(B_2)$ and $A \xrightarrow{b} F(A) \to F(B_2)$ are the same. Since $\hom(A, F(A)) \to \hom(A, F(B_2))$ is injective (we proved this in the previous step) this shows that both a and b are the same morphism.

Step c. The isomorphisms $\tilde{f}_A$ are functorial in A. Recall the second part of our inductive hypothesis: for any morphism $\phi : A \to C$ with length$(A)+$ length $(C) < N$ the isomorphisms $\tilde{f}_A$ and $\tilde{f}_C$ are functorial with respect to ϕ. Suppose $\phi : A \to C$ is a morphism in $D^b(\mathcal{A})$ with length$(A)+$ length $(C) = N$. By Proposition 1.3.4 we can assume that A and C are of the form:

$$\cdots \to A^{n-2} \to A^{n-1} \to A^n \to 0 \to \dots$$

$$\cdots \to C^{m-2} \to C^{m-1} \to C^m \to 0 \to \dots$$

First suppose that $m < n$. Choose a surjection $L_i^{\oplus k} \to A^n$ and consider the distinguished triangle

$$L_i^{\oplus k}[-n] \to A \to B \to L_i^{\oplus k}[1-n]$$

used to define $\tilde{f}_A$. Applying $\hom(-, C)$ to this we get an exact sequence from Corollary 1.1.8:

$$\hom(B, C) \to \hom(A, C) \to \hom(L_i^{\oplus k}[-n], C)$$

Since $\hom(L_i^{\oplus k}[-n], C) = 0$ (use the spectral sequence 1.3.9, AS2, and $m < n$) we can lift $A \to C$ to a morphism $B \to C$ giving the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
F(A) \arrow[r] \arrow[rr, bend right] & F(B) \arrow[r] & F(C)
\end{tikzcd}$$

Recalling that we saw that the length of B is one less than the length of A, by the inductive hypothesis it follows that the two inner squares commute. Hence, the outer square commutes as well.

Now suppose that $n \leq m$. We choose a surjection $L_i^{\oplus k} \to C^m$ and get the distinguished triangle used to define $\tilde{f}_C$

$$\begin{tikzcd}
L_i^{\oplus k}[-m] \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & L_i^{\oplus k}[1-m] \arrow[d] \\
F(L_i^{\oplus k})[-m] \arrow[r] & F(C) \arrow[r] & F(D) \arrow[r] & F(L_i^{\oplus k})[1-m]
\end{tikzcd}$$

Composing the morphism $A \to C$ with $C \to D$ we obtain the diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[rr, bend left] & C \arrow[r] \arrow[d] & D \arrow[d] \\
F(A) \arrow[r] \arrow[rr, bend right] & F(C) \arrow[r] & F(D)
\end{tikzcd}$$

Again, since the length of D is one less than C, by the inductive hypothesis the outer square and the rightmost square are both commutative. We wish to see that the left square is commutative. Using the two commutative squares we see that the two compositions

$$A \to F(A) \to F(C) \to F(D) \qquad \text{and} \qquad A \to C \to F(C) \to F(D)$$

are the same, so it is enough to show that the morphism $\hom(A, F(C)) \to \hom(A, F(D))$ is injective.

We have an isomorphism $\hom(A, L_i[-m]) \cong \hom(A, F(L_i)[-m])$ and since $n \leq m$, by the spectral sequence 1.3.9 we see that $\hom(A, L_i[-m]) = \hom(H^m(A), L_i)$ (recall that $\hom(H^{-q}(A), L_i[p]) = 0$ for $p < 0$ and $q < -n$). If $n < m$ then $H^m(A) = 0$ and if $n = m$ then $\hom(H^m(A), L_i) = 0$ for $i \ll 0$. Hence, $\hom(A, F(L_i)[-m]) = 0$ and so by inspecting the long exact sequence (Corollary 1.1.8) we get from image of the distinguished triangle $F(L_i^{\oplus k})[-m] \to F(C) \to F(D) \to F(L_i^{\oplus k})[1-m]$ under $\hom(A, -)$, we see that $\hom(A, F(C)) \to \hom(A, F(D))$ is injective. $\square$

Chapter 3

Postnikov systems

3.1 Postnikov systems

We now introduce Postnikov systems and prove some basic results about them that we will need. The material presented here is a reworking and embelishment of that appearing in [Orl96].

Postnikov systems can be thought of as a mechanism for turning a complex in a triangulated category into an object. The general idea is that a bounded complex in an abelian category can be "built" from small segments (possibly of length 1) by iterating the cone construction. This idea is developped more fully in the examples below.

Definition 3.1.1. Let $X^\bullet = \{X^c \xrightarrow{d^c} X^{c+1} \xrightarrow{d^{c+1}} \cdots \xrightarrow{d^{-1}} X^0\}$ be a bounded complex over a triangulated category $\mathcal{D}$ (so all compositions $d^{i+1} \circ d^1$ are equal to zero). A *left Postnikov system*, attached to $X^\bullet$ is a diagram

$$\begin{tikzcd}
X^c \arrow[rr, "d^c"] \arrow[dr, equal] & & X^{c+1} \arrow[rr, "d^{c+1}"] \arrow[dr, "i_{c+1}"] \arrow[d, phantom, "\triangle"] & & X^{c+2} \arrow[rr] \arrow[dr, "i_{c+2}"] \arrow[d, phantom, "\triangle"] & & \cdots \arrow[rr] & & X^0 \arrow[dr, "i_0"] \arrow[d, phantom, "\triangle"] & \\
& Y^c = X^c \arrow[ur, "d^c = j_c"] \arrow[u, phantom, "\circlearrowleft"] & & Y^{c+1} \arrow[ll, "{[1]}"'] \arrow[ur, "j_{c+1}"] \arrow[u, phantom, "\circlearrowleft"] & & Y^{c+2} \arrow[ll, "{[1]}"'] & \cdots \arrow[l] & Y^{-1} \arrow[l] \arrow[ur, "j_{-1}"] & & Y^0 \arrow[ll, "{[1]}"']
\end{tikzcd}$$

in which all the triangles marked with $\triangle$ are distinguished and triangles marked with $\circlearrowleft$ are commutative. An object E of $\mathcal{D}$ is called a left convolution of $X^\bullet$ if there exists a left Postnikov system, attached to $X^\bullet$ such that $E = Y^0$.

Example 3.1.2. 1. Let $X^c \to X^{c+1} \to \cdots \to X^0$ be a bounded complex in an abelian category $\mathcal{A}$. If we consider each object X^i as a complex concentrated in degree zero, each of the truncated complexes $Y^i = \{X^c \to X^{c+1} \to \cdots \to X^i\}$ can be obtained as the cone of the obvious morphism of complexes

$$\{X^c \to X^{c+1} \to \cdots \to X^{i-1}\} \to \{X^i\}$$

and these come with naturally equipped with morphisms $X^i \to Y^i \to X^{i+1}$ that factor the differentials $X^i \to X^{i+1}$. Hence, we obtain a Postnikov system in $K^b(\mathcal{A})$ the bounded homotopy category of $\mathcal{A}$.

2. Suppose that we have two bounded complexes $X^\bullet = \{X^c \to X^{c-1} \to \cdots \to X^{c'-1} \to X^{c'}\}$ and $Z^\bullet = \{Z^{c'} \to Z^{c'+1} \to \cdots \to Z^0\}$ in an abelian category $\mathcal{A}$ and a morphism $X^{c'} \to Z^{c'}$ such that the two compositions $X^{c'-1} \to X^{c'} \to Z^{c'}$ and $X^{c'} \to Z^{c'} \to Z^{c'+1}$ are zero. Then this defines a morphism of complexes $X^\bullet \to Z^\bullet$ and the cone of this morphism is the concatenated complex

$$Y^\bullet = \{X^c \to X^{c-1} \to \cdots \to X^{c'} \to Z^{c'} \to Z^{c'+1} \to \cdots \to Z^0\}$$

This gives a left Postnikov system:

$$\begin{tikzcd}
X^\bullet \arrow[rr] \arrow[dr, equal] & & Z^\bullet \arrow[dr] & \\
& X^\bullet \arrow[ur] & & Y^\bullet \arrow[ll]
\end{tikzcd}$$

Notice that in this situation the hom groups $\hom(X^\bullet, Z^\bullet[i])$ are zero for $i < 0$ (again we are working in the homotopy category of $\mathcal{A}$, not the derived category).

3. The previous example can be generalized to concatenate more than two bounded complexes in the same way the the first example concatenates multiple "bounded complexes of length 1". Since the index acrobatics required to explicitely describe the Postnikov system in this situation obscures the idea we don't go into detail. Roughly what happens in this example is that a bounded complex Y^0 is partitioned into disjoint segments X^i which are then appropriately translated so that the first nonzero term of X^i is in the same degree as the last nonzero term of X^{i+1}. The morphisms $X^i \to X^{i+1}$ are then induced by the appropriate differential of Y^0 and each Y^i is the concatenation of the first $i - c$ segments $(i, c < 0)$.

   This example should help to illuminate the condition (C1) below.

Definition 3.1.3. Let $X^\bullet, X_1^\bullet$ and $X_2^\bullet$ be bounded complexes in a triangulated category and let Y^0 and Y_2^0 be convolutions for $X^\bullet$ and $X_2^\bullet$ respectively. We will have cause to refer to the following conditions:

(C1) $\hom(X^a[i], X^{a+j}) = 0$ for all $i, j > 0$ and all a.

(C2) $\hom(X^a[i], Y^0) = 0$ for $i > 0$ and all a

(C3) $\hom(X_1^a[i], X_2^{a+j}) = 0$ for all $i, j > 0$ and all a.

(C4) $\hom(X_1^a[i], Y_2^0) = 0$ for $i > 0$ and all a

Lemma 3.1.4. *Let $X^\bullet = \{X^c \xrightarrow{d^c} X^{c+1} \to \cdots \to X^0\}$ be a complex in a triangulated category $\mathcal{D}$ satisfying (C1). Then there exists a diagram of the form*

$$\begin{tikzcd}
X^c \arrow[rr, "d^c"] \arrow[dr, equal] & & X^{c+1} \arrow[rr, "d^{c+1}"] \arrow[dr, "i_{c+1}"] \arrow[d, phantom, "\triangle"] & & X^{c+2} \arrow[r] & \cdots \arrow[r] & X^0 \\
& X^c \arrow[ur, "d^c"] & & Y^{c+1} \arrow[ll, "{[1]}"'] \arrow[ur, "j_{c+1}"'] \arrow[u, phantom, "\circlearrowleft"] & & &
\end{tikzcd}$$

where the triangle $\triangle$ is distinguished and the triangle $\circlearrowleft$ commutes. Furthermore, given any such diagram, $\{Y^{c+1} \xrightarrow{j_{c+1}} X^{c+2} \xrightarrow{d^{c+2}} \cdots \to X^0\}$ is also a complex satisfying (C1).

Proof. Set Y^{c+1} to be the cone of the morphism d^c and consider the diagram:

$$\begin{tikzcd}
X^c \arrow[r, "d^c"] & X^{c+1} \arrow[r, "i_{c+1}"] \arrow[d, "d^{c+1}"] & Y^{c+1} \arrow[r] & X^c[1] \\
0 \arrow[r] & X^{c+2} \arrow[r, "id"] & X^{c+2} \arrow[r] & 0[1]
\end{tikzcd}$$

By the definition of Y^{c+1} the top row is a distinguished triangle, and by TR1(a) and TR2 the bottom row is distinguished. Since $d^{c+1}d^c = 0$ it follows from Lemma 1.1.12 that there is a uniquely determined morphism j_{c+1} such that $d^{c+1} = j_{c+1}i_{c+1}$.

To see that $d^{c+2}j_{c+1} = 0$ consider the exact sequence

$$\hom(X^c[1], X^{c+3}) \longrightarrow \hom(Y^{c+1}, X^{c+3}) \xrightarrow{i_{c+1}^*} \hom(X^{c+1}, X^{c+3})$$

obtained from the distinguished triangle $X^c \to X^{c+1} \to Y^{c+1} \to X^c[1]$, TR(2) and Lemma 1.1.8. The morphism $d^{c+2}j_{c+1} \in \hom(Y^{c+1}, X^{c+3})$ gets mapped to $d^{c+2}j_{c+1}i_{c+1} = d^{c+2}d^{c+1} = 0$ and so is in the kernel of i_{c+1}^*. However, (C1) implies that $\hom(X^c[1], X^{c+3}) = 0$ and so i_{c+1}^* is injective. Hence $d^{c+2}j_{c+1} = 0$ and so $X_Y^\bullet = \{Y^{c+1} \to X^{c+2} \to \cdots \to X^0\}$ is a complex.

To see that $X_Y^\bullet$ satisfies (C1) we need only show that $\hom(Y^{c+1}[i], X^j) = 0$ for $i > 0$ and $j = c+2, \ldots, 0$ (since the triviality of the other hom groups follows from $X^\bullet$ satisfying (C1)). To see that this is satisfied consider the exact sequence

$$\hom(X^c[i+1], X^j) \longrightarrow \hom(Y^{c+1}[i], X^j) \longrightarrow \hom(X^c[i], X^j)$$

and note that the first and last terms are zero under the appropriate conditions on i and j as consequence of $X^\bullet$ satisfying (C1). □

Corollary 3.1.5. *Let $X^\bullet = \{X^c \xrightarrow{d^c} X^{c+1} \to \cdots \to X^0\}$ be a complex in a triangulated category $\mathcal{D}$ satisfying (C1). Then there exists a convolution for $X^\bullet$.*

Proof. We work by induction on the length of the complex. The case $c = -1$ is clear: take Y^0 to be the cone of d^{-1}. For the case $c < -1$ the inductive step is provided by Lemma 3.1.4 since the length of $\{Y^{c+1} \to X^{c+2} \to \cdots \to X^0\}$ is one less than the length of $X^\bullet$. $\square$

Lemma 3.1.6. *Let $X_1^\bullet = \{X_1^c \to \cdots \to X_1^0\}$ and $X_2^\bullet = \{X_2^c \to \cdots \to X_2^0\}$ be two complexes and let $(f_c, \dots, f_0)$ be a morphism between them:*

$$\begin{tikzcd} X_1^c \arrow[r, "d_1^c"] \arrow[d, "f_c"] & X_1^{c+1} \arrow[r] \arrow[d, "f_{c+1}"] & \cdots \arrow[r] & X_1^0 \arrow[d, "f_0"] \\ X_2^c \arrow[r, "d_c^2"] & X_2^{c+1} \arrow[r] & \cdots \arrow[r] & X_2^0 \end{tikzcd}$$

Suppose that $X_1^\bullet, X_2^\bullet$ satisfy condition (C3) and that they both satisfy condition (C1). Then:

1. *For objects Y_1^{c+1} and Y_2^{c+1} fitting into diagrams as in Lemma 3.1.4 there exists a morphism $Y_1^{c+1} \overset{g_{c+1}}{\to} Y_2^{c+1}$ which when combined with $(f_c, \dots, f_0)$ forms a morphism of diagrams.*

2. *For any such morphism,*

$$\begin{tikzcd} Y_1^{c+1} \arrow[r] \arrow[d, "g_{c+1}"] & X_1^{c+2} \arrow[r] \arrow[d, "f_{c+2}"] & \cdots \arrow[r] & X_1^0 \arrow[d, "f_0"] \\ Y_2^{c+1} \arrow[r] & X_2^{c+2} \arrow[r] & \cdots \arrow[r] & X_2^0 \end{tikzcd} \tag{3.1}$$

   is a morphism of two complexes satisfying (C3) (recall that both of the rows are complexes satisfying (C1) by Lemma 3.1.4).

