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Homology of Schemes

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Chapter 1

Introduction

1.1 Overview

In this mémoire, we will study the categories constructed by Voevodsky in the article [Voev], proposed as a triangulated category of mixed motives. This involves, for every noetherian base scheme S, a triangulated category $DM(S)$ and a functor $M : Sch/S \to DM(S)$ from the category of schemes of finite type over S to $DM(S)$.

The idea of a category of motives can be seen to originate from the large number of different cohomology theories and the relations between them. The term "motive" is used to denote an object in a hypothetical $\mathbb{Q}$–linear abelian category through which all cohomology theories factor. Initially, only cohomology theories on the category of smooth projective varieties were considered by Grothendieck and such a category is now referred to as a category of pure motives. If all algebraic varieties are considered the term "mixed motive" is used.

Were such a category to exist, it would explain many properties and relations between different cohomology theories of algebraic varieties as well as many conjectures. Unfortunately, such categories appear to be extremely difficult to construct. An alternative approach is to attempt to construct a triangulated category which behaves like the derived category of motives would behave, and then show that it is the derived category of an abelian category.

The category $DM(S)$ together with the functor M is a proposed solution to the first part of this compromise. The functor M satisfies the usual properties of homological theories and the pair

$$(DM_{ft}(S) \otimes \mathbb{Q},\ M_{\mathbb{Q}} : Sch/S \to DM_{ft}(S) \otimes \mathbb{Q}) \tag{1.1}$$

is claimed to be universal among functors from Sch/S to $\mathbb{Q}$–linear triangulated categories which satisfy some analog of the Eilenberg–Steenrod axioms for homological theories (although Voevodsky doesn't specify what analog).

The construction begins by taking each scheme to the corresponding representable sheaf of sets on the site of schemes over S with either the h or qfh–topologies. The category is made abelian by taking each sheaf of sets to the free sheaf of abelian groups corresponding to it. We then pass to the derived category in the usual way and then factor out all "contractible" objects where contractibility is defined using an "interval" object that comes from the original site, in this case the affine line.

A possibly surprising aspect of this construction is that the correspondences that are usually added as the first step of defining a category of motives do not appear (motives in the sense of Grothendieck are expected to be functorial not only with respect to morphisms but also correspondences). The lack of the need to formally add them can be explained by [Voev, Theorem 3.3.8] which gives the existence of transfer maps between the qfh–sheaves of abelian groups associated to X and Y where X is a normal connected scheme and $f : Y \to X$ is a finite surjective morphism of separable degree d, and the results of [SV, Section 6] which say that if a qfh–sheaf admits transfer maps on integral normal schemes, then it admits transfer maps on all schemes of finite type over a field.

This mémoire roughly follows the same outline as [Voev]. In emulation of [Voev2] we begin in Chapter 1 with a list of major results in an attempt to raise them out of their somewhat hidden position in [Voev]. Chapter 2 contains the material of [Voev] about the homological category of a site with interval. Chapter 3 encompasses relevant material of [Voev] relating to the h–topology and various kinds of sheaves. Most of the proofs in [Voev] that are of a distinct scheme theoretic nature have not been included as Voevodsky gives a suitably detailed account in [Voev] and there is nothing really to add. Chapter 4 contains the comparison results of [Voev] between h and qfh cohomology and étale cohomology. As in Chapter 2 and Chapter 5 an effort has been made to fill in as many of the missing details of [Voev] as possible. Chapter 5 deals with $DM_h(S)$ and $DM_{qfh}(S)$, the homological categories of interest and expounds some of its more easily proven properties. We then take a detour from [Voev] in Chapter 6 and outline briefly the paper [SV] and its relation to [Voev]. This is intended only as a brief outline and as such there are no detailed proofs. Chapter 7 returns again to [Voev] to state some properties with relatively involved proofs and then Chapter 8 contains a (very) brief overview of the categories constructed in [Voev2] and their relationship to $DM_h(k)$ (for a perfect field k which admits resolution of singularities).

I extend my sincere thanks to Bruno Kahn for accepting to direct this mémoire, for suggesting material which matched my interests so closely and for all of his kind patience and guidance throughout the year. I also thank Bruno Klinger for his course "Motifs de Voevodsky" which illuminated so much of the surrounding landscape for me.

1.2 Summary of main results

The main results about properties of the category $DM(S)$ that are proved in [Voev] are listed here.

1.2.1 Comparison of h, qfh and étale cohomology.

qfh–topology. [Voev, 3.4.1, 3.4.4] If either

1. X is a normal scheme and F is a qfh–sheaf of vector spaces, or
2. F is locally constant in the étale topology (in which case it is also a qfh–sheaf),

then

$$H^i_{qfh}(X, F) = H^i_{et}(X, F) \tag{1.2}$$

h–topology. [Voev, 3.4.5] If F is a locally constant torsion sheaf in the étale topology then F is an h–sheaf and

$$H^i_h(X, F) = H^i_{et}(X, F) \tag{1.3}$$

Dimension. [Voev, 3.4.6, 3.4.7, 3.4.8] Let X be a scheme of absolute dimension N. Then for any h–sheaf (resp. étale sheaf, resp. qfh–sheaf) of abelian groups and $i > N$ one has:

$$\begin{aligned} H^i_h(X, F) \otimes \mathbb{Q} &= 0 \\ \text{resp. } H^i_{et}(X, F) \otimes \mathbb{Q} &= 0 \\ \text{resp. } H^i_{qfh}(X, F) \otimes \mathbb{Q} &= 0 \end{aligned} \tag{1.4}$$

1.2.2 Basic properties.

Kunneth formula. [Voev, 2.1.2.4] There is a canonical isomorphism

$$M(X \times Y) = M(X) \otimes M(Y) \tag{1.5}$$

Mayer–Vietoris. [Voev, 4.1.2] For any open or closed cover $X = U \cup V$ there is an exact triangle in $DM(S)$:

$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1] \tag{1.6}$$

Homotopy invariance. The projection $M(\mathbb{A}^n \times X) \to M(X)$ is an isomorphism.

Blow–up distinguished triangle. [Voev, 4.1.5] Let Z be a closed subscheme of a scheme X and $p : Y \to X$ a proper surjective morphism of finite type which is an isomorphism outside Z. Then there is an exact triangle in $DM_h(S)$ of the form:

$$M_h(X)[1] \to M_h(p^{-1}(Z)) \to M_h(Z) \oplus M_h(Y) \to M_h(X) \tag{1.7}$$

1.2.3 Other homological properties.

Projective decomposition. [Voev, 4.2.7] Let X be a scheme and E a vector bundle on X. Denote $P(E)$ the projectivization of E. Then there is a natural isomorphism in DM

$$M(P(E)) \cong \bigoplus_{i=0}^{\dim\ E-1} M(X)(i)[2i] \tag{1.8}$$

Blow–up decomposition. [Voev, 4.3.4] Let $Z \subset X$ be a smooth pair over S. Then one has a natural isomorphism in $DM(S)$:

$$M(X_Z) = M(X) \oplus \left(\bigoplus_{i=1}^{codim\ Z-1} Z(i)[2i] \right) \tag{1.9}$$

Gysin exact triangle. [Voev, 4.4.1] Let $Z \subset X$ be a smooth pair over S and $U = X - Z$. Then there is a natural exact triangle in $DM(S)$ of the form

$$M(U) \to M(X) \to M(Z)(d)[2d] \to M(U)[1] \tag{1.10}$$

1.2.4 Other comparisons.

Singular homology. [Voev, 4.1.8] Let X be a scheme of finite type over $\mathbb{C}$. Then one has a canonical isomorphism of abelian groups

$$DM_h(\mathbb{Z}, M(X) \otimes (\mathbb{Z}/n\mathbb{Z})[k]) = H_k(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z}) \tag{1.11}$$

Other triangulated categories. [Voev2, 4.1.12] Let k be a field which admits resolution of singularities. Then the (canonical) functor

$$DM^{eff}_{-,et}(k) \to DM_h(k) \tag{1.12}$$

is an equivalence of triangulated categories. In particular, the categories $DM^{eff}_{-}(k) \otimes \mathbb{Q}$ and $DM_h(k) \otimes \mathbb{Q}$ are equivalent.

Chapter 2

Preliminaries

The construction of $DM(S)$ which appears in [Voev] is a specific case of the more general construction of the homological category of a site with interval. We begin with the definition of a site with interval.

2.1 Definition of the homological category

Definition 1. Let T be a site (with final object pt). An interval in T is an object I^+, such that there exist a triple of morphisms:

$$\begin{aligned} \mu : I^+ \times I^+ \to I^+ \\ i_0, i_1 : pt \to I^+ \end{aligned} \tag{2.1}$$

satisfying the conditions

$$\begin{aligned} \mu(i_0 \times Id) = \mu(Id \times i_0) = i_0 \circ p \\ \mu(i_1 \times Id) = \mu(Id \times i_1) = Id \end{aligned} \tag{2.2}$$

where $p : I^+ \to pt$ is the canonical morphism. It is assumed that $i_0 \amalg i_1 : pt \amalg pt \to I^+$ is a monomorphism.

The homological category of a site with interval is the final target of a sequence of functors, beginning with T. We list the intermediate categories now for the sake of notation.

T the site,

$Sets(T)$ the category of sheaves of sets on T,

$Ab(T)$ the category of sheaves of abelian groups on T,

$Ch(T)$ the category of bounded cochain complexes of $Ab(T)$,

$K(T)$ the category of bounded cochain complexes of $Ab(T)$ with homotopy classes of morphisms,

$D(T)$ the derived category of $Ab(T)$ using bounded complexes,

$H(T)$ the homological category of T.

The site T is mapped into $Sets(T)$ in the usual fashion via Yoneda (that is, an object X gets sent to the sheafification of the presheaf $\mathrm{Hom}\,(-,X)$). $Sets(T)$ is sent to $Ab(T)$ by associating to a sheaf of sets F the sheaf of abelian groups associated to $U \mapsto \mathbb{Z}\,F(U)$. The functor $T \to Ab(T)$ is denoted $\mathbb{Z}$. The category $Ab(T)$ is embedded in $Ch(T)$ by considering a sheaf of abelian groups F as a cochain complex concentrated in degree 0 and $Ch(T) \to K(T) \to D(T)$ are the natural projections. The category $H(T)$ is constructed from $D(T)$ by localizing with respect to a thick subcategory $Contr(T)$ which we will define presently.

There is a second way of mapping $Sets(T)$ into $H(T)$. We can take a sheaf of sets X to the kernel $\tilde{\mathbb{Z}}(X)$ of the natural morphism $\mathbb{Z}(X) \to \mathbb{Z}$. So we have two functors, which are denoted as follows:

$$\begin{aligned} M &: Sets(T) \to H(T) \\ \tilde{M} &: Sets(T) \to H(T) \end{aligned} \tag{2.3}$$

The first is the composition of the functors described above using $\mathbb{Z} : Sets(T) \to Ab(T)$ and the second is the composition using $\tilde{Z} : Sets(T) \to Ab(T)$.

We first need a slightly different "unit interval" object and we will have cause to use a related "circle" object.

Definition 2. Denote by I^1 the kernel of the morphism $\mathbb{Z}(I^+) \to \mathbb{Z}$ and I^n its nth tensor power. Consider the morphism

$$i = \mathbb{Z}(i_0) - \mathbb{Z}(i_1) : \mathbb{Z} \to I^1 \tag{2.4}$$

We denote the cokernel of i by S^1. Since $i_0 \amalg i_1$ is a monomorphism, i is a monomorphism and so S^1 is quasi–isomorphic to the cone of i. That is, in the derived category of $Ab(T)$ there is a canonical morphism

$$\partial : S^1 \to \mathbb{Z}[1] \tag{2.5}$$

The contractible objects that we are going to factor out are defined in terms of strictly contractible objects.

Definition 3. A sheaf of abelian groups F on T is called strictly contractible if there exists a morphism

$$\phi : F \otimes I^1 \to F \tag{2.6}$$

such that the composition

$$F \stackrel{id \otimes i}{\to} F \otimes I^1 \stackrel{\phi}{\to} F \tag{2.7}$$

is the identity morphism. A sheaf of abelian groups is called contractible if it has a resolution which consists of strictly contractible sheaves. $Contr(T)$ denotes the thick subcategory of $D(T)$ generated by contractible sheaves.

So the homological category is defined as follows:

Definition 4. The homological category $H(T)$ of a site with interval (T, I^+) is the localization of the category $D(T)$ with respect to the subcategory $Contr(T)$.

2.2 Examples of contractible sheaves

We collect here some examples of strictly contractible and contractible sheaves which will be used later on.

Lemma 5 ([Voev, 2.2.3]).

1. *The sheaf* $ker(\mathbb{Z}((I^+)^n) \to \mathbb{Z})$ *is strictly contractible for any* $n \geq 0$.

2. *If* G *is a strictly contractible sheaf (and* F *any sheaf) then both* $F \otimes G$ *and* $\underline{\text{Hom}}(G, F)$ *are strictly contractible.*

Proof.

1. Consider the object $(I^+)^{n+1}$ in the category T. Using the projections $pr_i : (I^+)^{n+1} \to I^+$ we define a morphism

$$\alpha = (\mu(pr_1, pr_{n+1}), \ldots, \mu(pr_n, pr_{n+1})) : (I^+)^{n+1} \to (I^+)^n \tag{2.8}$$

which satisfies

$$\begin{aligned} \alpha \circ \left(\text{Id}_{(I^+)^n} \times i_0\right) &= i_0 \circ p \\ \alpha \circ \left(\text{Id}_{(I^+)^n} \times i_1\right) &= \text{Id}_{(I^+)^n} \end{aligned} \tag{2.9}$$

Pushing these identities through the functor $\mathbb{Z}$ and using the isomorphism $\mathbb{Z}(X \times Y) = \mathbb{Z}(X) \otimes \mathbb{Z}(Y)$ we find that

$$\mathbb{Z}\alpha \circ (\text{Id}_{\mathbb{Z}(I^+)^n} \otimes i) = \mathbb{Z}i_0 \circ \mathbb{Z}p - \text{Id}_{\mathbb{Z}(I^+)^n} \tag{2.10}$$

We construct one more morphism. Denote $q = (i_0, i_0, \ldots, i_0) : pt \to (I^+)^n$ and note that $p \circ q = \text{Id}_{pt}$. As a consequence of this, the morphism $\text{Id}_{\mathbb{Z}(I^+)^n} - \mathbb{Z}q \circ \mathbb{Z}p$ composed with $\mathbb{Z}q$ is zero, and so factors through $\ker(\mathbb{Z}(I^+)^n \to \mathbb{Z})$ giving a retraction $\rho : \mathbb{Z}(I^+)^n \to K$ where $K = \ker(\mathbb{Z}(I^+)^n \to \mathbb{Z})$.

Now we have a diagram:

$$\begin{array}{ccccc} K & \xrightarrow{\text{Id}\otimes i} & K \otimes I^1 & \overset{\phi}{\dashrightarrow} & K \\ \downarrow & & \downarrow & & \downarrow \uparrow \rho \\ \mathbb{Z}(I^+)^n & \xrightarrow{\text{Id}\otimes i} & \mathbb{Z}(I^+)^{n+1} & \xrightarrow{\mathbb{Z}\alpha} & \mathbb{Z}(I^+)^n \end{array} \tag{2.11}$$

where ϕ is the composition of $\rho, \mathbb{Z}\alpha$ and the inclusion $K \otimes I^1 \to \mathbb{Z}(I^+)^{n+1}$. Now using Equation 2.10, the fact that $\mathbb{Z}p$ composed with the inclusion $K \to \mathbb{Z}(I^+)^n$ is zero and the fact that ρ is a retraction shows that $\phi \circ (\text{Id} \otimes i) = \text{Id}_K$. Hence, K is contractible.

2. Suppose $\phi : G \otimes I^1 \cong I^1 \otimes G \to G$ is a morphism corresponding to the strict contactibility of G. Then $id \otimes \phi$ defines a morphism which shows the contractibility of $F \otimes G$. Consider $\underline{\text{Hom}}(G, F)$. The functor $\underline{\text{Hom}}(G, -)$ is right adjoint to $- \otimes G$ so to define a morphism

$$\phi' : \underline{\text{Hom}}(G, F) \otimes I^1 \to \underline{\text{Hom}}(G, F) \tag{2.12}$$

it is enough to define a morphism

$$\psi : \underline{\mathrm{Hom}}(G, F) \otimes I^1 \otimes G \to F \tag{2.13}$$

We also have use of $ev : \underline{\mathrm{Hom}}(G, F) \otimes G \to F$, the morphism corresponding to the identity on $\underline{\mathrm{Hom}}(G, F)$. Define ψ as the composition of

$$\underline{\mathrm{Hom}}(G, F) \otimes (I^1 \otimes G) \xrightarrow{id \otimes \phi} \underline{\mathrm{Hom}}(G, F) \otimes G \xrightarrow{ev} F \tag{2.14}$$

and ϕ' the morphism corresponding to it under the adjunction. Then through naturality of the adjointness, the composition

$$\underline{\mathrm{Hom}}(G, F) \xrightarrow{id \otimes i} \underline{\mathrm{Hom}}(G, F) \otimes I^1 \xrightarrow{\phi'} \underline{\mathrm{Hom}}(G, F) \tag{2.15}$$

corresponds to

$$\underline{\mathrm{Hom}}(G, F) \otimes G \xrightarrow{id \otimes i \otimes id} \mathrm{Hom}(G, F) \otimes I^1 \otimes G \xrightarrow{\psi} F \tag{2.16}$$

whose composition is

$$\underline{\mathrm{Hom}}(G, F) \otimes G \xrightarrow{ev} F \tag{2.17}$$

which corresponds under the adjunction to the identity on $\underline{\mathrm{Hom}}(G, F)$. Hence, $\underline{\mathrm{Hom}}(G, F)$ is strictly contractible.

□

As a consequence of the lemma we have the following proposition which says that we can't move objects out of $Contr(T)$ by tensoring them with objects of $D(T)$. This helps to show that the tensor structure of $D(T)$ can be transfered to $H(T)$.

Proposition 6 ([Voev, 2.2.4]). *Let X be an object of $D(T)$ and Y be an object of $Contr(T)$. Then $X \otimes Y$ belongs to $Contr(T)$.*

Proof. Follows from the definitions and the tensor part of [Voev, 2.2.3]. □

Voevodsky provides one last example of a contractible sheaf, or more specifically provides a way of checking if a sheaf is contractible. This will be used to prove an isomorphism in $DM(S)$ ([Voev, 4.2.5]) which will in turn be used to show prove the projective decomposition theorem. The specific details will not be given here but we will outline what happens.

Voevodsky starts by defining a cosimplicial object in T using the $(I^+)^n$ which is denoted by a_{I^+}. This object is supposed to take the place of the usual simplicial object used in topology. This in term provides a complex of sheaves $C_*(F)$ for each sheaf of abelian groups F with terms $\underline{\mathrm{Hom}}(\mathbb{Z}(I^+)^n, F)$ and differentials the alternating sums of morphisms induced by the coface morphisms of a_{I^+}. We then have a criteria for F to be contractible.

Lemma 7 ([Voev, 2.2.5]). *Let F be a sheaf of abelian groups on T such that the complex $C_*(F)$ is exact. Then F is contractible.*

2.3 A result about morphisms in $H(T)$.

The last thing Voevodsky proves in the generality of homological category of a site with interval is the equivalence of the hom sets $\mathrm{Hom}_{H(T)}(X,Y)$ and $\mathrm{Hom}_{D(T)}(X,Y)$ when Y is strictly homotopy invariant. We show this now. The major technical result towards this end is [Voev, Proposition 2.2.6] which involves a different localization of $D(T)$.

Let E be the class of objects of $D(T)$ of the form $X \otimes I^1$ and $\mathcal{E}$ the thick subcategory of $D(T)$ it generates. The localization of $D(T)$ with respect to $\mathcal{E}$ is denoted $H_0(T)$ and the functors analogous to M and $\tilde{M}$ are denoted M_0 and $\tilde{M}_0$. We will also use $\mathcal{W}$ (which doesn't appear in [Voev]), the multiplicative system in $D(T)$ generated by the set of morphisms

$$W = \{\mathrm{Id}_X \otimes \partial^{\otimes n} : X \otimes S^n \to X[n] : n \geq 0, X \in ob(D(T))\} \tag{2.18}$$

Lemma 8. *The multiplicative system $\mathcal{W}$ and the thick subcategory $\mathcal{E}$ correspond to each other (see [SGA 4.5]). That is, localizing with respect to $\mathcal{W}$ is the same as localizing with respect to $\mathcal{E}$.*

Proof. Recall that in [SGA 4.5] two maps ϕ and ψ are given and are shown to be inverses of each other. The map ϕ takes thick subcategories to multiplicative systems and is defined by

$$\phi(E') = \{X \xrightarrow{s} Y \ : \ Cone(s) \in ob(E')\} \tag{2.19}$$

and the map ψ takes multiplicative systems to thick subcategories

$$\psi(S) = \{Cone(s) \ : \ s \in S\} \tag{2.20}$$

Since W generates $\mathcal{W}$ if we show that $\psi(W)$ generates $\mathcal{E}$ then because the correspondence given by ψ and ϕ is bijective this will show that $\psi(\mathcal{W}) = \mathcal{E}$.

We will show that $Cone(\mathrm{Id}_X \otimes \partial^n) \in ob(\mathcal{E})$ for all objects $X \in ob(D(T))$. Since $Cone(\mathrm{Id}_X \otimes f) = \mathrm{Id}_X \otimes Cone(f)$ we only need to consider the case where $X = \mathbb{Z}$. We proceed by induction. For $n = 0, 1$ the statement is true since $Cone(\mathrm{Id}_{\mathbb{Z}}) = 0$ and $Cone(\partial) = I^1[1] = I^1 \otimes \mathbb{Z}[1]$.

Assume that the statement is true for $n-1$, and consider the diagram

$$\begin{array}{ccccccc}
S^n & \xrightarrow{\mathrm{Id}_{S^1}\otimes\partial^{\otimes n-1}} & S^1 \otimes \mathbb{Z}[n-1] & \longrightarrow & S^1 \otimes Cone(\partial^{\otimes n-1}) & \longrightarrow & S^n[1] \\
\downarrow{\scriptstyle \partial\otimes\mathrm{Id}_{S^{n-1}}} & \searrow{\scriptstyle \partial^{\otimes n}} & \downarrow{\scriptstyle \partial\otimes\mathrm{Id}_{\mathbb{Z}[n-1]}} & & \downarrow & & \downarrow \\
S^{n-1}[1] & \xrightarrow[\mathrm{Id}_{\mathbb{Z}[1]}\otimes\partial^{\otimes n}]{} & \mathbb{Z}[n] & \longrightarrow & Cone(\partial^{\otimes n-1})[1] & \longrightarrow & S^{n-1}[2]
\end{array} \tag{2.21}$$

where the top row is the exact triangle $S^{n-1} \to \mathbb{Z}[n-1] \to Cone(\partial^{\otimes n-1})$ tensored with S^1, the bottom row is this triangle tensored with $\mathbb{Z}[1]$ and the morphism of triangles is the identity tensored with ∂. Consider what happens when we pass to the quotient category $H_0(T)$. The two vertical morphisms on the left become isomorphisms (since ∂ is an isomorphism in $H_0(T)$), the two terms that contain cones become zero (by induction) and so all the morphisms in the square on the left become isomorphisms. Hence, ∂^n is a isomorphism in $H_0(T)$ so its cone is zero. Therefore its cone in $D(T)$ is in $\mathcal{E}$. □

Lemma 9 (Contained in the proof of [Voev, 2.2.6]). *For any object Y of $\mathcal{E}$ there is some n such that* $\mathrm{Id}_Y \otimes \partial^{\otimes n} = 0$.

Proof. It is enough to show that the class E' of objects Y of $\mathcal{E}$ satisfying the property form a thick category of $D(T)$ which contains E.

Suppose $Y = X \otimes I^1$ for some object X. The morphism $\mathrm{Id}_{X\otimes I^1} \otimes \partial : X \otimes I^1 \otimes S^1 \to X \otimes I^1[1]$ fits into the exact triangle:

$$X \otimes I^1 \to X \otimes I^1 \otimes I^1 \to X \otimes I^1 \otimes S^1 \to X \otimes I^1[1] \tag{2.22}$$

The multiplication morphism $\mu : I^1 \otimes I^1 \to I^1$ splits the morphism $X \otimes I^1 \to X \otimes I^1 \otimes I^1$ and so $\mathrm{Id}_{X\otimes I^1} \otimes \partial = 0$. Hence, $E \subseteq E'$.

