On the Neron model of $y^2 = \pi(x^3 + \pi^3)$

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'O my body, make of me always a man who questions'
–Frantz Fanon

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1 Preliminaries

1.1 Understanding Flatness, Regularity and Smoothness

Flatness

Flatness is essentially an algebraic concept. We first encounter it while working with modules. Applied to morphisms between schemes, flatness allows for geometric interpretation. It ensures the 'continuity' of fibers in some sense. I state without proof a few standard definitions and results regarding flat modules, before discussing flat morphisms of schemes.

1.1.1 Definition. For a ring A, an A-module M is said to be *flat* if for any short exact sequence of A-modules:

$$0 \to N \to N' \to N'' \to 0$$

the sequence

$$0 \to M \otimes_A N \to M \otimes_A N' \to M \otimes_A N'' \to 0$$

is exact.

Since the tensor product is right exact in any case, defining M to be flat over A is equivalent to saying $M \otimes_A -$ is left exact.

1.1.2 Proposition. *Let A be a ring. Then:*

(a) Every free A-module is flat

(b) The tensor product of two flat A-modules is flat over A

(c) Let B be an A-algebra. If M is flat over A, then $M \otimes_A B$ is flat over B

(d) Let B be a flat A-algebra (i.e flat for its A-module structure,with corresponding homomorphism called a flat homomorphism). Then every B-module that is flat over B is flat over A.

(e) (Flatness is local) An A-module M is flat over $A \Leftrightarrow M_p = M \otimes_A A_p$ is flat over A_p for all p. On the other hand the ring homomorphism $\varphi : A \to B$ is flat $\Leftrightarrow \forall q \in$ Spec A, the homomorphism $A_{\varphi^{-1}(q)} \to B_q$ is flat

(f) Over a Noetherian local ring, a finitely generated module is flat $\Leftrightarrow$ it is free. (Note that this says something about $O_X - Modules$.

1.1.3 Definition. An A-module M is said to be *faithfully flat* over A if it is flat, and what's more, we have the implication: $M \otimes_A N = 0 \Rightarrow N = 0$ for every A-module N. The notion of faithful flatness is useful in that it imposes an *if and only if* condition on the preserving of exactness with tensoring. More precisely:

1.1.4 Lemma. *Let B be a faithfully flat A-algebra. Then an A-module M is flat over $A \Leftrightarrow B \otimes_A M$ is flat over M.*

Note: For finitely generated modules over Noetherian rings, flat and locally free are equivalent. Thus we have the following corollary to the Lemma above:

1.1.5 Corollary. *Let A be a Noetherian ring, B a faithfully flat A-algebra and M a finitely generated A-module. Then M is locally free $\Leftrightarrow$ $B \otimes_A M$ is locally free.*

1.1.6 Definition. A morphism of schemes $f : X \to Y$ is *flat* at $x \in X$ if the induced homomorphism on stalks $O_{Y,y} \to O_{X,x}$ where $y = f(x)$ is flat. The morphism f is said to be flat if it is flat at all $x \in X$. f is said to be *faithfully flat* if it is flat and surjective.

1.1.7 Proposition. *Properties of flat morphisms: (Result directly from the definition and properties of flat homomorphisms)*

(a) Open immersions are flat morphisms.

(b) Flat morphisms are stable under base change.

(c) The composition of two flat morphisms is flat.

(d) The fibered product of two flat morphisms is flat.

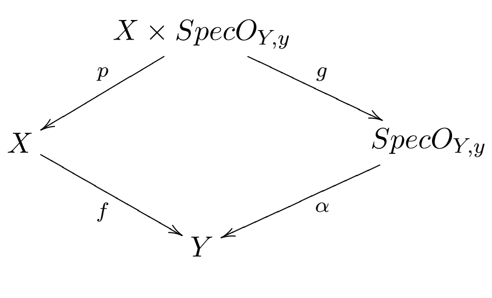
(e) Let $\varphi : A \to B$ be a ring homomorphism. Then the corresponding morphism of affine schemes $f : SpecB \to SpecA$ is flat $\Leftrightarrow$ $\varphi : A \to B$ is flat.