3. *If all the f_i's are isomorphisms then so is g_{c+1}.*

Proof. 1. Define $Y_1^{c+1} \overset{g_{c+1}}{\to} Y_2^{c+1}$ to be a morphism obtained from completing the diagram

$$\begin{tikzcd} X_1^c \arrow[r, "d_1^c"] \arrow[d, "f_c"] & X_1^{c+1} \arrow[r, "i_{1,c+1}"] \arrow[d, "f_{c+1}"] & Y_1^{c+1} \arrow[r] & X_1^c[1] \arrow[d, "f_c[1]"] \\ X_2^c \arrow[r, "d_2^c"] & X_2^{c+1} \arrow[r, "i_{2,c+1}"] & Y_2^{c+1} \arrow[r] & X_2^c[1] \end{tikzcd} \tag{3.2}$$

to a morphism of distinguished triangles using Axiom TR3. We wish to show that the square

$$\begin{tikzcd} Y_1^{c+1} \arrow[r, "j_{1,c+1}"] \arrow[d, "g_{c+1}"] & X_1^{c+2} \arrow[d, "f_{c+2}"] \\ Y_2^{c+1} \arrow[r, "j_{2,c+1}"] & X_2^{c+2} \end{tikzcd} \tag{3.3}$$

commutes. Denote by h the difference of the two diagonal morphisms $h = f_{c+2}j_{1,c+1} - j_{2,c+1}g_{c+1}$ and consider the exact sequence

$$\hom(X_1^c[1], X_2^{c+2}) \longrightarrow \hom(Y_1^{c+1}, X_2^{c+2}) \xrightarrow{i_{1,c+1}^*} \hom(X_1^{c+1}, X_2^{c+2})$$

obtained from the upper row of Diagram 3.2. We know that all the outside squares and end faces of the prism

$$\begin{tikzcd}
X_1^{c+1} \arrow[rr] \arrow[dr, "i_{1,c+1}"] \arrow[dd] & & X_1^{c+2} \arrow[dd] \\
& Y_1^{c+1} \arrow[ur] \arrow[dr, dotted, "h"] \arrow[dd] & \\
X_2^{c+1} \arrow[rr] \arrow[dr] & & X_2^{c+2} \\
& Y_2^{c+1} \arrow[ur] &
\end{tikzcd}$$

commute, except possibly the square containing the morphism h. Hence, h is in the kernel of $i_{1,c+1}^*$. But by assumption $\hom(X_1^c[1], X_2^{c+2})$ is zero and so $i_{1,c+1}^*$ is injective. So h is zero and the square 3.3 commutes.

2. The fact that $(g_{c+1}, f_{c+2}, \ldots, f_0)$ is a morphism of complexes is consequence of g_{c+1} forming a morphism of diagrams, which we have just proven. So we just need to show that $\hom(Y_1^{c+1}[i], X_2^j) = 0$ for $i > 0$ and $j = c+2, \ldots, 0$. To see that this is satisfied consider the exact sequence

$$\hom(X_1^c[i+1], X_2^j) \longrightarrow \hom(Y_1^{c+1}[i], X_2^j) \longrightarrow \hom(X_1^c[i], X_2^j)$$

and note that the first and last terms are zero under the appropriate conditions on i and j as consequence of $X_1^\bullet, X_2^\bullet$ satisfying (C3).

3. This is a consequence of the Five Lemma 1.1.10 since by definition (f_c, f_{c+1}, g_{c+1}) is a morphism of distinguished triangles.

□

Corollary 3.1.7. *Let $X_1^\bullet = \{X_1^c \to \cdots \to X_1^0\}$ and $X_2^\bullet = \{X_2^c \to \cdots \to X_2^0\}$ be two complexes and let $(f_c, \ldots, f_0)$ be a morphism between them as in Lemma 3.1.6. Suppose that $X_1^\bullet, X_2^\bullet$ satisfy condition (C3) and that they both satisfy condition (C1). Then:*

1. *For any Postnikov systems of $X_1^\bullet$ and $X_2^\bullet$, the morphisms f_i extend to a morphism of Postnikov systems.*

2. *If all the f_i's are isomorphisms then they extend to an isomorphism of Postnikov systems.*

3. *If Y_1^0 and Y_2^0 are convolutions of $X_1^\bullet$ and $X_2^\bullet$ respectively, and in addition (C4) is satisfied, then the morphism $f : Y_1^0 \to Y_2^0$ is unique.*

Proof. By Lemma 3.1.6 and induction, we obtain the first two statements, so we only need to show the uniqueness of f in the case (C4) is satisfied. Let Y_1^a be the elements of the Postnikov system attached to $X_1^\bullet$.

We first we show that $\hom(Y_1^a[i], Y_2^0) = 0$ for all a and $i > 0$. We do this by induction on a starting with $a = c$. Since $Y_1^c = X_1^c$ the case $a = c$ is true by condition (C4). For the induction step, we use the exactness of

$$\hom(Y_1^{a-1}[i+1], Y_2^0) \to \hom(Y_1^a[i], Y_2^0) \to \hom(X_1^a[i], Y_2^0)$$

which comes from applying $\hom(-, Y_2^0)$ to the distinguished triangle $Y_1^{a-1} \to X_1^a \to Y_1^a \to Y_1^{a-1}[1]$ (Corollary 1.1.8). The outside two groups are trivial for $i > 0$ by (C4) and the inductive hypothesis.

Now consider the rightmost distinguished triangle of the Postnikov systems. We have a diagram

$$\begin{tikzcd}
Y_1^{-1} \arrow[r] \arrow[d] & X_1^0 \arrow[r] \arrow[d] & Y_1^0 \arrow[r] \arrow[d, "f"] & Y_1^{-1}[1] \arrow[d] \\
Y_2^{-1} \arrow[r] & X_2^0 \arrow[r] & Y_2^0 \arrow[r] & Y_2^{-1}[1]
\end{tikzcd}$$

We have just shown that, in particular, $\hom(Y_1^{-1}[1], Y_2^0) = 0$ and so it follows from Lemma 1.1.12 that f is unique. □

Corollary 3.1.8. *Let $X^\bullet$ be a bounded complex in a triangulated category that satisfies (C1). Then all Postnikov systems (and therefore convolutions as well) are (non-cannonically) isomorphic. If (C2) is also satisfied then all Postnikov systems (and therefore convolutions as well) are canonically isomorphic.*

Proof. Let $X_1^\bullet = X^\bullet = X_2^\bullet$ and $(f_c, \ldots, f_0) = (id_{X^c}, \ldots, id_{X^0})$. Then Condition (C1) is the same as Condition (C3) and Condition (C2) is the same as Condition (C4). So the results follow from parts 3 and 2 of Corollary 3.1.7. □

Chapter 4

The Beilinson resolution

To prove Orlov's Theorem in [Orl96] Orlov explicitely builds a kernel. This is done using the Beilinson resolution of the structure sheaf of the diagonal $\mathcal{O}_\Delta$ on $\mathbb{P}^n \times \mathbb{P}^n$. This resolution is an extremely useful tool and so we develop it in some depth here (at least, more depth than is strictly necassary). In the first section we describe the resolution as well as a spectral sequence. In the second we apply the spectral sequence to obtain some lemmas that we will need for the main proof.

4.1 The resolution and spectral sequence.

Our account of the following proposition follows [Căl05] but the original result was published in [Bei78]. We use the notation

$$E \boxtimes F \overset{def}{=} pr_1^* E \otimes pr_2^* F$$

for objects $E \in D^b(X)$ and $F \in D^b(Y)$ where $pr_1 : X \times Y \to X$ and $pr_2 : X \times Y \to Y$ are the projections. The idea is to find a section of $\mathcal{O}(1) \boxtimes \mathcal{T}(-1)$ (where $\mathcal{T}$ is the tangent sheaf on $\mathbb{P}^n$) which has the diagonal for its locus. Then we take the associated Koszul resolution. The Koszul resolution is not mentioned, even briefly here as it will take us too far from the main topic. The clearest exposition of Koszul complexes (for this application) that the author knows of is [FL85, Chapter IV, Section 2]. It is covered in many other texts by [FL85] is particularly lucid.

Proposition 4.1.1. *The following is a locally free resolution of $\mathcal{O}_\Delta$ on $\mathbb{P}^n \times \mathbb{P}^n$:*

$$0 \to \mathcal{O}(-n) \boxtimes \Omega^n(n) \to \mathcal{O}(-n+1) \boxtimes \Omega^{n-1}(n-1) \to \dots$$

$$\dots \to \mathcal{O}(-1) \boxtimes \Omega^1(1) \to \mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}_\Delta \to 0$$

Proof. Fix a basis $y_0, \dots, y_n$ of $H^0(\mathbb{P}^n, \mathcal{O}(1))$ (we use the definition of $\mathcal{O}(1)$ given in [Har77] as the sheaf associated to the graded module $k[y_0, \dots, y_n]$ with degree shifted by

1). There is an exact sequence of vector bundles on $\mathbb{P}^n$ (taken, for example, as the dual to the sequence of [Har77, Theorem 8.13] twisted by $\mathcal{O}(-1)$):

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{n+1} \to \mathcal{T}(-1) \to 0$$

Since $\mathcal{O}(-1)$ has no global sections, we get an isomorphism of global sections $H^0(\mathbb{P}^n, \mathcal{O}^{n+1}) \cong H^0(\mathbb{P}^n, \mathcal{T}(-1))$. We take a basis as follows. The sheaf $\mathcal{O}^{n+1}$ in the sequence of [Har77, Theorem 8.13] has a natural basis, with e_i being the element of $H^0(\mathbb{P}^n, \mathcal{O}^{n+1})$ that gets taken to y_i. The $\mathcal{O}^{n+1}$ in our sequence above is actually $\hom(\mathcal{O}^{n+1}, \mathcal{O})$ and so we take $e_i^\vee$ to be the dual of e_i, and denote by $\partial/\partial y_i$ its image in $H^0(\mathbb{P}^n, \mathcal{T}(-1))$.

Consider the global section s of $\mathcal{O}(1) \boxtimes \mathcal{T}(-1)$ on $\mathbb{P}^n \times \mathbb{P}^n$ defined by

$$s = \sum_{i=0}^{n} x_i \boxtimes \frac{\partial}{\partial y_i}$$

where the x_i's and y_i's are coordinates on the first and second $\mathbb{P}^n$, respectively. We want to show that the zeros of s are the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$. Since the argument for each coordinate patch is the same, we only give it once, in the case when $x_0 \neq 0, y_0 \neq 0$.

Let $B = k[\frac{y_1}{y_0}, \dots, \frac{y_n}{y_0}]$. On the open set $D_+(y_0) = \operatorname{Spec} B$ of the second $\mathbb{P}^n$, we follow [Har77] and give $\Gamma(D_+(y_0), \Omega^1(1))$ the basis $y_0 d\frac{y_i}{y_0} = (e_i - \frac{y_i}{y_0} e_0)$ for $i \neq 0$. The restriction of the $\partial/\partial y_i$ for $i \neq 0$ give a dual basis to the $y_0 d\frac{y_i}{y_0}$ and since $\frac{\partial}{\partial y_0}(e_i - \frac{y_i}{y_0} e_0) = -\frac{y_i}{y_0}$ we have $\frac{\partial}{\partial y_0} = -\sum_{i=1}^n \frac{y_i}{y_0}\frac{\partial}{\partial y_i}$ on $D_+(y_0) \subset \mathbb{P}^n$. So our section s restricts to

$$s = \sum_{i=0}^{n} x_i \boxtimes \frac{\partial}{\partial y_i} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i} - \sum_{i=1}^{n} x_0 \frac{y_i}{y_0}\frac{\partial}{\partial y_i} = \sum_{i=1}^{n} \left(\frac{x_i}{x_0} - \frac{y_i}{y_0}\right) x_0 \frac{\partial}{\partial y_i}$$

Its zero locus is therefore the closed subset defined by the ideal generated by the $\frac{x_i}{x_0} - \frac{y_i}{y_0}$, that is, the diagonal.

We then take the Koszul resolution of this section to get the result. $\square$

Theorem 4.1.2 (Beilinson). *For any coherent sheaf F on $\mathbb{P}^N$ there exist two natural spectral sequences:*

$$E_1^{r,s} = H^s(F(r)) \otimes \Omega^{-r}(-r) \implies E^{r+s} = \begin{cases} F & r+s=0 \\ 0 & \text{otherwise} \end{cases}$$

$$E_1^{r,s} = H^s(F \otimes \Omega^{-r}(-r)) \otimes \mathcal{O}(r) \implies E^{r+s} = \begin{cases} F & r+s=0 \\ 0 & \text{otherwise} \end{cases}$$

Remark 4.1.3. Since $\Omega^{-r}(-r)$ is the $-r$th exterior product of a localy free sheaf of rank N, it is trivial for $r \notin [-N, \dots, 0]$. Furthermore, the sth cohomology of a coherent sheaf on $\mathbb{P}^N$ is trivial for $s \notin [0, \dots, N]$ [Gro57]. This means both spectral sequences are concentrated in the rectangle $[-N, \dots, 0] \times [0, \dots, N]$.

Proof. The proof of this theorem uses the Beilinson resolution of the diagonal described above and the spectral sequence $E_1^{r,s} = R^sF(A^r) \implies R^{r+s}F(A)$.

Let L be the resolution of the diagonal considered above (Proposition 4.1.1); that is, $L^i = \mathcal{O}(-i) \boxtimes \Omega^i(i)$ for $i < 0$, and $L^i = 0$ for $i > 0$. Let A denote the tensor product $pr_1^*(F) \otimes L$ (note that the tensor product need not be derived as each term of L is a locally free sheaf). So we now have $A^i = pr_1^*(F) \otimes L^i = F(-i) \boxtimes \Omega^i(i)$ for $i < 0$ and $A^0 = pr_1^*(F) \otimes L^0 = pr_1^*(F)$. Now consider the functor $pr_{2*} : D^b(\mathbb{P}^N \times \mathbb{P}^N) \to D^b(\mathbb{P}^N)$. We have

$$\begin{aligned} pr_{2*}(A^r) &= pr_{2*}\Big(F(r) \boxtimes \Omega^{-r}(-r)\Big) \\ &= pr_{2*}\Big(pr_1^*F(r) \otimes pr_2^*\Omega^{-r}(-r)\Big) \\ \text{(projection formula 1.4.5)} &= \Big(pr_{2*}pr_1^*F(r)\Big) \otimes \Omega^{-r}(-r) \\ \text{(flat base change 1.4.4)} &= \Gamma(F(r)) \otimes \Omega^{-r}(-r) \end{aligned}$$

So $R^s(pr_{2*})(A^r) = R^s\Gamma(F(-r)) \otimes \Omega^r(r) = H^s(F(r)) \otimes \Omega^{-r}(-r)$.