We will show now that E' is a triangulated subcategory. Let $X \to Y \to Z \to X[1]$ be an exact triangle such that there exist m and n such that $\mathrm{Id}_X \otimes \partial^{\otimes n}$ and $\mathrm{Id}_Y \otimes \partial^{\otimes m}$. We have the diagram:

$$\begin{array}{ccccc} Y \otimes S^n & \longrightarrow & Z \otimes S^n & \xrightarrow{f} & X[1] \otimes S^n \\ \downarrow & & \downarrow & \overset{\alpha}{\swarrow} & \\ Y[n] & \longrightarrow & Z[n] & & \end{array} \tag{2.23}$$

The dotted arrow exists because the upper row is part of an exact triangle and $Y \otimes S^n \to Y[n] = 0$. Now because $\mathrm{Id}_Z \otimes \partial^{\otimes n}$ factors as $\alpha \circ f$ we can factor $\mathrm{Id}_Z \otimes \partial^{\otimes n+m}$ as

$$\mathrm{Id}_Z \otimes \partial^{\otimes(n+m)} = (\mathrm{Id}_Z \otimes \partial^{\otimes n}) \otimes \partial^{\otimes m} = (\alpha \otimes \partial^m) \circ (f \otimes \mathrm{Id}_{S^m}) \tag{2.24}$$

and now the morphism $\alpha \otimes \partial^m$ can be factored as

$$\alpha \otimes \partial^m = \alpha[m] \circ (\mathrm{Id}_{X[1]\otimes S^n} \otimes \partial^m) = 0 \tag{2.25}$$

To finish the proof we need to show that E' is closed under direct summands. Let $X = X_0 \oplus X_1$ with $X \in E'$. Then there is some n such that $\mathrm{Id}_X \otimes \partial^n = 0$. But $\mathrm{Id}_X = \mathrm{Id}_{X_0} \oplus \mathrm{Id}_{X_1}$ and so

$$0 = \mathrm{Id}_X \otimes \partial^n = (\mathrm{Id}_{X_0} \oplus \mathrm{Id}_{X_1}) \otimes \partial^n = (\mathrm{Id}_{X_0} \otimes \partial^n) \oplus (\mathrm{Id}_{X_1} \otimes \partial^n) \tag{2.26}$$

and so $0 = (\mathrm{Id}_{X_0} \otimes \partial^n)$. Hence, X_0 is an object in E' and so E' is closed under direct summands.

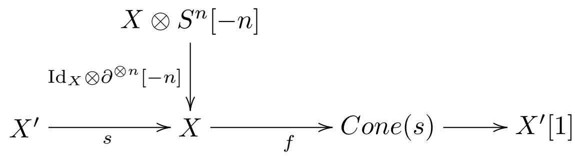
□

The set W falls short of being a multiplicative system. We do however, have the following lemma, which allows us to use $\lim_{n\to\infty}(X \otimes S^n, Y[n])$ to calculate the morphism groups in $H_0(T)$.

Lemma 10. *Let $s : X' \to X$ be a morphism in $\mathcal{W}$. Then there exists n together with a morphism $X \otimes S^n[-n] \to X'$ such that the following diagram commutes:*

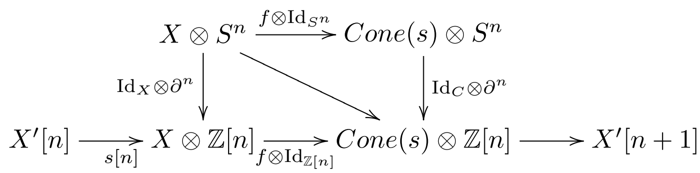
$$\begin{array}{ccc} & & X \otimes S^n[-n] \\ & \swarrow^{\mathrm{Id}_X \otimes \partial^{\otimes n}[-n]} & \downarrow \\ & & X' \\ X & \overset{s}{\longleftarrow} & \end{array} \tag{2.27}$$

Proof. It follows from Lemma 8 that the cone $C = Cone(s)$ of s lies in $\mathcal{E}$ and so by Lemma 9 there is some n such that $Id_C \otimes \partial^n = 0$. Consider the following diagram:



(2.28)

If we tensor all the objects with $\mathbb{Z}[n]$ (and the morphisms with $\mathrm{Id}_{\mathbb{Z}[n]}$) then we obtain the following commutative diagram:



(2.29)

Since $\mathrm{Id}_C \otimes \partial^n = 0$ the diagonal morphism is zero. Since the lower row is an exact triangle, $\mathrm{Hom}\,(X \otimes S^n, -)$ sends it to a long exact sequence, and so the diagonal begin zero implies that the morphism $\mathrm{Id}_X \otimes \partial^n$ factors through $X'[n]$ and so we obtain $X \otimes S^n \xrightarrow{g} X'[n]$. Now $g[-n] : X \otimes S^n[-n] \to X'$ gives the desired morphism. □

Proposition 11 ([Voev, 2.2.6]). *Let $X, Y \in ob(D(T))$. Then one has*

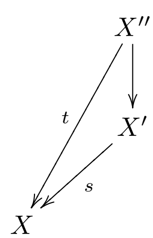
$$Hom_{H_0(T)}(X, Y) = \lim_{n \to \infty} Hom_{D(T)}(X \otimes S^n, Y[n]) \tag{2.30}$$

where the direct system on the right hand side is defined by tensor multiplication of morphisms with $\partial : S^1 \to \mathbb{Z}[1]$.

Proof. Using calculus of fractions, the morphism group $\mathrm{Hom}_{H_0(T)}(X, Y)$ can be written as

$$\lim_{X' \xrightarrow{s} X \in \mathcal{W}} \mathrm{Hom}_{D(T)}(X', Y) \tag{2.31}$$

where the limit is over the category whose objects are morphisms in $\mathcal{W}$ with target X and morphisms between say $X' \xrightarrow{s} X$ and $X'' \xrightarrow{t} X$ are commutative triangles



(2.32)

Lemma 10 shows that every morphism in $\mathcal{W}$ is equivalent to a morphism of the form

$$\mathrm{Id}_X \otimes \partial^{\otimes n}[-n] : X \otimes S^n[-n] \to X \tag{2.33}$$

and so we can calculate the hom groups in $H_0(T)$ by taking the limit over the category with objects morphisms $\mathrm{Id}_X \otimes \partial^{\otimes n}[-n]$ and morphisms commutative triangles:

$$\begin{array}{ccc} & X \otimes S^m[-m] & \\ & \downarrow {\scriptstyle \mathrm{Id}_{X\otimes S^{m-n}} \otimes (\partial^{\otimes(m-n)}[-1])} & \\ {\scriptstyle \mathrm{Id}_X \otimes \partial^{\otimes m}[-m]} \swarrow & X \otimes S^n[-n] & \\ X \quad \xleftarrow{\mathrm{Id}_X \otimes \partial^{\otimes n}[-n]} & & \end{array} \tag{2.34}$$

where $m > n$. In other words, we have groups $\mathrm{Hom}_{D(T)}(X \otimes S^n[-n], Y)$ and for $m > n$, a morphism from $\mathrm{Hom}_{D(T)}(X \otimes S^m[-m], Y)$ to $\mathrm{Hom}_{D(T)}(X \otimes S^n[-n], Y)$ given by composition with $\mathrm{Id}_{X\otimes S^{m-n}} \otimes (\partial^{m-n}[-1]) : X \otimes S^m[-m] \to X \otimes S^n[-n]$. Since translation is an automorphism of $D(T)$, we can replace $\mathrm{Hom}_{D(T)}(X \otimes S^n[-n], Y)$ by $\mathrm{Hom}_{D(T)}(X \otimes S^n, Y[n])$ and the morphism from $\mathrm{Hom}_{D(T)}(X \otimes S^m, Y[m])$ to $\mathrm{Hom}_{D(T)}(X \otimes S^n, Y[n])$ is now given by tensoring with $\partial^{\otimes(m-n)}$.

□

Corollary 12 ([Voev, 2.2.7]). *Let X, Y be a pair of objects of $D(T)$ such that for any n and m we have $Hom_{D(T)}(X \otimes I^n, Y[m]) = 0$. Then*

$$Hom_{H_0(T)}(X, Y[m]) = Hom_{D(T)}(X, Y[m]) \tag{2.35}$$

Proof. If we show that the morphisms

$$\mathrm{Hom}_{D(T)}(X, Y[m]) \to \mathrm{Hom}_{D(T)}(X \otimes S^n, Y[m+n]) \tag{2.36}$$

are isomorphisms for every n then the result will follow from the previous proposition. We will show this by induction on n.

For $n = 0$ the result is trivial. Assume that the result holds for $n - 1$ and consider the exact triangle

$$X \otimes S^{n-1} \to X \otimes I^1 \otimes S^{n-1} \to X \otimes S^n \to X \otimes S^{n-1}[1] \tag{2.37}$$

As this is an exact triangle, it gets sent by $\mathrm{Hom}_{D(T)}(-, Y[m])$ to a long exact sequence of abelian groups. Now since X satisfies the conditions of the proposition, $X \otimes I^1$ does as well and so by the inductive hypothesis,

$$\mathrm{Hom}_{D(T)}(X \otimes I^1 \otimes S^{n-1}, Y[m]) \cong \mathrm{Hom}_{D(T)}(X \otimes I^1, Y[m-n]) = 0 \tag{2.38}$$

So from the long exact sequence of hom groups we find that

$$\begin{aligned} \mathrm{Hom}_{D(T)}(X \otimes S^n, Y[m]) &= \mathrm{Hom}_{D(T)}(X \otimes S^{n-1}[1], Y[m]) \\ &= \mathrm{Hom}_{D(T)}(X \otimes S^{n-1}, Y[m-1]) \\ &= \mathrm{Hom}_{D(T)}(X, Y[m-1-(n-1)]) \\ &= \mathrm{Hom}_{D(T)}(X, Y[m-n]) \end{aligned} \tag{2.39}$$

So the morphism of equation 2.36 is an isomorphism for all n and so the result follows from the previous proposition. □

Something else which does not appear in [Voev] is that localization with respect to $Contr(T)$ and $\mathcal{E}$ is actually the same. This means that a lot of the results of [Voev] comparing hom groups actually hold more generally.

Lemma 13. *The thick subcategories $\mathcal{E}$ and $Contr(T)$ are the same. That is, $H_0(T) = H(T)$.*

Proof. Any sheaf F of the form $F \otimes I^1$ is contractible and so the associated object of $D(T)$ is in $Contr$. Conversely, if a sheaf has a resolution of strictly contractible sheaves then it is equivalent in $D(T)$ to a strictly contractible sheaf. For a sheaf to be strictly contractible there has to exist a ϕ such that the composition of $F \xrightarrow{i} F \otimes I^1 \xrightarrow{\phi} F$ is the identity of F. That is, F is a direct summand of $F \otimes I^1$. Since thick categories are closed under direct summand this means F is in $\mathcal{E}$.

Now $D(T)$ is generated by complexes concentrated in degree zero so $\mathcal{E}$ is generated by objects of the form $F \otimes I^1$. So $\mathcal{E} \subset Contr(T)$. Conversely, every object in $Contr(T)$ is equivalent to an object of $\mathcal{E}$ so $Contr(T) \subset \mathcal{E}$. Hence, they are the same. □

Proposition 14. *Let Y be an object of $D(T)$ such that $Hom(X \otimes I^1, Y) = 0$ for all other objects X of $D(T)$. Then for all X in $D(T)$*

$$Hom_{H_0(T)}(X, Y) = Hom_{D(T)}(X, Y) \tag{2.40}$$

Proof. This is actually a corollary of [Voev, 2.2.7]. If $\mathrm{Hom}_{D(T)}(X \otimes I^1, Y) = 0$ for all objects X then this includes objects of the form $X = Z[-m] \otimes I^{n-1}$. Now $Z[-m] \otimes I^{n-1} \otimes I^1 = Z[-m] \otimes I^n = (Z \otimes I^n)[-m]$. So the hypothesis implies that $\mathrm{Hom}_{D(T)}((Z \otimes I^n)[-m], Y) = 0$ for every n and m. This group is isomorphic to $\mathrm{Hom}_{D(T)}(Z \otimes I^n, Y[m])$ and so the hypothesis of [Voev, 2.2.7] is satisfied. Taking the case $m = 0$ gives the desired result. □

Corollary 15 ([Voev, 2.2.9]). *Let Y be an object of $D(T)$ such that $Hom(X \otimes I^1, Y) = 0$ for all other objects X of $D(T)$. Then for all X in $D(T)$*

$$Hom_{H(T)}(X, Y) = Hom_{D(T)}(X, Y) \tag{2.41}$$

Proof. Follows from the previous proposition and Lemma 13. □

Definition 16. An object X of $D(T)$ is called an object of finite dimension if there exists N such that for any $F \in ob(Ab(T))$ and any $n > N$ one has

$$\mathrm{Hom}_{D(T)}(X, F[n]) = 0 \tag{2.42}$$

Chapter 3

The h–topology on the category of schemes

3.1 Definitions, examples and coverings of normal form

In this section we present the h and qfh topologies together with some examples and state a characterization of them by "coverings of normal form". This is intended as an overview only and does not attempt to prove this characterization as is done in Section 3.1 of [Voev].

Definition 17. A morphism of schemes $p : X \to Y$ is called a *topological epimorphism* if the underlying topological space of Y is a quotient space of the underlying topological space of X. That is, p is surjective and a subset U of Y is open if and only if $p^{-1}U$ is open in X.

A topological epimorphism $p : X \to Y$ is called a *universal topological epimorphism* if for any morphism $f : Z \to Y$ the projection $Z \times_Y X \to Z$ is a topological epimorphism.

Example 1.

1. Any open or closed surjective morphism is a topological epimorphism. This is fairly straightforward from the definitions.

2. Any surjective flat morphism is a topological epimorphism (at least when it is locally of finite–type) and in fact is a universal topological epimorphism. That a surjective flat morphism is a topological epimorphism follows from the property that flat morphisms are open (at least if it is locally of finite–type [Mil, Theorem 2.12]). That a surjective flat morphism is a universal topological epimorphism follows from the property that both surjectiveness and flatness are preserved by base change [Mil, Proposition 2.4].

3. By definition, proper morphisms are universally closed and hence a surjective proper morphism is a universal topological epimorphism.

4. Any composition of (universal) topological epimorphisms is a (universal) topological epimorphism.

Definition 18. The h–topology on the category of schemes is the Grothendieck topology with coverings of the form $\{p_i : U_i \to X\}$ where $\{p_i\}$ is a finite family of morphisms of finite type such that the morphisms $\coprod p_i : \coprod U_i \to X$ is a universal topological epimorphism.

The qfh–topology on the category of schemes is the Grothendieck topology with coverings of the form $\{p_i : U_i \to X\}$ where $\{p_i\}$ is a finite family of *quasi–finite* morphisms of finite type such that the morphisms $\coprod p_i : \coprod U_i \to X$ is a universal topological epimorphism.

The first thing that should be noted is that unlike other grothendieck topologies frequently used in algebraic geometry (Zariski, Nisnevich, étale, flat) the h and qfh–topologies are not subcanonical. That is, representable presheaves are not necessarily sheaves [Voev, 3.2.11].

Example 2.

1. Any flat covering is a h–covering as well as a qfh–covering.

2. Any surjective proper morphism of finite type is an h–covering.

3. Let X be a scheme X with a finite group G acting on it. If the categorical quotient X/G exists then the canonical projection $p : X \to X/G$ is a qfh–covering [SGA. 1, 7, ex. 5 n.1].

4. For an example of a surjective morphism that is not an h–covering Voevodsky uses the blowup $p : X_x \to X$ of a surface X with center in a closed point $x \in X$. By removing a closed point $x_0 \in p^{-1}(x)$ in the preimage of x a surjective morphism

$$p_U : U = X_x - x_0 \to X \tag{3.1}$$

can be constructed (the morphism induced by p). Now consider a curve C through x in X such that the preimage $p^{-1}C$ of C is the union of a curve $\tilde{C}$ which intersects the exceptional divisor $p^{-1}(x)$ in the point x_0, and the exceptional divisor. That is,

$$\begin{aligned} p^{-1}C &= \tilde{C} \cup p^{-1}x \\ \tilde{C} \cap p^{-1}x &= \{x_0\} \end{aligned} \tag{3.2}$$

Then the preimage of $C - \{x\}$ under p_U is a closed subset of U but $C - \{x\}$ isn't.

5. Consider a morphism of the form

$$\coprod U_i \xrightarrow{p_i} \overline{U} \xrightarrow{f} X_Z \xrightarrow{s} X \tag{3.3}$$

where $\{p_i\}$ is an open cover of $\overline{U}$, the morphism f is finite and surjective, and s is the blowup of a closed subscheme in X. Then the composition of these three morphisms is an h–covering (at least when all the schemes are finite type [Lev, 6.2]).

6. Consider a morphism of the form

$$\coprod U_i \xrightarrow{p_i} \overline{U} \xrightarrow{f} X \tag{3.4}$$

where $\{p_i\}$ is an open cover of $\overline{U}$ and f is finite and surjective. Then the composition of these two morphisms is a qfh–covering (at least when all the schemes are finite type [Lev, 6.2]).

The final two illuminating examples are not found in [Voev] but taken from [Lev].

Definition 19. A finite family of morphisms $\{U_i \xrightarrow{p_i} X\}$ is called an h–covering of normal form if the morphism $\coprod p_i$ admits a factorization as in the fifth example above.

The next theorem is the main result of Section 3.1 in [Voev]. To prove this Voevodsky restricts his attention to noetherian excellent schemes. For interest, some basic material on excellent schemes from [EGA. 4, 7.8] (which Voevodsky references but omits) can be found in Appendix D. The definition of an excellent scheme makes it quite obvious why Voevodsky omits the definition. He does however recall the following properties of excellent schemes:

1. Any scheme of the form $X = \mathrm{Spec}\,(A)$ where A is a field or a Dedekind domain with field of fractions of characteristic zero is excellent.

2. If X is an excellent scheme and $Y \to X$ is a morphism of finite type, then Y is excellent.

3. Any localization of an excellent scheme is excellent.

4. If X is an excellent integral scheme and L is a finite extension of the field of functions K of X, the normalization of X in L is finite over X.

Theorem 20 ([Voev, 3.1.9]). *Let $\{U_i \xrightarrow{p_i} X\}$ be an h–covering of an excellent reduced noetherian scheme X. Then there exists an h–covering of normal form, which is a refinement of $\{p_i\}$.*

3.2 Representable sheaves of sets

In section 3.2 of [Voev] Voevodsky develops some of the properties of representable sheaves of sets, being mostly concerned with finding information about morphisms between them. We will restrict ourselves to presenting some criteria for when an induced morphism $L(X) \to L(Y)$ is a monomorphism, epimorphism or isomorphism.

For this section Sch/S will denote the category of separated schemes of finite type over a noetherian excellent scheme S. The term scheme (resp. morphism) will refer to an object (resp. morphism) of Sch/S.

Following Voevodsky we denote by L the functor $Sch/S \to Sets_h(S)$ which takes a scheme to the sheafification of the corresponding representable presheaf in the h–topology on Sch/S. The notation L_{qfh} is used for the corresponding functor with respect to the qfh–topology.

We first need the following two lemmas.

Lemma 21 ([Voev, 3.2.1]). *Let X be a scheme and X_{red} its maximal reduced subscheme. Then the natural morphism $L_{qfh}(i) : L_{qfh}(X_{red}) \to L_{qfh}(X)$ is an isomorphism.*

Proof. Since $i : X_{red} \to X$ is a monomorphism in the category of schemes and L is left exact (that is, it preserves inverse limits), $L(i)$ is also a monomorphism. It is an epimorphism because i is a qfh–cover. Hence, i is an isomorphism. □

Lemma 22 ([Voev, 3.2.2]). *Let X be a reduced scheme and $U \to X$ an h–covering. Then it is an epimorphism in the category of schemes. In particular for any reduced X and any Y the natural map $Hom_S(X, Y) \to Hom(L(X), L(Y))$ is injective.*

Proof. Since $U \to X$ is surjective on the underlying topological space it is an epimorphism in the category of schemes. Now consider two morphisms $f, g : X \to Y$ and assume that they induce the same morphism of sheaves $L(f), L(g) : L(X) \to L(Y)$. The identity morphism $X \to X$ represents a section of $L(X)(X)$ which gets mapped by $L(f)$ to the section represented by f in $L(Y)(X)$ and by $L(g)$ to the section represented by g. Since $L(f) = L(g)$ this means that f and g represent the same section. From the construction of associated sheaf, this means that there is an h–covering $p : U \to X$ such that $f \circ p = g \circ p$. But p is an epimorphism so $f = g$. □

The following result gives criteria for a morphism of finite type between schemes to induce monomorphisms, epimorphisms and isomorphisms. To describe these we first need a couple of definitions. Note the similarity of these definitions to that of a universal topological epimorphism.

Definition 23. Let $f : X \to Y$ be a morphism of finite type. The morphism f is called radicial if for any scheme $Z \to Y$ over Y the pullback $X \times_Y Z \to Z$ induces an immersion of the underlying topological spaces.

The morphism f is called a universal homeomorphism if for any scheme $Z \to Y$ over Y the pullback $X \times_Y Z \to Z$ induces a homeomorphism of the underlying topological spaces.

Proposition 24 ([Voev, 3.2.5]). *Let $f : X \to Y$ be a morphism of finite type. Then one has*

1. *The morphism $L(f)$ (resp. $L_{qfh}(f)$) is a monomorphism if and only if f is radical.*
2. *The morphism $L(f)$ is an epimorphism if and only if f is a topological epimorphism.*
3. *The morphism $L(f)$ (resp. $L_{qfh}(f)$) is an isomorphism if and only if f is a universal homeomorphism.*

Proof. [Voev, Lemma 3.2.1] allows us to assume that the schemes X and Y are reduced.

1. If f is radicial (and X is reduced) then it is a monomorphism in the category of schemes and so since L is left exact $L(f)$ is a monomorphism. Now suppose that $L(f)$ is a monomorphism. We use [Voev, 3.2.2] and the result that a morphism is radicial if and only if it induces monomorphisms on the sets of geometric points [EGA. 1]. Suppose there are two geometric points $p_1, p_2 : \bullet \to X$ such that $f \circ p_1 = f \circ p_2$. Then $L(f) \circ L(p_1) = L(f) \circ L(p_2)$ and so $L(p_1) = L(p_2)$ (since $L(f)$ is a monomorphism by assumption) and so by [Voev, 3.2.2] $p_1 = p_2$. Hence, f induces a monomorphism on the set of geometric points.

2. If f is a topological epimorphism it is a cover by definition and so $L(f)$ is surjective. Suppose that $L(f)$ is surjective. This means that the sheafification of the image $Im_P(f)$ of $L(f)$ in the category of presheaves is isomorphic $L(Y)$. Since $Im_P(f)$ is a subpresheaf of $L(Y)$ it is separated and so each element of $L(Y)$ is represented by a pair $(U \to Y, s)$ where U is a cover and $s \in Im_P(f)(U)$. Consider $\mathrm{Id} \in \mathrm{Hom}\,(Y, Y)$ and let $(U \to Y, s)$ represent it. There is some $t \in L(X)(U)$ which gets mapped to s and this t can be represented by $(V \to U, V \xrightarrow{t'} X)$. So we have found a cover $V \to U \to Y$ which factors through f. This implies that $X \to Y$ is a topological epimorphism.

3. If f is a universal homeomorphism then it is a qfh–covering so $L_{qfh}(f)$ is an epimorphism. $L(f)$ is an epimorphism by (2). Both $L(f)$ and $L_{qfh}(f)$ are monomorphisms by (1). Conversely suppose that $L(f)$ (resp. $L_{qfh}(f)$) is an isomorphism. (1) and (2) imply then that f is a radicial universal epimorphism and therefore a topological homeomorphism.

□

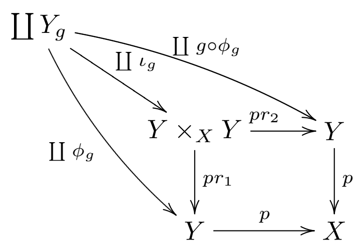
3.3 "Representable" sheaves of groups

In this section we develop some of the properties of sheaves of abelian groups of the form $\mathbb{Z}(X)$. The results are mainly focused on things needed to prove the main results in the next section.

For a scheme X over S, in this section $\mathbb{Z}(X)$ (resp. $\mathbb{Z}_{qfh}$) will denote the h–sheaf (resp. qfh–sheaf) of abelian groups freely generated by the sheaf of sets $L(X)$. Analogously, $\mathbb{N}(X)$ and $\mathbb{N}_{qfh}(X)$ will denote the sheaves of monoids freely generated by the sheaf of sets $L(X)$. For an abelian monoid A its group completion is denoted A^+.

Proposition 25 ([Voev, 3.3.2]). *Let X be a normal connected scheme and let $p : Y \to X$ be the normalization of X in a Galois extension of its field of functions. Then for any qfh–sheaf F of abelian monoids the image of $p^* : F(X) \to F(Y)$ coincides with the submonoid $F(Y)^G$ of Galois invariant elements in $F(Y)$.*

Proof. Each automorphism of Y that preserves Y gives a morphism of $F(Y)$ which preserves the image of $F(X)$ in $F(Y)$ so $p^*F(X) \subset F(Y)^G$. Let $a \in F(Y)^G$ and consider the scheme $Y \times_X Y$. The irreducible components of $Y \times_X Y$ can be labeled by G in such a way that the first projection induces isomorphisms $\phi_g : Y_g \to Y$ with Y and the the second induces $g \circ \phi_g$ the isomorphism composed with the automorphism defined by $g \in G$. Now we have a commutative diagram



(3.5)

where $\coprod \iota_g$ and p are both qfh–coverings. The element $a \in F(Y)$ was chosen to be G–invariant so $(\coprod \iota_g)^* \circ pr_1^*(a) = (\coprod \iota_g)^* \circ pr_2^*(a)$ and since $\coprod \iota_g$ is a qfh–covering this implies $pr_1^* a = pr_2^* a$. But now since p is a qfh–covering this implies that a is in the image of p^*. Hence, $F(Y)^G = p^*F(X)$. □

Proposition 26 ([Voev, 3.3.6]). *Let X be a scheme over S such that there exists symmetric powers S^nX of X over S. Then the sheaves $\mathbb{N}(X)$ and $\mathbb{N}_{qfh}(X)$ are representable by the (ind–)scheme $\coprod_{n\geq 0} S^nX$.*

Proof. Voevodsky claims that it is sufficient to prove the proposition for the case of the qfh–topology.