1.1.8 Remark. Given a Noetherian scheme X and a coherent sheaf $\mathcal{F}$ over X, $\mathcal{F}$ is flat over X (i.e for any $x \in X \mathcal{F}_x$ is flat over $O_{Y,f(x)}$) $\Leftrightarrow$ it is locally free. (This is simply a translation of the remark for modules)

1.1.9 Examples. Examples and non Examples:

- Any algebra over a field is flat. Thus the structural morphism of an algebraic variety over a field k is flat.
- Closed immersions are generally not flat. In fact, given a closed immersion into a Noetherian scheme, we find that it is flat $\Leftrightarrow$ it is also open. Consider the closed immersion $Speck(x)/(x) \to Speck(x)$. Intuitively it is clear that this morphism isn't flat since away from the origin all fibers are empty. Algebraically too, this morphism corresponds to the homomorphism $k(x) \to k(x)/(x)$ and since $k(x)/(x)$ is not torsion free and $k(x)$ is a PID, it follows from Propostion 1.5 (e) that the given morphism isn't flat.
- $\mathbb{Z}/n\mathbb{Z}$ is not flat over $\mathbb{Z}$, since over a PID flat is equivalent to torsion free
- Consider $f : X := Speck[t, x, y]/(ty - x^2) \to Y := Speck[t]$ This is a flat morphism since $k[t, x, y]/(ty - x^2)$ is torsion free as a $k[t]$-module.

1.1.10 Remark. Consider the following diagram where X and Y are schemes and $y = f(x)$:



Note that α is the natural map arising from the canonical map $O_Y(U) \to O_{Y,y}$ where U is an open containing y. This map corresponds to the map $\varphi : SpecO_{Y,y} \to U$ under the anti-equivalence of categories, and φ composed with the open immersion $U \hookrightarrow Y$ gives us α. Now, since flat morphisms are stable under base change, saying that $f : X \to Y$ is flat at $x \in X$ is the same as saying $g : X \times SpecO_{Y,y} \to SpecO_{Y,y}$ is flat at the inverse image of x under the projection$p : X \times SpecO_{Y,y} \to X$.
This provides an intuitive glimpse into the geometric implications of flatness with respect to fibers.

1.1.11 Lemma. *Let X and Y be schemes with Y irreducible. Let $f : X \to Y$ be a flat morphism. Then every non-empty open subset U of X dominates Y. If X has a finite number of irreducible components then each of these dominates Y.*

Proof : First we ma assume $Y = SpecA$ and $U = SpecB$ to be affine. Then the map $SpecB \to SpecA$ is an open immersion $U \hookrightarrow X$ composed with f. Thus since the composition of two flat morphisms is flat, we have that U is flat over Y. Let ξ be the generic point of Y and N the nilradical of A. By the flatness of B we have the following:

$$B/NB = B \otimes_A (A/N) \subseteq B \otimes_A Frac(A/N) = B \otimes_A k(\xi) = O(U_\xi)$$

If $U_\xi = \emptyset$, then $B = NB$ is nilpotent, implying $U = \emptyset$, contradiction to our choice of U. Thus $\xi \in f(U)$, giving the result. The second statement follows from the fact that every irreducible component contains a non-empty open set.

1.1.12 Proposition. *Let $f : X \to Y$ be a flat morphism of schemes with Y having only a finite number of irreducible components. If Y is reduced (respectively irreducible, integral), and if the generic fibers of f are also reduced (respectively irreducible, integral), then X is reduced (respectively irreducible, integral).*

1.1.13 Definition. A scheme X is said to be *Dedekind* if it is a locally Noetherian, normal scheme of dimension 0 or 1.

1.1.14 Proposition. *Let $f : X \to Y$ be a morphism with Y a Dedekind scheme and X reduced. Then f is flat $\Leftrightarrow$ every irreducible component of X dominates Y.*