On the other hand, L is quasi-isomorphic to the structure sheaf of the diagonal $\mathcal{O}_\Delta$ and so, if we notate $\delta : \mathbb{P}^N \to \mathbb{P}^N \times \mathbb{P}^N$ the diagonal inclusion, we have

$$\begin{aligned} pr_{2*}(A) &= pr_{2*}(pr_1^*(F) \otimes \mathcal{O}_\Delta) \\ &= pr_{2*}(pr_1^*(F) \otimes \delta_*\mathcal{O}_{\mathbb{P}^N}) \\ \text{(projection formula 1.4.5)} &= pr_{2*}\delta_*(\delta^*pr_1^*(F) \otimes \mathcal{O}_{\mathbb{P}^N}) \\ &= pr_{2*}\delta_*(\delta^*pr_1^*(F) \otimes \mathcal{O}_{\mathbb{P}^N}) \\ &= F \end{aligned}$$

This proves the first spectral sequence. The proof of the second is the same but with the rôles of pr_1 and pr_2 swapped. □

4.2 Some immediate applications.

Corollary 4.2.1. *For $k \geq 0$ we have a left resolution of $\mathcal{O}(k)$ on $\mathbb{P}^N$ of the form*

$$\Big\{V_N \otimes \Omega^N(N) \to \cdots \to V_1 \otimes \Omega^1(1) \to V_0 \otimes \mathcal{O}\Big\} \overset{q.i.}{\to} \mathcal{O}(k)$$

where V_i is the vector space $H^0(\mathcal{O}(k-i))$. For $k < N$ there is a right resolution of $\mathcal{O}(k)$ on $\mathbb{P}^N$ of the form

$$\mathcal{O}(k) \overset{q.i.}{\to} \Big\{V_0 \otimes \mathcal{O} \to V_1 \otimes \mathcal{O}(1) \to \cdots \to V_N \otimes \mathcal{O}(N) \to 0\Big\}$$

where $V_i = H^N(\Omega^{N-i}(k-i))$.

Proof. For the first statement we use the first spectral sequence from Theorem 4.1.2 with $F = \mathcal{O}(k)$. So (r,s) term is $H^s(\mathcal{O}(k+r)) \otimes \Omega^{-r}(-r)$. We already know that the spectral sequence is concentrated in the rectangle $[-N,\dots,0]\times[0,\dots,N]$ (Remark 4.1.3). We obtain more vanishing inside this rectangle by noting (from [Har77, Theorem II.5.1] for example) that

$$H^s(\mathcal{O}(r+k)) \cong \begin{cases} S_{r+k} & \text{if } s=0 \text{ and } r+k\geq 0 \\ S_{-N-r-k-1} & \text{if } s=N \text{ and } -N-r-k-1 \geq 0 \\ 0 & \textit{otherwise} \end{cases}$$

where $S_\bullet$ is the graded k-algebra $k[X_0,\dots,X_N]$. Since we are assuming $k\geq 0$, the sheet E_1 is zero everywhere except in degrees $(-N,0),(1-N,0),\dots,(0,0)$ where it is the sequence

$$H^0(\mathcal{O}(k-N))\otimes\Omega^N(N)\to\cdots\to H^0(\mathcal{O}(k-1))\otimes\Omega^1(1)\to H^0(\mathcal{O}(k))\otimes\mathcal{O}$$

Clearly the spectral sequence collapses in the second sheet to $E_\infty^{0,0}=\mathcal{O}(k)$ and $E_\infty^{r,s}=0$ otherwise and so we obtain our desired resolution.

For the second statment we use similar reasoning but with the second spectral sequence of Theorem 4.1.2 and using $F=\mathcal{O}(k-N)$. We claim that this spectral sequence vanishes everywhere except the Nth row, between the $-N$th and 0th columns, where it is

$$H^N(\Omega^N(k))\otimes\mathcal{O}(-N)\to H^N(\Omega^{N-1}(k-1))\otimes\mathcal{O}(1-N)\to\cdots\to H^N(\Omega^0(k-N))\otimes\mathcal{O}$$

As before, this shows that the spectral sequence collapses in the second sheet to $E_\infty^{0,0}=\mathcal{O}(k-N)$ and $E_\infty^{r,s}=0$ otherwise and so, after tensoring everything with $\mathcal{O}(N)$, we obtain our desired resolution.

Proof of Claim: Recall that there is a short exact sequence $0\to\Omega^1(1)\to\mathcal{O}^{N+1}\to\mathcal{O}(1)\to 0$ [Har77, Theorem II.8.13]. In general, for a short exact sequence $0\to E'\to E\to L\to 0$ of locally free sheaves with L a line bundle there are short exact sequences $0\to\wedge^p E'\to\wedge^p E\to L\otimes\wedge^{p-1}E\to 0$. Applying this to our short exact sequence and tensoring with $\mathcal{O}(k-N)$ we get

$$0\to\Omega^{-r}(k-N-r)\to\mathcal{O}(k-N)^{\binom{N+1}{-r}}\to\mathcal{O}(k-N-1)^{\binom{N+1}{-r-1}}\to 0$$

The long exact sequence of cohomology groups associated to this short exact sequence, together with the description of $H^s(\mathcal{O}(r+k))$ above, and the assumption that $k<N$ shows that $H^s(\Omega^{-r}(k-N-r))$ vanishes when $s\neq N$. ☐

Lemma 4.2.2. *Let X be a smooth projective variety over a field k and let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on X. Then in the derived category,* $\hom^p_{D^b(X)}(\mathcal{F},\mathcal{G})=0$ *unless* $p\in[0,\dim\ X]$.

Proof. Using Serre Duality 1.4.2 we note that $\hom_{D^b(X)}(\mathcal{F},\mathcal{G}[i])=\hom_{D^b(X)}(\mathcal{G},\mathcal{F}\otimes\omega_X[i-n])^\vee$ where $n=\dim X$. If $i<0$ or $i-n<0$ then these are zero. ☐

Definition 4.2.3. Let F be an exact functor from a derived category $D^b(\mathcal{A})$ to a derived category $D^b(\mathcal{B})$. The functor F is called *bounded* if there exists $z \in \mathbb{Z}, n \in \mathbb{N}$ such that for any $A \in \mathcal{A}$ the cohomology objects $H^i(F(A))$ vanish for $i \notin [z, z+n]$.

Lemma 4.2.4 ([Orl96, Lemma 2.4])**.** *Let X and Y be smooth projective varieties. If an exact functor $F : D^b(X) \to D^b(Y)$ has a left adjoint then it is bounded.*

Proof. Let $G : D^b(Y) \to D^b(X)$ be a left adjoint functor to F. Take a very ample invertible sheaf $\mathcal{L}$ on Y and its associated embedding $Y \hookrightarrow \mathbb{P}^N$. On $\mathbb{P}^N$ for any $i < 0$ we have a right resolution of $\mathcal{O}(i)$ in terms of the sheaves $\mathcal{O}(j)$ for $j = 0, 1, 2, \ldots, N$ (Corollary 4.2.1) and this resolution is of the form

$$\mathcal{O}(i) \xrightarrow{\cong} \left\{ V_0 \otimes \mathcal{O} \to V_1 \otimes \mathcal{O}(1) \to \cdots \to V_N \otimes \mathcal{O}(N) \to 0 \right\}$$

where the V_k are vector spaces. Restricting this to Y gives a resolution of $\mathcal{L}^i$ in terms of the sheaves $\mathcal{L}^j$ (again $j = 0, 1, \ldots, N$). Since G is exact (Proposition 1.2.2) we have a spectral sequence 1.3.10

$$E_1^{p,q} = V_p \otimes H^q(G(\mathcal{L}^p)) \implies H^{p+q}(G(\mathcal{L}^i))$$

where all non-zero terms of E_1 are contained in a finite rectangle. The point is not that this rectangle is finite (which should be fairly obvious), but that it is the same rectangle for all $i < 0$, and so the nonzero cohomology objects of $G(\mathcal{L}^i)$ are contained in some finite interval, say, $[z', z' + n']$ (for $z' \in \mathbb{Z}, n' \in \mathbb{N}$), and this interval is independent of i.

We now repeatedly use the second spectral sequence from Proposition 1.3.9. If A is a (coherent) sheaf on X, we have the spectral sequence:

$$E_2^{p,q} = \hom_{D^b(X)}(H^{-q}(G(\mathcal{L}^i)), A[p]) \implies \hom_{D^b(X)}(G(\mathcal{L}^i), A[p+q])$$

Lemma 4.2.2 implies that $E_2^{p,q} = 0$ for $p \notin [0, \dim\ X]$ and the previous discussion shows that $E_2^{p,q} = 0$ for $q \notin [-z' - n', -z']$. Hence, $\hom_{D^b(X)}(G(\mathcal{L}^i), A[j]) = 0$ for $j \notin [z, z+n]$ where $z = -z' - n'$ and $n = \dim X + n'$. The functor G is left adjoint to F and so this implies that $\hom_{D^b(Y)}(\mathcal{L}^i, F(A)[j]) = 0$ for $j \notin [z, z+n]$.

We now use the fact that the $\mathcal{L}^i$ form an ample sequence in $D^b(Y)$ (Example 2.1.3). By Lemma 2.1.5 we have $\hom_{D^b(Y)}(\mathcal{L}^i, F(A)[j]) = \hom_{\mathcal{A}}(\mathcal{L}^i, H^j(F(A)))$ for $i \ll 0$ and since $\mathcal{L}^i$ is an ample sequence, the morphism $\hom(\mathcal{L}^i, H^j(F(A))) \otimes \mathcal{L}^i \to H^j(F(A))$ is surjective for $i \ll 0$. These two facts combined with $\hom_{D^b(Y)}(\mathcal{L}^i, F(A)[j]) = 0$ show that $H^j(F(A)) = 0$ for $j \notin [z, z+n]$. $\square$

Corollary 4.2.5. *Given two smooth projective varieties X and Y over a field and an exact functor $F : D^b(X) \to D^b(Y)$ with a left adjoint, we can assume that the cohomology objects $H^i(F(\mathcal{F}))$ for a coherent sheaf $\mathcal{F}$ are nonzero only if $i \in [-a, 0]$ for some fixed $a \in \mathbb{N}$ that depends only on X, Y and F.*

Proof. The previous lemma says that F is bounded. That is, for a coherent sheaf $\mathcal{F}$ the cohomology objects of $F(\mathcal{F})$ are nonzero only if $i \in [z, z+n]$ for some $z \in \mathbb{Z}, n \in \mathbb{N}$ independent of the sheaf $\mathcal{F}$. Since $H^i(\mathcal{F}[j]) = H^{i+j}(\mathcal{F})$ for any object $\mathcal{F}$ in any derived category, if we replace F be the functor $F' = F(-)[-z-n]$ then the cohomology objects $H^i(F'(\mathcal{F}))$ of $F'(\mathcal{F})$ are nonzero only if $i \in [-n, 0]$. $\square$

Chapter 5

Fourier-Mukai transforms

5.1 Fourier-Mukai transforms

Given two smooth projective varieties X and Y over a field k, out of the many triangulated categories we can construct we are interested in three in particular: the bounded derived categories of coherent sheaves $D^b(X), D^b(Y)$ and $D^b(X \times Y)$. Of the many functors between these three categories, we take two in particular – the derived pullback $D^b(X) \stackrel{pr_1^*}{\to} D^b(X \times Y)$ and the derived pushforward $D^b(X \times Y) \stackrel{pr_{2*}}{\to} D^b(Y)$ where pr_1 and pr_2 are the projections to the first and second components from $X \times Y$ to X and Y.

Using these two functors, we have a functor $D^b(X \times Y) \to \hom(D^b(X), D^b(Y))$ where an object $E \in D^b(X \times Y)$ is taken to the functor

$$\Phi_E(-) \stackrel{def}{=} pr_{2*}(E \otimes pr_1^*(-))$$

Anything in the essential image of $D^b(X \times Y) \to \hom(D^b(X), D^b(Y))$ is called a *Fourier-Mukai transform.*

Example 5.1.1. 1. For any morphism $f : X \to Y$ we have the structure sheaf $\mathcal{O}_\Gamma$ of its graph over $X \times Y$. Then $\Phi_{\mathcal{O}_\Gamma} : D^b(X) \to D^b(Y)$ is isomorphic to the pushforward f_*.

2. Any line bundle $\mathcal{L}$ on X defines an autoequivalence $-\otimes\mathcal{L} : D^b(X) \to D^b(X)$. This is isomorphic to the Fourier-Mukai transform with kernel $i_*(\mathcal{L})$ where $i : X \to X \times X$ is the diagonal embedding.

3. The shift functor $D^b(X) \to D^b(X)$ can be described as a Fourier-Mukai transform with kernel $\mathcal{O}_\Delta[1]$ the shifted structure sheaf of the diagonal on $X \times X$.

4. The Serre functor $F \mapsto F \otimes \omega_X[n]$ with $n = \dim X$ is Fourier-Mukai transform with kernel $i_*\omega_X[n]$ where again $i : X \to X \times X$ is the diagonal embedding.

5. Let P be a coherent sheaf on $X \times Y$, flat over X and consider the associated Fourier-Mukai transform $\Phi_P : D^b(X) \to D^b(Y)$. If $x \in X$ is a closed point with residue

field $k(x)$ isomorphic to the base field k then the image of the skyskraper sheaf $k(x)$ under the Fourier-Mukai transform is

$$\Phi(k(x)) \cong P_x$$

where $P_x = P|_{\{x\}\times Y}$ is considered as a sheaf on Y via pr_{2*}. This explains the choice of the notation M for the first scheme in [Orl96] as we can think of P as a family of sheaves on Y parametrized by X (the M is for Moduli). To obtain the sheaf corresponding to x we evaluate the Fourier-Mukai transform Φ_P at the skyscraper sheaf of x.

6. As a particular case of the previous example we have the Poincaré bundle on $A \times \hat{A}$ for an abelian variety A.

Remark 5.1.2. The analogy to the classical Fourier transform is most prominant in the case $\Phi_P : D^b(A) \to D^b(\hat{A})$ where P is the Poincaré bundle on the product of an abelian variety A and its dual $\hat{A}$. Căldăraru ([Căl05]) describes the analogy as:

Fourier transform	Fourier-Mukai transform
L^2-function	object E of $D^b(X)$
product with an integral kernel	tensor of pr_1^*E with a kernel in $D^b(X \times Y)$
integral	pushforward pr_{2*}

We now list some interesting results about Fourier-Mukai transforms. As our main task is to prove Orlov's Theorem and we leave them all unproven. The proofs can all be found in [Huy06, Chapter 5.1]; they are not terribly difficult and, in fact, Proposition 5.1.5 is an exercise and Proposition 5.1.3 is a direct consequence of Grothendieck-Verdier duality (which can be found in [Har66] for example).

Proposition 5.1.3 ([Muk81]). *Let $E \in D^b(X \times Y)$ and define*

$$E_L \stackrel{def}{=} E^\vee \otimes pr_1^*\omega_Y[\dim Y] \qquad \text{and} \qquad E_R \stackrel{def}{=} E^\vee \otimes pr_2^*\omega_X[\dim X]$$

Then $\Phi_{P_L} \dashv \Phi_P \dashv \Phi_{P_R}$.

Proposition 5.1.4 ([Muk81]). *Consider $E \in D^b(X \times Y)$ and $F \in D^b(Y \times Z)$ together with the projections*

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "pr_{12}"'] \arrow[d, "pr_{13}"] \arrow[dr, "pr_{23}"] & \\ X \times Y & X \times Z & Y \times Z \end{tikzcd}$$

and define $G \stackrel{def}{=} pr_{13*}(pr_{12}^*E \otimes pr_{23}^*F)$. Then the composition $\Phi_F \circ \Phi_E$ is isomorphic to Φ_G.