The sheaf $\mathbb{N}_{qfh}(X)$ is characterized by the universal property that any morphism $L(X) \to G$ from $L(X)$ to a qfh–sheaf G of abelian monoids factors uniquely through $\mathbb{N}_{qfh}(X)$. We will show that $L(\coprod S^n X)$ satisfies this property. Using Yoneda's Lemma we restate the universality property as follows:

For any abelian monoid G and any element $a \in G(X)$ there is a unique element

$$f \in G(\amalg S^n X) = \operatorname{Hom}\left(L(\amalg S^n X), G\right) \tag{3.6}$$

which is a morphism of sheaves of abelian monoids such that f restricted to X is a.

Let $a \in G(X)$ and let $y_n = \sum_{i=1}^n pr_i^* a \in G(X^n)$ where $pr_i : X^n \to X$ is the ith projection. The element y_n is invariant under the action of the symmetric group and so using a similar argument to the proof of [Voev, 3.3.2] above with $q : X^n \to S^n X$ in place of $p : Y \to X$ we find an element $f_n \in G(S^n X)$. Then f can be defined as the sum of the f_n

$$f = 1 \oplus f_1 \oplus f_2 \oplus \cdots \in \bigoplus_{n \geq 0} G(S^n X) = G\left(\coprod_{n \geq 0} S^n X\right) \tag{3.7}$$

In the case $n = 1$ we have $f_1 = a$ and so f restricted to X is indeed a.

We now show that f induces a morphism of sheaves of abelian monoids by showing that for every U, the induced morphism $\operatorname{Hom}(U, \amalg S^n X) \to G(U)$ is a morphism of abelian monoids. Consider two morphisms $g_1, g_2 : U \to \amalg S_n X$. The morphism $\operatorname{Hom}(U, \amalg S^n X) \to G(U)$ defined by f sends $g \mapsto g^* f$ so we are trying to show that $(g_1 + g_2)^* f = g_1^* f + g_2^* f$.

We can reduce to the case where U is connected. Since U is connected each of these morphisms have their image in one of the $S^i X$ so we are considering $g_1 : U \to S^i X$ and $g_2 : U \to S^j X$ and we have commutative diagrams:

$$\begin{array}{ccccc}
X^i, X^j & \xleftarrow{pr_1, pr_2} & X^i \times X^j & = & X^{i+j} \\
\downarrow & & \downarrow & & \downarrow \\
S^i X, S^j X & \xleftarrow{pr_1, pr_2} & S^i X \times S^j X & \xrightarrow{\alpha} & S^{i+j} X \\
& \nwarrow^{g_1, g_2} & \uparrow (g_1, g_2) & \nearrow^{g_1 + g_2} & \\
& & U & &
\end{array} \tag{3.8}$$

$$\begin{array}{ccccc}
G(X^i) \oplus G(X^j) & \xrightarrow{pr_1^* + pr_2^*} & G(X^i \times X^j) & = & G(X^{i+j}) \\
\uparrow & & \uparrow & & \uparrow \\
G(S^i X) \oplus G(S^j X) & \longrightarrow & G(S^i X \times S^j X) & \longleftarrow & G(S^{i+j} X) \\
& \searrow_{g_1^* + g_2^*} & \downarrow & \swarrow_{(g_1 + g_2)^*} & \\
& & G(U) & &
\end{array}$$

where α is the morphism induced by the monoid multiplication of $\amalg S^n X$. We are now trying to show that the image of $(f_i, f_j) \in G(S^i X) \oplus G(S^j X)$ in $G(U)$ is the same as the image of $f_{i+j} \in G(S^{i+j} X)$ in $G(U)$. Since the diagram commutes it is enough to show that the two corresponding elements of $G(S^i X \times S^j X)$ are the same.

To see this recall that we have

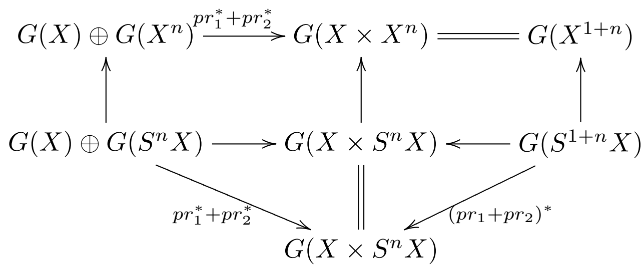
$$G(S^n X) \cong G(X^n)^{S_n}$$
$$G(S^i X \times S^j X) \cong G(X^i \times X^j)^{S_i \times S_j} \tag{3.9}$$

where S_n is the symmetric group on n elements and $S_i \times S_j$ the obvious subgroup. The result now follows from the definition of the f_n.

The last thing to check is uniqueness. We will work by induction. Suppose

$$f' = 1 \oplus f'_1 \oplus f'_2 \oplus \cdots \in \bigoplus_{n \geq 0} G(S^n X) \tag{3.10}$$

satisfies the required conditions. Since the restriction to $G(X)$ must be a we have that $f'_1 = a = f_1$. Now suppose that $f'_i = f_i$ for $i \leq n$ where $n \geq 1$. Consider the lower diagram of 3.8 with $U = S^i X \times S^j X$, $i = 1$ and $j = n$:



(3.11)

Since f' is a morphism of sheaves of abelian monoids, we have $(pr_1 + pr_2)^* f'_{1+n} = pr_1^* f'_1 + pr_2^* f'_n$. By the inductive hypothesis $f'_1 = f_1$ and $f'_n = f_n$ so since $f'_{1+n} \in G(S^{1+n} X)$ is uniquely determined by its image in $G(X^{1+n})$ and the diagram above commutes, we see that $f'_{1+n} = f_{1+n}$. So $f' = f$. □

Proposition 27 ([Voev, 3.3.7]). *Let Z be a closed subscheme of a scheme X and $p : Y \to X$ be a proper surjective morphism of finite type which is an isomorphism outside Z. Then the kernel of the morphism of qfh–sheaves*

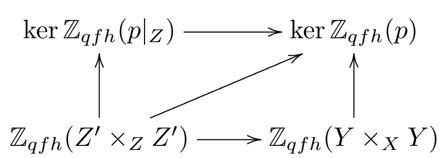
$$\mathbb{Z}_{qfh}(p) : \mathbb{Z}_{qfh}(Y) \to \mathbb{Z}_{qfh}(X) \tag{3.12}$$

is canonically isomorphic to the kernel of the morphism

$$\mathbb{Z}(p|_Z) : \mathbb{Z}_{qfh}(p^{-1}(Z)) \to \mathbb{Z}_{qfh}(Z) \tag{3.13}$$

Proof. The morphism ker $\mathbb{Z}_{qfh}(p|_Z) \to$ ker $\mathbb{Z}_{qfh}(p)$ is a monomorphism since $p^{-1}Z \to Y$ is so we need only show that it is an epimorphism. For ease of notation let $Z' = p^{-1}Z$.

Consider the diagram:



(3.14)

The two exact sequences from [Voev, 2.1.4] corresponding to the morphisms $Y \to X$ and $Z' \to Z$ show that the columns are epimorphisms. Now

$$\Delta \amalg i : Y \amalg Z' \times_Z Z' \to Y \times_X Y \tag{3.15}$$

is a qfh cover and so the morphism $L(\Delta \amalg i)$ is an epimorphism (c.f. [Voev, 3.2.5.2]) and since $\mathbb{Z}_{qfh}$ is right exact ([Voev, 2.1.2.1]) the morphism $\mathbb{Z}_{qfh}(\Delta \amalg i) = \mathbb{Z}_{qfh}(\Delta) \oplus \mathbb{Z}_{qfh}(i)$ is an epimorphism. Thus, its composition with the right column in the diagram is an epimorphism. However, the composition

$$\mathbb{Z}_{qfh}(Y) \stackrel{\mathbb{Z}_{qfh}\Delta}{\longrightarrow} \mathbb{Z}_{qfh}(Y \times_X Y) \longrightarrow \mathbb{Z}_{qfh}(Y) \tag{3.16}$$

is zero so the diagonal morphism in the diagram is a epimorphism. Hence, the top row is an epimorphism. □

Theorem 28 ([Voev, 3.3.8]). *Let X be a normal connected scheme and let $f : Y \to X$ be a finite surjective morphism of separable degree d. Then there is a morphism*

$$\mathrm{tr}(f) : \mathbb{Z}_{qfh}(X) \to \mathbb{Z}_{qfh}(Y) \tag{3.17}$$

such that $\mathbb{Z}_{qfh}(f) \circ \mathrm{tr}(f) = d\ \mathrm{Id}_{\mathbb{Z}_{qfh}(X)}$.

Proof. If Y is not the normalization of X in a finite extension of the field of functions on X, consider the field of functions $K(Y)$ of Y. Since $Y \to X$ is finite and surjective, $K(Y)$ is a finite extension of $K(X)$, the function field of X. Let Z be the normalization of X in $K(Y)$ so that we have maps $Z \stackrel{n}{\to} Y \stackrel{f}{\to} X$ where $f \circ n$ is the natural map $Z \to X$. If the theorem holds for $Z \to X$ then we can define $tr(f) = \mathbb{Z}_{qfh}(n) \circ tr(f \circ n)$ and so the theorem holds for $Y \to X$ as well. So we can assume that Y is the normalization of X in a finite extension of the field of functions on X.

Every finite field extension can be decomposed into a separable and a purely inseparable extension so there is a decomposition $f = f_0 \circ f_1$ where f_1 corresponds to a separable and f_0 a purely inseparable extension. By [Voev, 3.1.7] f_0 is a universal homeomorphism and so by [Voev, 3.2.5] $L_{qfh}(f_0)$ is an isomorphism. So we can assume that $f_0 = \mathrm{Id}$.

Let $\tilde{Y} \to X$ be the normalization of X in a Galois extension of $K(X)$ which contains $K(Y)$, let $G = Gal(\tilde{Y}/X)$ be the Galois group of $\tilde{Y}$ over X, let $H = Gal(\tilde{Y}/Y)$ be the subgroup corresponding to Y and consider the natural morphism $\tilde{f} : \tilde{Y} \to Y$. This morphism corresponds to a section in $\mathbb{Z}_{qfh}(Y)(\tilde{Y})$ and using it we can construct a section

$$a = \sum_{g \in G/H} g(\tilde{f}) \tag{3.18}$$

which is G–invariant. By [Voev, 3.3.2] (and [Voev, 3.3.3]) this corresponds to a section $a' \in \mathbb{Z}_{qfh}(Y)(X)$ which corresponds (via Yoneda) to a morphism $tr(f) : \mathbb{Z}_{qfh}(X) \to \mathbb{Z}_{qfh}(Y)$.

We will describe $tr(f)$ more explicitly to check that it satisfies the required property. A section of $\mathbb{Z}_{qfh}(X)(U)$ for some U is represented by a pair $(\{U_i \stackrel{p_i}{\to} U\}, \{\sum_j h_{ij}\})$ consisting of a qfh–cover $\{p_i : U_i \to U\}$ and a formal sum of morphisms $h_{ij} : U_i \to X$ for each element of the cover. Since $f \circ \tilde{f} : \tilde{Y} \to X$ is a qfh–cover of X, for each $h_{ij} : U_i \to X$, the pullback $\tilde{Y} \times_X U_i \to U_i$ is a cover of U_i and so the set of compositions $\tilde{Y} \times_X U_i \to U_i \to U$ is a cover of U, the point being that we can assume each h_{ij} fits into a commutative diagram:

$$\begin{array}{ccc} \tilde{Y} & \stackrel{\tilde{f}}{\longrightarrow} & Y \\ {\scriptstyle q_{ij}}\uparrow & & \downarrow{\scriptstyle f} \\ U_i & \stackrel{h_{ij}}{\longrightarrow} & X \end{array} \tag{3.19}$$

Each of the morphisms $h_{ij} : U_i \to X$ induces a morphism $\mathbb{Z}_{qfh}(Y)(h_{ij}) : \mathbb{Z}_{qfh}(Y)(X) \to \mathbb{Z}_{qfh}(Y)(U_i)$ and using these we obtain an element $\sum_j \mathbb{Z}_{qfh}(Y)(h_{ij})(a') \in \mathbb{Z}_{qfh}(Y)(U_i)$ for every U_i. Since the $\sum_j h_{ij} \in \mathbb{Z}_{qfh}(X)(U_i)$ agreed on the restrictions, the $\sum_j \mathbb{Z}_{qfh}(Y)(h_{ij})(a')$ agree on the restrictions and so we have a section:

$$\left(\{U_i \stackrel{p_i}{\to} U\}, \left\{\sum_j \mathbb{Z}_{qfh}(Y)(h_{ij})(a')\right\}\right) \in \mathbb{Z}_{qfh}(Y)(U) \tag{3.20}$$

which is the image of our original section $(\{U_i \stackrel{p_i}{\to} U\}, \{\sum_j h_{ij}\}) \in \mathbb{Z}_{qfh}(X)(U)$ under $tr(f)(U)$. Using the above factorization of h_{ij} through $\tilde{Y}$ and the fact that a' corresponds to $\sum g(\tilde{f})$ via $\mathbb{Z}_{qfh}(Y)(f)$

$$\begin{array}{ccc} \mathbb{Z}_{qfh}(Y)(\tilde{Y}) & & \\ {\scriptstyle \mathbb{Z}_{qfh}(Y)(q_{ij})}\downarrow & \nwarrow & \\ \mathbb{Z}_{qfh}(Y)(U_i) & \xleftarrow[\mathbb{Z}_{qfh}(Y)(h_{ij})]{} & \mathbb{Z}_{qfh}(Y)(X) \end{array} \qquad \begin{array}{ccc} \sum g(\tilde{f}) & & \\ \downarrow & \nwarrow & \\ \mathbb{Z}_{qfh}(Y)(h_{ij})(a') & \longleftarrow & a' \end{array} \tag{3.21}$$

we see that each $\mathbb{Z}_{qfh}(Y)(h_{ij})(a') \in \mathbb{Z}_{qfh}(Y)(U_i)$ can actually be written as $\sum g(\tilde{f}) \circ q_{ij}$. So we have

$$\begin{aligned} tr(f)(U) : \mathbb{Z}_{qfh}(X)(U) \quad &\to \quad \mathbb{Z}_{qfh}(Y)(U) \\ (\{U_i \stackrel{p_i}{\to} U\}, \{\sum_j h_{ij}\}) \quad &\mapsto \quad \left(\{U_i \stackrel{p_i}{\to} U\}, \left\{\sum_j \sum_{g \in G/H} g(\tilde{f}) \circ q_{ij}\right\}\right) \end{aligned} \tag{3.22}$$

Now pushing this back through $\mathbb{Z}_{qfh}(f)$ we obtain

$$\left(\{U_i \stackrel{p_i}{\to} U\}, \left\{\sum_j \sum_{g \in G/H} f \circ g(\tilde{f}) \circ q_{ij}\right\}\right) = \left(\{U_i \stackrel{p_i}{\to} U\}, \left\{\sum_j \sum_{g \in G/H} h_{ij}\right\}\right) = (\{U_i \stackrel{p_i}{\to} U\}, \{\sum_j dh_{ij}\}) \tag{3.23}$$

which is d times our original section. So the formula holds. □

Chapter 4

Comparison of h, qfh and to étale cohomologies

In this section we discuss the comparison results of Section 3.4 of [Voev]. These results compare the cohomology of various sheaves over the h, qfh and étale topologies. The results of this section highlight the connection between these three topologies. They are also used to calculate certain hom groups in the derived categories of motives $DM_h(S)$ and $DM_{qfh}(S)$ which turn out to be isomorphic to étale cohomology groups. Conversely, these étale cohomology groups can then be calculated from the related hom groups in $DM(S)$.

Some of these results are also used in [Voev2] to compare $DM_h(k)$ to $DM^{eff}_{-,et}(k)$, one of the categories constructed in that paper. The main result to that end is the following, which we will discuss in greater detail later.

Theorem 29 ([Voev2, 4.1.12]). *Let k be a field which admits resolution of singularities. Then the functor*

$$DM^{eff}_{-,et}(k) \otimes \mathbb{Q} \to DM_h(k) \otimes \mathbb{Q}$$

is an equivalence of triangulated categories.

The first comparison theorem we discuss relates the qfh and étale topologies.

Theorem 30 ([Voev, 3.4.1]). *Let X be a normal scheme and F be a qfh–sheaf of $\mathbb{Q}$–vector spaces, then one has*

$$H^i_{qfh}(X, F) = H^i_{et}(X, F)$$

Proof. Step 1: Reduction to showing $H^i_{qfh}(Spec(R), F) = 0$. Since every étale covering is also a qfh–covering the identity functor gives morphism of sites $\pi : X_{qfh} \to X_{et}$ (see [Mil] for material related to morphisms of sites). The Leray spectral sequence ([Mil, III.1.18]) of π is

$$H^p_{et}(X, R^q\pi_*F) \implies H^{p+q}_{qfh}(X, F) \tag{4.1}$$

where in this case $\pi_* : Sh(X_{qfh}) \to Sh(X_{et})$ is the inclusion functor. Since π_* is the inclusion functor, $R^q\pi_*F$ is the sheaf associated with $U \mapsto H^q_{qfh}(U, F)$. If we show that $U \mapsto H^q_{qfh}(U, F)$ is the zero functor for $q \geq 1$ then the spectral sequence will collapse and we will be left with the desired result. To show that $U \mapsto H^q_{qfh}(U,F)$ is zero it is enough to show that it is zero on stalks, so we can restrict our attention to the case where $U = Spec(R)$ for a normal strictly local ring R.

Step 2: Reduction to $H^1_{qfh}(Spec(R),F) = 0$. By embedding F in an injective sheaf I we get a short exact sequence $0 \to F \to I \to I/F \to 0$ of sheaves which gives a long exact sequence of cohomology groups

$$\begin{array}{l} \cdots \longrightarrow H^{i+1}_{qfh}(Spec(R),F) \longrightarrow H^{i+1}_{qfh}(Spec(R),I) \longrightarrow H^{i+1}_{qfh}(Spec(R),I/F) \\ \longrightarrow H^{i}_{qfh}(Spec(R),F) \longrightarrow H^{i}_{qfh}(Spec(R),I) \longrightarrow H^{i}_{qfh}(Spec(R),I/F) \\ \longrightarrow \cdots \qquad \cdots \longrightarrow H^{1}_{qfh}(Spec(R),I/F) \\ \longrightarrow H^{0}_{qfh}(Spec(R),F) \longrightarrow H^{0}_{qfh}(Spec(R),I) \longrightarrow H^{0}_{qfh}(Spec(R),I/F) \end{array} \tag{4.2}$$

The terms $H^i_{qfh}(Spec(R),I)$ are zero for $i > 0$ (since I is injective) and so $H^{i+1}_{qfh}(Spec(R),I/F) \cong H^i_{qfh}(Spec(R),F)$ for $i > 0$. Hence, if the first cohomology groups $H^1_{qfh}(U,F)$ are zero for an arbitrary sheaf (including I/F) then so are the higher cohomology groups $H^i_{qfh}(U,F)$.

Step 3: Reduction to the splitting of $\mathbb{Q}(\coprod U_i) \to \mathbb{Q}(Spec(R))$. Let $a \in H^1_{qfh}(Spec(R),F)$ be a cohomological class. Then there exists a qfh–covering $\{U_i \to Spec(R)\}$ and a Cech cocycle $a_{ij} \in \oplus F(U_i \times_{Spec(R)} U_j)$ which represents a. Let $U = \coprod U_i$. Over the site with base scheme $Spec(R)$, we have isomorphisms

$$\mathbb{Q}(U \times \cdots \times U) \cong \mathbb{Q}(U) \otimes \cdots \otimes \mathbb{Q}(U) \tag{4.3}$$

and $\mathbb{Q}(X) \otimes \mathbb{Q}(R) \cong \mathbb{Q}(X)$ for an arbitrary scheme X over R. If we have a splitting $\mathbb{Q}(R) \xrightarrow{s} \mathbb{Q}(U)$ of $\mathbb{Q}(U) \to \mathbb{Q}(R)$ we can construct a homotopy between the identity and zero of the complex

$$\ldots \longrightarrow \mathbb{Q}(U) \otimes \mathbb{Q}(U) \otimes \mathbb{Q}(U) \longrightarrow \mathbb{Q}(U) \otimes \mathbb{Q}(U) \longrightarrow \mathbb{Q}(U) \longrightarrow \mathbb{Q} \tag{4.4}$$

by defining

$$s_n = \mathrm{Id} \otimes s : \underbrace{\mathbb{Q}(U) \otimes \cdots \otimes \mathbb{Q}(U)}_{n \text{ times}} \otimes \mathbb{Q}(R) \to \underbrace{\mathbb{Q}(U) \otimes \cdots \otimes \mathbb{Q}(U)}_{n+1 \text{ times}} \tag{4.5}$$

Using the adjoint discussed in [Voev, 2.1.5] this becomes a homotopy between zero and the identity of the complex

$$\ldots \longrightarrow \mathbb{Q}(U \times_R U \times_R U) \longrightarrow \mathbb{Q}(U \times_R U) \longrightarrow \mathbb{Q}(U) \longrightarrow \mathbb{Q}(R) \tag{4.6}$$

and so any representative $a' \in \mathrm{Hom}(\mathbb{Q}(U \times_R U), F)$ of our cocycle $a \in H^1_{qfh}(Spec(R),F)$ is cohomologous to a coboundary. Hence, $a = 0$.

Final step. The result now follows from [Voev, 3.3.8] and [Voev, 3.4.2]. The lemma [Voev, 3.4.2] gives a finite surjective morphism $s : V \to X$ which factors through $\coprod U_i$ and [Voev, Theorem 3.3.8] provides a splitting of s (after composing with the automorphism $1/d$ where d is the separable degree of s). Hence $\mathbb{Q}_{qfh}(\amalg U_i) \to \mathbb{Q}_{qfh}(Spec(R))$ splits, so our 1–cocycle is a coboundary, so the first cohomology is zero, so all the cohomology groups are zero so the sequence degenerates so the isomorphism in the theorem holds. □

Lemma 31 ([Voev, 3.4.2]). *Let X be the spectrum of a strictly local (normal) ring and let $\{p_i : U_i \to X\}$ be a qfh–covering. Then there exists a finite surjective morphism $p : V \to X$ which factors through $\amalg U_i$.*

Proof. We can assume that $U_1 \to X$ is finite and the image of all the other U_i do not contain the closed point of X (because [Mil, I.4.2(c)] states that if A is Henselian and $f : Y \to Spec(A)$ is quasi–finite and separated then $Y = Y_0 \amalg \cdots \amalg Y_n$ where $f(Y_0)$ does not contain the closed point of $Spec(A)$ and $Y_i \to Spec(A)$ is finite for $i > 0$). Since X is a local ring, if we can show that $U_1 \to X$ is surjective then we have obtained the desired splitting. We do this by induction on the dimension of X.

Voevodsky claims that the result is obvious if $\dim X < 2$. Suppose that U_1 is surjective if $\dim X < n$ where $n > 1$. Let $x \in X$ be a point of dimension one and $\overline{x}$ its closure. Consider the base change:

$$\begin{array}{ccc} p_1^{-1}\overline{x} & \longrightarrow & U_1 \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle p_1} \\ \overline{x} & \longrightarrow & X \end{array} \tag{4.7}$$

Since U_1 is a component of a universal topological epimorphism q_1 is a part of a topological epimorphism. So q_1 is surjective and x is in the image of U_1. So the image of U_1 contains all points of dimension one. Voevodsky claims that since its image is closed this implies that it is surjective. □

Lemma 32 ([Voev, 3.4.3]). *Let k be a separably closed field. Then for any qfh–sheaf of abelian groups F and any $i > 0$ one has*

$$H^i_{qfh}(Spec(k), F) = 0 \tag{4.8}$$

Proof. Since k is separable closed, the site over base scheme $Spec(k)$ is trivial. Hence, all the nonzero cohomology groups vanish. □

Theorem 33 ([Voev, 3.4.4]). *Let X be a scheme and F be a locally constant in the étale topology sheaf on Sch/X, then F is a qfh–sheaf and one has*

$$H^i_{qfh}(X, F) = H^i_{et}(X, F) \tag{4.9}$$

Proof. The fact that F is a qfh–sheaf is obvious. Following the same line of reasoning as the first step of the proof to [Voev, 3.4.1] above, to prove the comparison statement it is sufficient to show that if X is a strictly henselian scheme then $H^q_{qfh}(X, F) = 0$ for $q > 0$. Denote $Finite(X)$ the site whose objects are schemes finite over X and coverings are surjective families of morphisms. There is a morphism of sites

$$\gamma : (Sch/X)_{qfh} \to Finite(X) \tag{4.10}$$

induced by the inclusion functor. Lemma 3.4.2 says that every qfh–cover has a $Finite(X)$ refinement and so by [Mil, Proposition III.3.3] the morphism of sites

$$\gamma : (Sch/X)_{qfh} \to Finite(X) \tag{4.11}$$

induces isomorphisms

$$H^i_{finite}(X, \gamma_*(F)) = H^i_{qfh}(X, F) \tag{4.12}$$

so it is sufficient to show that $H^i_{finite}(X, \gamma_*(F)) = 0$ for $i > 0$.