Proof : Suppose f is flat, then Y Dedekind $\Rightarrow$ irreducible and thus the result is a direct consequence of the Lemma above. Now suppose that every irreducible component of X dominates Y.For the generic point ξ of Y, $O_{Y,\xi} = K(Y)$ is a field. Thus $O_{X,x}$ is flat over $O_{Y,\xi}$ for $\xi = f(x)$. We now only need to check for points $y \in Y$ closed. Choose such a $y \in Y$. Then $O_{Y,y}$ is a DVR. Let its uniformising parameter be π. The image of π in $O_{X,x}$, say t, will not be contained in any minimal prime ideal, because if it was, then y would be the image of the generic point of one of the irreducible components of X, and hence would be the generic point of Y, contrary to our choice of y. There is a result which states that in a reduced ring, every zero-divisor is contained in a minimal prime (It is easy to prove if we look at the localization homomorphism from A to A_a, where a is some zero-divisor in A. Thus the image of π in $O_{X,x}$ is not a zero divisor. Thus $O_{X,x}$ is torsion free which is equivalent to saying it is flat over $O_{Y,y}$ since the latter is a PID.

1.1.15 Corollary. *Let Y be Dedekind and X integral. Then if $f : X \to Y$ is non-constant ($f(X)$ not reduced to a point), then f is flat.*

Proof : Y Dedekind $\Rightarrow$ it is irreducible and of dimension 1. Therefore we have that $f(X)$ is dense in Y. Thus by above, f is flat.

Flatness and fibers

One main consequence of flat morphisms is that the dimension of their fibers behave well in some sense. The following theorem formulates this more precisely.

1.1.16 Theorem. *Let $f : X \to Y$ be a flat morphism between locally Noetherian schemes. For $x \in X$ and $y = f(x)$, we denote $X_y = X \otimes_Y Speck(y)$ to be the fiber of X at y. Then we have:*

$$dim_x X_y = dim_x X - dim_y Y$$

1.1.17 Corollary. *Let $f : X \to Y$ be a flat, surjective morphisms between schemes of finite type over a field k. Suppose Y is irreducible, and X pure (i.e all irreducible components are equidimensional). Then $\forall y \in Y$, the fiber X_y is pure with*

$$dim X_y = dim X - dim Y$$

1.1.18 Remark. In particular, what the corollary implies is that for a flat morphism between irreducible algebraic varieties, the dimension of fibers is constant.

Note that while dimension is one of the properties that is preserved with flatness, there are others which are not: For example, consider the same morphism we did earlier $f : Speck[t,x,y]/(ty - x^2) \to Speck[t]$. $Y = Speck[t]$ is nothing but $\mathbb{A}^1_k$, the affine line over k. So each element $a \in k$ corresponds to a closed point on Y and in turn to a fiber X_a of f. When $a \neq 0$, $X_a = (ax = y^2) \subseteq \mathbb{A}^2$ is a nonsingular curve and hence reduced. However at $a = 0$ the fiber $X_0 = (x^2 = 0)$ is a double line and hence not reduced. Thus the property reduced is not preserved by flatness.

Regularity

1.1.19 Definitions. Let A be a Noetherian local ring with maximal ideal $\mathbf{m}$, and residue class field k. Then A is said to be *regular* if $dim_k(\mathbf{m}/\mathbf{m}^2) = dimA$., where $dimA$ is the Krull dimension of A. By Nakayama's lemma, this is equivalent to saying that A is regular $\Leftrightarrow$ $\mathbf{m}$ is generated by $dimA$ elements.
Let $dimA = d$. Then any system of generators for $\mathbf{m}$ with d elements is called a *system of parameters*. For $d = 1$, the single generator is called a *uniformising parameter.*

Note: The localization of a regular ring at a prime ideal is again regular.

1.1.20 Proposition. *Let $(A, \boldsymbol{m})$ be a regular Noetherian local ring. Let $f \in A \backslash \{0\}$. Then A/fA is regular if and only if $f \notin \boldsymbol{m}/\boldsymbol{m}^2$*

This follows from the fact that A is infact an integral domain.

1.1.21 Definitions. Let X be a locally Noetherian scheme. Let $x \in X$ be a point, Then X is regular at x if $O_{X,x}$ is regular. We say that the scheme X is regular, if all its local rings are regular, or equivalently if all the local rings at closed points of the affine open subschemes in some affine open cover are regular. (by Note above).
A point that is not regular is called a *singular* point, similarly a scheme that is not regular is called singular.