Proposition 5.1.5 ([Orl03]). *Let $R \in D^b(X_1 \times X_2), E_1 \in D^b(X_1 \times Y_1), E_2 \in D^b(X_2 \times Y_2)$ and $S = \Phi_{E_1 \boxtimes E_2}(R)$. Then the following diagram commutes*

$$\begin{tikzcd}
D^b(X_1) \arrow[d, "\Phi_R"'] & D^b(Y_1) \arrow[l, "\Phi_{E_1}"'] \arrow[d, "\Phi_S"] \\
D^b(X_2) \arrow[r, "\Phi_{E_2}"'] & D^b(Y_2)
\end{tikzcd}$$

Chapter 6

The proof

In this section we arrive at the focal point of this work – the proof of Orlov's Theorem – which we reproduce here for convenience. The reference for this section is [Orl96] although the reader should be warned that the material has been restructured and in some places embellished.

Orlov's Theorem. *Let $F : D^b(X) \to D^b(Y)$ be an exact functor between the bounded derived categories of coherent sheaves on two smooth projective varieties X and Y over a field. Suppose F is full and faithful and has a right (and consequently, a left) adjoint functor.*

Then there exists an object $E \in D^b(X \times Y)$ such that F is isomorphic to the Fourier-Mukai transform Φ_E and this object is unique up to isomorphism.

Even after all the background material that we have developed solely for use in this proof, the argument in this part of the proof is still relatively involved. It can be roughly outlined as follows:

1. Construct an object $E \in Ob\ D^b(X \times Y)$ and a natural isomorphism $\overline{f} : F|_{\mathcal{C}} \xrightarrow{\sim} \Phi_E|_{\mathcal{C}}$ where $\mathcal{C} \subset D^b(X)$ is a full subcategory generated by an ample sequence.

2. Using the right adjoint $F^!$ to F we obtain a natural isomorphism of the endomorphisms
$$F^!(\overline{f}) : F^! \circ F|_{\mathcal{C}} \cong id_{\mathcal{C}} \xrightarrow{\sim} F^! \circ \Phi_E|_{\mathcal{C}}$$
using Proposition 2.2.1 we extend this to a natural isomorphism $id \xrightarrow{\sim} F^! \circ \Phi_E$ of endomorphisms of $D^b(X)$.

3. Since $F^!$ is the right adjoint to F, we obtain an extension f of $\overline{f}$. We show that this is a natural isomorphism.

To construct and work with the natural isomorphism $\overline{f} : F|_{\mathcal{C}} \xrightarrow{\sim} \Phi_E|_{\mathcal{C}}$ however, it is more convenient to be in $D^b(\mathbb{P}^N)$ for some embedding $j : X \hookrightarrow \mathbb{P}^N$ and then transfer certain properties to $D^b(X)$. This is not just a once off procedure however, and for all of this section we will be moving backwards and forwards between the functor $F : D^b(X) \to D^b(Y)$ and the induced functor $F' \stackrel{def}{=} F \circ j^* : D^b(\mathbb{P}^N) \to D^b(Y)$.

In Section 6.1 we build the objects $E \in D^b(X \times Y)$ and $E' \in D^b(\mathbb{P}^N \times Y)$ that will be the kernels of the Fourier-Mukai transforms F and F' respectively. In Section 6.2 we construction the natural isomorphisms $\overline{f} : F|_{\mathcal{C}} \xrightarrow{\sim} \Phi_E|_{\mathcal{C}}$ and $f : F'|_{\mathcal{C}'} \xrightarrow{\sim} \Phi_{E'}|_{\mathcal{C}'}$ on the full subcategories $\mathcal{C}$ and $\mathcal{C}'$ whose objects are ample sequences. In Section 6.3 we tie everything together and give the final details of the proof of Orlov's Theorem.

6.1 Construction of the Fourier–Mukai kernel.

In this section we construct the objects $E' \in D^b(\mathbb{P}^N \times Y)$ and $E \in D^b(X \times Y)$ which will eventually show to be the Fourier-Mukai kernels of F' and F respectively. We will also show that $(j \times id)_* E \cong E'$.

First we set some notation. Choose a very ample invertible sheaf $\mathcal{L}$ on X such that $H^i(\mathcal{L}^k) = 0$ for all $k > 0$ when $i \neq 0$. Associated to $\mathcal{L}$ (by definition of very ample) we have a closed embedding $j : X \to \mathbb{P}^N$ such that $\mathcal{L} = j^*\mathcal{O}(1)$. We set notation for this section:

$F' =$ the composition $D^b(\mathbb{P}^N) \xrightarrow{j^*} D^b(X) \xrightarrow{F} D^b(Y)$.

$\mathcal{C}' =$ the full subcategory of $D^b(\mathbb{P}^N)$ with $Ob\,\mathcal{C} = \{\mathcal{O}(k) | k \in \mathbb{Z}\}$.

$\mathcal{C} =$ the full subcategory of $D^b(X)$ with $Ob\,\mathcal{C} = \{\mathcal{L}^k | k \in \mathbb{Z}\}$.

Construction 6.1.1. Consider the resolution of the diagonal on the product $\mathbb{P}^N \times \mathbb{P}^N$ from Proposition 4.1.1:

$$\left\{\mathcal{O}(-N) \boxtimes \Omega^N(N) \to \cdots \to \mathcal{O}(-1) \boxtimes \Omega^1(1) \xrightarrow{d_{-1}} \mathcal{O} \boxtimes \mathcal{O}\right\} \xrightarrow{q.i.} \mathcal{O}_\Delta \tag{6.1}$$

We define a complex

$$C \stackrel{def}{=} \left\{\mathcal{O}(-N) \boxtimes F'(\Omega^N(N)) \xrightarrow{d'_{-N}} \cdots \to \mathcal{O}(-1) \boxtimes F'(\Omega^1(1)) \xrightarrow{d'_{-1}} \mathcal{O} \boxtimes F'(\mathcal{O})\right\} \tag{6.2}$$

over the derived category $D^b(\mathbb{P}^N \times Y)$ where the differentials

$$\mathcal{O}(-i) \boxtimes F'(\Omega^i(i)) \xrightarrow{d'_{-1}} \mathcal{O}(-i+1) \boxtimes F'(\Omega^{i+1}(i+1))$$

are defined by following d_{-i} from the resolution 6.1 through the following sequence of maps. Notice that they are all bijections except possibly for F'.

$$
\begin{aligned}
&\hom_{D^b(\mathbb{P}^N\times\mathbb{P}^N)}\Big(\mathcal{O}(-i)\boxtimes\Omega^i(i),\quad \mathcal{O}(-i+1)\boxtimes\Omega^{i-1}(i-1)\Big) \xrightarrow{pr_1^*\mathcal{O}(i)\otimes -}\\
&\hom_{D^b(\mathbb{P}^N\times\mathbb{P}^N)}\Big(\mathcal{O}\boxtimes\Omega^i(i),\mathcal{O}(1)\boxtimes\Omega^{i-1}(i-1)\Big) \xrightarrow{pr_{2*}}\\
&\hom_{D^b(\mathbb{P}^N)}\Big(\Omega^i(i),H^0(\mathcal{O}(1))\otimes\Omega^{i-1}(i-1)\Big) \xrightarrow{F'}\\
&\hom_{D^b(Y)}\Big(F'(\Omega^i(i)),H^0(\mathcal{O}(1))\otimes F'(\Omega^{i-1}(i-1))\Big) \xleftarrow{pr_{2*}}\\
&\hom_{D^b(\mathbb{P}^N\times Y)}\Big(\mathcal{O}\boxtimes F'(\Omega^i(i)),\mathcal{O}(1)\boxtimes F'(\Omega^{i-1}(i-1))\Big) \xrightarrow{pr_1^*\mathcal{O}(-i)\otimes -}\\
&\hom_{D^b(\mathbb{P}^N\times Y)}\Big(\mathcal{O}(-i)\boxtimes F'(\Omega^i(i)),\mathcal{O}(-i+1)\boxtimes F'(\Omega^{i-1}(i-1))\Big)
\end{aligned}
$$

Lemma 6.1.2. *The composition $d_{-i+1}\circ d_{-i}$ is equal to zero.*

Proof. Ommitted as it is short, tedious, and uninsightful. □

Lemma 6.1.3. *There exists a convolution of the complex C determined up to (non-unique) isomorphism.*

Proof. To see this we show that the complex satisfies the condition (C1) of Section 3 and then apply Corollary 3.1.5. That is, we just have to show that $\hom(C^a, C^{a+j}[\ell]) = 0$ for all $j > 0$, $\ell < 0$ and all a.

For $\ell < 0$ we have the following isomorphisms

$$
\begin{aligned}
&\hom\Big(\mathcal{O}(-i)\boxtimes F'(\Omega^i(i)),\quad \mathcal{O}(-k)\boxtimes F'(\Omega^k(k))[\ell]\Big) \xrightarrow{pr_1^*\mathcal{O}(i)\otimes -}\\
&\hom\Big(\mathcal{O}\boxtimes F'(\Omega^i(i)),\quad \mathcal{O}(i-k)\boxtimes F'(\Omega^k(k))[\ell]\Big) \xrightarrow{pr_{2*}}\\
&\hom\Big(F'(\Omega^i(i)),\quad H^0(\mathcal{O}(i-k))\otimes F'(\Omega^k(k))[\ell]\Big) \xleftarrow{F}\\
&\hom\Big(j^*(\Omega^i(i)),\quad H^0(\mathcal{O}(i-k))\otimes j^*(\Omega^k(k))[\ell]\Big) = 0
\end{aligned}
$$

Hence, we can apply corollary 3.1.5 to obtain a convolution of C determined up to non-unique isomorphism. □

Definition 6.1.4. We define the object E' to be one of the (isomorphic) convolutions of C and γ_0 the morphism $\mathcal{O}\boxtimes F'(\mathcal{O}) \xrightarrow{\gamma_0} E'$.

We now set about constructing an object E over $D^b(X\times Y)$ which will be the kernel of the Fourier-Mukai transform F. There is rather a lot of notation that Orlov uses in this part of the proof and so we collect all of this together now. Recall that we have chosen a very ample invertible sheaf $\mathcal{L}$ on X such that $H^i(\mathcal{L}^k) = 0$ for all $k > 0$ when $i \neq 0$.

$A =$ the graded k-algebra $\bigoplus_{i=0}^{\infty} H^0(X, \mathcal{L}^i) = \bigoplus_{i=0}^{\infty} \Gamma(X, \mathcal{L}^i)$.

$B =$ the graded module defined recursively as follows:

$$B_0 = k \qquad B_1 = A_1 \qquad B_m = \ker(B_{m-1} \otimes_k A_1 \to B_{m-2} \otimes A_2)$$

$\mathcal{R} =$ the graded $\mathcal{O}_X$-module defined by $\mathcal{R}_0 = \mathcal{O}_X$ and $\mathcal{R}_m = \ker(B_m \otimes_k \mathcal{O}_X \to B_{m-1} \otimes_k \mathcal{L})$.

$\mathcal{T} =$ the graded $\mathcal{O}_X$-module defined by $\mathcal{T}_k = \ker(A_{k-n} \otimes \mathcal{R}_n \to A_{k-n+1} \otimes \mathcal{R}_{n-1})$

We present the following example in the hope that it will provide meaning to these definitions and highlight the analogy between the construction of E and E'. The extra subtlety in constructing E' comes from the non-vanishing of $\mathcal{T}$ (described below) in the general case.

Example 6.1.5. In the case where $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}(1)$ we have the following:

$A \cong$ the polynomial algebra $k[X_0, \dots, X_n]$.

$B \cong$ the exterior product algebra $\Lambda^\bullet A_1$.

$\mathcal{R}_m \cong \Omega^m(m)$.

$\mathcal{T}_m \cong 0$.

Construction 6.1.6. For a given n we can find a $\mathcal{L}$ such that the following conditions hold ([IM94]):

1. The following sequence is exact (i.e. A is n-Koszul):

$$B_n \otimes_k A \to B_{n-1} \otimes_k A \to \cdots \to B_1 \otimes_k A \to A \to k \to 0$$

2. The following complex on $X \times X$ is exact:

$$\mathcal{L}^{-n} \boxtimes \mathcal{R}_n \to \cdots \to \mathcal{L}^{-1} \boxtimes \mathcal{R}_1 \to \mathcal{O}_M \boxtimes \mathcal{R}_0 \to \mathcal{O}_\Delta$$

3. The following sequences on M are exact for any $k \geq 0$ (by definition, if $k - 1 < 0$ then $A_{k-i} = 0$).

$$A_{k-n} \otimes \mathcal{R}_n \to A_{k-n+1} \otimes \mathcal{R}_{n-1} \to \cdots \to A_{k-1} \otimes \mathcal{R}_1 \to A_k \otimes \mathcal{R}_0 \to \mathcal{L}^k \to 0$$

We consider the following complex over $D^b(X \times Y)$ which is analogous to the complex in Equation (6.2):

$$\mathcal{L}^{-n} \boxtimes F(\mathcal{R}_n) \to \cdots \to \mathcal{L}^{-1} \boxtimes F(\mathcal{R}_1) \to \mathcal{O}_M \boxtimes F(\mathcal{R}_0) \tag{6.3}$$

where $\mathcal{L}^{-k} \boxtimes F(\mathcal{R}_k) \to \mathcal{L}^{-k+1} \boxtimes F(\mathcal{R}_{k-1})$ is induced by $\mathcal{R}_k \to A_1 \otimes \mathcal{R}_{k-1}$ via

$$\hom\Big(\mathcal{L}^{-k} \boxtimes F(\mathcal{R}_k), \mathcal{L}^{-k+1} \boxtimes F(\mathcal{R}_{k-1})\Big) \cong \hom\Big(F(\mathcal{R}_k), H^0(\mathcal{L}) \otimes F(\mathcal{R}_{k-1})\Big) \cong \hom\Big(\mathcal{R}_k, A_1 \otimes \mathcal{R}_{k-1}\Big)$$

Proposition 6.1.7.

1. *Complex (6.3) has a convlution G, uniquely determined up to isomorphism.*
2. $pr_{2*}(G \otimes pr_1^*(\mathcal{L}^k)) \cong F(\mathcal{T}_k[n] \oplus \mathcal{L}^k)$
3. *$G \cong C \oplus E$ where E and C are objects of $D^b(X \times Y)$ such that $H^i(E) = 0$ for $i \notin [-a, 0]$ and $H^i(C) = 0$ for $i \notin [-n-a, -n]$.*

Proof. 1. It follows from Lemma 3.1.5 that Complex 6.3 has a convolution, uniquely determined up to isomorphism to check that condition (C1) is satisfied the argument is the same as the proo fof Lemma 6.1.3.

2. By applying $pr_{2*}(- \otimes pr_1^*(\mathcal{L}^k))$ to Complex 6.3 and recalling that convolutions are preserved under exact functors, we see that $pr_{2*}(G \otimes pr_1^*(\mathcal{L}^k))$ is a convolution of the complex:

$$A_{k-n} \otimes F(\mathcal{R}_n) \to A_{k-n+1} \otimes F(\mathcal{R}_{n-1}) \to \cdots \to A_k \otimes F(\mathcal{R}_0) \tag{6.4}$$

Now consider the complex

$$A_{k-n} \otimes \mathcal{R}_n \to A_{k-n+1} \otimes \mathcal{R}_{n-1} \to \cdots \to A_k \otimes \mathcal{R}_0 \tag{6.5}$$

Viewing it as an object R in $D^b(X)$ and using reasoning as in the proof of Proposition 1.3.4 we have a distinguished triangle $T_k[n] \to R \to \mathcal{L}^k \to T_k[n+1]$. But for $n \gg 0$ there are no morphisms $\mathcal{L}^k \to T_k[n+1]$ (Example 2.1.8) so $R \cong \mathcal{L}^k \oplus T_k[n]$ by Lemma 1.1.9. From the first example of Example 3.1.2 we see then that $\mathcal{T}_k[n] \oplus \mathcal{L}^k \cong R$ is a convolution of Complex 6.5.