Let $x : Spec(k) \to X$ be the closed point of X. For any finite morphism $Y \to X$ the scheme Y is a disjoint union of strictly henselian schemes (if A is a henselian ring and B a finite A–algebra then $B = \prod_{i=1}^n B_{\mathfrak{m}_i}$ by [Mil, I.4.2.b] and each B_{m_i} is henselian by [Mil, I.4.3], furthermore if A is strictly henselian then the residue field $A/\mathfrak{m}$ is separably algebraically closed and so each $B_{\mathfrak{m}_i}/\mathfrak{m}_i B_{\mathfrak{m}_i}$ is a finite extension of a separably algebraically closed field and therefore, also separably algebraically closed). It follows from this that the number of connected components of Y coincides with the number of connected components of the fiber $Y_x \to Spec(k)$. So the canonical morphism

$$\gamma_*(F) \to x_*(\gamma_*(F)) \tag{4.13}$$

of sheaves on the finite sites is an isomorphism. Lemma 3.4.3 states $H^i_{qfh}(Spec(k), F) = 0$ and so combining these results we obtain

$$\begin{aligned} H^i_{qfh}(X, \gamma_*(F)) &= H^i_{finite}(X, \gamma_*(F)) && \text{[Voev, 3.4.2], [Mil, III.3.3]} \\ &= H^i_{finite}(Spec(k), \gamma_*(F)) && \\ &= H^i_{qfh}(Spec(k), F) && \text{[Voev, 3.4.2], [Mil, III.3.3]} \\ &= 0 && \text{[Voev, 3.4.3]} \end{aligned} \tag{4.14}$$

for all $i > 0$. Thus the theorem is proved. □

Theorem 34 ([Voev, 3.4.5]). *Let X be a scheme and F be a locally constant torsion sheaf in étale topology on Sch/X. Then F is an h–sheaf and for any $i \geq 0$ one has a canonical isomorphism*

$$H^i_h(X, F) = H^i_{et}(X, F) \tag{4.15}$$

Proof. Rather than give a proof, [Voev] cites [SV] where the proof is found in the appendix on h–cohomology. The appendix from [SV] begins with the evident site morphisms

$$(Sch/S)_h \xrightarrow{\alpha} (Sch/S)_{qfh} \xrightarrow{\beta} (Sch/S)_{et} \tag{4.16}$$

where S is noetherian and Sch/S is the category of schemes of finite type over S. Towards the end of [SV, 10.7] it is proven that if S is excellent then $(\beta\alpha)_*(\mathbb{Z}/n) = \mathbb{Z}/n$ and $R^q(\beta\alpha)_*(\mathbb{Z}/n) = 0$ for $q, n > 0$ with the corollaries

$$\begin{aligned} \alpha_*(\mathbb{Z}/n) &= \mathbb{Z}/n \\ R^q\alpha_*(\mathbb{Z}/n) &= 0 \quad \text{for} \quad q > 0 \end{aligned} \tag{[SV, 10.9]}$$

and

$$\begin{aligned} Ext^*_{et}(F, \mathbb{Z}/n) &= Ext^*_{qfh}(\beta^* F, \mathbb{Z}/n) \\ Ext^*_{qfh}(G, \mathbb{Z}/n) &= Ext^*_h(\alpha^* G, \mathbb{Z}/n) \end{aligned} \tag{[SV, 10.10]}$$

where F is an étale sheaf and G is a qfh–sheaf. Applying these last isomorphisms with F and G the free abelian sheaves represented by a scheme X and using the isomorphism

$$Ext^*(\mathbb{Z}(X), F) = H^*(X, F) \tag{4.17}$$

we obtain the desired result. □

Voevodsky remarks in [Voev] that this result

> ... is false for sheaves which are not torsion sheaves, but it can be shown that it is still valid for arbitrary locally constant sheaves if X is a smooth scheme of finite type over a field of characteristic zero (we need this condition only to be able to use the resolution of singularities.

Theorem 35 ([Voev, 3.4.6]). *Let X be a scheme of (absolute) dimension N, then for any h–sheaf of abelian groups and any $i > N$ one has:*

$$H^i_h(X, F) \otimes \mathbb{Q} = 0 \tag{4.18}$$

To prove this, Voevodsky first proves a similar result for the étale topology:

Lemma 36 ([Voev, 3.4.7]). *Let X be a scheme of absolute dimension N, then for any étale sheaf of abelian groups F and any $i > N$ one has:*

$$H^i_{et}(X, F) \otimes \mathbb{Q} = 0 \tag{4.19}$$

Proof. The proof follows the same lines as the proof to [Mil, VI.1.1]. It moves by induction on N. If $N = 0$ then the statement is obvious. Let $x_1, \dots, x_k$ be the set of generic points of the irreducible components of X and let $in_j : Spec(K_j) \to X$ be the corresponding inclusions. There is a natural morphism of sheaves on X_{et}:

$$F \to \oplus_{j=1}^k (in_j)_*(in_j)^*(F) \tag{4.20}$$

The kernel and cokernel of this morphism have support in codimension at least one and so their cohomology vanishes in dimensions greater than $N-1$ by the inductive hypothesis. So we have reduced the problem to considering sheaves of the form $(in_j)_*G$. At this point Voevodsky leaves the reader to complete the proof suggesting only that they should use the Leray spectral sequence of the inclusions in_j. Milnor completes all the details and these can be applied directly to this situation although they will not be reproduced. He proves that $R^q(in_j)_*F$ has support in dimension $\leq N-q$ [Mil, VI.1.2] and then uses the Leray spectral sequence for each in_j

$$H^p_{et}(X, R^q(in_j)_*F) \implies H^{p+q}_{et}(Spec(K_j), F) \tag{4.21}$$

By the inductive hypothesis $H^p_{et}(X, R^q(in_j)_*F) = 0$ for $q > 0, p > N$ and so the Leray spectral sequence gives isomorphisms $H^i_{et}(X, (in_j)_*F) \cong H^i_{et}(Spec(K_j), F)$ for $i > N$ and the latter group is zero. ☐

Proof of [Voev, 3.4.6]. With this lemma it now follows from [Voev, 3.4.1] that if X is a normal scheme of dimension N then for $i > N$ we have $H^i_{qfh}(X, F) = 0$.

Using the spectral sequence connecting the Cech and usual cohomology we reduce to the Cech cohomology groups $\check{H}^i_h(X, F) \otimes \mathbb{Q}$. Consider a cohomology class $a \in \check{H}^i_h(X, F)$ and an h–cover of normal form $\{U_i \to \overline{U} \to X_Z \to X\}$ which realises it. We can assume that X_Z is normal be passing to a refinement. Now since $\{U_i \to \overline{U} \to X_Z\}$ is a qfh–covering the restriction of a to X_Z is zero by the above result for the qfh–topology. Now we have two long exact sequences, one from [Voev, 3.3.7]

$$\dots \to Ext^{i-1}(G, F) \to H^i_h(X, F) \to H^i_h(X_Z, F) \to Ext^i(G, F) \to \dots \tag{4.22}$$

and the other from [Voev, 2.1.3]

$$\cdots \to Ext^{i-1}(G,F) \to H^i_h(Z,F) \to H^i_h(PN_Z,F) \to Ext^i(G,F) \to \dots \tag{4.23}$$

Since the dimension of Z and PN_Z are smaller than X we use the inductive hypothesis to obtain $H^i_h(Z,F) = H^i_h(PN_Z,F) = 0$ for all $i > N-1$. From exactness of the sequence this implies $Ext^i(G,F) = 0$ for all $i > N-1$ and so in the first exact sequence $H^i_h(X,F) \cong H^i_h(X_Z,F)$ for all $i > N$. We have shown however that $H_h(X_Z,F) = 0$ and so $H^i_h(X,F) = 0$. □

Corollary 37 ([Voev, 3.4.8]). *Let X be a scheme of absolute dimension N. Then for any qfh–sheaf of abelian groups F on Sch/X and any $i > N$ one has*

$$H^i_{qfh}(X,F) = 0 \tag{4.24}$$

Chapter 5

The categories $DM(S)$ and basic properties

In this section we discuss some of the results from Section 4.1 of [Voev]. The forth section of [Voev] is where all of the main results of the paper are contained and apart from the definition of the homological category of a site with interval, everything preceding this section is, in a way, supporting lemmas and definitions for the results in this section.

Voevodsky begins by defining the site with interval to be used, defining a cosimplicial object that is isomorphic to a_{I^+} (the cosimplicial object used in the general case presented in [Voev, Section 2.2]) and then defining the categories $DM_h(S)$ and $DM_{qfh}(S)$. He then goes on to prove various properties of these categories that follow either directly from the definitions or from some results established earlier. There is also mention at the end of [Voev, Section 4.1] of some results that come from [SV].

We begin with the site with interval.

Definition 38. Let Sch/S denote the category of schemes over a base S as a site with either the h or qfh topology. Set $I^+ = \mathbb{A}^1_S$ and let μ, i_0, i_1 be the multiplication morphism ($\mathbb{Z}[x] \to \mathbb{Z}[x]\otimes\mathbb{Z}[x]$, defined by $x \mapsto x\otimes x$) and the points 0 and 1 ($\mathbb{Z}[x] \to \mathbb{Z}$, $x \mapsto 0, 1$) respectively. Then $(Sch/S, \mathbb{A}^1_S)$ with the morphisms (μ, i_0, i_1) satisfy the necessary conditions to be a site with interval.

In this category with interval, there is a cosimplicial object which is easier to work with then the cosimplicial object a_{I^+} defined in [Voev, Section 2.2]. It is constructed in a way which closely resembles the "usual" simplicial object in singular homology.

Definition 39. Let Δ^n_S denote the scheme

$$\Delta^n_S = S \times_S Spec\left(\mathbb{Z}[x_0, \ldots, x_n]/\left(\sum_{i=0}^{n} x_i = 1\right)\right) \tag{5.1}$$

For any morphism $f : [n] \to [m]$ in the standard simplicial category Δ (that is, any non–decreasing set morphism $\{0, \ldots, n\} \to \{0, \ldots, m\}$) we construct a morphism $a'(f) : \Delta^n_S \to \Delta^m_S$ via the ring homomorphisms

$$\begin{aligned} \mathbb{Z}[x_0, \ldots, x_m] &\to \mathbb{Z}[x_0, \ldots, x_n] \\ x_i &\mapsto \begin{cases} \sum_{j \in f^{-1}(i)} x_j & \text{if } f^{-1}(i) \neq \varnothing \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{5.2}$$

In this way we obtain a cosimplicial object $a' : \Delta \to \mathrm{Aff}/S$.

Proposition 40 ([Voev, 4.1.1]). *The cosimplicial object a' is isomorphic to the cosimplicial object a_{I^+} of the site with interval $((Aff/S), \mathbb{A}^1_S)$.*

The proof is omitted as it is uninsightful.

We now come to the definition of $DM(S)$ and some immediate consequences of the definitions. Recall that Sch/S is being used to denote the category of separated schemes of finite type over a noetherian excellent scheme S.

Definition 41. Denote by $DM_h(S)$ (resp. $DM_{qfh}(S)$) the homological category of the site with interval $((Sch/S)_h, \mathbb{A}^1_S)$ (resp. $((Sch/S)_{qfh}, \mathbb{A}^1_S)$) and let $M_h, \tilde{M}_h : Sch/S \to DM_h(S)$ (resp. $M_{qfh}, \tilde{M}_{qfh}$) be the corresponding functors.

The categories $DM_h(S)$ and $DM_{qfh}(S)$ are denoted $DM(S)$ whenever a result holds for both topologies. More explicitly, the categories $DM(S)$ are the target of the sequence of functors:

$$\begin{array}{c} Sch/S \\ \Big\downarrow {\scriptstyle X \mapsto \text{the sheaf represented by } X} \\ Sets(Sch/S) \\ \Big\downarrow {\scriptstyle F \mapsto \text{the free abelian group sheaf generated by } F} \\ Ab(Sch/S) \\ \Big\downarrow {\scriptstyle \text{the usual embedding of an abelian category into its derived category}} \\ D(Sch/S) \\ \Big\downarrow {\scriptstyle \text{the usual projection}} \\ D(Sch/S)/Contr(Sch/S) \end{array} \tag{5.3}$$

Sheaves of abelian groups on Sch/S are identified with the corresponding object in $DM(S)$ and schemes with their corresponding representable sheaves of sets.

It follows immediately from the construction that the categories $DM(S)$ are tensor triangulated categories. Furthermore, for any morphism of schemes $S_1 \to S_2$ there is an exact, tensor functor $f^* : DM(S_2) \to DM(S_1)$ such that for a scheme X over S_2 it holds that $f^*(M(X)) = M(X \times_{S_2} S_1)$. The properties of $\mathbb{Z}(-)$ (cf. [Voev, 2.1.2]) imply that for any schemes X, Y over S it holds that

$$M(X \amalg Y) = M(X) \oplus M(Y) \tag{5.4}$$

$$M(X \times_S Y) = M(X) \otimes M(Y) \tag{5.5}$$

The rest of this section involves standard results of cohomological theories as well as a couple of results about the hom sets in the categories $DM(S)$.

Mayer–Vietoris ([Voev, 4.1.2]). *Let $X = U \cup V$ be an open or closed covering of X. Then there is a natural exact triangle in $DM(S)$ of the form*

$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1] \tag{5.6}$$

Proof. We will prove that the cone of $M(U \cap V) \to M(U) \oplus M(V)$ is isomorphic to $M(X)$ by showing that they are both isomorphic to a sequence coming from [Voev, 2.1.4].

Relabel U and V to U_1 and U_2, denote the intersection $\cap_{j=1}^{k} U_{i_j}$ where $i_j = 1$ or 2 by $U_{i_1 \dots i_k}$ and let $Y = U_1 \coprod U_2$. Consider the morphism $f : Y \to X$. This corresponds to a morphism in $Sets(Sch/S)$ and from this we obtain the sequence

$$\dots \longrightarrow \mathbb{Z}(Y \times_X Y) \stackrel{\mathbb{Z}(pr_1)-\mathbb{Z}(pr_2)}{\longrightarrow} \mathbb{Z}(Y) \stackrel{\mathbb{Z}(f)}{\longrightarrow} \mathbb{Z}(X) \longrightarrow 0 \tag{5.7}$$

which [Voev, 2.1.4] says is a resolution for *coker* $\mathbb{Z}(f)$. Since $Y \to X$ is a covering, this cokernel is zero and so the sequence is exact. Then by dropping the $\mathbb{Z}(X)$ term we obtain a sequence

$$\dots \longrightarrow \mathbb{Z}(Y \times_X Y \times_X Y) \longrightarrow \mathbb{Z}(Y \times_X Y) \stackrel{\mathbb{Z}(pr_1)-\mathbb{Z}(pr_2)}{\longrightarrow} \mathbb{Z}(Y) \longrightarrow 0 \tag{5.8}$$

which is quasi–isomorphic to $\mathbb{Z}(X) = coker\ \mathbb{Z}(pr_1) - \mathbb{Z}(pr_2)$ considered as a sequence concentrated in degree zero.

To obtain the other quasi–isomorphism, notice that $\underbrace{Y \times_X \cdots \times_X Y}_{n \text{ times}} = \coprod_{\mathbf{i} \in \{1,2\}^n} U_{\mathbf{i}}$ and rewrite sequence 5.8 as

$$\dots \longrightarrow \bigoplus_{\mathbf{i} \in \{1,2\}^3} \mathbb{Z}(U_{\mathbf{i}}) \longrightarrow \bigoplus_{\mathbf{i} \in \{1,2\}^2} \mathbb{Z}(U_{\mathbf{i}}) \longrightarrow U_1 \oplus U_2 \to 0 \tag{5.9}$$

Each of the morphisms $\bigoplus_{\mathbf{i} \in \{1,2\}^{n+1}} \mathbb{Z}(U_{\mathbf{i}}) \longrightarrow \bigoplus_{\mathbf{i} \in \{1,2\}^n} \mathbb{Z}(U_{\mathbf{i}})$ is defined termwise by the alternating sum

$$\sum_{j=1}^{n+1} (-1)^j \sigma_j : \mathbb{Z}(U_{i_1 \dots i_{n+1}}) \to \bigoplus_{\mathbf{i} \in \{1,2\}^n} \mathbb{Z}(U_{\mathbf{i}}) \tag{5.10}$$

where σ_j are the morphisms induced by the inclusions

$$U_{i_1 \dots i_{n+1}} \to U_{i_1 \dots i_{j-1} i_{j+1} \dots i_{n+1}} \tag{5.11}$$

We want to define a quasi–isomorphism

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \bigoplus_{\mathbf{i} \in \{1,2\}^3} \mathbb{Z}(U_{\mathbf{i}}) & \longrightarrow & \bigoplus_{\mathbf{i} \in \{1,2\}^2} \mathbb{Z}(U_{\mathbf{i}}) & \longrightarrow & \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}(U_{12}) & \xrightarrow{-\sigma_1 + \sigma_2} & \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) & \longrightarrow & 0
\end{array} \tag{5.12}$$

Choose the right vertical morphism to be the identity and the middle vertical morphism to be the sum of

$$\begin{aligned}
0 &: \mathbb{Z}(U_{11}) \to \mathbb{Z}(U_{12}) \\
id &: \mathbb{Z}(U_{12}) \to \mathbb{Z}(U_{12}) \\
-id &: \mathbb{Z}(U_{21}) \to \mathbb{Z}(U_{12}) \\
0 &: \mathbb{Z}(U_{22}) \to \mathbb{Z}(U_{12})
\end{aligned} \tag{5.13}$$

It can be verified (with some patience) that this is indeed a morphism of chain complexes. Since the top complex is exact everywhere except at the last term we see that this morphism induces isomorphisms on homology for every term in degree higher than one. Since the morphisms $U_{12} \to U_1, U_2$ are inclusions, the corresponding sheaves are injective and $\mathbb{Z}(U_{12}) \to \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2)$ is injective so the complex morphism induces an isomorphism on the homology in degree one. Now we consider the morphism $\bigoplus_{\mathbf{i}\in\{1,2\}^2} \mathbb{Z}(U_{\mathbf{i}}) \to \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2)$. On each term this morphism is

$$\begin{aligned} 0 : \mathbb{Z}(U_{11}) &\to \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) \\ -\sigma_1 + \sigma_2 : \mathbb{Z}(U_{12}) &\to \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) \\ \sigma_1 - \sigma_2 : \mathbb{Z}(U_{21}) &\to \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) \\ 0 : \mathbb{Z}(U_{22}) &\to \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) \end{aligned} \tag{5.14}$$

and so its image is the same as the image of $\mathbb{Z}(U_{12}) \to \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2)$. Hence, the complex morphism induces isomorphisms on homology in degree zero. So the complex morphism is a quasi–isomorphism.

Hence the cone of $M(U_{12}) \to M(U_1) \oplus M(U_2)$ is isomorphic to $M(X)$ and therefore $M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1]$ is an exact triangle in $DM(S)$. □

Homotopy invariance. *Let X be a scheme in Sch/S. Then for $n \geq 0$*

$$M(X \times_S \mathbb{A}^n_S) \cong M(X) \tag{5.15}$$

Proof. This is a specific case of the more general statement that $\mathbb{Z}(X) \otimes \mathbb{Z}(I^+)^n$ becomes isomorphic to $\mathbb{Z}(X)$ in the homological category of any site with interval where X is an object of the underlying category T. To see this consider the exact sequence

$$0 \to J \to \mathbb{Z}(I^+)^n \to \mathbb{Z} \to 0 \tag{5.16}$$

where J is the kernel of $\mathbb{Z}(I^+)^n \to \mathbb{Z}$. By [Voev, 2.2.3] the sheaf J is strictly contractible so the second morphism becomes an isomorphism in the homological category. By [Voev, 2.1.2] the sheaf $\mathbb{Z}(X)$ is flat and so tensoring the exact sequence with $\mathbb{Z}(X)$ leaves it exact. So $\mathbb{Z}(X) \otimes J$ is the kernel of $\mathbb{Z}(X) \otimes \mathbb{Z}(I^+)^n \to \mathbb{Z}(X)$ and $\mathbb{Z}(X) \otimes J$ becomes zero in the homological category. Hence, $\mathbb{Z}(X) \otimes \mathbb{Z}(I^+)^n \to \mathbb{Z}(X)$ becomes an isomorphism. □

Proposition 42 ([Voev, 4.1.3]). *Let $p : Y \to X$ be a locally trivial (in Zariski topology) fibration whose fibers are affine spaces. Then the morphism $M(p) : M(Y) \to M(X)$ is an isomorphism.*

Proof. This can be proven by induction on the size of the trivializing cover of X. If $Y = X \times \mathbb{A}^1$ then the result is a corollary of the previous result on homotopy invariance. Suppose that $\cup_{i=1}^n U_i = X$ is a trivializing cover and set $U = \cup_{i=1}^{n-1}$ and $V = U_n$. There is a morphism of exact triangles:

$$\begin{array}{ccccccc} M(p^{-1}(U \cap V)) & \to & M(p^{-1}U) \oplus M(p^{-1}V) & \to & M(Y) & \to & M(p^{-1}(U \cap V))[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M(U \cap V) & \longrightarrow & M(U) \oplus M(V) & \longrightarrow & M(X) & \longrightarrow & M(U \cap V)[1] \end{array} \tag{5.17}$$

The first two horizontal morphism are isomorphism by the trivialization and induction respectively and so the third horizontal morphism is also an isomorphism. □

Proposition 43 ([Voev, 4.1.4]). *Let $f : Y \to X$ be a finite surjective morphism of normal connected schemes of the separable degree d. Then there is a morphism $tr(f) : M(X) \to M(Y)$ such that $M(f)tr(f) = d\ id_{M(X)}$.*

Proof. It follows directly from [Voev, 3.3.8] □

Blow–up distinguished triangle ([Voev, 4.1.5]). *Let Z be a closed subscheme of a scheme X and $p : Y \to X$ a proper surjective morphism of finite type which is an isomorphism outside Z. Then there is an exact triangle in $DM_h(S)$ of the form*

$$M_h(X)[1] \to M_h(p^{-1}(Z)) \to M_h(Z) \oplus M_h(Y) \to M_h(X) \tag{5.18}$$

Proof. The proof runs along exactly the same lines as the proof to [Voev, 4.1.2] above except with the use of the morphism $Z \amalg Y \to X$. Recall that we used the fact that $U_1 \coprod U_2 \to X$ was a covering to prove the quasi–isomorphism between the sequence 5.8 and $\mathbb{Z}(X)$ and so we need the fact that $Z \amalg Y \to X$ is a h–covering. □

Since $Z \amalg Y \to X$ is not necessarily a qfh–covering the above proposition does not necessarily hold for the qfh–topology.

We now come to some results about the hom groups in $DM(S)$. First, we describe the relationship between cohomology groups and certain hom groups in $DM(S)$.

Proposition 44 ([Voev, 4.1.6]). *Let F be a locally free in étale topology sheaf of torsion prime to the characteristic of S. Then for any scheme X one has a natural isomorphism*

$$DM(M(X), F[i]) = H^n_{et}(X, F) \tag{5.19}$$

Proof. We can use [Voev, 2.2.9] (which says $\text{Hom}_{H(T)}(X, Y) = \text{Hom}_{D(T)}(X, Y)$ for a site with interval T) to reduce to the derived category of sheaves of abelian groups. On the right hand side we can use [Voev, 3.4.4] and [Voev, 3.4.5] to change the right hand side to the h or qfh topology. The result then follows from the homological algebra result that $H^n(X, F) \cong \text{Hom}_{D(Sch/S)}(\mathbb{Z}(X), F[n])$. More explicitly, we have a sequence of isomorphisms:

$$H^n_{et}(X, F) \tilde{\to} H^n(X, F) \tilde{\to} \text{Hom}_{D(Sch/S)}(\mathbb{Z}(X), F[n]) \tilde{\to} DM(M(X), F[n]) \tag{5.20}$$

□

Proposition 45 ([Voev, 4.1.7]). *Let S be a scheme of characteristic $p > 0$. Then the category $DM(S)$ is $\mathbb{Z}[1/p]$ linear.*

Proof. Since we have a canonical exact triangle $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to \mathbb{Z}[1]$ if we show that $\mathbb{Z}/p$ is zero then it implies that the morphism $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ of multiplication by p is an isomorphism. Multiplication of a morphism $f : X \to Y$ by p is the same as tensoring f with $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ and so showing that $\mathbb{Z}/p$ is zero will prove the result.

Consider the Artin–Shrier exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{G}_a \xrightarrow{F-1} \mathbb{G}_a \to 0 \tag{5.21}$$

where $\mathbb{G}_a$ is the sheaf of abelian groups represented by $\mathbb{A}^1$ and F is the geometrical Frobenius morphism (that is, F is the morphism corresponding to the ring morphism $(a \mapsto a^{1/p}) \otimes id_{\mathbb{Z}[t]}$: $k \otimes \mathbb{Z}[t] \to k \otimes \mathbb{Z}[t]$). Since $\mathbb{G}_a$ is strictly contractible it is isomorphic to zero in $DM(S)$ and so we obtain an exact triangle $\mathbb{Z}/p \to 0 \to 0 \to \mathbb{Z}/p[1]$ which implies that $\mathbb{Z}/p$ is zero. $\square$

Chapter 6

“Singular homology of abstract algebraic varieties”

The last two results of Section 4.1 of [Voev] follow from [SV] and so we give a brief description of this paper before stating and proving these results.