1.1.22 Examples.

- Every DVR is regular.
- Any normal, locally Noetherian scheme of dimension 1 is regular
- An algebraic curve over a field k is regular $\Leftrightarrow$ it is normal.

1.1.23 Theorem.*(Jacobian criterion for Regularity)*
Let k be a field, and $X = V(I)$, a closed subvariety of $\mathbb{A}_k^n = Spec\ k[T_1, ...T_n]$ with $f_1, ...f_r$ a system of generators for I. Let $x \in X(k)$ be a rational point. Consider the Jacobian matrix:

$$J_x = (\partial f_i(x)/\partial T_j)_{1\leq i\leq r, 1\leq j\leq n}$$

Then X is regular at $x \Leftrightarrow$

$$rank J_x = n - dim O_{X,x}$$

.

Proof : Define a map $D_x : k[T_1...T_n] \to (k^n)^\vee$ by sending $D_x P : (t_1, ...t_n) \mapsto \sum_{1\leq i\leq n} \partial f_i(x) t_i/\partial T_j$. Then the restriction on D_x to $\mathbf{m}$, the maximal ideal corresponding to x, gives us an isomorphism $\mathbf{m}/\mathbf{m}^2 \simeq (k^n)^\vee$. Let $X = V(I)$ be a closed subvariety of $\mathbb{A}_k^n$ defined by I. And let $x \in X(k)$. We have a canonical map induced by the closed immersion $X \hookrightarrow \mathbb{A}_k^n$, which is as follows: $(\mathbf{m}/\mathbf{m}^2)^\vee \to (\mathbf{m'}/\mathbf{m'}^2)^\vee$, where $\mathbf{m'}$ is the maximal ideal associated to x when seen as a point in $\mathbb{A}_k^n$. Identifying $(\mathbf{m}/\mathbf{m}^2)^\vee$ with k^n, this map gives us an isomorphism from $(\mathbf{m}/\mathbf{m}^2)^\vee$ to $(D_x I)^\perp$, allowing us to identify $(\mathbf{m}/\mathbf{m}^2)^\vee$ with the set

$$\{(t_1...t_n) \in k^n | \sum_{1\leq i\leq n} \partial P(x) t_i/\partial T_j = 0\ \forall P \in I\}.$$

Thus $dim \mathbf{m}/\mathbf{m}^2 = dim(D_x I)^\perp = n - (D_x I)$. But $(D_x I)$ is generated by the line vectors of our Jacobian. $\Rightarrow rank J_x = dim(D_x I)$, giving us the result.

Smoothness

We first encounter smooth morphisms in the context of algebraic varieties over a field k- An algebraic variety X is said to be smooth over a field k if $X_{\overline{k}} = X \times_k \overline{k}$ is regular. We now extend this concept to define smooth morphisms between general schemes.

1.1.24 Definitions. Let $A \to B$ be a morphism of rings. We say that B is *finitely presented* as an A-algebra if B is finitely generated as an A-algebra, (i.e if there exists a surjective homomorphism $A[x_1, ..., x_n] \to B$) with the property that the kernel of this homomorphism is a finitely generated $A[x_1, ..., x_n]$-ideal. If A is Noetherian, clearly the conditions finitely presented and finitely generated are equivalent for algebras over A.

A morphism $f : X \to Y$ of schemes is called *locally of finite presentation* at a point $x \in X$ if there exist affine neighbourhoods $U = Spec A$, $V = Spec B$ of $f(x)$ and x respectively, such that $f(V) \subseteq U$ and B is a finitely presented A-algebra. The morphism f is called locally of finite presentation if it is locally of finite presentation at every point of X. That finitely presented $\Rightarrow$ finite type is clear. On X Noetherian, the conditions are equivalent.

1.1.25 Definitions. Let $f : X \to Y$ be a morphism of schemes. Then f is called *smooth* at a point $x \in X$ if the following conditions hold:
(a) f is flat at x
(b) f is locally of finite presentation at x

(c) the fiber $X_{f(x)}$ is geometrically regular at x, i.e. all the localizations of the semi-local ring $O_{X,x} \otimes_{O_{Y,f(x)}} \overline{k(f(x))}$ are regular, where $k(f(x))$ denotes the residue field of $O_{Y,f(x)}$ and $\overline{k(f(x))}$ its algebraic closure.
We say that f is smooth if it is smooth at every point of X.This amounts to saying that
(a) f is flat
(b) f is locally of finite presentation
(c) the fibers of f are geometrically regular.