Now, again, since convolutions are preserved under exact functors, applying F to Complex 6.5 we see that $F(\mathcal{T}_k[n] \oplus \mathcal{L}^k)$ is a convolution of Complex 6.4. But $pr_{2*}(G \otimes pr_1^*(\mathcal{L}^k))$ is a convolution of Complex 6.4 and so there is an isomorphism

$$pr_{2*}(G \otimes pr_1^*(\mathcal{L}^k)) \cong F(\mathcal{T}_k[n] \oplus \mathcal{L}^k)$$

3. It follows from Corollary 4.2.5 that the cohomology sheaves $H^i(F(\mathcal{T}_k[n])) \oplus H^i(F(\mathcal{L}^k))$ fall in the interval $[-n-a, -n] \cup [-a, 0]$ for any $k > 0$ where a is defined in Corollary 4.2.5. Furthermore, recall that we can choose n such that $n > \dim\ X + \dim\ Y + a$ to ensure that these two intervals don't overlap. Hence, the cohomology sheaves of $pr_{2*}(G \otimes pr_1^*(\mathcal{L}^k)) \cong F(\mathcal{T}_k[n] \oplus \mathcal{L}^k)$ also fall in this interval, and since $- \otimes pr_1^*(\mathcal{L}^k)$ and pr_{2*} are exact (this can be checked locally), the cohomology sheaves of G fall in this interval as well. We now use the same argument as in the proof of Proposition 1.3.4 to obtain the result.

□

Lemma 6.1.8.

$$j_*(E) \cong E'$$

Proof. Consider the following morphism of complexes in $D^b(\mathbb{P}^N \times Y)$ (note in particular that the first term is indexed by n and not N):

$$\begin{tikzcd}
\mathcal{O}(-n) \boxtimes F'(\Omega^n(n)) \arrow[r] \arrow[d, "can \boxtimes F(f_n)"] & \cdots \arrow[r] & \mathcal{O} \boxtimes F'(\mathcal{O}) \arrow[d, "can \boxtimes F(f_0)"] \\
j_*(\mathcal{L}^{-n}) \boxtimes F(\mathcal{R}_n) \arrow[r] & \cdots \arrow[r] & j_*(\mathcal{O}_M) \boxtimes F(\mathcal{R}_0)
\end{tikzcd}$$

The conditions of Lemma 3.1.7 are satisfied so we have a morphim of convolutions $\phi : E'_n \to j_*(G)$ where E'_n is a convolution of the upper complex. If $n < N$ then E'_n is not isomorphic to $E' = E'_N$, but there is a distinguished triangle

$$S \to E'_n \to E' \to S[1]$$

for some S (explicitely, S is isomorphic to $Cone(E'_n \to E')[-1]$). The cohomology sheaves $H^i(S)$ of S are nonzero only when $i \in [-n-a, -n]$. Since $\hom(S, j_*(E)) = 0$ and $\hom(S[1], j_*(E)) = 0$, there is a uniquely determined morphism $\psi : E' \to j_*(E)$ such that the following diagram commutes

$$\begin{tikzcd}
E'_n \arrow[r, "\phi"] \arrow[d] & j_*(G) \arrow[d] \\
E' \arrow[r, "\psi"] & j_*(E)
\end{tikzcd}$$

We know that $pr_{2*}(E' \otimes pr_1^*(\mathcal{O}(k)) \cong F(\mathcal{L}^k) \cong pr_{2*}(E \otimes pr_1^*(\mathcal{L}^k))$. Let ψ_k be the morphism $pr_{2*}(E' \otimes pr_1^*(\mathcal{O}(k))) \to pr_{2*}(E \otimes pr_1^*(\mathcal{L}^k))$ induced by ψ. The morphism ψ_k can be included in the following commutative diagram

$$\begin{tikzcd}
S^k A_1 \otimes F(\mathcal{O}) \arrow[r, "can"] \arrow[d, "can"] & F(\mathcal{L}^k) \arrow[r, "\sim"] & pr_{2*}(E' \otimes pr_1^*(\mathcal{O}(k))) \arrow[d, "\psi_k"] \\
A_k \otimes F(\mathcal{O}) \arrow[r, "can"] & F(\mathcal{L}^k) \arrow[r, "\sim"] & pr_{2*}(E \otimes pr_1^*(\mathcal{L}^k))
\end{tikzcd}$$

Thus we see that ψ_k is an isomorphism for any $k \geq 0$. Hence, ψ is an isomorphism too. $\square$

6.2 The restricted natural isomorphism $\overline{f} : F|_{\mathcal{C}} \xrightarrow{\sim} \Phi_E|_{\mathcal{C}}$

In this section we build the natural isomorphism $\overline{f} : F|_{\mathcal{C}} \xrightarrow{\sim} \Phi_E|_{\mathcal{C}}$.

Recall the full subcategory $\mathcal{C}' \subset D^b(\mathbb{P}^N)$ with $Ob\ \mathcal{C}' = \{\mathcal{O}(k) | k \in \mathbb{Z}\}$.

Lemma 6.2.1. *There exists a canonically defined natural isomorphism*

$$f : F'|_{\mathcal{C}'} \xrightarrow{\sim} \Phi_{E'}|_{\mathcal{C}'}$$

Proof. We begin by defining an isomorphism $F'(\mathcal{O}(k)) \xrightarrow{f_k} \Phi_{E'}(\mathcal{O}(k))$ for each $k \geq 0$. Consider again the resolution of sheaves on $\mathbb{P}^N$ described in Corollary 4.2.1:

$$\{H^0(\mathcal{O}(k-N)) \otimes \Omega^N(N) \to \cdots \to H^0(\mathcal{O}(k-1)) \otimes \Omega^1(1) \to H^0(\mathcal{O}(k)) \otimes \mathcal{O}\} \xrightarrow{\delta_k} \mathcal{O}(k)$$

This is a resolution of $\mathcal{O}(k)$ and so $\mathcal{O}(k)$ is its convolution (see the first example in Example 3.1.2). Since convolutions are preserved by exact functors we see that $F'(\mathcal{O}(k))$ is a convolution of the complex D_k:

$$H^0(\mathcal{O}(k-N)) \otimes F'(\Omega^N(N)) \to \cdots \to H^0(\mathcal{O}(k-1)) \otimes F'(\Omega^1(1)) \to H^0(\mathcal{O}(k)) \otimes F'(\mathcal{O})$$

over $D^b(Y)$.

Now consider the complex C of Equation (6.2) defined in Construction 6.1.1, its convolution E' in $D^b(\mathbb{P}^N \times Y)$, and the morphism $C^0 \xrightarrow{\gamma} E'$. Pushing these through the functor $pr_{2*}(- \otimes pr_1^*\mathcal{O}(k))$ we see that $pr_{2*}(C \otimes pr_1^*\mathcal{O}(k)) = D_k$, and so $F'(\mathcal{O}(k))$ and $pr_{2*}(E' \otimes pr_1^*\mathcal{O}(k)) = \Phi_{E'}(\mathcal{O}(k))$ are both convolutions of the same complex.

By assumption, F is fully faithful and so for locally free sheaves G, H on $\mathbb{P}^N$ we have

$$\hom_{D^b(Y)}(F'(G), F'(H)[i]) = \hom_{D^b(X)}(F \circ j^*(G), F \circ f^*(H)[i]) =$$

$$\hom_{D^b(X)}(j^*(G), j^*(H)[i]) = 0 \text{ for } i < 0$$

Since the sheaves $\Omega^k(k)$ and $\mathcal{O}(k)$ are locally free, it follows that the complex D_k satisfies the conditions of Corollary 3.1.8. Hence, there exists a uniquely defined isomorphism $f_k : F'(\mathcal{O}(k)) \xrightarrow{\sim} \Phi_{E'}(\mathcal{O}(k))$ completing the commutative diagram:

$$\begin{tikzcd}
H^0(\mathcal{O}(k)) \otimes F'(\mathcal{O}) \arrow[r, "F'(\delta_k)"] \arrow[d, "id"] & F'(\mathcal{O}(k)) \arrow[d, "f_k"] \\
H^0(\mathcal{O}(k)) \otimes F'(\mathcal{O}) \arrow[r, "pr_{2*}(\gamma \otimes pr_1^*\mathcal{O}(k))"] & \Phi_{E'}(\mathcal{O}(k))
\end{tikzcd}$$

Now we show that these isomorphisms are functorial. Consider a morphism $\alpha : \mathcal{O}(k) \to \mathcal{O}(\ell)$ (still assuming that $\ell, k \geq 0$). We have the following diagram

$$\begin{tikzcd}
H^0(\mathcal{O}(k)) \otimes F'(\mathcal{O}) \arrow[r, "F'(\delta_k)"] \arrow[d, "H^0(\alpha) \otimes id"] \arrow[rr, bend left, "pr_{2*}(\gamma \otimes pr_1^*\mathcal{O}(k))"] & F'(\mathcal{O}(k)) \arrow[r, "f_k"] \arrow[d, "F'(\alpha)"] & \Phi_{E'}(\mathcal{O}(k)) \arrow[d, "\Phi_{E'}(\alpha)"] \\
H^0(\mathcal{O}(\ell)) \otimes F'(\mathcal{O}) \arrow[r, "F'(\delta_\ell)"] \arrow[rr, bend right, "pr_{2*}(\gamma \otimes pr_1^*\mathcal{O}(\ell))"'] & F'(\mathcal{O}(\ell)) \arrow[r, "f_\ell"] & \Phi_{E'}(\mathcal{O}(\ell))
\end{tikzcd}$$

where we know that the two "triangles" commute (they are just the previous commutative square) as well as the outside rectangle and left square. It follows that the following two paths from the upper left object to the lower right object are the same:

$$f_\ell \circ F'(\alpha) \circ F'(\delta_k) = \phi_{E'}(\alpha) \circ f_k \circ F'(\delta_k)$$

Since the complexes D_k and D_ℓ satisfy the conditions of Lemma 3.1.7 there is only one morphism $h : F'(\mathcal{O}(k)) \to \phi_{E'}(\mathcal{O}(\ell))$ such that

$$h \circ F'(\delta_k) = \left(pr_{2*}(\gamma \otimes pr_1^*\mathcal{O}(\ell))\right) \circ (H^0(\alpha) \otimes id)$$

and hence the rightmost square in the diagram above commutes.

Now consider the case $k < N$. We consider the resolution

$$\mathcal{O}(k) \xrightarrow{\sim} \{V_0^k \otimes \mathcal{O} \to \cdots \to V_N^k \otimes \mathcal{O}(N)\}$$

from the Corollary 4.2.1. It follows from Corollary 3.1.7 that the morphism

$$\begin{tikzcd} V_0^k \otimes F'(\mathcal{O}) \arrow[r] \arrow[d, "id\otimes f_0"', "\wr"] & \cdots \arrow[r] & V_N^k \otimes F'(\mathcal{O}(N)) \arrow[d, "id\otimes f_N"', "\wr"] \\ V_0^k \otimes \Phi_{E'}(\mathcal{O}) \arrow[r] & \cdots \arrow[r] & V_N^k \otimes \Phi_{E'}(\mathcal{O}(N)) \end{tikzcd}$$

of complexes over $D^b(Y)$ determines uniquely a morphism $f_k : F'(\mathcal{O}(k)) \to \Phi_{E'}(\mathcal{O}(k))$. It is not too hard to see that these morphisms are functorial. $\square$

Recall the full subcategory $\mathcal{C} \subset D^b(X)$ with $Ob\ \mathcal{C} = \{\mathcal{L}^k | k \in \mathbb{Z}\}$.

Lemma 6.2.2. *There exists a canonically defined natural isomorphism*

$$\overline{f} : F|_{\mathcal{C}} \xrightarrow{\sim} \Phi_E|_{\mathcal{C}}$$

Proof. The result follows more or less from the natural isomorphism $f : F'|_{\mathcal{C}'} \xrightarrow{\sim} \Phi_{E'}|_{\mathcal{C}'}$ of Lemma 6.2.1 and the property $(j \times id)_*(E) \cong E'$ of Lemma 6.1.8.

The functor

$$- \circ j_* : \mathrm{hom}(D^b(\mathbb{P}^N), D^b(Y)) \to \mathrm{hom}(D^b(X), D^b(Y))$$

takes F to F' by definition of F'. Consider what it does to $\Phi_{E'}$. For an object $\mathcal{F} \in D^b(X)$ we have

$$\begin{array}{rcl} \Phi_{E'} \circ j_*(\mathcal{F}) & = & pr_{2*}(E' \otimes pr_1^* j_*(\mathcal{F})) \\ \text{(flat base change 1.4.4)} & \cong & pr_{2*}(E' \otimes (j \times id)_* pr_1^*(\mathcal{F})) \\ & \cong & pr_{2*}\left((j \times id)_*(E) \otimes (j \times id)_* pr_1^*(\mathcal{F})\right) \\ & = & pr_{2*}(j \times id)_*(E \otimes pr_1^*\mathcal{F}) \\ & = & pr_{2*}(E \otimes pr_1^*\mathcal{F}) \\ & = & \Phi_E(\mathcal{F}) \end{array}$$

Since the isomorphisms are natural we see that the image of $\Phi_{E'}$ under $- \circ j_*$ is Φ_E and so using the adjunction morphisms $\mathcal{O}(k) \to j_*j^*\mathcal{O}(k) = j_*\mathcal{L}^k$ we get our natural isomorphism. $\square$

6.3 Proof of the main theorem.

In this section we finally give the proof of Orlov's Theorem. We recall the notation we have been using.

$F : D^b(X) \to D^b(Y)$ an exact full and faithful functor between derived categories of smooth projective varieties with a right (and consequently, a left) adjoint functors.

$F^*, F^!$ the left and right adjoints to F, respectively.

$\mathcal{L}$ a very ample invertible sheaf on M such that $H^i(X, \mathcal{L}^k) = 0$ for any $k > 0$ and $i \neq 0$.

$j : X \hookrightarrow \mathbb{P}^N$ the embedding associated to $\mathcal{L}$.

$\mathcal{C} \subset D^b(X)$ the full subcategory whose objects are $\mathcal{L}^i$ for $i \in \mathbb{Z}$.

First we collect together some properties of the functors and natural transformations involved to make the argument of the proof clearer.

Lemma 6.3.1. *Recall that Φ_E has a left adjoint Φ_E^* 5.1.3. The following hold:*

1. *Φ_E is full and faithful.*

2. *$\Phi_E^* \circ F \dashv F^! \circ \Phi_E$, and therefore both of these functors have both left and right adjoints.*

3. *$F^! \circ \Phi_E$ and $\Phi_E^* \circ F$ are full and faithful.*

4. *$F^! \circ \Phi_E$ and $\Phi_E^* \circ F$ are equivalences.*

Proof. 1. The category $Coh(X)$ has finite homological dimension (Example 2.1.8) and Φ_E is an exact functor with left and right adoints (Proposition 5.1.3). Furthermore, since F is fully faithful and there is a natural isomorphism $F|_{\mathcal{C}} \to \Phi_E|_{\mathcal{C}}$ the maps $\hom(\mathcal{L}^i, \mathcal{L}^j[k]) \to \hom(\Phi_E(\mathcal{L}^i), \Phi_E(\mathcal{L}^j)[k])$ are isomorphisms. So we can apply Lemma 2.1.10 and deduce that Φ_E is full and faithful.