In [SV] Suslin and Voevodsky use the qfh–topology to resolve a conjecture relating the (topological) singular homology of a variety X over $\mathbb{C}$ to $H_i^{sing}(X)$, groups defined for any scheme of finite type over a field k in a way that emulates the topological singular homology. They also compare the related cohomology groups $H^i_{sing}(X, \mathbb{Z}/n)$ to the qfh and étale cohomology.

6.1 Dold–Thom and singular homology of schemes.

The paper begins with a theorem of Dold and Thom. This theorem says that the topological singular homology groups of a CW–complex X coincide with the homotopy groups of

$$\mathrm{Hom}_{top}\Big(\Delta^{\bullet}_{top}, \coprod_{d=0}^{\infty} S^d X\Big)^+ \tag{6.1}$$

where $S^d(X)$ is the dth symmetric power of X, Δ^i_{top} are the usual topological simplicies and the $+$ denotes the group completion of an abelian monoid. This notation–laden statement is actually quite natural.

Consider an “effective” chain in one of the groups $C_i = \mathbb{Z}\,\mathrm{Hom}_{top}(\Delta^i_{top}, X)$ used to define the topological singular homology. That is, a formal finite sum $\sum_{j=1}^n a_j c_j$ of maps $c_j : \Delta^i_{top} \to X$ where all the coefficients $a_j \in \mathbb{Z}$ are positive. If we allow duplicates we can write this sum as $\sum_{k=1}^N c_k$ where $N = \sum a_j$ and from this we obtain a tuple $c = (c_1, \ldots, c_N)$ which actually gives a map $c : \Delta^i_{top} \to X^N$ (which depends on the order chosen for the c_k). We compose c with the projection $X^N \to S^N X$ and since the order of the c_k doesn't matter now, we have defined a map from all effective cycles such that $\sum a_j = N$ to $\mathrm{Hom}_{top}(\Delta^i_{top}, S^N X)$.

Now if we move from $\mathrm{Hom}_{top}(\Delta^i_{top}, S^N X)$ to $\mathrm{Hom}_{top}(\Delta^i_{top}, \coprod_{d=0}^{\infty} S^d X)$ we can extend this map to all elements of $\mathbb{N}\,\mathrm{Hom}_{top}(\Delta^i_{top}, X)$. It can be seen that this is a monoid homomorphism and hence passes to a group homomorphism

$$\mathbb{Z}\,\mathrm{Hom}_{top}(\Delta^i_{top}, X) \to \mathrm{Hom}_{top}(\Delta^i_{top}, \coprod_{d=0}^{\infty} S^d X)^+ \tag{6.2}$$

This description of the topological singular homology groups is much easier to emulate in the algebraic setting than the usual definition that uses formal sums of maps from Δ^i_{top} to X. Take the simplicies Δ^i as defined in Definition 39 of the previous section (with $S = Spec\ k$) and we can immediately define:

Definition 46. The groups $H_i^{sing}(X)$ for a scheme of finite type over k are the homotopy groups

$$H_i^{sing}(X) = \pi_i\left(\mathrm{Hom}\left(\Delta^\bullet, \coprod_{d=0}^{\infty} S^d X\right)^+\right) \tag{6.3}$$

The conjecture that is resolved in [SV] is the following:

Theorem 47. *If X is a variety over $\mathbb{C}$ then the evident homomorphism*

$$Hom\left(\Delta^\bullet, \coprod_{d=0}^{\infty} S^d(X)\right)^+ \to Hom_{top}\left(\Delta^\bullet_{top}, \coprod_{d=0}^{\infty} S^d(X)\right)^+ \tag{6.4}$$

induces isomorphisms

$$H_i^{sing}(X, \mathbb{Z}/n) \cong H_i(X(\mathbb{C}), \mathbb{Z}/n) \tag{6.5}$$

6.2 Transfer maps and the rigidity theorem.

One of the main tools in resolving the above conjecture is a rigidity theorem [SV, 4.4] which is a general version of a rigidity theorem of Suslin, Gabber, Gillet and Thomason. To state it we need the concept of transfer maps.

Definition 48. A presheaf $\mathcal{F}$ is said to admit transfer maps if for any finite surjective morphism $p : X \to S$ in Sch/k, where X is reduced and irreducible and S is irreducible and regular we are given a homomorphism:

$$Tr_{X/S} : \mathcal{F}(X) \to \mathcal{F}(S) \tag{6.6}$$

such that various conditions hold.

The conditions will not be stated explicitly here but instead an interpretation of them will be given. In many of the constructions of categories of motives the first step is to add extra morphisms to the category Sm/k of smooth schemes over a field k. The new hom sets are composed of closed irreducible subschemes of $X \times Y$ such that the projection $X \times Y \to X$ is finite and surjective. These can be intuitively thought of as multivalued functions from X to Y. The original hom sets are contained inside the new ones by taking the graph $\Gamma_f \subset X \times Y$ of any morphism $f : X \to Y$.

Now given a presheaf on Sm/k we can ask if it extends to a presheaf on the category of Sm/k with the added morphisms. That is, for every closed irreducible subscheme of $X \times Y$ that is finite and surjective over X we need a morphism $\mathcal{F}(Y) \to \mathcal{F}(X)$ and these morphisms need to compose in a certain way. This is a rough way to interpret the conditions in the definition of a presheaf that admits transfer maps.

We need two more definitions to state the rigidity theorem.

Definition 49. A presheaf $\mathcal{F}$ on Sch/k is homotopy invariant if $\mathcal{F}(X) = \mathcal{F}(X \times \mathbb{A}^1)$ for all $X \in Sch/k$ and it is called n–torsion if $n\mathcal{F}(X) = 0$ for all $X \in Sch/k$ where n is prime to the exponential characteristic of k.

Now we can state the rigidity theorem. We actually use the statement [Lev, Theorem 8.2] as it is easier to remove from its context (the version in [SV] has the preconditions scattered throughout the section and actually uses the henselization of affine space at the origin instead of the henselization of a k–variety at a smooth point).

Theorem 50 ([SV, 4.4], [Lev, 8.2]). *Let $\mathcal{F}$ be a presheaf on Sch/k which*

1. *is homotopy invariant,*
2. *has transfers, and*
3. *is n–torsion.*

Let x be a smooth point on a k–variety X, let X^h_x be the henselization of X at x and let i_x : Spec $k \to X^h_x$ be the inclusion. Then

$$i_0^* : \mathcal{F}(X^h_x) \to \mathcal{F}(Spec\ k) \tag{6.7}$$

is an isomorphism.

The rigidity theorem is used to prove:

Theorem 51 ([SV, 4.5]). *Assume k is an algebraically closed field of characteristic zero. Assume further that $\mathcal{F}$ is a homotopy invariant presheaf on Sch/k equipped with transfer maps. Denote $\mathcal{F}^\sim_h$ (resp. $\mathcal{F}^\sim_{qfh}, \mathcal{F}^\sim_{et}$) the sheaf associated with $\mathcal{F}$ in the h–topology (resp. qfh-topology, étale topology). Then for any $n > 0$ there are canonical isomorphisms:*

$$Ext^*_{et}(\mathcal{F}^\sim_{et}, \mathbb{Z}/n) = Ext^*_{qfh}(\mathcal{F}^\sim_{qfh}, \mathbb{Z}/n) = Ext^*_h(\mathcal{F}^\sim_h, \mathbb{Z}/n) = Ext^*_{Ab}(\mathcal{F}(Spec(k)), \mathbb{Z}/n) \tag{6.8}$$

6.3 Theorem 7.6 and comparison of cohomology groups.

We now come to the main theorem of [SV]. This needs some notational preparation to state. Let $\mathcal{F}$ be a presheaf on Sch/k. Suslin and Voevodsky use the following notation in Section 7 of [SV].

$\mathcal{F}^\sim$ The qfh–sheaf associated to $\mathcal{F}$.

$C_*(\mathcal{F})$ The simplicial abelian group obtained by applying $\mathcal{F}$ to $\Delta^\bullet$.

$\mathcal{F}_*$ The simplicial presheaf of abelian groups whose components are the presheaves $U \mapsto \mathcal{F}(U \times \Delta^q)$.

The definition of algebraic singular (co)homology is extended here to presheaves of abelian groups. Suslin and Voevodsky define (for any abelian group A):

$H^{sing}_*(\mathcal{F})$ The homology of the complex $(C_*(\mathcal{F}), d = \sum(-1)^i\partial_i)$.

$H^{sing}_*(\mathcal{F}, A)$ $= H_*(C_*(\mathcal{F}) \overset{L}{\otimes} A)$

$H^*_{sing}(\mathcal{F}, A) \ = H^*(RHom(C_*(\mathcal{F}), A))$

We recover the original definition of $H^{sing}_*(Z)$ for a scheme Z as follows. We have seen ([Voev, 3.3.6]) that the sheaves of monoids $\mathbb{N}_{qfh}(Z)$ (and $\mathbb{N}_h(Z)$) are representable by the (ind–)scheme $\coprod_{n\geq 0} S^n Z$ so $\mathbb{N}_{qfh}(Z)(\Delta^\bullet) = \mathrm{Hom}\,(\Delta^\bullet, \coprod_{n\geq 0} S^n Z)$. Furthermore, since each Δ^k is normal we can use [Voev, 3.3.3] to obtain $\mathbb{Z}_{qfh}(Z)(\Delta^\bullet) = \mathbb{N}_{qfh}(Z)(\Delta^\bullet)^+ = \mathrm{Hom}\,(\Delta^\bullet, \coprod_{n\geq 0} S^n Z)^+$. So

$$H^{sing}_*(\mathbb{Z}_{qfh}(Z)) = H^{sing}_*(Z) \tag{6.9}$$

It is perhaps insightful to note that if $\mathcal{F}$ is representable by, say X, and $\mathrm{Hom}\,(\Delta^i, X)$ were to exist in Sch/k then $\mathcal{F}(- \times \Delta^i) = \mathrm{Hom}\,(- \times \Delta^i, X) = \mathrm{Hom}\,(-, \mathrm{Hom}\,(\Delta^i, X))$. That is, $\mathcal{F}_q$ would be representable by $\mathrm{Hom}\,(\Delta^q, X)$. Thus, keeping in mind the Yoneda embedding of a category into the category of presheaves of sets or abelian groups on it, the simplicial presheaf $\mathcal{F}_*$ comes to resemble the simplicial object $\mathrm{Hom}_{\,top}(\Delta^\bullet_{top}, X)$ that gives the simplicial homology groups in the topological setting.

These three objects are intimately related. Firstly, the complex $C_*(\mathcal{F})$ coincides with the complex of global sections of $\mathcal{F}_*$. Also, we can apply $\sim$ to each element of $\mathcal{F}_*$ to obtain a complex of sheaves $(\mathcal{F}_*)^\sim$. Lastly, we confuse each $C_q(\mathcal{F})$ with the constant sheaf it defines.

Now each of the projections $X \times \Delta^i \to \Delta^i$ becomes a morphism $\mathcal{F}(X \times \Delta^i) = \mathcal{F}_i(X) \to C_i(\mathcal{F}) = \mathcal{F}(\Delta^i)$ and also, $\mathcal{F}_0(X) = \mathcal{F}(X \times \mathbb{A}^0) = \mathcal{F}(X)$ so we have two morphism of complexes:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F}(\Delta^2) & \longrightarrow & \mathcal{F}(- \times \Delta^2)^\sim & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F}(\Delta^1) & \longrightarrow & \mathcal{F}(- \times \Delta^1)^\sim & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F}(\Delta^0) & \longrightarrow & \mathcal{F}(- \times \Delta^0)^\sim & \longleftarrow & \mathcal{F}^\sim(-)
\end{array} \tag{6.10}$$

Theorem 7.6 of [SV] says that both of these morphisms induce isomorphisms on Ext groups.

Theorem 52 ([SV, 7.6]). *Let k be an algebraically closed field of characteristic zero. Let $\mathcal{F}$ be a presheaf on Sch/k which admits transfer maps. Then both arrows in the diagram*

$$C_*(\mathcal{F}) \to (\mathcal{F}_*)^\sim \leftarrow \mathcal{F}^\sim \tag{6.11}$$

induce isomorphisms on $Ext^*_{qfh}(-, \mathbb{Z}/n)$.

The isomorphism induced by $(\mathcal{F}_*)^\sim \leftarrow \mathcal{F}^\sim$ is shown using the spectral sequence

$$I^{p,q}_1 = Ext^p((\mathcal{F}_q)^\sim, \mathbb{Z}/n) \implies Ext^{p+q}((\mathcal{F}_*)^\sim, \mathbb{Z}/n) \tag{6.12}$$

and the isomorphism induced by $C_*(\mathcal{F}) \to (\mathcal{F}_*)^\sim$ is shown using the spectral sequence

$$II^{p,q}_2 = Ext^p(H_q((\mathcal{F}_*)^\sim), \mathbb{Z}/n) \implies Ext^{p+q}((\mathcal{F}_*)^\sim, \mathbb{Z}/n) \tag{6.13}$$

The rigidity theorem comes up in showing the degeneracy of the second spectral sequence.

As a corollary of [SV, Theorem 7.6] and using some results about free qfh–sheafs of the form $\mathbb{Z}(X)$ we obtain:

Corollary 53 ([SV, 7.8]). *Let X be a separated scheme of finite type over an algebraically closed field k of characteristic zero. Then*

$$H^*_{sing}(X, \mathbb{Z}/n) = H^*_{qfh}(X, \mathbb{Z}/n) = H^*_{et}(X, \mathbb{Z}/n) \tag{6.14}$$

To obtain the proof of the conjecture stated at the start of [SV], Suslin and Voevodsky consider the category CW of triangulable topological spaces. They equip this category with a the Grothendieck topology defined by local homeomorphisms. After proving a topological version of [SV, Theorem 7.6] (which is much easier to prove) they look at sheaves represented by schemes/topological spaces of the form $\coprod_{d=0}^{\infty} S^d Z$ for a sheaf/topological space Z and obtain morphisms (for an object $Z \in Sch/\mathbb{C}$)

$$\begin{aligned} H_*^{sing}(Z, \mathbb{Z}/n) &\to H_*(Z(\mathbb{C}), \mathbb{Z}/n) \\ H^*(Z(\mathbb{C}), \mathbb{Z}/n) &\to H^*_{sing}(Z, \mathbb{Z}/n) \end{aligned} \tag{6.15}$$

Finally, using [SV, Theorem 7.6] they prove:

Theorem 54 ([SV, 8.3]). *For any separated scheme $Z \in Sch/\mathbb{C}$ the above homomorphisms are isomorphisms.*

6.4 Connections to $DM_h(S)$.

The last two theorems of Section 4.1 in [Voev] are proved using results from [SV]. We state them without proof.

Theorem 55 ([Voev, 4.1.8]). *Let X be a scheme of finite type over $\mathbb{C}$. Then one has canonical isomorphisms of abelian groups*

$$DM_h(\mathbb{Z}, M(X) \otimes \mathbb{Z}/n[k]) = H_k(X(\mathbb{C}), \mathbb{Z}/n) \tag{6.16}$$

Definition 56. An object X of $DM_h(S)$ is called a torsion object if there exists $N > 0$ such that $N \, \mathrm{Id}_X = 0$.

Theorem 57 ([Voev, 3.2.12]). *Let k be a field of characteristic zero. Denote by D_k the derived category of the category of torsion sheaves of abelian groups on the small étale site of $Spec(k)$. Then the canonical functor*

$$\tau : D_k \to DM_h(Spec(k)) \tag{6.17}$$

is a full embedding and any torsion object in $DM_h(Spec(k))$ is isomorphic to an object of the form $\tau(K)$ for some $K \in ob(D_k)$.

Chapter 7

Other homological properties of $DM(S)$

7.1 Projective decomposition

In Section 4.2 of [Voev] Voevodsky introduces the Tate motive in the categories $DM(S)$ and develops some properties of it. He then uses these properties to prove the decomposition of the motive of the projectivization of a vector bundle.

Definition 58 ([Voev, 4.2.1]). The Tate motive $\mathbb{Z}(1)$ is the object of the category DM which corresponds to the sheaf $\mathbb{G}_m$ shifted by minus one, i.e.

$$\mathbb{Z}(1) = \mathbb{G}_m[-1] \tag{7.1}$$

We denote by $\mathbb{Z}(n)$ the n–tensor power of $\mathbb{Z}(1)$ and for any object X of DM by $X(n)$ the tensor product $X \otimes \mathbb{Z}(n)$.

Proposition 59 ([Voev, 4.2.2]). *For any n and k there exists an exact triangle of the form*

$$\mathbb{Z}(n) \xrightarrow{k} \mathbb{Z}(n) \longrightarrow \mu_k^{\otimes n} \longrightarrow \mathbb{Z}(n)[1] \tag{7.2}$$

Proof. Since $\mathbb{Z}/k\mathbb{Z}$ is the cokernel of the injective morphism $\mathbb{Z} \xrightarrow{k} \mathbb{Z}$ there is an exact triangle

$$\mathbb{Z} \xrightarrow{k} \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z} \to \mathbb{Z}[1] \tag{7.3}$$

Tensoring this with $\mathbb{Z}(n)$ we obtain the exact triangle $\mathbb{Z}(n) \xrightarrow{k} \mathbb{Z}(n) \to \mathbb{Z}(n) \otimes \mathbb{Z}/k\mathbb{Z} \to \mathbb{Z}(n)[1]$. So if $\mathbb{Z}(n) \otimes \mathbb{Z}/k\mathbb{Z} \cong \mu_k^{\otimes n}$ then the proposition is proven. After replacing $\mathbb{Z}(n)$ by $(\mathbb{G}_m[-1])^{\otimes n}$ and shifting by n, the isomorphism we are trying to prove becomes $\mathbb{G}_m^{\otimes n} \otimes \mathbb{Z}/k\mathbb{Z} \cong \mu_k^{\otimes n}[n]$.

We first note that by definition μ_k is the kernel of the sheaf morphism $\kappa : \mathbb{G}_m \to \mathbb{G}_m$ corresponding to the morphism of schemes $(\mathbb{A}^1 - 0) \to (\mathbb{A}^1 - 0)$ defined by the ring morphism $k[t, t^{-1}] \to k[t, t^{-1}]$ which sends $t \mapsto t^k$. This scheme morphism is an h–covering and so the corresponding morphism from the sheaf $\mathbb{G}_m = L(\mathbb{A}^1 - \{0\})$ to itself is an epimorphism. So the cone of κ is isomorphic to $\mu_k[1]$ in the derived category giving an exact triangle:

$$\mathbb{G}_m \xrightarrow{\kappa} \mathbb{G}_m \longrightarrow \mu_k[1] \longrightarrow \mathbb{G}_m[1] \tag{7.4}$$

On the other hand, κ can be written as $(id_{\mathbb{G}_m}) \otimes k : \mathbb{G}_m \otimes \mathbb{Z} \to \mathbb{G}_m \otimes \mathbb{Z}$ so tensoring the exact triangle of Equation 7.3 with $\mathbb{G}_m$ gives us the exact triangle:

$$\mathbb{G}_m \xrightarrow{\kappa} \mathbb{G}_m \longrightarrow \mathbb{G}_m \otimes \mathbb{Z}/k\mathbb{Z} \longrightarrow \mathbb{G}_m[1] \tag{7.5}$$

Comparing 7.4 with 7.5 gives us the isomorphism $\mathbb{G}_m \otimes \mathbb{Z}/k\mathbb{Z} \cong \mu_k[1]$ and so tensoring this n times and noticing the isomorphism $(\mathbb{Z}/k\mathbb{Z})^{\otimes n} \cong \mathbb{Z}/k\mathbb{Z}$ gives the desired isomorphism $\mathbb{G}_m^{\otimes n} \otimes \mathbb{Z}/k\mathbb{Z} \cong \mu_k^{\otimes n}[n]$.

To finish the proof of the proposition one should show that $\mu_k^{\otimes n} \otimes^L \mathbb{G}_m \cong \mu_k^{\otimes(n+1)}[1]$. □

In this paper Voevodsky defines the motivic cohomology to be the groups

$$H^p(X, \mathbb{Z}(q)) = DM(M(X), \mathbb{Z}(q)[p]) \tag{7.6}$$

using the notation $H_h^p(X, \mathbb{Z}(q))$ and $H_{qfh}^p(X, \mathbb{Z}(q))$ to specify a topology if necessary. He notes that there is a multiplication

$$H^p(X, \mathbb{Z}(q)) \otimes H^{p'}(X, \mathbb{Z}(q')) \to H^{p+p'}(X, \mathbb{Z}(q+q')) \tag{7.7}$$

and that the direct sum

$$\bigoplus_{p,q} H^p(X, \mathbb{Z}(q)) \tag{7.8}$$

has a natural structure of a bigraded ring, which is commutative as a bigraded ring.

Through the triangle of [Voev, 4.2.2] and the isomorphism [Voev, 4.1.6] there is a relationship between the étale cohomology groups with coefficients in $\mu_k^{\otimes n}$ and the motivic cohomology groups in the form of a long exact sequence

Proposition 60 ([Voev, 4.2.3]). *Let X be a scheme. For any q and any k prime to the characteristic of X one has a long exact sequence of the form:*

$$\cdots \to H^p(X, \mathbb{Z}(q)) \xrightarrow{k} H^p(X, \mathbb{Z}(q)) \to H_{et}^p(X, \mu_k^{\otimes n}) \to H^{p+1}(X, \mathbb{Z}(q)) \to \dots \tag{7.9}$$

Proof. Since $\mu_k^{\otimes n}$ is locally free in the étale topology over $Spec(\mathbb{Z}[1/k])$ we can use [Voev, 4.1.6] to write $H_{et}^p(X, \mu_k^{\otimes n})$ as $DM(M(X), \mu_k^{\otimes n}[p])$. Now writing $H^p(X, \mathbb{Z}(q))$ as $DM(M(X), \mathbb{Z}(q)[p])$ we can rewrite the above long exact using the hom groups. The fact that the sequence is exact follows from the triangle [Voev, 4.2.2] (and of course for every exact triangle $A \to B \to C \to A[i]$ in a triangulated category the related long exact sequence $\cdots \to \mathrm{Hom}\,(X, A[i]) \to \mathrm{Hom}\,(X, B[i]) \to \mathrm{Hom}\,(X, C[i]) \to \mathrm{Hom}\,(X, A[i+1]) \to \dots$ is exact). □

Proposition 61 ([Voev, 4.2.4]). *Let X be a regular scheme of exponential characteristic p. Then for any $i \geq 0$ one has a canonical isomorphism*

$$H_{qfh}^i(X, \mathbb{Z}(1)) = H^{i-1}(X, \mathbb{G}_m) \otimes \mathbb{Z}[1/p] \tag{7.10}$$

Proof. The homology groups can be expressed in terms of hom groups which we have seen [Voev, 4.1.6] are equivalent to étale cohomology $H_{qfh}^i(X, \mathbb{Z}(1)) = DM(X, \mathbb{Z}(1)[i]) = DM(X, \mathbb{G}_m[i-1]) = H_{et}^{i-1}(X, \mathbb{G}_m)$. We also have that the category $DM(S)$ is $\mathbb{Z}[1/p]$–linear [Voev, 4.1.7] and so tensoring the hom groups with $\mathbb{Z}[1/p]$ does nothing. □

Now we begin leading up to (one of) the main theorem(s) of this section. Showing that the projectivization of a vector bundle decomposes.

Theorem 62 ([Voev, 4.2.5]). *The tautological section of the sheaf* $\mathbb{G}_m$ *over* $\mathbb{A}^1 - \{0\}$ *defines an isomorphism in* DM

$$\tilde{M}(\mathbb{A}^1 - \{0\}) \cong \mathbb{Z}(1)[1] \tag{7.11}$$

Proof. The morphism $\tilde{M}(\mathbb{A}^1 - \{0\}) \to \mathbb{Z}(1)[1] = \mathbb{G}_m = L(\mathbb{A}^1 - \{0\})$ is defined via the map $\mathbb{Z}(\mathbb{A}^1 - \{0\}) \to L(\mathbb{A}^1 - \{0\})$ defined by the tautological section. Explicitly, on the level of presheaves, for each U we have

$$\begin{gathered} \mathbb{Z}\ \mathrm{Hom}\,(U, \mathbb{A}^1 - \{0\}) \longrightarrow \mathrm{Hom}\,(U, \mathbb{A}^1 - \{0\}) \\ \sum a_i f_i \mapsto \prod f_i^{a_i} \end{gathered} \tag{7.12}$$

where $\mathrm{Hom}\,(U, \mathbb{A}^1 - \{0\})$ inherits the group structure of $\mathbb{A}^1 - \{0\}$. This morphism is an epimorphism and so showing that its kernel is contractible will imply that it becomes an isomorphism in $DM(S)$ (recall that sheafification is an exact functor). To do this we explicitly find a scheme that represents it and then see that it is contractible.

By [Voev, 3.3.6] the sheaf $\mathbb{N}(\mathbb{A}^1 - \{0\})$ is representable by $\amalg S^n(\mathbb{A}^1 - 0)$ and so $\mathbb{Z}(\mathbb{A}^1 - \{0\})$ is the sheaf of abelian groups associated to this representable sheaf of monoids.

The scheme $S^n(\mathbb{A}^1 - \{0\})$ is isomorphic to $(\mathbb{A}^1 - \{0\}) \times \mathbb{A}^{n-1}$ via

$$\begin{gathered} (\mathbb{A}^1 - \{0\}) \times \mathbb{A}^{n-1} \longleftarrow S^n(\mathbb{A}^1 - \{0\} \\ \mathbb{Z}[t_1, t_2, \ldots, t_{n-1}, t_n, t_n^{-1}] \longrightarrow \mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_n, t_n^{-1}]^{S_n} \\ t_i \mapsto \sigma_i \end{gathered} \tag{7.13}$$

where S_n is the symmetric group on n letters, A^G indicates the G invariant elements of the ring A for a group action of G on A, and $\sigma_i = \sum_{j_1 < j_2 < \cdots < j_i} t_{j_1} t_{j_2} \ldots t_{j_i}$ is the ith symmetric polynomial. The morphism of monoid schemes $\amalg S^n(\mathbb{A}^1 - \{0\}) \to \mathbb{A}^1 - \{0\}$ is given by the ring homomorphism

$$\begin{gathered} \mathbb{A}^1 - \{0\} \longleftarrow \coprod S^n(\mathbb{A}^1 - \{0\}) \\ \mathbb{Z}[t, t^{-1}] \longrightarrow \prod_{n \geq 0} \mathbb{Z}[x_1, x_2, \ldots, x_n, x_n^{-1}] \\ t \mapsto (1, x_1, x_2, x_3, \ldots) \end{gathered} \tag{7.14}$$

and it has kernel (coalgebra cokernel)

$$\begin{gathered} \coprod S^n(\mathbb{A}^1 - \{0\}) \longleftarrow \coprod \mathbb{A}^{n-1} \\ \prod_{n \geq 0} \mathbb{Z}[x_1, x_2, \ldots, x_n, x_n^{-1}] \longrightarrow \prod_{n \geq 0} \mathbb{Z}[x_1, x_2, \ldots, x_n, x_n^{-1}]/(x_n - 1) \end{gathered} \tag{7.15}$$

where the monoid structure on $\amalg \mathbb{A}^n$ is given by

$$\begin{gathered} \mathbb{A}^{n+m} \longleftarrow \mathbb{A}^n \times \mathbb{A}^m \\ \mathbb{Z}[x_1, x_2, \ldots, x_{n+m}] \longrightarrow \mathbb{Z}[x_1, \ldots, x_n] \otimes \mathbb{Z}[x_1, \ldots, x_m] \\ x_k \mapsto \sum_{i+j=k} x_i \otimes x_j \end{gathered} \tag{7.16}$$

Hence, the kernel of 7.12 is contractible. □

Corollary 63 ([Voev, 4.2.6]). *The morphism $\tilde{M}(\mathbb{P}^1_S) \to \mathbb{G}_m[1]$ which corresponds to the cohomological class in $H^1(\mathbb{P}^1, \mathbb{G}_m)$ represented by the line bundle $\mathcal{O}(-1)$ is an isomorphism in $DM_{qfh}(S)$.*

Proof. Consider the open cover of $\mathbb{P}^1$ consisting of two affine lines. Using Mayer–Vietoris we have a diagram:

$$\begin{array}{ccccccc} \tilde{M}(\mathbb{A}^1 - 0) & \longrightarrow & \tilde{M}(\mathbb{A}^1) \oplus \tilde{M}(\mathbb{A}^1) & \longrightarrow & \tilde{M}(\mathbb{P}^1) & \longrightarrow & \tilde{M}(\mathbb{A}^1 - 0)[1] \\ \downarrow \wr & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{G}_m & \longrightarrow & 0 & \longrightarrow & \mathbb{G}_m[1] & \xrightarrow{\sim} & \mathbb{G}_m[1] \end{array} \tag{7.17}$$

where the rows are exact triangles. The first and last vertical morphism is the isomorphism from [Voev, 4.2.5], the second vertical morphism is the sum of the structural morphisms of $\mathbb{A}^1$ which are isomorphisms by homotopy invariance. The third vertical morphism is the morphism in question and since the rows are exact and the other vertical morphisms are isomorphisms, this must also be an isomorphism. □

Projective decomposition ([Voev, 4.2.7]). *Let X be a scheme and E be a vector bundle on X. Denote by $P(E) \to X$ the projectivization of E. One has a natural isomorphism in DM*

$$M(P(E)) \cong \bigoplus_{i=0}^{\dim E-1} M(X)(i)[2i] \tag{7.18}$$

Proof. Step 1: Construction of the morphism. Suppose that X is the base scheme and let $\mathcal{O}(-1)$ be the tautological line bundle on $P(E)$. Any line bundle on a scheme Y defines a cohomology class $H^1(Y, \mathbb{G}_m)$. Since $\mathbb{G}_m = \mathbb{Z}(1)[1]$ and $H^i(Y, F) = DM(M(Y), F[i])$, the bundle $\mathcal{O}(-1)$ defines a morphism $a : M(P(E)) \to \mathbb{Z}(1)[2]$ in the category $DM(X)$. Using the diagonal morphism $M(P(E)) \to M(P(E)) \otimes M(P(E))$ we can define a^i recursively as the composition

$$M(P(E)) \xrightarrow{\delta} M(P(E)) \otimes M(P(E)) \xrightarrow{a^1 \otimes a^{n-1}} \mathbb{Z}(1)[2] \otimes \mathbb{Z}(n-1)[2n-2] \tag{7.19}$$

We can now define ϕ as

$$\phi : \bigoplus_{i=0}^{\dim E-1} a^i : M(P(E)) \longrightarrow \bigoplus_{i=0}^{\dim E-1} \mathbb{Z}(i)[2i] \tag{7.20}$$

Step 2: Reduction to the case of a trivial bundle. This step proceeds via Mayer–Vietoris and induction on the size of a trivializing open cover. Suppose that $\cup_{i=1}^n U_i = X$ is a trivializing open cover and that the isomorphism holds for covers of size $n-1$. Let $V = \cup_{i=1}^{n-1}$. We have a morphism of exact triangles:

$$\begin{array}{ccc} M(U \cap V) & \longrightarrow & \bigoplus M(U \cup V)(i)[2i] \\ \downarrow & & \downarrow \\ M(U) \oplus M(V) & \longrightarrow & \bigoplus M(U)(i)[2i] \oplus \bigoplus M(V)(i)[2i] \\ \downarrow & & \downarrow \\ M(X) & \longrightarrow & \bigoplus M(X)(i)[2i] \\ \downarrow & & \downarrow \\ M(U \cap V)[1] & \longrightarrow & \bigoplus M(U \cup V)(i)[2i+1] \end{array} \tag{7.21}$$

where all the large direct sums are over $i = 0, \dots, \dim E - 1$. The first two horizontal arrows are isomorphisms by the inductive hypothesis so the only remaining thing to prove is that for the case of a vector bundle with a trivializing cover of size 1, that is a trivial bundle, the theorem holds.

Step 3: The case of a trivial bundle. Recall that we are now trying to prove that the morphism

$$\phi = \bigoplus_{i=0}^{n} a_n^i : \mathbb{P}^n \to \bigoplus_{i=0}^{n} \mathbb{Z}(i)[2i] \tag{7.22}$$

is an isomorphism where $a_n^1 : \mathbb{P}^n \to \mathbb{Z}(1)[2] = \mathbb{G}_m[1]$ corresponds to the line bundle $\mathcal{O}(-1)$. This final step proceeds by induction on n. For $n = 0$ the statement is trivial. Consider the covering

$$\mathbb{P}^n = (\mathbb{P}^n - \{0\}) \cup \mathbb{A}^n \tag{7.23}$$

where $\{0\}$ is the origin of $\mathbb{A}^n \subset \mathbb{P}^n$. Mayer–Vietoris gives the exact triangle

$$M(\mathbb{A}^n - 0) \longrightarrow M(\mathbb{P}^n - 0) \oplus M(\mathbb{A}^n) \longrightarrow M(\mathbb{P}^n) \longrightarrow M(\mathbb{A}^n - 0)[1] \tag{7.24}$$

Voevodsky constructs a morphism from this triangle to an exact triangle of the form

$$\mathbb{Z}(n)[2n-1] \oplus \mathbb{Z} \longrightarrow \oplus_{i=0}^{n-1} \mathbb{Z}(i)[2i] \oplus \mathbb{Z} \longrightarrow \oplus_{i=0}^{n} \mathbb{Z}(i)[2i] \longrightarrow \mathbb{Z}(n)[2n] \oplus \mathbb{Z}[1] \tag{7.25}$$

and shows that it is an isomorphism on the first two terms which implies that it is an isomorphism of exact triangles.

Step 3a: The first isomorphism. The first morphism comes from a cohomology class $\psi \in H^{n-1}(\mathbb{A}^n - \{0\}, \mathbb{G}_m^{\otimes n})$. Consider the covering of $\mathbb{A}^n - \{0\}$ given by $\mathbb{A}^n - \{0\} = \cup_{i=1}^n (\mathbb{A}^n - H_i)$ where H_i is the hyperplane $x_i = 0$. A Cech cocycle in $Z^{n-1}(\mathbb{A}^n - \{0\}, \mathbb{G}_m^{\otimes n})$ with respect to this covering is a section of the sheaf $\mathbb{G}_m^{\otimes n}$ over $\cap_{i=1}^n (\mathbb{A}^n - H_i)$. So define ψ to be the cohomological class corresponding to the section $x_1 \otimes \cdots \otimes x_n$. This cohomology class defines a morphism $M(\mathbb{A}^n - \{0\}) \to \mathbb{G}_m^{\otimes n}[n-1] = \mathbb{Z}(n)[2n-1]$ and we take the other component to be the structural morphism $M(\mathbb{A}^n - \{0\}) \to \mathbb{Z}$. This gives $f : M(\mathbb{A}^n - \{0\}) \to \mathbb{Z}(n)[2n-1] \oplus \mathbb{Z}$.

Voevodsky suggests that we show f is an isomorphism by using induction starting with [Voev, 4.2.5]. Certainly, in the case $n = 1$, the morphism f is constructed using the tautological section of $\mathbb{G}_m$ over $\mathbb{A}^1 - \{0\}$ and the structural morphism and so the morphism $\tilde{M}(\mathbb{A}^1 - \{0\}) \to \mathbb{Z}(1)[1]$ defined by $f : M(\mathbb{A}^1 - \{0\}) \to \mathbb{Z}(1)[1] \oplus \mathbb{Z}$ is the same as the one shown to be an isomorphism in [Voev, 4.2.5]. For the inductive step consider the Mayer–Vietoris exact triangle given by the covering:

$$\mathbb{A}^n - \{0\} = \left(\mathbb{A}^n - \mathbb{A}^{n-1}\right) \cup \left(\mathbb{A}^n - \mathbb{A}^1\right) \tag{7.26}$$

where the affine line $\mathbb{A}^1$ is the intersection of all the "coordinate" hyperplanes except the one denoted by $\mathbb{A}^{n-1}$. By homotopy invariance first and then the inductive hypothesis we have

$$\begin{aligned} M(\mathbb{A}^n - \mathbb{A}^{n-1}) &\cong M(\mathbb{A}^1 - \{0\}) \cong \mathbb{Z}(1)[1] \oplus \mathbb{Z} \\ M(\mathbb{A}^n - \mathbb{A}^1) &\cong M(\mathbb{A}^{n-1} - \{0\}) \cong \mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z} \end{aligned} \tag{7.27}$$

which gives the second term of the triangle. The first term comes from the intersection of the two open sets in the cover, i.e.

$$\mathbb{A}^n - \mathbb{A}^1 - \mathbb{A}^{n-1} = (\mathbb{A}^{n-1} - \{0\}) \times (\mathbb{A}^1 - \{0\}) \tag{7.28}$$

which we know, again by the inductive hypothesis, to be:

$$M(\mathbb{A}^{n-1} - \{0\}) \otimes M(\mathbb{A}^1 - \{0\}) \cong (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z}) \otimes (\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \tag{7.29}$$

Now a careful consideration of the morphism between these first two terms in the Mayer–Vietoris exact triangle reveals the cone to be isomorphic to $\mathbb{Z}(n)[2n-1] \oplus \mathbb{Z}$.

Step 3b: The second morphism. The morphism

$$g : M(\mathbb{P}^n - \{0\}) \oplus M(\mathbb{A}^n) \to \bigoplus_{i=0}^{n-1} \mathbb{Z}(i)[2i] \oplus \mathbb{Z} \tag{7.30}$$

is defined using the natural projection $p : \mathbb{P}^n - \{0\} \to \mathbb{P}^{n-1}$ in the first component and the structural morphism in the second. Explicitly, g is given by $g = \phi \circ M(p) \oplus (M(\mathbb{A}^n) \to \mathbb{Z})$. Since p has $\mathbb{A}^1$ as fibers it becomes an isomorphism in DM and we already know that the structural morphism of $\mathbb{A}^n$ is an isomorphism in DM. The morphism $\phi : M(\mathbb{P}^n - 1) \to \oplus \mathbb{Z}(i)[2i]$ is an isomorphism under the inductive hypothesis.

From the construction of f and g it can be seen that together with ϕ they form a morphism between the triangles 7.24 and 7.25. Since f and g are isomorphisms it follows that ϕ is an isomorphism. So the theorem is proved. □

7.2 Blowup decomposition

The main result of section 4.3 in [Voev] is the decomposition of the blowup of a smooth closed subscheme of a smooth scheme. To state and prove this theorem we need some notation.

For a scheme X and a closed subscheme Z we denote by X_Z the blowup of X with center in Z and $p_Z : X_Z \to X$ the corresponding projection. The projectivization of the normal cone to Z in X is denoted PN_Z and $p : PN_Z \to Z$ its corresponding projection. Finally [Voev] uses $O_X(Z)$ for the kernel of qfh–sheaves

$$\mathbb{Z}_{qfh}(p) : \mathbb{Z}_{qfh}(PN_Z) \to \mathbb{Z}_{qfh}(Z) \tag{7.31}$$

The justification for the notation $O_X(Z)$ is that it incorporates both X and Z and we will have cause to consider the above kernel for various X and Z.

The main theorem of this section is the following:

Blow–up decomposition ([Voev, 4.3.4]). *Let $Z \subset X$ be a smooth pair over S. Then one has a natural isomorphism in $DM(S)$:*

$$M(X_Z) = M(X) \oplus \left(\bigoplus_{i=1}^{codim\ Z-1} Z(i)[2i] \right) \tag{7.32}$$

As a preliminary step though it is necessary to show that

$$O_X(Z) \to M(X_Z) \to M(X) \to O_X(Z)[1] \tag{7.33}$$

is an exact triangle in $DM(S)$ and this requires quite a bit of work. Assuming this result, the decomposition follows quite easily using the projective bundle decomposition of the previous section.

Proof of the Blow–up decomposition. By [Voev, 4.3.1] the triangle 7.33 is exact and by the projective decomposition

$$M(PN_Z) \cong \bigoplus_{i=0}^{codim\ Z-1} Z(i)[2i] \tag{7.34}$$

so since $M(Z) = Z(0)[0]$

$$O_X(Z) \cong \bigoplus_{i=1}^{codim\ Z-1} Z(i)[2i] \tag{7.35}$$

so to prove the theorem it is enough to construct a splitting of the triangle 7.33. To do this we consider the diagram (the morphisms are numbered for convenience):

$$\begin{CD}
O_X(Z) @>{1}>> O_{X\times\mathbb{A}^1}(Z\times\{0\}) \\
@V{3}VV @VV{4}V \\
M(X_Z) @>{2}>> M\Big((X\times\mathbb{A}^1)_{Z\times\{0\}}\Big) \\
@V{M(p_Z)}VV @VV{M(p_{Z\times\{0\}})}V \\
M(X) @>{M(i_0)}>> M(X\times\mathbb{A}^1) \\
@VVV @VVV \\
O_X(Z)[1] @>>> O_{X\times\mathbb{A}^1}(Z\times\{0\})[1]
\end{CD} \tag{7.36}$$

The morphism $i_1 : X \to X \times \mathbb{A}^1$ which embeds X as $X \times \{1\}$ lifts to $i_1' : X \to (X \times \mathbb{A}^1)_{Z\times\{0\}}$ and since $M(i_1)$ is an isomorphism in DM, and equal to $M(i_0)$ (where $i_0 : X \to X \times \mathbb{A}^1$ embeds X as $X \times \{0\}$) we obtain a splitting of $M(p_{Z\times\{0\}})$ by $M(i_1') \circ M(i_1)^{-1}$. Since the triangle on the right is exact, the morphism (4) splits as well. To transfer this splitting to the triangle in the left column it is sufficient to split the injective morphism (1). This splitting comes from the projective decomposition as follows. By definition $O_{X\times\mathbb{A}^1}(Z \times \{0\})$ is the kernel of the projection of the projectivized normal cone of the blowup of $Z \times \{0\}$ in $X \times \mathbb{A}^1$.

$$\mathbb{Z}_{qfh}(PN_{Z\times\{0\}}) \to \mathbb{Z}_{qfh}(Z \times \{0\}) \tag{7.37}$$

By the same line of reasoning that was used at the start of the proof we have an isomorphism $O_{X\times\mathbb{A}^1}(Z \times \{0\}) \cong \bigoplus_{i=1}^{codim\ Z} Z(i)[2i]$ where the upper bound of the sum has changed since the codimension of $Z \times \{0\}$ in $X \times \mathbb{A}^1$ is one more than the codimension of Z in X. So the morphism (1) is the same as the morphism:

$$\bigoplus_{i=1}^{codim\ Z-1} Z(i)[2i] \to \bigoplus_{i=1}^{codim\ Z} Z(i)[2i] \tag{7.38}$$

which splits canonically. Combining (2) with the splitting of (4) and (1) we obtain the splitting of (3). Since the triangle on the left is exact, this means $M(p_Z)$ splits as well.

□

We will finish this section by outlining the proof of the theorem that supports the blowup decomposition:

Theorem 64 ([Voev, 4.3.1]). *Let $Z \subset X$ be a smooth pair over S. Then the sequence of sheaves*

$$O_X(Z) \to \mathbb{Z}_{qfh}(X_Z) \to \mathbb{Z}_{qfh}(X) \tag{7.39}$$

defines an exact triangle in $DM(S)$ of the form

$$O_X(Z) \to M(X_Z) \to M(X) \to O_X(Z)[1] \tag{7.40}$$

Equivalently, the cokernel of the morphism $\mathbb{Z}_{qfh}(p_Z)$ is isomorphic to zero in $DM(S)$.

Outline of proof of [Voev, 4.3.1]. The theorem is proved by showing that the cokernel of the morphism $\mathbb{Z}_{qfh}(p_Z)$ is zero.

1. *Reduction to the local case.* The first step in this proof and the first of a series of reductions is to show that it is equivalent to prove the theorem for each open set in a well chosen open cover. This is done by choosing a cover $\{U_i\}$ of X, denoting the corresponding cover of X_Z using $V_i = p^{-1}U_i$ and considering the morphism of long exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_{qfh}(\cap V_i) & \longrightarrow \cdots \longrightarrow & \oplus\mathbb{Z}_{qfh}(V_i) & \longrightarrow & \mathbb{Z}_{qfh}(X_Z) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_{qfh}(\cap U_i) & \longrightarrow \cdots \longrightarrow & \oplus\mathbb{Z}_{qfh}(U_i) & \longrightarrow & \mathbb{Z}_{qfh}(X) & \longrightarrow & 0 \end{array} \tag{7.41}$$

   This appears as [Voev, Lemma 4.3.2] which states that the complex which is the cokernel of this morphism is exact.

2. *Reduction to the relative case.* The next reduction is to give a cover and show that instead of the cokernel of $\mathbb{Z}_{qfh}(U_{Z\cap U}) \to \mathbb{Z}_{qfh}(U)$ we can consider the cokernel of

$$\mathbb{Z}_{qfh}(U_{Z\cap U})/\mathbb{Z}_{qfh}(U-Z) \to \mathbb{Z}_{qfh}(U)/\mathbb{Z}_{qfh}(U-Z) \tag{7.42}$$

   The cover is given using the SGA result [SGA. 1, 2.4.9] that for any smooth pair $Z \subset X$ there is a covering $X = \cup U_i$ such that, for any i, there is an étale morphism $f_i : U_i \to A^N$ satisfying $Z \cap U_i = f^{-1}(\mathbb{A}^k)$ where $N = \dim_S X$ and $k = \dim_S Z$.

3. *Reduction to the affine case.* To reduce to the affine case Voevodsky first states and proves [Voev, Lemma 4.3.3] which says that if $Z \to X$ is a closed embedding and $f : U \to X$ is an étale surjective morphism such that $U \times_X Z \to Z$ is an isomorphism then there is a natural isomorphism of sheaves

$$\mathbb{Z}(U)/\mathbb{Z}(U - f^{-1}(Z)) \cong \mathbb{Z}(X)/\mathbb{Z}(X-Z) \tag{7.43}$$

   We can apply this lemma to the morphisms $\mathbb{A}^{N-k} \times (Z \cap U) \to \mathbb{A}^{N-k} \times f(Z \cap U)$ (after replacing U by $f^{-1}(\mathbb{A}^{N-k} \times f(Z \cap U))$) and so now we are trying to show that the cokernel of

$$\mathbb{Z}_{qfh}\Big(Y \times (\mathbb{A}^{N-k}_{\{0\}}/\mathbb{A}^{N-k} - \{0\}))\Big) \to \mathbb{Z}_{qfh}\Big(Y \times (\mathbb{A}^{N-k}/(\mathbb{A}^{N-k} - \{0\}))\Big) \tag{7.44}$$

   is zero and therefore have reduced the problem to showing that the cokernel of $\mathbb{Z}p_{\{0\}}$: $\mathbb{Z}_{qfh}(\mathbb{A}^n_{\{0\}}) \to \mathbb{Z}_{qfh}(\mathbb{A}^n)$ is zero in $DM(S)$.

4. *Proof of the affine case.* Showing that the cokernel of $\mathbb{Z}p_{\{0\}}$ is zero in $DM(S)$ is equivalent to showing that *ker* $\mathbb{Z}p_{\{0\}}$ is isomorphic to $(cone\ \mathbb{Z}p_{\{0\}})[-1]$ in $DM(S)$. From [Voev, 3.3.7] the kernel of $\mathbb{Z}p_{\{0\}} : \mathbb{Z}_{qfh}(\mathbb{A}^n_{\{0\}}) \to \mathbb{Z}_{qfh}(\mathbb{A}^n)$ is canonically isomorphic to the kernel of $\mathbb{Z}_{qfh}(\mathbb{P}^{n-1}) \to \mathbb{Z}_{qfh}$ which is isomorphic to $\mathbb{Z}_{qfh}(\mathbb{P}^{n-1}) \to \mathbb{Z}_{qfh}$ as a complex concentrated in degree 0 and -1. Since $\mathbb{A}^n_{\{0\}}$ is isomorphic to the total space of the bundle $\mathcal{O}(-1)$ on $\mathbb{P}^{n-1}$ and $\mathbb{Z}_{qfh}(\mathbb{A}^n)$ is isomorphic to $\mathbb{Z}_{qfh}$ in $DM(S)$ the cone of $\mathbb{Z}_{qfh}(\mathbb{A}^n_{\{0\}}) \to \mathbb{Z}_{qfh}(\mathbb{A}^n)$ is isomorphic to the cone of $\mathbb{Z}_{qfh}(\mathbb{P}^{n-1}) \to \mathbb{Z}_{qfh}$, that is, $\mathbb{Z}_{qfh}(\mathbb{P}^{n-1}) \to \mathbb{Z}_{qfh}$ considered as a complex concentrated in degrees 1 and 0.

□

7.3 Gysin exact triangle

We now come to the last major result of [Voev].

Gysin exact triangle ([Voev, 4.4.1]). *Let $Z \subset X$ be a smooth pair over S and $U = X - Z$. Then there is defined a natural exact triangle in $DM(S)$ of the form*

$$M(U) \to M(X) \to M(Z)(d)[2d] \to M(U)[1] \tag{7.45}$$

where d is the codimension of Z. Equivalently, there is a natural isomorphism

$$M(X/U) \cong M(Z)(d)[2d] \tag{7.46}$$

Proof. The theorem is proved by constructing a morphism $G_{X,Z} : M(X/U) \to M(Z)(d)[2d]$ in $DM(S)$ and showing that it is an isomorphism.

1. *Construction of the morphism $G_{X,Z}$.* Consider again the diagram

$$\begin{array}{ccc} O_X(Z) & \xrightarrow{1} & O_{X\times\mathbb{A}^1}(Z\times\{0\}) \\ {\scriptstyle 3}\downarrow & & \downarrow{\scriptstyle 4} \\ M(X_Z) & \xrightarrow{2} & M\Big((X\times\mathbb{A}^1)_{Z\times\{0\}}\Big) \\ {\scriptstyle M(p_Z)}\downarrow & & \downarrow{\scriptstyle M(p_{Z\times\{0\}})} \\ M(X) & \xrightarrow{M(i_0)} & M(X\times\mathbb{A}^1) \\ \downarrow & & \downarrow \\ O_X(Z)[1] & \longrightarrow & O_{X\times\mathbb{A}^1}(Z\times\{0\})[1] \end{array} \tag{7.47}$$

from the proof of the blow–up decomposition [Voev, 4.3.4]. We consider two morphisms $M(X_Z) \to M\Big((X\times\mathbb{A}^1)_{Z\times\{0\}}\Big)$. The first morphism $M(\tilde{i_0})$ comes from the morphism $\tilde{i_0} : X_Z \to (X\times\mathbb{A}^1)_{Z\times\{0\}}$ which corresponds to the embedding of X in $X\times\mathbb{A}^1$ as $X\times\{0\}$. The second $M(\tilde{i_1})$ comes from lifting the embedding $i_1 : X \to X\times\mathbb{A}^1$ of X as $X\times\{1\}$ to $X \to (X\times\mathbb{A}^1)_{Z\times\{0\}}$ and then composing it with p_Z.