2. $\hom(\Phi_E^* F\mathcal{F}, \mathcal{G}) \cong \hom(F\mathcal{F}, \Phi_E\mathcal{G}) \cong \hom(\mathcal{F}, F^!\Phi_E\mathcal{G})$. We then use Proposition 1.4.3.

3. We again use Lemma 2.1.10. The functors Φ_E and F are fully faithful and so the adjunction natural transformations $id \to F^! \circ F$ and $\Phi_E^* \circ \Phi_E \to id$ are isomorphisms. The images of the natural isomorphism $F|_{\mathcal{C}} \xrightarrow{\sim} \Phi_E|_{\mathcal{C}}$ under Φ_E^* and $F^!$ are still natural isomorphisms, and so the compositions $\Phi_E^* \circ F|_{\mathcal{C}} \xrightarrow{\sim} \Phi_E^* \circ \Phi_E|_{\mathcal{C}} \xrightarrow{\sim} id|_{\mathcal{C}}$ and $id|_{\mathcal{C}} \xrightarrow{\sim} F^! \circ F|_{\mathcal{C}} \xrightarrow{\sim} F^! \circ \Phi_E|_{\mathcal{C}}$ are natural isomorphisms. Hence, the maps $\hom(\mathcal{L}^i, \mathcal{L}^j[k]) \to \hom(F^! \circ \Phi_E(\mathcal{L}^i), F^! \circ \Phi_E(\mathcal{L}^j)[k])$ are isomorphisms and similarly for $\Phi_E^* \circ F$. We have just shown (Item 2. of this lemma) that both of these functors have both left and right adjoints and so we apply Lemma 2.1.10 to obtain the result.

4. $\Phi_E^* \circ F \dashv F^! \circ \Phi_E$ by 2. and both functors are full and faithful by 3. Hence the adjunction natural transformations are isomorphisms and so both functors are essentially surjective.

$\square$

Proof of Orlov's Theorem. Existence. Consider the natural isomorphism $id|_{\mathcal{C}} \xrightarrow{\sim} F^! \circ F|_{\mathcal{C}} \xrightarrow{\sim} F^! \circ \Phi_E|_{\mathcal{C}}$. By Lemma 6.3.1 4. $F^! \circ \Phi_E$ is an equivalence and by Lemma 6.3.1 2. it has both left and right adjoints. Hence, we can invoke Proposition 2.2.1 and obtain an extension $id \xrightarrow{\sim} F^! \circ \Phi_E$. Since $F^!$ is the right adjoint to F, this corresponds to a natural morphism $F \to \Phi_E$. We will now show that this is actually a natural isomorphism.

Let C_Z be the cone of $f_Z : F(Z) \to \Phi_E(Z)$ for some object $Z \in D^b(M)$; so we have a distinguished triangle $F(Z) \xrightarrow{f_Z} \Phi_E(Z) \to C_Z \to F(Z)[1]$ in $D^b(X)$. Since $F^!(f_Z)$ is an isomorphism $F^!(C_Z) = 0$ (Lemma 1.1.11) so $\hom(F(Y), C_Z) \cong \hom(Y, F^!(C_Z)) = 0$ for any object Y. Using the isomorphisms $F(\mathcal{L}^k) \cong \Phi_E(\mathcal{L}^k)$ we have

$$\hom(\mathcal{L}^k, \Phi_E^!(C_Z)[i]) = \hom(\Phi_E(\mathcal{L}^k), C_Z[i]) = \hom(F(\mathcal{L}^k), C_Z[i]) = 0$$

for al k and i. Hence, Lemma 2.1.6 tells us that $\Phi_E^!(C_Z) = 0$ which implies that $\hom(\Phi_E(Z), C_Z) = 0$ and so $F(Z) = C_Z[-1] \oplus \Phi_E(Z)$ (Lemma 1.1.9). But we know that $\hom(F(Z), C_Z[-1]) \cong \hom(Z, F^!(C_Z)[1]) = 0$ (recall that $F^!(C_Z) = 0$) and so $C_Z = 0$ and hence, f is an isomorphism (Lemma 1.1.11).

Uniqueness. Suppose that we are given an object E_1 and a natural isomorphism $F \cong \Phi_{E_1}$. We will find an isomorphism $E_1 \cong E$ where E is the object constructed above. First recall Complex 6.3, reproduced below, and that it has a convolution G, unique up to isomorphism, with $G \cong E \oplus C$ where $H^i(E) = 0$ for $i \notin [-a, 0]$ and $H^i(C) = 0$ for $i \notin [-n-a, -n]$ for $n \gg 0$

$$\mathcal{L}^{-n} \boxtimes F(\mathcal{R}_n) \to \cdots \to \mathcal{L}^{-1} \boxtimes F(\mathcal{R}_1) \to \mathcal{O}_X \boxtimes F(\mathcal{R}_0) \tag{6.3}$$

Recalling the definitions in Construction 6.1.6 there is an obvious morphism $A_1 \otimes \mathcal{R}_0 \to \mathcal{L}$ whose composition with $\mathcal{R}_1 \to A_1 \otimes \mathcal{R}_0$ is zero. Following this through the chain

$$\mathcal{R}_1 \longrightarrow A_1 \otimes \mathcal{R}_0 \longrightarrow \mathcal{L}$$

$$F(\mathcal{R}_1) \longrightarrow F(A_1 \otimes \mathcal{R}_0) \longrightarrow F(\mathcal{L}) \cong pr_{2*}(E_1 \otimes pr_1^*\mathcal{L})$$

$$\mathcal{O}_X \boxtimes F(\mathcal{R}_1) \longrightarrow \mathcal{L}^1 \boxtimes F(\mathcal{R}_0) \longrightarrow E_1 \otimes \mathcal{L}$$

$$\mathcal{L}^{-1} \boxtimes F(\mathcal{R}_1) \longrightarrow \mathcal{O}_X \boxtimes F(\mathcal{R}_0) \longrightarrow E$$

(used to define the last differential of Complex 6.3) it can be seen that E_1 can be added to the end of Complex 6.5 to obtain a new complex. Furthermore, the same argument used to show that Complex 6.5 satisfies (C1) (the argument from Lemma 6.1.3) can be used to show that this new complex also satisfies (C1). So it too has a convolution, say C_1 unique up to isomorphism. Since the entire Postnikov system is unique up to isomorphism, the rightmost part of this Postnikov system looks like this:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{O}_X \boxtimes F(\mathcal{R}_0) \arrow[rr] \arrow[dr] & & E_1 \arrow[dr] & \\
\cdots \arrow[ur] & & G \arrow[ll] \arrow[ur] & & C_1 \arrow[ll]
\end{tikzcd}$$

Apply $pr_{2*}(- \otimes pr_1^*\mathcal{L}^k)$ to this Postnikov system and compare it to the image of following Postnikov system (which is the Postnikov system from Compex 6.5 with $\mathcal{L}^k$ attached to the right) after applying F:

$$\begin{tikzcd}
\cdots \arrow[r] & A_k \otimes \mathcal{R}_0 \arrow[rr] \arrow[dr] & & \mathcal{L}^k \arrow[dr] & \\
\cdots \arrow[ur] & & T_k[n] \oplus \mathcal{L}^k \arrow[ll] \arrow[ur] & & \mathcal{T}_k[n+1] \arrow[ll]
\end{tikzcd}$$

Since the Postnikov systems must be isomorphic (Corollary 3.1.8) we obtain an isomorphism of the rightmost distinguished triangles

$$\begin{tikzcd}
F(T_k[n]) \oplus F(\mathcal{L}^k) \arrow[r] \arrow[d] & F(\mathcal{L}^k) \arrow[r] \arrow[d] & F(\mathcal{T}_k[n+1]) \arrow[r] \arrow[d] & F(T_k[n])[1] \oplus F(\mathcal{L}^k)[1] \arrow[d] \\
pr_{2*}(G \otimes pr_1^*\mathcal{L}^k) \arrow[r] & pr_{2*}(E_1 \otimes pr_1^*\mathcal{L}^k) \arrow[r] & pr_{2*}(C_1 \otimes pr_1^*\mathcal{L}^k) \arrow[r] & pr_{2*}(G \otimes pr_1^*\mathcal{L}^k)[1]
\end{tikzcd}$$

Now consider where the cohomology of these objects vanishes. The cohomology of $F(\mathcal{L}^k)$ and $F(\mathcal{T}_k[n+1])$ is concentrated in the intervals $[-a, 0]$ and $[-n-a-1, -n-1]$ respectively, and so the same is true of objects isomorphic to them. So we see that the cohomology of E_1 and C_1 are concentrated in these two intervals, respectively. Consider the long exact sequence of cohomology objects (1.3.8) associated to the distinguished triangle $G \to E_1 \to C_1 \to G[1]$. The cohomology of C_1 vanishes in the interval $[-a-1, 0]$ and so the induced morphisms $H^i(G) \to H^i(E_1)$ in these intervals are isomorphisms. But recall that $G \cong E \oplus C$ where $H^i(E) = 0$ for $i \notin [-a, 0]$ and $H^i(C) = 0$ for $i \notin [-n-a, -n]$ for $n \gg 0$. So the composition $E \oplus C \cong G \to E_1$ induces an isomorphism $E \cong E_1$. $\square$

Chapter 7

Motives

This chapter is devoted to some conjetures of Orlov which relates Fourier-Mukai transforms to morphisms of motives. Simply put, a Fourier-Mukai transform between the derived categories of two varieties produces a morphism between their motives via suitable Chern classes of the kernel of the transform. The three conjectures that appear in [Orl05] then ask whether certain properties (equivalence, fully faithfullness) of the Fourier-Mukai transform give information about the morphism of motives (isomorphism, split injective).

All five of the conjectures presented in this section are still open, although a partial answer to Conjecture 1 is given in Corollary 7.2.9 by a simple application of some theorems of Bondal and Orlov. The first three conjectures are taken directly from [Orl05] whereas the Conjecture 4 and Conjecture 5 were communicated to the author [Orl08] after a suggestions from Luca Barbieri-Viale and Paolo Stellari to consider the case of abelian varieties.

We begin with a (very) brief review of the two categories of motives considered, followed by the details of the conjectures. We then discuss some special cases of the third conjecture.

The references for the material in Section 7.1 on motives are [Sch94] for pure motives and [Voe00] for Voevodsky motives. The material presented in Section 7.2 that relates Fourier-Mukai transforms to motives is from [Orl05] and finally, the material of Section 7.3 that we present about derived equivalences between abelian varieties is taken from [Orl02].

7.1 Categories of motives.

The category of motives is a hypothetical universal category through which every cohomological functor factorizes. Of course, this is wonderfully vague as we haven't specified what we mean by a cohomological functor or which categories these cohomological functors go between.

We begin with the "classical" category motives – the category of pure Chow motives – which was hoped to be a universal with respect to Weil cohomology theories defined on smooth projective varieties. Our account is based mostly on [Sch94] although the reader should be warned that to maintain coherence with our account of $DM_{gm}(k)$ the notation will more closely resemble that of [Voe00].

We denote by $SmProj(k)$ be the category of smooth projoective k-schemes.

Definition 7.1.1. Let $A^\bullet(X)$ denote the Chow ring of a scheme X, graded by codimension. For smooth projective k-schemes X, Y with X of pure dimension d we define $Corr^r(X,Y) = A^{d+r}(X \times Y)$. If X is not of pure dimension we break it into its irreducible components $X = \coprod X_i$ and define $Corr^r(X,Y) = \oplus Corr^r(X_i, Y) \subset A^\bullet(X \times Y)$. Correspondences can be composed via the projections and the intersection product:

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "pr_{12}"'] \arrow[d, "pr_{13}"] \arrow[dr, "pr_{23}"] & \\ X \times Y & X \times Z & Y \times Z \end{tikzcd} \qquad \begin{tikzcd} Corr^r(X,Y) \otimes Corr^s(Y,Z) \arrow[r] & Corr^{r+s}(X,Z) \\ f \otimes g \arrow[r, mapsto] & pr_{13*}(pr_{12}^* f \cdot pr_{23}^* g) \end{tikzcd}$$

For a morphism $f : X \to Y$ we will use $\Gamma_f \subset X \times Y$ to denote its graph and often use f to refer also to the cycle in $Corr^0(X,Y)$ represented by Γ_f. The reader should note that composition of morphisms is preserved under this abuse of notation.

The category $Chow(k)$ of *Chow motives* over k is then defined as follows: The objects are triples (X, p, n) where $X \in SmProj(k)$, $p \in Corr^0(X,X)$ satisfies $p^2 = p$, and $n \in \mathbb{Z}$. Morphisms are

$$\hom_{Chow(k)}((X,p,n),(Y,q,m)) \overset{def}{=} pCorr^{n-m}(X,Y)q \subset Corr^\bullet(X,Y)$$

This category comes with a tensor product

$$(X,p,n) \otimes (Y,q,m) \overset{def}{=} (X \times Y, p \otimes q, n+m)$$

and if $n = m$ we can define a direct sum

$$(X,p,n) \oplus (Y,q,n) = (X \coprod Y, p \oplus q, n)$$

which extends to a direct sum on $Chow(k)$ in general.

The construction of $Chow(k)$ is usually broken into a sequence of steps using the following two subcategories:

1. The full subcategory with objects $(X, id_X, 0)$. This is denoted simply by $\mathcal{C}_0$ in [Voe00]. The $\mathcal{C}$ possibly stands for $\mathcal{C}$orrespondences or $\mathcal{C}$how.

2. The full subcategory $Chow^{eff}(k)$ of *effective Chow motives* which has objects $(X, p, 0)$.

Remark 7.1.2. The functor $SmProj(k) \to \mathcal{C}_0$ amounts heuristically to altering the hom sets to include "multivalued" morphisms (c.f. $SmCor(k)$ below), making the category additive, and then removing excess morphisms (via rational equivalence). The category $Chow^{eff}(k)$ is then the idempotent completion of this category (so we formally adjoin kernels to idempotent morphisms), and the inclusion $Chow^{eff}(k) \to Chow(k)$ formally adjoins the tensor inverse $(\operatorname{Spec} k, id_{\operatorname{Spec} k}, 1)$ of the *Leftschetz motive* $\mathbb{L} \stackrel{def}{=} (\operatorname{Spec} k, id_{\operatorname{Spec} k}, -1)$ (the integer n in (X, π, n) formally records how many copies of $\mathbb{L}^{-1}$ are tensored with $(X, \pi, 0) \in Chow^{eff}(k)$).

The Leftschetz motive is usually defined as the kernel of the idempotent $\pi_2 : \mathbb{P}^1 \to \mathbb{P}^1$ which is the composition of $\mathbb{P}^1 \to \operatorname{Spec} k \to \mathbb{P}^1$ for some chosen point of $\mathbb{P}^1$; this gives rise to a decomposition $\mathbb{P}^1 = \mathbf{1} \oplus \mathbb{L}$ where $\mathbf{1} = (\operatorname{Spec} k, id_{\operatorname{Spec} k}, 0)$ is the tensor unit.

Extending the category of motives to include more general geometric objects such as singular, nonprojective schemes runs into some problems and in fact it is known that in important cases a category of motives proper doesn't exist. However, a lot of important information could still be obtained from the derived category of a hypothetically existant category of motives and so even if a category of motives doesn't exist, having a triangulated category that behaves similarly to how its derived category would behave is still a useful tool. This is the impetus for the definition of various triangulated categories of motives by Voevodsky, one of which we discuss now. We follow [Voe00].

Definition 7.1.3. Denote $Sm(k)$ the category of smooth k-schemes (k still a field). We now define the category of *smooth correspondences* $SmCor(k)$ to be the category with the same objects as $Sm(k)$ and morphisms as follows. The group $\hom_{SmCor(k)}(X, Y)$ is the free abelian group generated by integral closed subschemes W in $X \times Y$ which are finite and surjective over a connected component of X. Composition is defined in the same way as above

$$f \circ g \stackrel{def}{=} pr_{13*}(pr_{12}^* f \cdot pr_{23}^* g)$$

where here $\cdot$ is the scheme theoretic intersection.

Consider the homotopy cateogry $\mathcal{H}^b(SmCor(k))$ of bounded complexes over $SmCor(k)$ (so morphisms are homotopy classes of morphisms of complexes) and recall that this is a triangulated category [Wei94]. Denote by T the class consisting of complexes of the form

$$[X \times \mathbb{A}^1] \xrightarrow{[\pi]} [X] \qquad\qquad [U \cap V] \xrightarrow{[j_U] \oplus [j_V]} [U] \oplus [V] \xrightarrow{[i_U] \oplus (-[i_V])} [X]$$

for smooth varieties X and open covers $U \cup V = X$ where i_U, i_V, j_U, j_V are the obvious inclusions and π is the projection. The degree of the complex that these objects are placed in doesn't matter as we are concerned with the minimal thick subcategory of $\mathcal{H}^b(SmCor(k))$ containing T. Following Voevodsky we denote this thick subcategory by $\overline{T}$. The triangulated category $DM_{gm}^{eff}(k)$ of effective geometric motives over k is then defined as the idempotent completion of the localization $\overline{T}^{-1}\mathcal{H}^b(SmCor(k))$. Voevodsky points out however that the idempotent completion is taken for the sole purpose of making comparisons with classical results about $Chow^{eff}(k)$ and $Chow(k)$ more elegant.

As in the category of effective Chow motives we formally adjoin the tensor inverse of

Figure 7.1: Constructing categories of motives.

$$
\begin{tikzcd}
SmProj(k) \arrow[r] \arrow[d, "\substack{\text{replace sets of morphisms}\\ \text{with Chow rings}}"] & Sm(k) \arrow[d, "\substack{\text{replace sets of morphisms}\\ \text{with groups of correspondences}}"] \\
\mathcal{C}_0 \arrow[ddd, "\text{add kernels of idempotents}"] & SmCor(k) \arrow[d, "\text{embed in the homotopy category}"] \\
 & \mathcal{H}^b(SmCor(k)) \arrow[d, "\substack{\text{impose Mayer-Vietoris}\\ \text{and homotopy invariance}}"] \\
 & \overline{T}^{-1}\mathcal{H}^b(SmCor(k)) \arrow[d, "\text{add kernels of idempotents}"] \\
Chow^{eff}(k) \arrow[r] \arrow[d, "\text{invert } \mathbb{L}"] & DM^{eff}_{gm}(k) \arrow[d, "\text{invert } \mathbb{Z}(1)"] \\
Chow(k) & DM(k)
\end{tikzcd}
$$

an object. In $DM^{eff}_{gm}(k)$ the *Tate object* is defined to be the image of the complex:

$$
\mathbb{Z}(1) \overset{def}{=} \left\{ \underset{2}{[\mathbb{P}^1]} \longrightarrow \underset{3}{[\mathrm{Spec}\, k]} \right\}
$$

Its tensor inverse is formally added to $DM^{eff}_{gm}(k)$ to obtain $DM_{gm}(k)$ the triangulated category of *geometric motives*.

The sequence of alterations and associated functors is summarized on the right of Figure 7.1.

7.2 Morphisms of motives from Fourier-Mukai transforms.

In this section we discuss the way morphisms of motives are obtained from Fourier-Mukai transforms. There exists a full embedding $Chow^{eff}(k) \to DM^{eff}_{gm}(k)$ if k admits resolution of singularities [Voe00, Corollary 4.2.6] and thus, it doesn't matter in which category motives of smooth projective varieties are considered. We will denote by $M(X)$ the motive associated to the smooth projective variety X. We also need to state the following definition.

Definition 7.2.1. If we tensor all the hom groups of any of the above categories with the rational numbers, we get the category of *pure motives with rational coefficients* $Chow(k) \otimes \mathbb{Q}$ and the category of *geometric motives with rational coefficients* $DM_{gm}(k) \otimes \mathbb{Q}$.

Remark 7.2.2. To emphasize that a motive is being considered in a category with rational coefficients it will be denoted $M(X)_{\mathbb{Q}}$ but in general, everything that follows from now on will be with rational coefficients.

The first two conjectures of [Orl05] are the following.

Conjecture 1 (Orlov [Orl05])**.** Let X and Y be smooth projective varieties, and let $D^b(X) \cong D^b(Y)$. Then the motives $M(X)_{\mathbb{Q}}$ and $M(Y)_{\mathbb{Q}}$ are isomorphic in $Chow^{eff}(k) \otimes \mathbb{Q}$ (and in $DM^{eff}_{gm}(k) \otimes \mathbb{Q}$).

Conjecture 2 (Orlov [Orl05])**.** Let X and Y be smooth projective varieties and let $F : D^b(X) \to D^b(Y)$ be a fully faithful functor. Then the motive $M(X)_{\mathbb{Q}}$ is a direct summand of the motive $M(Y)_{\mathbb{Q}}$.

We now discuss some evidence for why these might be true.

Given a Fourier-Mukai transform $\Phi_E : D^b(X) \to D^b(Y)$ we obtain an element in the Chow ring $A^\bullet(X \times Y)$ via the Mukai vector:

$$v(E) \overset{def}{=} \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}_{X \times Y}}$$

Consider the decomposition of this into its graded components $v(E) = a_0 + \cdots + a_{n+m}$ with $n = \dim X$ and $m = \dim Y$. Each component a_q defines a morphism in $DM_{gm}(k)$.

$$\alpha_q : M_{gm}(X) \to M_{gm}(Y)(q-m)[2(q-m)]$$

In particular, the degree m part defines a morphism of motives $M(X) \to M(Y)$.

Definition 7.2.3. Let $E \in D^b(X \times Y)$ with X, Y smooth projective varieties of pure dimension n and m respectively. We define $\Phi^M_E : M(X) \to M(Y)$ to be the morphism of motives induced by the degree m component of the Mukai vector $v(E) = \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}(X \times Y)}$.

Remark 7.2.4. As far as the author is aware, there is no standard notation for this induced morphism of motives and so the notation Φ^M_E was chosen as an analogy with that of the cohomological Fourier-Mukai transform Φ^H_E associated to E. The reader should be warned however that the assignment $E \to \Phi^M_E$ does not respect composition (whereas $E \to \Phi^H_E$ does).

Example 7.2.5.
1. Consider $f_* : D^b(X) \to D^b(Y)$ for some morphism $f : X \to Y$. This is a Fourier-Mukai transform with kernel $\mathcal{O}_{\Gamma_f}$ the structure sheaf of the graph $\Gamma_f \subset X \times Y$ of f. That is, the kernel is $\gamma_* \mathcal{O}_X$ where $\gamma = (id_X, f) : X \to X \times Y$ embedds X as Γ_f. Applying the Grothendieck-Riemann-Roch Theorem to γ we see that
$$\mathrm{td}_{X \times Y} \cdot \mathrm{ch}(\gamma_* \mathcal{O}_X) = \gamma_*(\mathrm{td}_X \cdot \mathrm{ch}(\mathcal{O}_X)) = \gamma_*(\mathrm{td}_X)$$

and so $v(\gamma_*\mathcal{O}_X) = \gamma_*(\mathrm{td}_X)/\sqrt{\mathrm{td}_{X\times Y}}$. Consider the decompoition into graded components $\gamma_*(\mathrm{td}_X) = g_0 + \cdots + g_{n+m}$ and $\mathrm{td}_{X\times Y} = t_0 + \cdots + t_{n+m}$. We know that $t_0 = 1$ (by definition) and so the degree zero term of $1/\sqrt{\mathrm{td}_{X\times Y}}$ is also 1. From the definition of the push forward on Chow groups, it can be seen that actually $\gamma_*(\mathrm{td}_X) = [\Gamma_f] + g_{m+1} + \cdots + g_{n+m}$. So the degree m part of $v(\mathcal{O}_{\Gamma_f})$ is the class of the graph $[\Gamma_f]$. So the induced morphism of motives $M(X) \to M(Y)$ is that induced by f.

2. Since the pullback $f^* : D^b(Y) \to D^b(X)$ has the same kernel as the pushforward, but used as a Fourier-Mukai transform in the opposite direction, we see from the previous example that the induced morphism of motives is that defined by the transpose $[\Gamma_f]^t$ of the graph of f.

3. Consider the shift automorphism $D^b(X) \to D^b(X)$. It is a Fourier-Mukai transform with kernel $\mathcal{O}_\Delta[1]$ the structure sheaf of the diagonal shifted by one. Since the diagonal is the graph of the identity id_X, and $\mathrm{ch}(F[1]) = -\mathrm{ch}(F)$ for any object $F \in D^b(X \times X)$, we see that the induced morphism of motives is $-id : M(X) \to M(X)$.

4. Now consider the automorphism $\mathcal{L} \otimes - : D^b(X) \to D^b(X)$ where $\mathcal{L}$ is a line bundle. This is a Fourier-Mukai transform with kernel $\delta_*\mathcal{L}$ where $\delta : X \to X \times X$ is the diagonal embedding. Using the same reasoning as in the first example together with the fact that $\mathrm{ch}(\mathcal{L}) = 1 + \ldots$ we see that the induced morphism of motives is the identity.

   In fact, for an arbitrary locally free sheaf $F \in D^b(X)$, the functor $F \otimes -$ is Fourier-Mukai with kernel δ_*F and so since $\mathrm{ch}(F) = r + \ldots$ (where $r = \mathrm{rank}\, F$) the induced morphism of motives is r times the identity $r\, id : M(X) \to M(X)$.

To summarize:

Φ_E	Φ_E^M
$f_* : D^b(X) \to D^b(Y)$	$f : M(X) \to M(Y)$
$f^* : D^b(Y) \to D^b(X)$	$f^t : M(Y) \to M(X)$
$(-)[1] : D^b(X) \to D^b(X)$	$-id : M(X) \to M(X)$
$F \otimes - : D^b(X) \to D^b(X)$ (with F a loc. free sheaf)	$(\mathrm{rank}\, F) \cdot id : M(X) \to M(X)$

In [Orl05], Orlov provides us with the following useful theorem.

Theorem 7.2.6. *Let X and Y be smooth projective varieties of dimension n, and let $\Phi_E : D^b(X) \to D^b(Y)$ be a fully faithful Fourier-Mukai transform, such that the support of E in $X \times Y$ has dimension n. Then $M(X)$ is a direct summand of $M(Y)$. If, in addition, Φ_E is an equivalence, then the motives $M(X)$ and $M(Y)$ are isomorphic.*

Proof. Consider the object $E' \in D^b(X \times Y)$ which represents the left adjoint Fourier-Mukai transform to Φ_E. As above we consider the decomposition of the Mukai vector

$v(E')$ into its components graded by codimension $b_0 + b_1 + \cdots + \ldots b_{2n}$ but this time we consider the induced morphism

$$\beta : \bigoplus_{i=-n}^{n} M(Y)_{\mathbb{Q}}(i)[2i] \to M(X)_{\mathbb{Q}}$$

We can now compose this with the morphism

$$\alpha : M(X)_{\mathbb{Q}} \to \bigoplus_{i=-n}^{n} M(Y)_{\mathbb{Q}}(i)[2i]$$

obtained from the Mukai vector of E. With a little thought, it can be seen that the composition $M(X) \to M(X)$ is the morphism induced by the degree n part of $pr_{13*}(pr_{12}^*(v(E)) \cdot pr_{23}^*(v(E')))$ where $X \times Y \times X \overset{pr_{12}}{\to} X \times Y$ is the projection and similarly for pr_{23} and pr_{13}. To see this, first note that the composition $M(X) \to M(X)$ is the sum of the the compositions $\beta_{2n-q} \circ \alpha_q$ for $q = 0, \ldots, 2n$ where α_q and β_{2n-q} are the morphisms induced by a_q and b_{2n-q} respectively. Next, the degree n part of $pr_{13*}(pr_{12}^*(v(E)) \cdot pr_{23}^*(v(E')))$ is the sum of $pr_{13*}(pr_{12}^*(a_q) \cdot pr_{23}^*(b_{2n-q}))$ for $q = 0, \ldots, 2n$. So we just need to see that $pr_{13*}(pr_{12}^*(a_q) \cdot pr_{23}^*(b_{2n-q}))$ induces the morphism $\beta_{2n-q} \circ \alpha_q$. To prove this we need to have discussed in greater depth the way cycles induce morphisms, which can be found in [Voe00].

Since Φ_E is fully faithful, the composition $\Phi_{E'} \circ \Phi_E$ is isomorphic to the identity. It can be shown (see [Muk78]) that this composition is isomorphic to the Fourier-Mukai transform with kernel $pr_{13*}(pr_{12}^*E \otimes pr_{23}^*E')$. Since the kernel of a Fourier-Mukai equivalence is unique up to isomorphism, we see that this is isomorphic to the structure sheaf of the diagonal $\mathcal{O}_\Delta$ in $D^b(X \times X)$. It can also be shown that

$$v(pr_{13*}(pr_{12}^*E \otimes pr_{23}^*E')) = pr_{13*}(pr_{12}^*(v(E)) \cdot pr_{23}^*(v(E')))$$

holds in general, and so the degree n component of $pr_{13*}(pr_{12}^*(v(E)) \cdot pr_{23}^*(v(E'))$ is nothing other than the class of the diagonal in $X \times X$. This induces an identity of motives so we see that $\beta \circ \alpha = id_{M(X)}$.

Now we use the assumption on the dimension of the support. Since the dimension of the support of E is dimension n, the induced morphisms α_i are zero for $i < 0$. The kernel of the adjoint E' can be taken to be $E' = E^\vee \otimes pr_2^*\omega_Y[n]$ (see [Muk78]) and so we see that E' also has support in a dimension n subscheme of $X \times Y$. So the morphisms β_i are zero for $i > 0$. Hence, the composition $\beta \circ \alpha$ is actually none other than $\beta_0 \circ \alpha_0$. So $M(X)_{\mathbb{Q}}$ is a direct summand of $M(Y)_{\mathbb{Q}}$.

If Φ_E is an equivalence then we can switch the schemes X, Y and the kernels E, E' and see that $M(Y)_{\mathbb{Q}}$ is also a direct summand of $M(X)_{\mathbb{Q}}$, hence they are isomorphic. □

An immediate application of this theorem is to answer a special case of Conjecture 1. We first state two theorems of Bondal and Orlov.

Theorem 7.2.7 (Bondal, Orlov [BO01]). *Let X and Y be smooth projective varieties and assume that the (anti-)canonical bundle of X is ample. If there exists an exact equivalence $D^b(X) \cong D^b(Y)$, then X and Y are isomorphic. In particular, the (anti)-canonical bundle of Y is also ample.*

Theorem 7.2.8 (Bondal, Orlov [Huy06, Proposition 4.17]). *Let X be a smooth projective variety with ample (anti-)canonical bundle. The group of autoequivalences of $D^b(X)$ is generated by: i) automorphisms of X, ii) the shift functor T, and iii) twists by line bundles. In other words, one has*

$$Aut(D^b(X)) \cong \mathbb{Z} \times (Aut(X) \ltimes Pic(X))$$

As a direct consequence of these two theorems we can give a partial answer to Conjecture 1. The author cannot give a reference for this corollary as he has not seen it anywhere.