Consider the difference $M(\tilde{i_1}) - M(\tilde{i_0})$. Since the composition of $M(\tilde{i_1})$ and $M(\tilde{i_0})$ with $M(p_{Z\times\{0\}})$ is the same, the composition of the difference with $M(p_{Z\times\{0\}})$ is zero so $M(\tilde{i_1}) - M(\tilde{i_0})$ lifts to a morphism $M(X_Z) \to O_{X\times\mathbb{A}^1}(Z\times\{0\})$. The composition

$$O_X(Z) \longrightarrow M(X_Z) \longrightarrow O_{X\times\mathbb{A}^1}(Z\times\{0\}) \longrightarrow O_{X\times\mathbb{A}^1}(Z\times\{0\})/O_X(Z) \tag{7.48}$$

is zero and so the composition $M(X_Z) \to O_{X\times\mathbb{A}^1}(Z\times\{0\})/O_X(Z)$ descends to a morphism

$$M(X) \to O_{X\times\mathbb{A}^1}(Z\times\{0\})/O_X(Z) \tag{7.49}$$

Now we start again with slightly different exact triangles. Letting $U = X - Z$ we have a diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & O_X(Z) & \longrightarrow & O_X(Z) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M(U) & \longrightarrow & M(X_Z) & \longrightarrow & M(X_Z/U) & \longrightarrow & M(U)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M(U) & \longrightarrow & M(X) & \longrightarrow & M(X/U) & \longrightarrow & M(U)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & O_X(Z)[1] & \longrightarrow & O_X(Z)[1] & \longrightarrow & 0
\end{array} \tag{7.50}$$

We know that it commutes, that every column except possibly the third is a distinguished triangle, that the middle two rows are distinguished triangles and so we can apply [May, Lemma 2.6] and see that the third column is a distinguished triangle. Similarly,

$$O_{X\times\mathbb{A}^1}(Z\times\{0\}) \to M\Big((X\times\mathbb{A}^1)_{Z\times\{0\}}/U\times\mathbb{A}^1\Big) \to M(X\times\mathbb{A}^1/U\times\mathbb{A}^1) \to O_{X\times\mathbb{A}^1}(Z\times\{0\})[1] \tag{7.51}$$

is a distinguished triangle. Repeating the argument above with these two new distinguished triangles and the obvious morphism between them in place of diagram 7.47 results in a morphism $M(X/U) \to O_{X\times\mathbb{A}^1}(Z\times\{0\})/O_X(Z)$ in place of $M(X) \to O_{X\times\mathbb{A}^1}(Z\times\{0\})/O_X(Z)$.

Applying [Voev, 4.2.7] we have isomorphisms

$$\begin{aligned} O_X(Z) &\cong \oplus_{i=1}^{d-1} M(Z)(i)[2i] \\ O_{X\times\mathbb{A}^1}(Z\times\{0\}) &\cong \oplus_{i=1}^{d} M(Z)(i)[2i] \end{aligned} \tag{7.52}$$

and therefore $O_{X\times\mathbb{A}^1}(Z\times\{0\})/O_X(Z) \cong M(Z)(d)[2d]$ so we have found our morphism $M(X/U) \to M(Z)(d)[2d]$.

2. *Showing that $G_{X,Z}$ is an isomorphism: The case $X = \mathbb{P}^n, Z = \{x\}$.* To show that $G_{X,Z}$ is an isomorphism we first deal with the case of an S-point in projective space. We claim that

in this case the diagram:

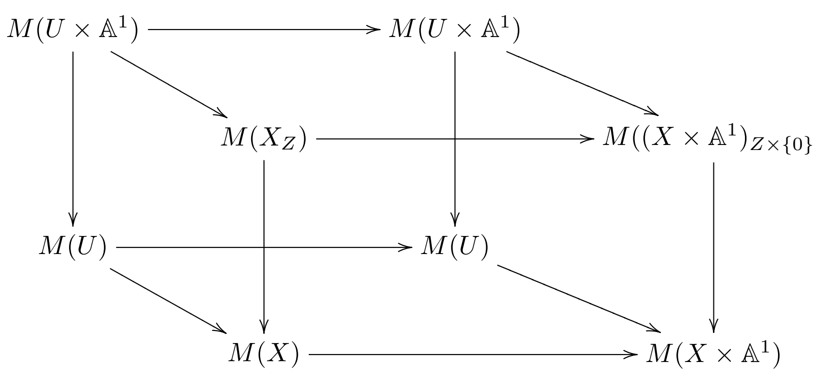
$$\begin{array}{ccc} O_X(Z) & \longrightarrow & O_{X\times\mathbb{A}^1}(Z\times\{0\}) \\ \downarrow & & \downarrow \\ M(X_Z/U\times\mathbb{A}^1) & \longrightarrow & M\Big((X\times\mathbb{A}^1)_{Z\times\{0\}}/U\times\mathbb{A}^1\Big) \\ \downarrow & & \downarrow \\ M(X/U) & \xrightarrow{M(i_0)} & M(X\times\mathbb{A}^1/U\times\mathbb{A}^1) \end{array} \tag{7.53}$$

is isomorphic to:

$$\begin{array}{ccc} \bigoplus_{j=1}^{n-1}\mathbb{Z}(j)[2j] & \xrightarrow{1} & \bigoplus_{j=1}^{n}\mathbb{Z}(j)[2j] \\ \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 4} \\ \mathbb{Z}(n)[2n]\oplus\Big(\bigoplus_{j=1}^{n-1}\mathbb{Z}(j)[2j]\Big) & \xrightarrow{2} & \mathbb{Z}(n)[2n]\oplus\Big(\bigoplus_{j=1}^{n}\mathbb{Z}(i)[2i]\Big) \\ \downarrow & & \downarrow \\ \mathbb{Z}(n)[2n] & \longrightarrow & \mathbb{Z}(n)[2n] \end{array} \tag{7.54}$$

with $M(\tilde{i_0})$ corresponding to the inclusion in place of morphism (2) and $M(\tilde{i_1})$ corresponding to the projection to $\mathbb{Z}(n)[2n]$ followed by the obvious inclusion. We start with the lower square.

There is a cube:

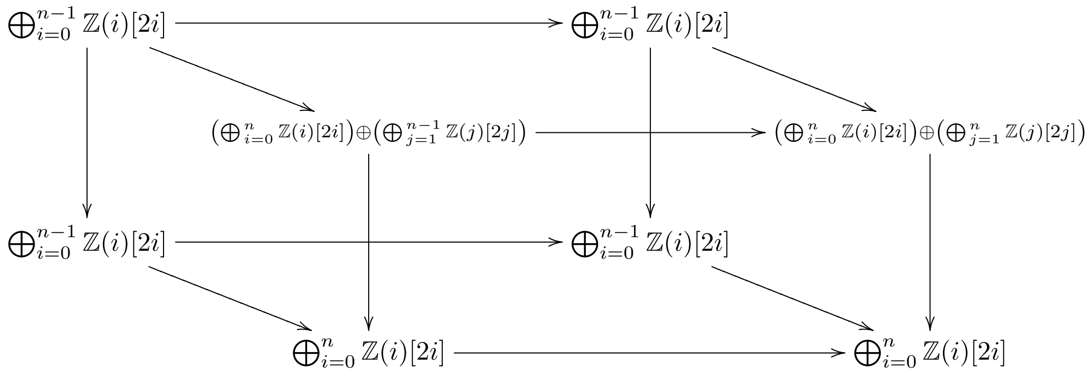


(7.55)

Since $X=\mathbb{P}^n$ and $U=\mathbb{P}^n-\{x\}=\mathbb{P}^{n-1}\times\mathbb{A}^1$ we already have morphisms from [Voev, 4.2.7]

$$\begin{aligned} M(X)=M(\mathbb{P}^n) &\to \oplus_{i=0}^{n}\mathbb{Z}(i)[2i] \\ M(X\times\mathbb{A}^1)\cong M(X)=M(\mathbb{P}^n) &\to \oplus_{i=0}^{n}\mathbb{Z}(i)[2i] \\ M(U)=M(\mathbb{P}^{n-1}\times\mathbb{A}^1)\cong M(\mathbb{P}^{n-1}) &\to \oplus_{i=0}^{n-1}\mathbb{Z}(i)[2i] \\ M(U\times\mathbb{A}^1)=M(\mathbb{P}^{n-1}\times\mathbb{A}^2)\cong M(\mathbb{P}^{n-1}) &\to \oplus_{i=0}^{n-1}\mathbb{Z}(i)[2i] \end{aligned} \tag{7.56}$$

defined using the tautological bundles on the corresponding projective spaces. These morphisms become isomorphisms in $DM(S)$ and furthermore, fit together with the morphisms of the cube to give us half of a morphism of diagrams from 7.55 to



(7.57)

The other two morphisms come from [Voev, 4.3.4] together with [Voev, 4.2.7] and are constructed from classes

$$\begin{aligned} a, b &\in H^1((\mathbb{P}^n \times \mathbb{A}^1)_{\{x\}\times\{0\}}, \mathbb{G}_m) \\ a_0, b_0 &\in H^1((\mathbb{P}^n_{\{x\}}), \mathbb{G}_m) \end{aligned} \tag{7.58}$$

corresponding to $p^{-1}_{\{x\}\times\{0\}}(\mathbb{P}^{n-1} \times \mathbb{A}^1)$ and the special divisor (for a and b) and $p^{-1}_{\{x\}}(\mathbb{P}^{n-1})$ and the special divisor (for a_0 and b_0). Now completing the "diagonal" morphisms to triangles gives the isomorphisms required.

Showing the the upper square of 7.53 is isomorphic to the upper square of 7.54 is done in a similar way, recalling the isomorphism of [Voev, 4.2.7] and then looking at the way the vertical morphisms were constructed to show everything commutes.

So now using the diagram 7.54 to construct the morphism $G_{X,Z}$ we end up with the obvious isomorphism:

$$\mathbb{Z}(n)[2n] \to \left(\bigoplus_{j=1}^{n} \mathbb{Z}(j)[2j] \right) / \left(\bigoplus_{j=1}^{n-1} \mathbb{Z}(j)[2j] \right) \tag{7.59}$$

3. *Showing that $G_{X,Z}$ is an isomorphism: The general case.* To show that $G_{X,Z}$ is an isomorphism in the general case Voevodsky suggests that we follow a similar localizing argument as the one that is used in the proof of [Voev, 4.3.1].

□

Chapter 8

"Triangulated categories of motives over a field"

8.1 Overview

The category constructed in [Voev] is just one of many categories proposed as possible derived categories of mixed motives. Another, possibly more standard one is constructed in [Voev2]. We give an overview of the construction here together with its relationship to the category $DM_h(S)$ constructed in [Voev] in the case where $S = \operatorname{Spec} k$ for a perfect field k of characteristic zero.

8.2 The categories $DM_{gm}^{eff}(k)$ and $DM_{-}^{eff}(k)$.

In this section we outline the construction of $DM_{gm}^{eff}(k)$ and $DM_{-}^{eff}(k)$ and state the embedding theorem of $DM_{gm}^{eff}(k)$ in $DM_{-}^{eff}(k)$.

The first construction presented in [Voev2] is the more natural category of effective geometric motives over the field k. This proceeds basically by starting with the category of smooth schemes over k, and then successively altering it to satisfy all of the relevant properties that are expected of a category of derived motives.

This category $DM_{gm}^{eff}(k)$ is not so easy to work in however, so Voevodsky constructs another category $DM_{-}^{eff}(k)$ using Nisnevich sheaves with transfers. He then provides a functor $DM_{gm}^{eff}(k) \to DM_{-}^{eff}(k)$ and shows that it is a full embedding with dense image.

The series of categories and functors between them constructed in order to define $DM_{gm}^{eff}(k)$ and $DM_{-}^{eff}(k)$ are the following:

$$\begin{array}{ccccc} Sm(k) & & Sm(k) & & \\ \downarrow & & \downarrow & & \\ SmCor(k) & & SmCor(k) & & \\ \downarrow & & \downarrow{\scriptstyle \text{Yoneda}} & & \\ \mathcal{H}^b(SmCor(k)) & & Shv_{Nis}(SmCor(k)) & \nwarrow{\scriptstyle\text{inclusion}} & \\ \downarrow & & \downarrow & & \\ \mathcal{H}^b(SmCor(k))/\overline{T} & & D^-(Shv_{Nis}(SmCor(k))) & & HI(k) \\ \downarrow & & {\scriptstyle\text{inclusion}}\uparrow\ \downarrow{\scriptstyle \mathbf{R}C} & & \\ DM^{eff}_{gm}(k) & & DM^{eff}_{-}(k) & \swarrow & \end{array} \tag{8.1}$$

Notation.

$Sm(k)$ The category of smooth schemes over a field k.

$SmCor(k)$ The category whose objects are smooth schemes over a field k and morphism groups are as follows: for two smooth schemes X, Y the group $\text{Hom}\,(X, Y)$ is the free abelian group generated by integral closed subschemes W of $X \times Y$ such that the projection of W to X is finite and there is a connected component of X for which the projection $W \to X$ is surjective. Composition $\text{Hom}\,(X, Y) \times \text{Hom}\,(Y, Z) \to \text{Hom}\,(X, Z)$ is defined via pullbacks and pushforwards between the triple product $X \times Y \times Z$ and the products $X \times Y, Y \times Z$ and $X \times Z$. See [Voev2] for more details.

The object of $SmCor(k)$ corresponding to the scheme X is denoted $[X]$. Note that there is a canonical functor $Sm(k) \to SmCor(k)$ taking a scheme X to $[X]$ and a morphism $X \xrightarrow{f} Y$ to its graph in $X \times Y$.

$\mathcal{H}^b(SmCor(k))$ The homotopy category of bounded complexes over $SmCor(k)$.

T The class of complexes of the forms:

1. $[X \times \mathbb{A}^1] \overset{[pr_1]}{\to} [X]$ where X is a smooth scheme and $pr_1 : X \times \mathbb{A}^1 \to X$ is projection to the first component.
2. $[U \cap V] \xrightarrow{[j_U] \oplus [j_V]} [U] \oplus [V] \xrightarrow{[i_U] \oplus -[i_V]} [X]$ where $U \cup V = X$ is an open covering of a smooth scheme X and j_U, i_U, j_V, i_V are the obvious embeddings.

$\overline{T}$ The minimal thick subcategory of $\mathcal{H}^b(SmCor(k)$ containing T.

$DM^{eff}_{gm}(k)$ The pseudo–abelian envelope of the localization $\mathcal{H}^b(SmCor(k))/\overline{T}$. The pseudo–abelian envelope is taken so that comparison results involving "classical motives" are more elegant.

$Shv_{Nis}(SmCor(k))$ The category of Nisnevich sheaves with transfers. A presheaf with transfers is an additive contravariant functor from $SmCor(k)$ to the category of abelian groups. It is called a Nisnevich sheaf with transfers if the corresponding presheaf of abelian groups is a sheaf on Sm/k with the Nisnevich topology. For any element $[X]$ of $SmCor(k)$ it turns out that the corresponding representable presheaf is a Nisnevich sheaf with transfers and so Yoneda's lemma provides an embedding of $SmCor(k)$ to $Shv_{Nis}(SmCor(k))$.

$HI(k)$ The full subcategory of $Shv_{Nis}(SmCor(k))$ consisting of homotopy invariant sheaves. That is, sheaves F for which the morphism $F(X) = F(X \times \mathbb{A}^1)$ induced by the projection is an isomorphism. If k is perfect then $HI(k)$ is abelian and the inclusion functor is exact [Voev2, 3.1.13].

$D^-(Shv_{Nis}(SmCor(k)))$ The derived category of $Shv_{Nis}(SmCor(k))$ constructed via complexes that are bounded from above.

$DM_-^{eff}(k)$ The full subcategory of $D^-(Shv_{Nis}(SmCor(k))$ consisting of complexes with homotopy invariant cohomology sheaves.

$\mathbf{R}C$ The functor induced by the functor $\underline{C}_*(-) : Shv_{Nis}(SmCor(k)) \to \{$ Complexes bounded from above $\}$ which we describe now. For a Nisnevich sheaf with transfers F we have a complex of presheaves $\underline{C}_*(F)$ where $\underline{C}_n(F)(X) = F(X \times \Delta^n)$ and $\Delta^\bullet$ is the cosimplicial object in Sm/k which we have considered in earlier sections of this mémoire. The boundary morphisms are constructed in the usual way using alternating sums of morphisms induced by the face maps of $\Delta^\bullet$. If F is representable by X then the notation $\underline{C}_*(X)$ is used. It turns out [Voev2, 3.2.1] that for any presheaf with transfers F the cohomology sheaves of $\underline{C}_*(X)$ are homotopy invariant and so we actually have a functor

$$Shv_{Nis}(SmCor(k)) \to DM_-^{eff}(k) \tag{8.2}$$

Voevodsky shows [Voev2, 3.2.3] that this functor can be extended to a functor

$$\mathbf{R}C : D^-(Shv_{Nis}(SmCor(k))) \to DM_-^{eff}(k) \tag{8.3}$$

which is left adjoint to the natural embedding and in fact identifies $DM_-^{eff}(k)$ with localization of $D^-(Shv_{Nis}(SmCor(k)))$ with respect to the localizing subcategory generated by complexes of the form

$$L(X \times \mathbb{A}^1) \stackrel{L(pr_1)}{\to} L(X) \tag{8.4}$$

where L is used to denote the Yoneda embedding.

The Yoneda embedding $L : SmCor(k) \to Shv_{Nis}(SmCor(k))$ composed with the natural projection $Shv_{Nis}(SmCor(k)) \to D^-(Shv_{Nis}(SmCor(k)))$ extends to a functor $\mathcal{H}^b(SmCor(k)) \to D^-(Shv_{Nis}(SmCor(k)))$ which is also denoted by L.

The main technical result used to study the category $DM_{gm}^{eff}(k)$ is the following:

Theorem 65 ([Voev2, 3.2.6]). *Let k be a perfect field. Then there is a commutative diagram of tensor triangulated functors of the form*

$$\begin{array}{ccc} \mathcal{H}^b(SmCor(k)) & \xrightarrow{L} & D^-(Shv_{Nis}(SmCor(k))) \\ \downarrow & & \downarrow \mathbf{R}C \\ DM_{gm}^{eff}(k) & \xrightarrow{i} & DM_-^{eff}(k) \end{array} \tag{8.5}$$

such that the following conditions hold:

1. *The functor i is a full embedding with a dense image.*

2. *For any smooth scheme X over k the object $\mathbf{R}C(L(X))$ is canonically isomorphic to $\underline{C}_*(X)$.*

8.3 The category $DM^{eff}_{-,et}(k)$, motives of schemes of finite type and relationships between $DM^{eff}_{-}(k)$, $DM^{eff}_{-,et}(k)$ and $DM_h(k)$.

In Section 3.3 of [Voev2] Voevodsky considers $DM^{eff}_{-,et}(k)$, an analogue of $DM^{eff}_{-}(k)$ using the étale topology and compares the two categories. In Section 4.1 of [Voev2] he extends the functor $L : Sm/k \to PreShv(SmCor(k))$ to schemes of finite type over k and compares the three categories $DM^{eff}_{-,et}(k), DM^{eff}_{-}(k)$ and $DM_h(k)$. We outline all of this briefly here.

The construction of $DM^{eff}_{-}(k)$ can be repeated using the étale topology instead of the Nisnevich topology and all of the same arguments hold except for the construction of $\mathbf{R}C$ which follows from the results of [Voev3]. Voevodsky assumes that k has finite étale cohomological dimension to avoid technical difficulties when he does this. The associated sheaf functor provides a functor $Shv_{Nis}(SmCor(k)) \to Shv_{et}(SmCor(k))$ [Voev, 3.3.1]. Using results of [Voev3] it can be seen that $DM^{eff}_{-,et}(k)$ (the full subcategory of $D^-(Shv_{et}(SmCor(k)))$ consisting of complexes with homotopy invariant cohomology sheaves) is a triangulated subcategory, the analog of [Voev2, 3.2.3] holds and the associated sheaf functor gives a functor

$$DM^{eff}_{-}(k) \to DM^{eff}_{-,et}(k) \tag{8.6}$$

It turns out [Voev2, 3.3.2] that after tensoring with $\mathbb{Q}$ this functor gives an equivalence of triangulated categories.

$$DM^{eff}_{-}(k) \otimes \mathbb{Q} \xrightarrow{\sim} DM^{eff}_{-,et}(k) \otimes \mathbb{Q} \tag{8.7}$$

To complement this result involving rational coefficients, Voevodsky makes brief mention of finite coefficients. We have the following new notation:

Notation.

$Shv_{et}(SmCor(k), \mathbb{Z}/n\mathbb{Z})$ The abelian category of étale sheaves of $\mathbb{Z}/n\mathbb{Z}$–modules with transfers.

$DM^{eff}_{-,et}(k, \mathbb{Z}/n\mathbb{Z})$ The category constructed in the same way as the category $DM^{eff}_{-,et}(k)$ from the category $Shv_{et}(SmCor(k), \mathbb{Z}/n/ZZ)$.

$Shv(Spec(k)_{et}, \mathbb{Z}/n\mathbb{Z})$ The category of sheaves of $\mathbb{Z}/n\mathbb{Z}$–modules on the small étale site $Spec(k)_{et}$.

The result is:

Proposition 66 ([Voev, 3.3.3]). *Denote by p the exponential characteristic of the field k. Then one has:*

1. *Let $n \geq 0$ be an integer prime to p. Then the functor*

$$DM^{eff}_{-,et}(k, \mathbb{Z}/n\mathbb{Z}) \to D^-(Shv(Spec(k)_{et}, \mathbb{Z}/n\mathbb{Z})) \tag{8.8}$$

which takes a complex of sheaves on Sm/k to its restriction to $Spec(k)_{et}$ is an equivalence of triangulated categories.

2. *For any $n \geq 0$ the category $DM^{eff}_{-,et}(k, \mathbb{Z}/p^n\mathbb{Z})$ is equivalent to the zero category.*

For its proof, Voevodsky refers the reader to the rigidity theorem [Voev3, Theorem 5.25] for the first statement and [Voev] for the second one, where it is proven that $\mathbb{Z}/p\mathbb{Z} = 0$ in $DM^{eff}_{-,et}(k)$.

Now to motives of singular varieties. For X a scheme of finite type over a field k and any smooth scheme U over k Voevodsky defines $L(X)(U)$ to be the free abelian group generated by closed integral subschemes Z of $X \times U$ such that Z is finite over U and dominant over an irreducible component of U. These can be used to define a Nisnevich sheaf with transfers $L(X)$ on Sm/k. When X is smooth over k this notation agrees with the previously used notation for a representable sheaf with transfers. The presheaves $L(X)$ are covariantly functorial with respect to X and so we obtain a functor

$$L(-) : Sch/k \to PreShv(SmCor(k)) \tag{8.9}$$

which extends the functor $L(-) : Scm/k \to PreShv(SmCor(k))$ considered earlier.

Similarly, the functor $\underline{C}_*(-) : Sm/k \to DM^{eff}_{-}(k)$ extends to a functor

$$\underline{C}_*(-) : Sch/k \to DM^{eff}_{-}(k) \tag{8.10}$$

Using a result (Theorem 5.5(2)) of [FrVoev] Voevodsky proves the following proposition about blow–ups:

Proposition 67 ([Voev2, 4.1.3]). *Consider a Cartesian square of morphisms of schemes of finite type over k of the form*

$$\begin{array}{ccc} p^{-1}(Z) & \longrightarrow & X_Z \\ \downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{i} & X \end{array} \tag{8.11}$$

such that p is proper, i is a closed embedding and $p^{-1}(X - Z) \to X$ is an isomorphism. Then there is a canonical distinguished triangle in $DM^{eff}_{-}(k)$ of the form

$$\underline{C}_*(p^{-1}(Z)) \to \underline{C}_*(Z) \oplus \underline{C}_*(X_Z) \to \underline{C}_*(X) \to \underline{C}_*(p^{-1}(Z))[1] \tag{8.12}$$

From which it follows that:

Corollary 68 ([Voev2, 4.1.4]). *If k is a field which admits resolution of singularities then for any scheme X of finite type over k the object $\underline{C}_*(X)$ belongs to $DM^{eff}_{gm}(k)$.*

To end Section 4.1 of [Voev2], Voevodsky points out that there is canonical functor

$$DM^{eff}_{-,et}(k) \to DM_h(k) \tag{8.13}$$

and that the comparison results of [Voev] for sheaves of $\mathbb{Q}$–vector spaces can be used to show that this functor is an equivalence of categories after tensoring with $\mathbb{Q}$. Furthermore, this functor is also an equivalence for finite coefficients. Hence:

Theorem 69 ([Voev2, 4.1.12]). *Let k be a field which admits resolution of singularities. Then the functor*

$$DM^{eff}_{-,et}(k) \to DM_h(k) \tag{8.14}$$

is an equivalence of triangulated categories. In particular, the categories $DM^{eff}_{-}(k) \otimes \mathbb{Q}$ and $DM_h(k) \otimes \mathbb{Q}$ are equivalent.