Corollary 7.2.9. *Suppose that X and Y are smooth projective varieties and assume that the (anti-)canonical bundle of X is ample. Then every exact equivalence $D^b(X) \cong D^b(Y)$ induces an isomorphism of motives.*

Moreover, the isomorphism $M(X)_{\mathbb{Q}} \cong M(Y)_{\mathbb{Q}}$ is that induced by an isomorphism $X \cong Y$, up to sign.

Proof. By Theorem 7.2.7 we see that $X \cong Y$ and so the exact equivalence $D^b(X) \cong D^b(Y)$ can be considered as an automorphism of $D^b(X)$. It then follows from Theorem 7.2.8 that this automorphism is a product of shifts, automorphisms of X, and line bundle tensors. We have seen in the examples that the kernels of these Fourier-Mukai transforms all have support in dimension n and so it follows from Theorem 7.2.6 that they all induce isomorphisms of motives. For the last statement recall that we saw that the shift $F \mapsto F[1]$ induced the morphism $-id_{M(X)}$ and tensoring with a line bundle $\mathcal{L} \otimes -$ induced the identity $id_{M(X)}$. □

It is not always the case that for an equivalence $\Phi_E : D^b(X) \to D^b(Y)$ the map of motives $\Phi_E^M : M(X)_{\mathbb{Q}} \to M(Y)_{\mathbb{Q}}$ is an isomorphism (consider for example the Poincaré line bundle on $A \times \hat{A}$ for an abelian variety A). Orlov has conjectured that the following refinement of Conjecture 1 may be true.

Conjecture 3. Let E be an object of $D^b(X \times Y)$, for which $\Phi_E : D^b(X) \to D^b(Y)$ is an equivalence. Then there exist line bundles L and M on X and Y respectively such that $\Phi^M_{p_1^* L \otimes E \otimes p_2^*} : M(X)_{\mathbb{Q}} \to M(Y)_{\mathbb{Q}}$ is an isomorphism of motives.

7.3 Derived equivalences between abelian varieties.

To fully answer Conjecture 1 we need to deal with the case where X doesn't have ample (anti-)canonical bundle. As a first step in this direction we could consider the case when

X and Y are abelian varieties. In this section we discuss some results which may help. Most of this section is developed in [Orl02].

The main tool is an isometric isomorphism $f_E : A\times\hat{A} \to B\times\hat{B}$ obtained from a derived equivalence $D^b(A\times\hat{A}) \to D^b(B\times\hat{B})$ associated to Φ_E. This is used to completely describe the automorphism group of $D^b(A)$.

We list here some results of [Orl02] which help describe the situation.

1. [Orl02, Theorem 2.10] Let $\mathcal{J}(E) \in D^b(A \times \hat{A} \times B \times \hat{B})$ denote the kernel of the Fourier-Mukai transform decribed in Construction 7.3.1 below. Then there is an isomorphism $f_E : A \times \hat{A} \xrightarrow{\sim} B \times \hat{B}$ of abelian varieties and a line bundle L_E on $A \times \hat{A}$ such that
$$\mathcal{J}(E) \cong (id, f_E)_*(L_E)$$
2. [Orl02, Corollary 2.13] Consider points $(a, \alpha) \in A \times \hat{A}$ and $(b, \beta) \in B \times \hat{B}$. Write $T_a : A \to A$ for translation by a, write $P_\alpha \in Pic(A)$ for the line bundle determined by $\alpha \in \hat{A}$, and finally write $\Phi_{(a,\alpha)} = T_{a*}(-) \otimes P_\alpha : D^b(A) \to D^b(A)$. Then the following three are equivalent:
   (a) $f_E(a, \alpha) = (b, \beta)$
   (b) $\Phi_{(b,\beta)} \circ \Phi_E \cong \Phi_E \circ \Phi_{(a,\alpha)}$
   (c) $T_{b*}E \otimes P_\beta \cong T_{-a*}E \otimes P_\alpha = T_a^*E \otimes P_\alpha$
3. [Orl02, Proposition 2.18] The isomorphism f_E is isometric. That is, if we write it as the matrix on the left then its inverse is given by the matrix on the right.
$$f_E = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \qquad \tilde{f} = \begin{pmatrix} \hat{f}_4 & -\hat{f}_2 \\ -\hat{f}_3 & \hat{f}_1 \end{pmatrix}$$
4. [Orl02, Proposition 4.11, 4.12] For every isometric isomorphism $f : A \times \hat{A} \to B \times \hat{B}$ there is a Fourier-Mukai transform Φ_E with $f_E = f$.
5. [Orl02, Theorem 4.14] Suppose A is an abelian variety over an algebriacally closed field of characteristic 0. Then the group $\operatorname{Auteq} D^b(A)$ of exact autoequivalences of the derived category may be included into the following short exact sequence of groups:
$$0 \to \mathbb{Z} \oplus (A \times \hat{A})_k \to \operatorname{Auteq} D^b(A) \to U(A \times \hat{A}) \to 1$$
   where $U(A \times \hat{A})$ is the group of isometric isomorphisms (see Item 3 above).

   Elements of $U(A \times \hat{A})$ act via Item 4 above, elements of $\mathbb{Z}$ act via shift, and the k-points of $A \times \hat{A}$ act via $\Phi_{(a,\alpha)}$ (see Item 2).

As a consequence of Item 5 (and the examples above 7.2.5), the resolution of Conjecture 3 for the case when $X = Y$ is an abelian variety rests on exact autoequivalences Φ_E that have nontrivial f_E. In an attempt to understand these we present the constructions that build f_E from E and conversely, build E from an isometric isomorphism.

The isometric isomorphism is obtained by the following construction.

Construction 7.3.1. To a Fourier-Mukai transform between derived categories of abelian varieties $\Phi_E : D^b(A) \to D^b(B)$ of dimension n we associate the Fourier-Mukai transform

$$F_E : D^b(A \times \hat{A}) \to D^b(B \times \hat{B})$$

defined as the composition

$$\begin{tikzcd}
D^b(A \times \hat{A}) \arrow[r, "F_E"] \arrow[d, "\mathrm{id} \times \Phi_{\mathcal{P}_A}"'] & D^b(B \times \hat{B}) \\
D^b(A \times A) \arrow[d, "\mu_A *"'] & D^b(B \times B) \arrow[u, "(\mathrm{id} \times \Phi_{\mathcal{P}_B})^{-1}"'] \\
D^b(A \times A) \arrow[r, "\Phi_E \times \Phi_{E_R}"'] & D^b(B \times B) \arrow[u, "\mu_B^*"']
\end{tikzcd}$$

where $\mathcal{P}_A$ is the Poincaré bundle on A, the morphism μ_A is

$$\mu_A : A \times A \to A \times A, \qquad (a_1, a_2) \mapsto (a_1 + a_2, a_2),$$

similarly for $\mathcal{P}_B$ and μ_B, and the object $E_R = E^\vee[n]$ is the kernel of the adjoint Fourier-Mukai transformation to Φ_E (left and right since the canonical bundle on an abelian variety is trivial and $\dim A = \dim B$).

Remark 7.3.2. The definition of F_E is actually quite natural in the light of Item 2 of the above list of results. Consider the structure sheaf $\mathcal{O}_{(a,\alpha)}$ of a closed point in $D^b(A \times \hat{A})$. The composition of the two left vertical morphisms in Construction 7.3.1 send $\mathcal{O}_{(a,\alpha)}$ to $(\mathcal{O} \otimes P_\alpha) \otimes \mathcal{O}_{\Gamma_{-a}}$, which is the object of $D^b(A \times A)$ representing the autoequivelance $\Phi_{(-a,\alpha)}$ ([Huy06, Example 9.33]). The horizontal morphism then takes this object to the object of $D^b(B \times B)$ representing the autoequivalence $D^b(B) \to D^b(B)$ induced by Φ_E. That is, the kernel of the transform $D^b(B) \overset{\Phi_E^{-1}}{\to} D^b(A) \overset{\Phi_{(-a,\alpha)}}{\to} D^b(A) \overset{\Phi_E}{\to} D^b(A)$ ([Huy06, Exercise 5.13]). It miraculously turns out that this is also of the form $\Phi_{(-b,\beta)}$ and so the right most vertical morphisms take this back to the structure sheaf $\mathcal{O}_{(b,\beta)}$ in $D^b(B \times \hat{B})$.

$$\begin{tikzcd}
\mathcal{O}_{a,\alpha} \arrow[r, mapsto] \arrow[d, mapsto] & \mathcal{O}_{b,\beta} \\
\Phi_{(-a,\alpha)} \arrow[r, mapsto] & \Phi_{(-b,\beta)} = \Phi_E \circ \Phi_{(-a,\alpha)} \circ \Phi_E^{-1} \arrow[u, mapsto]
\end{tikzcd}$$

To go the other way, as mentioned in Item 4 in the above list, we will use the following two morphisms. First we have an embedding of the Néron-Severi group of an abelian variety D:

$$NS(D) \to \hom(D, \hat{D}) \qquad [\mathcal{L}] \mapsto \left(\phi_{\mathcal{L}} : d \mapsto T_d^* \mathcal{L} \otimes \mathcal{L}^{-1}\right)$$

where T_d is translation by d on D. Secondly we use this to assign to $\mu = \frac{[\mathcal{L}]}{\ell} \in NS(D) \otimes_{\mathbb{Z}} \mathbb{Q}$ a correspondence

$$\Phi_\mu = im(D \overset{(\ell, \phi_{\mathcal{L}})}{\to} D \times \hat{D})$$

Construction 7.3.3 ([Orl02])**.** Choose an isometric isomorphism $f : A \times \hat{A} \to B \times \hat{B}$ and denote its graph by $\Gamma \subset A \times \hat{A} \times B \times \hat{B}$. We write f in matrix form as

$$f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$$

Supposing that $f_2 : \hat{A} \to B$ is an isogeny, we obtain

$$g = \begin{pmatrix} f_2^{-1} f_1 & -f_2^{-1} \\ -\hat{f}_2^{-1} & f_4 f_2^{-1} \end{pmatrix} \in \hom(A \times B, \hat{A} \times \hat{B}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

which determines a correspondence on $(A \times B) \times (\hat{A} \times \hat{B})$. As a consequence of f being isometric, we find that $\hat{g} = g$, and so g is the image of some $\frac{[\mathcal{L}]}{\ell} \in NS(A \times B) \otimes \mathbb{Q}$ inside $\hom(A \times B, \hat{A} \times \hat{B})$. The result [Orl02, Proposition 4.6] reproduced from [Muk78, Theorem 7.10] then says that there is a simple semihomogeneous vector bundle E with $\frac{[\det E]}{\operatorname{rank} E} = \frac{[\mathcal{L}]}{\ell}$ in $NS(A \times B) \otimes \mathbb{Q}$.

Orlov proves (see Item 2 above) that for E obtained in this way, Φ_E is an equivalence [Orl02, Proposition 4.11] and $f_E = f$ [Orl02, Proposition 4.12].

Given the close relationship between a Fourier-Mukai transform between abelian varieties A and B, and the associated isomorphism $A \times \hat{A} \to B \times \hat{B}$ we might be led to specify Conjecture 3 in this case.

Conjecture 4 ([Orl08])**.** Let A and B be abelian varieties of dimension n and $\Phi_E : D^b(A) \to D^b(B)$ a Fourier-Mukai transform which is an equivalence. Then the map $f_1 : A \to B$ mentioned in Item 3 of the above list is an isogeny if and only if Φ_E^M is an isomorphism of motives.

To obtain the above conjecture in the case of abelian varieties we should also have:

Conjecture 5 ([Orl08])**.** Let A and B be abelian varieties of dimension n and $\Phi_E : D^b(A) \to D^b(B)$ a Fourier-Mukai transform which is an equivalence. Then there exist ample line bundles L and M on A and B respectively such that the map f_1 (see Item 3 above) obtained from $\Phi^M_{p_1^* L \otimes E \otimes p_2^* M}$ is an isogeny.

Bibliography

[Bei78] AA Beilinson, *Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra*, Functional Analysis and Its Applications **12** (1978), no. 3, 214–216.

[BK90] AI Bondal and MM Kapranov, *Representable functors, Serre functors, and mutations*, Izvestiya: Mathematics **35** (1990), no. 3, 519–541.

[BO01] A. Bondal and D. Orlov, *Reconstruction of a Variety from the Derived Category and Groups of Autoequivalences*, Compositio Mathematica **125** (2001), no. 03, 327–344.

[BVdB02] A. Bondal and M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Arxiv preprint math.AG/0204218 (2002).

[Căl05] A. Căldăraru, *Derived categories of sheaves: a skimming*, Snowbird Lectures in Algebraic Geometry: Proceedings of an Ams-IMS-Siam Joint Summer Research Conference on Algebraic Geometry–Presentations by Young Researchers, July 4-8, 2004, Snowbird, Utah **388** (2005).

[CS07] A. Canonaco and P. Stellari, *Twisted Fourier–Mukai functors*, Advances in Mathematics **212** (2007), no. 2, 484–503.

[FL85] W. Fulton and S. Lang, *Riemann-Roch Algebra*, Springer Verlag, 1985.

[Gro57] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J **9** (1957), no. 1.1, 11.

[Har66] R. Hartshorne, *Residues and duality*, Springer-Verlag New York, 1966.

[Har77] ———, *Algebraic Geometry*, Springer, 1977.

[Huy06] D. Huybrechts, *Fourier-Mukai Transforms in Algebraic Geometry*, Oxford University Press, USA, 2006.

[IM94] SP Inamdar and VB Mehta, *Frobenius splitting of Schubert varieties and linear syzygies*, Amer. J. Math **116** (1994), no. 6, 1569–1586.

[Kaw02] Y. Kawamata, *Equivalences of derived catgories of sheaves on smooth stacks*, Arxiv preprint math.AG/0210439 (2002).

[May01] JP May, *The Additivity of Traces in Triangulated Categories*, Advances in Mathematics **163** (2001), no. 1, 34–73.

[Muk78] S. Mukai, *Semi-homogeneous vector bundles on an abelian variety*, J. Math. Kyoto Univ **18** (1978), no. 2, 239–272.

[Muk81] ———, *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J **81** (1981), 153–175.

[Noo07] B. Noohi, *Lectures on derived and triangulated categories*, Arxiv preprint arXiv:0704.1009 (2007).

[Orl96] D. Orlov, *Equivalences of derived categories and $k3$ surfaces*, 1996.

[Orl02] D.O. Orlov, *Derived categories of coherent sheaves on Abelian varieties and equivalences between them*, Izvestiya: Mathematics **66** (2002), no. 3, 569–594.

[Orl03] ———, *Derived categories of coherent sheaves and equivalences between them*, Russian Mathematical Surveys **58** (2003), no. 3, 511–591.

[Orl05] ———, *Derived categories of coherent sheaves and motives*, Russian Mathematical Surveys **60** (2005), no. 6, 1242–1244.

[Orl08] D. Orlov, private communication, 2008.

[Sch94] AJ Scholl, *Classical motives*, Motives (1994), 163–187.

[Toë07] B. Toën, *The homotopy theory of dg-categories and derived Morita theory*, Inventiones Mathematicae **167** (2007), no. 3, 615–667.

[Ver96] J.L. Verdier, *Des categories derivees des categories abeliennes*, Asterisque **239** (1996), 253.

[Voe00] V. Voevodsky, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories **143** (2000), 188–238.

[Wei94] C.A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1994.