Appendix A

Freely generated sheaves

We present here the preliminary results that appear in [Voev, Section 2.1] as they seem out of place at the beginning of the paper.

Let T be a site and $\underline{R}$ the sheafification of the constant presheaf of rings R on T. The category of sheaves of $\underline{R}$–modules on T is denoted $\underline{R} - mod(T)$.

Proposition 70 ([Voev, 2.1.1]). *There exists a functor*

$$R : Sets(T) \to \underline{R} - mod(T) \tag{A.1}$$

which is left adjoint to the forgetful functor $\underline{R} - mod(T) \to Sets(T)$.

Proof. For any sheaf of sets X on T define $R(X)$ to be the sheafification of the presheaf of freely generated R–modules

$$U \mapsto \underline{R}(U)X(U) \tag{A.2}$$

which we will denote by $R'(X)$.

Now consider a sheaf of $\underline{R}$–modules M and natural transformation of sheaves of sets $\phi : X \to M$. For each U we obtain a morphism of modules $R'(X)(U) \to M(U)$ by extending $\phi(U)$ linearly and this defines a natural transformation $R'(X) \to M(U)$. Since M is a sheaf this factors through the sheafification $\underline{R}(X)$ of $R'(X)$.

Now we define a natural transformation $\psi : X \to \underline{R}(X)$ of sheaves of sets that will define the inverse map:

$$\mathrm{Hom}_{\,\underline{R}-mod(T)}(\underline{R}(X), M) \longrightarrow \mathrm{Hom}_{\,Sets(T)}(X, M) \tag{A.3}$$

For an object U take $x \in X(U)$ to $1 \cdot x \in R'(U)$ where 1 is the identity of $\underline{R}(U)$. This defines a natural transformation $X \to R'(X)$ which can then be composed with the natural transformation $R'(X) \to \underline{R}(X)$ coming from the sheafification. □

Proposition 71 ([Voev, 2.1.2]).

1. *The functor* $\underline{R}$ *is right exact, i.e. it takes direct limits in* $Sets(T)$ *to direct limits in* $\underline{R} - mod(T)$. *In particular it preserves epimorphisms.*

2. *The functor* $\underline{R}$ *preserves monomorphisms.*

3. *Sheaves of the form $\underline{R}(X)$ are flat.*

4. *For a pair X, Y of sheaves of sets T one has a canonical isomorphism*

$$\underline{R}(X \times Y) \cong \underline{R}(X) \otimes \underline{R}(Y) \tag{A.4}$$

Proof.

1. This is a property of all functors that have right adjoints. Let $F : C \to D$ be a functor and $G : D \to C$ a right adjoint. Let I be a small category, $\alpha : I \to C$ a functor and $\beta : \bullet \to C$ its limit. That is, β is a functor from the category with a single object and morphism to C together with a natural transformation $\phi : \alpha \to \beta$ such that any other functor $\beta' : \bullet \to C$ and natural transformations $\psi : \alpha \to \beta'$ factors through ϕ.

   Composing α and β with F we get a diagram $F \circ \alpha$ and an element $F \circ \beta$ in D together with a natural transformation $F \circ \phi : F \circ \alpha \to F \circ \beta$. If there is another functor $\beta' : \bullet \to D$ and natural transformation $\psi : F \circ \alpha \to \beta'$ under the adjunction this corresponds to a natural transformation $\psi' : \alpha \to G \circ \beta'$ which then factors through ϕ:

$$\begin{array}{ccc} \alpha & \xrightarrow{\psi'} & G \circ \beta' \\ {\scriptstyle \phi}\downarrow & \nearrow{\scriptstyle \varphi} & \\ \beta & & \end{array} \qquad\qquad \begin{array}{ccc} F \circ \alpha & \xrightarrow{\psi} & \beta' \\ {\scriptstyle F\circ\phi}\downarrow & \nearrow & \\ F \circ \beta & & \end{array} \tag{A.5}$$

   This φ corresponds to a natural transformation $F \circ \beta \to \beta'$ which gives a factoring of ψ. Hence $F \circ \beta$ is the colimit of the diagram $F \circ I$.

2. It is not so difficult to see that the functor R' from the previous proof preserves monomorphisms. The sheafification functor is exact and so $\underline{R}$ which is the composition of the two also preserves monomorphisms.

3. First note that for any $\underline{R}$–module M, the tensor product with a $\underline{R}$–modules of the form $\underline{R}(X)$ is the sheafification of the presheaf of $\underline{R}$–modules $(R'h_X) \otimes M$. That is, the presheaf:

$$U \mapsto \Big(\underline{R}(U)\mathrm{Hom}\,(U, X)\Big) \otimes_{\underline{R}(U)} M(U) \tag{A.6}$$

   Since $(R'h_X)(U)$ is free for every U it is flat for every U so it can be seen that $R'h_X$ is flat. Since sheafification is exact, the composition of the two functors $R'h_X \otimes -$ followed by sheafification is exact. That is, the functor $\underline{R}(X) \otimes -$ from $\underline{R} - mod(T)$ to itself is exact and so $\underline{R}(X)$ is flat.

4. The tensor product $M_1 \otimes M_2$ of two $\underline{R}$–modules can be defined as the sheaf satisfying the property that every bilinear map $M_1 \oplus M_2 \to M_3$ of sheaves of $\underline{R}$–modules factors through $M_1 \otimes M_2$. Bilinear means bilinear on each module of sections. Now consider the modules $\underline{R}(X)$ and $\underline{R}(Y)$. These have sub–presheaves $R'X$ and $R'Y$ and $R'X \oplus R'Y$ is a sub–presheaf of $\underline{R}(X) \oplus \underline{R}(Y)$. There is a morphism

$$R'X \oplus R'Y \longrightarrow R'(X \times Y) \tag{A.7}$$

defined in the obvious way and every bilinear morphism

$$R'X \oplus R'Y \to M \tag{A.8}$$

to a sheaf of $\underline{R}$–modules factors through it. It now follows from adjointness of the sheafification functor that every bilinear morphism of sheaves of $\underline{R}$–modules $\underline{R}(X) \oplus \underline{R}(Y) \to M$ factors through $\underline{R}(X \times Y)$. Hence, $\underline{R}(X \times Y) \cong \underline{R}(X) \otimes \underline{R}(Y)$.

□

Proposition 72 ([Voev, 2.1.3]). *Let $L(X)$ denote the sheaf of sets associated to $Hom(-, X)$ for an object X of T. For any sheaf F of $\underline{R}$–modules and $n \geq 0$ there is a canonical isomorphism:*

$$H^n(X, F) = Ext_{\underline{R}-mod}(\underline{R}(L(X)), F) \tag{A.9}$$

Proof. The definition of the cohomology groups is as the left derived functors of the global sections functor. Choose an injective resolution $I^\bullet$ of F in the category of $\underline{R}$–modules. By Yoneda, applying the the global sections functor to the complex $I^\bullet$ is the same as applying the functor $\mathrm{Hom}_{PreSh}(\mathrm{Hom}_T(-, X), -)$. By the adjointness of the sheafification functor this is the same as the functor $\mathrm{Hom}_{Sets(T)}(L(X), -)$ and by the adjointness of $\underline{R}$ this is the same as applying the functor $\mathrm{Hom}_{\underline{R}-mod}(\underline{R}(L(X)), -)$.

So the cohomology groups $H^n(X, F)$ are the cohomology groups of the complex $\mathrm{Hom}_{\underline{R}-mod}(\underline{R}(L(X)), I^\bullet)$. It follows from the definitions that these groups are the hom groups $\mathrm{Hom}(\underline{R}(L(X)), I^\bullet[n])$ in the homotopy category of bounded chain complexes. It can be shown (see Corollary 75 for example) that this hom group in the homotopy category is the same as the hom group in the derived category.

The resolution $F \to I^\bullet$ is a quasi–isomorphism and so in the derived category of $\underline{R}-mod$ (constructed using bounded complexes) $F \cong I^\bullet$ and so $Ext_{\underline{R}-mod}(\underline{R}(L(X)), F) = \mathrm{Hom}(\underline{R}(L(X)), I^\bullet)$ the hom group in the derived category. Hence, $H^n(X, F) = Ext_{\underline{R}-mod}(\underline{R}(L(X)), F)$. □

Appendix B

Some homological algebra

These lemmas support [Voev, 2.1.3]. They are taken from [Wei]. Here for an abelian category A we denote $K(A)$ the homotopy category of bound cochain complexes and $D(A)$ the corresponding derived category.

Lemma 73 ([Wei, 2.2.6]). *Let X, Y be objects of A, let $0 \to X \to I^0 \to X^1 \to \dots$ be a cochain complex with I^n injective and let $f' : Y \to X$ be a morphism in A. Then for every resolution $Y \to J^\bullet$ of Y there is a chain map $f : J^\bullet \to I^\bullet$ lifting f' that is unique up to homotopy.*

Proof. The chain morphism is defined inductively using the property that the I^n are injective. For the uniqueness up to homotopy, consider another lifting g. We want to show that there are morphisms $s^n : J^{n+1} \to I^n$ such that $ds^n + s^{n+1}d = f - g$. Let $h = f - g$.

If $n < 0$ then $I^n = 0$ so we set $s^n = 0$. Consider $n = 0$. We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle h^0} & & \downarrow{\scriptstyle h^1} & & \\ 0 & \longrightarrow & X & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \end{array} \tag{B.1}$$

By commutativity, h^0 maps the image of Y to zero. Since the top row is a resolution, this means the kernel of $J^0 \to J^1$ gets sent to zero and so h^0 passes to a morphism from $J_0/Y \cong im(J^0 \to J^1)$ to I^0. Since I^0 is injective and $im(J^0 \to J^1)$ is mapped injectively into J^1, the morphism h^0 lifts to a morphism $s^0 : J^1 \to I^0$ that satisfies $h^0 = s^0 d = s^0 d + ds^{-1}$ (recall that $s^{-1} = 0$).

Suppose the maps s^i are given for $i < n$ and consider $h^n - ds^{n-1}$. We use the same reasoning. The map $h^n - ds^{n-1}$ is zero on the image of $J^{n-1} \to J^n$ and so it passes to a map from $J^n/imd \cong im(J^n \to J^{n+1})$ to I^n. Since this is mapped injectively into J^{n+1} and I^n is injective, this lifts to a map $s^n : J^{n+1} \to I^n$ that satisfies the desired condition. □

Lemma 74 ([Wei, 10.4.6]). *If I is a bounded below complex of injectives then any quasi–isomorphism $s : I \to X$ is a split injection in $K(A)$.*

Proof. Consider the mapping cone $Cone(s)$ and the natural map $Cone(s) \to I[1]$. Since $s : X \to I$ is a quasi–isomorphism the mapping cone $Cone(s)$ is exact and since we are in the category of bounded cochain complexes, we can consider it as a resolution of the zero object. The natural map

$\phi : Cone(s) \to I[1]$ can now be thought of as a lifting of the zero map (by considering degree low enough so that the objects of both $Cone(s)$ and $I[1]$ are zero). The zero map $0 : Cone(s) \to I[1]$ would also be a lifting and by the previous result they are homotopic. So ϕ is null–homotopic, say, by a chain homotopy $v = (k, t)$ from $I[1] \oplus X \to I$. Now using the definition of the differential of $Cone(s) = I[1] \oplus X$, the definition of ϕ and $dv + vd = \phi$ we can explicitly see that t is a morphism of complexes and $t \circ s$ is homotopic to the identity via k. □

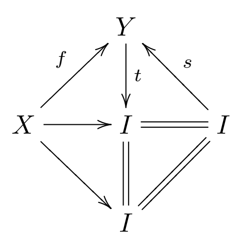
Corollary 75 ([Wei, 10.4.7]). *If I is a bounded below cochain complex of injectives, then*

$$Hom_{D(A)}(X, I) = Hom_{K(A)}(X, I) \tag{B.2}$$

for any object X.

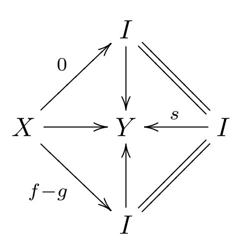
Proof. There is a natural morphism $\mathrm{Hom}_{K(A)}(X, I) \to \mathrm{Hom}_{D(A)}(X, I)$, we want to show that it is an isomorphism.

To see that it is a surjection we will show that any morphism in $\mathrm{Hom}_{D(A)}(X, I)$ is equivalent to one that comes from $\mathrm{Hom}_{K(A)}(X, I)$. The morphisms in $\mathrm{Hom}_{D(A)}(X, I)$ can be considered to be left fractions $X \xrightarrow{f} Y \xleftarrow{s} I$ where s and f are morphisms in $K(A)$ and s is a quasi–isomorphism. By the previous lemma this means that s is a split injection. Let t be a left inverse. The following diagram shows that $s^{-1}f$ is equivalent to $t \circ f$:



(B.3)

To see that the above morphism is an injection we will show that to morphism $f, g \in \mathrm{Hom}_{K(A)}(X, I)$ become equivalent in $\mathrm{Hom}_{D(A)}(X, I)$ if and only if there is a quasi–isomorphism $s : I \to Y$ in $K(A)$ such that $s \circ f = s \circ g$ (since these quasi–isomorphisms are split injectives, this will show that $f = g$ in $K(A)$). The morphisms f and g become equivalent in $D(A)$ if and only if $f - g$ becomes equivalent to 0. This happens if and only if there is some Y and a quasi–isomorphism $s \in \mathrm{Hom}_{K(A)}(I, Y)$ such that the following diagram commutes:



(B.4)

Hence, f becomes equivalent to g if and only if there is a quasi–isomorphism $s : I \to Y$ in $K(A)$ such that $s \circ f = s \circ g$. □

Appendix C

Localization of triangulated categories

In [SGA 4.5, Appendix] two methods of localization in triangulated categories are presented an shown to be equivalent. The first is the usual localization by a multiplicative system. The second, which is the one used in [Voev] is localization by a thick subcategory.

Since this version of localization is not as common it is outlined here together with a brief outline of the correspondence to localization by a multiplicative system. We also present an alternative criteria for a subcategory to be thick.

C.1 Localization by a multiplicative system.

Definition 76. A set of morphisms S in a triangulated category $\mathcal{T}$ is said to be a multiplicative system if it has the following properties:

1. If $f, g \in S$ are composable then $f \circ g \in S$. For every object $X \in ob(\mathcal{T})$ the identity of X is in S.

2. Every diagram like the one on the left can be completed to a commutative diagram like the one on the right.

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle s\in S} \\ Z & \xrightarrow{f} & X \end{array} \qquad\qquad \begin{array}{ccc} P & \xrightarrow{g} & Y \\ {\scriptstyle t\in S}\downarrow & & \downarrow{\scriptstyle s\in S} \\ Z & \xrightarrow{f} & X \end{array} \tag{C.1}$$

3. For any two morphisms f, g there following two properties are equivalent:

   (a) There is $s \in S$ such that $s \circ f = s \circ g$.

   (b) There is $t \in S$ such that $f \circ t = g \circ t$.

4. For every $s \in S$ the morphism $s[1]$ is in S.

5. If $X \to Y \to Z \to X[1]$ and $X' \to Y' \to Z' \to X'[1]$ are exact triangles and there are two morphisms f, g making the diagram commute, then there is a third morphism $h \in S$

completing the diagram to a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array} \tag{C.2}$$

C.2 Localization by a thick subcategory.

Definition 77. A subcategory B of a triangulated category A is said to be thick if B is a full subcategory of A and if moreover B satisfies:

1. For every split monomorphism $f : X \to Y$ if X and $Cone(f)$ are in B then Y is in B, and
2. for every split epimorphism $f : X \to Y$ if Y and $Cone(f)$ are in B then so is X.

We will shortly prove that the condition for B to be a thick subcategory is equivalent to saying that B is closed under direct sum.

Recall that a localization of a triangulated category A by a multiplicative system S is defined to be a universal triangulated category $S^{-1}A$ and a functor $A \to S^{-1}A$ such that every morphism in S becomes an isomorphism in $S^{-1}A$.

Definition 78. A localization of a triangulated category A by a thick subcategory is a universal triangulated category A/B and a functor $A \to A/B$ such that every object in B becomes isomorphic to zero in A/B.

The relationship between thick subcategories and systems is given in [SGA 4.5] by a map ϕ that takes thick subcategories to (saturated) multiplicative systems and an inverse ψ.

Definition 79. Let B be a thick subcategory of a triangulated category A. Define $\phi(B)$ to be the set of morphisms f that are contained in a distinguished triangle:

$$X \xrightarrow{f} Y \to Z \to X[1] \tag{C.3}$$

where Z is an object of B.

Let S be a (saturated) multiplicative system in a triangulated category A. Define $\psi(S)$ to bee the full subcategory generated by the objects Z contained in a distinguished triangle:

$$X \xrightarrow{f} Y \to Z \to X[1] \tag{C.4}$$

where f is an element of S.

An explicit description of localization by a thick subcategory, analogous to the calculus of fractions is not given in [SGA 4.5]. However, the equivalence of $S^{-1}A$ and $A/\psi(S)$ can be used.

C.3 An alternate description of thick subcategories.

Proposition 80. *Let B be a full subcategory of a triangulated category. Then the following two conditions are equivalent:*

1. *B is a thick subcategory.*

2. *B is closed under direct sum.*

Proof.

(1) $\implies$ (2): Straightforward. Take the split monomorphism $X \to X \oplus Z$ or the split epimorphism $Y \oplus Z \to Y$. The cone of $X \to X \oplus Z$ is isomorphic to Z (it is homotopic in the category of chain complexes) and so if X and Z are in B then so is $X \oplus Z$. In the other case, the cone of $Y \oplus Z \to Y$ is isomorphic to Z (again, homotopic in the category of chain complexes) and so if Z and Y are in B then so is $Y \oplus Z$.

(2) $\implies$ (1): Suppose that we have a morphism $X \to Y$ with left inverse. Then it follows from Lemma 84 that $Y \cong X \oplus Z$ for the Z completing the exact triangle. By assumption Z is an object of B and B is closed under direct sum, hence, Y is an object of B. The case where f has a right inverse is similar. ☐

Lemma 81. *Let X and Z be objects in a triangulated category and U the third object in the completed triangle $X[-1] \xrightarrow{0} Z \to U \to X$ (Axiom 1b). Then U is isomorphic to $X \oplus Z$.*

Proof. Consider the dotted morphisms completing the morphisms of triangles (Axiom 3) in the following diagram.

$$\begin{array}{ccccccc}
X[-1] & \longrightarrow & 0 & \longrightarrow & X & = & X \\
\| & & \downarrow & & \vdots & & \| \\
X[-1] & \xrightarrow{0} & Z & \longrightarrow & U & \longrightarrow & X \\
\downarrow & & \downarrow & & \vdots & & \downarrow \\
0 & \longrightarrow & Z & = & Z & \longrightarrow & 0
\end{array}$$

These morphism split $Z \to U \to X$ and hence the result. ☐

Lemma 82. *Let $X \xrightarrow{f} Y$ be a morphism in an additive category and suppose that there are morphisms $g, g' : Y \to X$ such that $f \circ g = id_Y$ and $g' \circ f = id_X$. Then $g = g'$.*

Proof. Firstly, $0 = f - f = f \circ id_x - id_Y \circ f = f \circ g \circ f - f \circ g' \circ f = f \circ (g - g') \circ f$. It then follows that $0 = g' \circ 0 \circ g = g' \circ f \circ (g - g') \circ f \circ g = g - g'$. ☐

Corollary 83. *Let $0 \to U \to V \to 0$ be an exact triangle in a triangulated category. Then $U \cong V$.*

Proof. Consider the two morphisms completing the triangle morphisms in the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & V & = & V & \longrightarrow & 0 \\
\downarrow & & \vdots & & \| & & \downarrow \\
0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & 0 \\
\downarrow & & \| & & \vdots & & \downarrow \\
0 & \longrightarrow & U & = & U & \longrightarrow & 0
\end{array}$$

The result now follows from the previous lemma. ☐

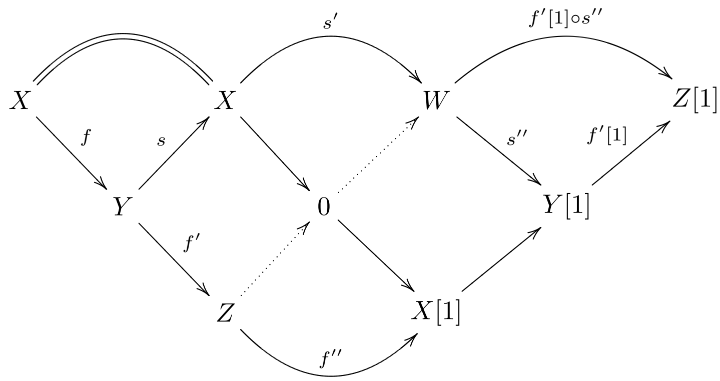
Lemma 84. *Suppose that the following equivalent conditions hold:*

1. *$X \xrightarrow{f} Y$ is a morphism in a triangulated category with left inverse s (that is, $s \circ f = id_X$), or*
2. *$Y \xrightarrow{s} X$ is a morphism in a triangulated category with right inverse f (that is, $s \circ f = id_X$).*

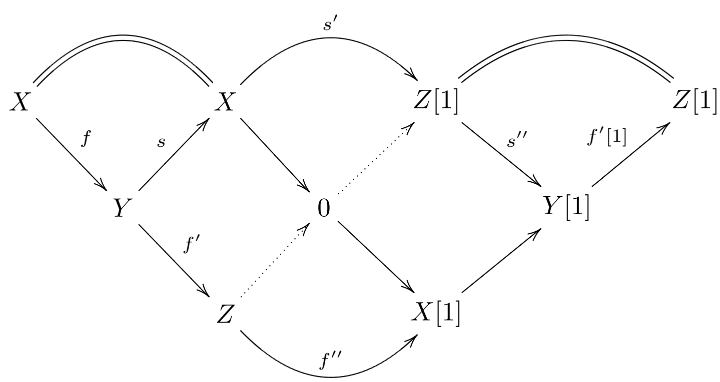
Then $Y \cong X \oplus Z$ where

1. *Z is the object that completes the triangle $X \xrightarrow{f} Y \to Z \to X[1]$ (Axiom 1b), or in the other case*
2. *Z is the object that completes the triangle $Y \xrightarrow{s} X \to Z[1] \to Y[1]$ (Axiom 1b).*

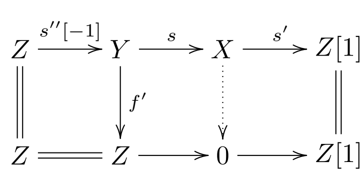
Proof. Axiom 4 gives



for some objects Z, W. Since the dotted morphisms $Z \to 0 \to W \to Z[1]$ form an exact triangle (Axiom 4) it follows from the previous corollary that $W \to Z[1]$ is an isomorphism. So we can replace W by $Z[1]$ in the diagram and obtain



Now since the following commutative diagram can be completed to a morphism of triangles (Axiom 3) we see that $s' = 0$.



So now we have an exact triangle of the form

$$X[-1] \xrightarrow{0} Z \xrightarrow{s''[-1]} Y \to X$$

and so it follows from Lemma 81 that $Y \cong X \oplus Z$. $\square$

Appendix D

Excellent schemes

Definition 85. A ring A is called *catenarian* if for all $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} A$ with $p \subset \mathfrak{q}$ there is a maximal chain of prime ideals of A between $\mathfrak{p}$ and $\mathfrak{q}$ and every such chain has the same length.

A ring A is *universally catenarian* if it is noetherian and every finitely generated A–algebra is catenarian.

Definition 86. Let A be a ring and I an ideal. The *completion* of A at I is the inverse limit of the system

$$A \to A/I \to A/I^2 \to A/I^3 \to A/I^4 \to \dots$$

If A is a local ring with maximal ideal $\mathfrak{m}$ the completion of A is its completion at $\mathfrak{m}$ and is denoted $\hat{A}$.

Definition 87. Let A be a local noetherien ring. The *formal fibers* of A are the fibers of the canonical morphism

$$f : \operatorname{Spec}(\hat{A}) \to \operatorname{Spec}(A)$$

Definition 88. A *radical tower* is a field extension L/F which has a filtration

$$F = L_0 \subset L_1 \subset \cdots \subset L_n = L \tag{D.1}$$

where for each i, $0 \leq i < n$ there exists an element $\alpha_i \in L_{i+1}$ and a natural number n_i such that $L_{i+1} = L(\alpha_i)$ and $\alpha_i^{n_i} \in L_i$.

A *radical extension* is a field extension K/F for which there exists a radical tower L/F with $L \supset K$.

Definition 89. A ring A is said to be *excellent* if it is noetherian and satisfies the following conditions:

1. A is universally catenarian (or equivalently, for every prime $\mathfrak{p}$, $A_{\mathfrak{p}}$ is universally catenarian).
2. For every prime $\mathfrak{p}$, the formal fibers of $A_{\mathfrak{p}}$ are geometrically regular.
3. For every integral quotient ring B of A and every finite radical extension K' of the field of fractions K of B, there is a finite sub–B–algebra B' of K' containing B having K' for its field of fractions, such that the set of regular points of $\operatorname{Spec} B'$ contains a nonempty open set.

Definition 90. A scheme is said to be *excellent* if there is an open affine cover $\operatorname{Spec} A_i$ where each A_i is excellent.

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