Note: Smooth morphisms are stable under base change. It is important to note that geometric regularity is a property of schemes over a field , whereas regularity is a property of schemes in general. Thus smoothness is a relative notion, while regularity is absolute.

1.1.26 Examples.

- Define $f : X \to Spec\mathbb{Z}$ with $X = Spec\mathbb{Z}[t]/(t^2 - 2)$. f is induced by the injection $\mathbb{Z} \hookrightarrow \mathbb{Z}[t]/(t^2 - 2)$. Thus f is finite and flat since $\mathbb{Z}$ is regular of dimension 1, and f is dominant. But f is not smooth since the fiber of a closed point in $\mathbb{Z}$ corresponding to the prime ideal $2\mathbb{Z}$ is $Spec(\mathbb{Z}/2t)/t^2$ which is not regular.

- Let X and Y be Noetherian, regular schemes. For example, let $X = Y = Speck[x]$, where k is algebraically closed. Define $f : X \to Y$ by sending $x \mapsto x^2$. Then f is clearly surjective (Indeed $f(X)$ contains all $\overline{k}$ points of X and also the generic point). However, f is not smooth since the fiber $X_{y=0} = Speck[x] \times_k Speck(y) = Spec(k(x)/x^2)$ which is not regular.

- For schemes of finite type over a perfect field, the notions of smooth and regular are equivalent. However this is not true for schemes of finite type over non-perfect fields - Let k_0 be a field of characteristic $p \geq 2$. Set $k = k_0(t)$. Let $X \subseteq \mathbb{A}^2_k$ defined by $y^2 = x^p - t$. It is easy to see with the Jacobian criterion that X is regular everywhere, but not smooth.

- Take the affine scheme X over a DVR R defined by the equation $x^2 = t$, where t is a uniformising parameter.

$$X = Spec(R[x]/(x^2 - t))$$

  It is integral and dominates R, hence it is flat. It is regular (normal and of dimension 1) but not smooth over R because the special fiber, namely, the fiber over the closed point of R is given by $x^2 = 0$ is not regular. A similar example with similar argument is that of $X = R[x, y]/(xy - t)$.
  Both these examples serve to demonstrate our assertion about regularity being absolute while smoothness is relative.

1.2 Divisors, Invertible sheaves and Maps to Projective Space

Divisors

In order to study the intrinsic geometry of a variety or scheme, it is sometimes useful to look at it's closed subschemes of dimension strictly smaller than the scheme itself. We thus introduce the notion of Divisors, which are in some sense a generalization of closed subschemes of codimension 1. There are several ways to define Divisors, but the two most useful for our purpose are Weil and Cartier divisors, the notions of which, as we will show, co-incide in certain situations.

Weil Divisors

Weil Divisors, although geometrically more intuitive than Cartier Divisors, are restricted to Noetherian, integral schemes and therefore do not suffice to study curves which are, say, not reduced.

1.2.1 Definitions. Let X be a Noetherian, integral, separated scheme, regular in co-dimension 1 (i.e For every point x of codimension 1, the local ring $O_{X,x}$ is regular.) A *prime divisor* on X is defined to be a closed integral subscheme of codimension 1. A *Weil Divsor* is an element of $DivX$, the free abelian group generated by the prime divisors. It is thus a formal sum $\sum n_i Y_i$ where the Y_i's are prime divisors and the n_i's $\in \mathbb{Z}$, with only finitely many n_i's different from 0. If all the n_i's are ≥ 0, the divisor is called *effective*.

Note that a prime divisor Y, being irreducible and of codimension 1, has a unique generic point η and the the ring $O_{X,\eta}$ is a discrete valuation ring, with field of fractions, say K. The corresponding discrete valuation v is called the valuation of Y. For any non-zero rational function $f \in K^*$, $v_Y(f) \in \mathbb{Z}$ and $v_Y(f) = 0$ for all but finitely many Y. If $v_Y(f)$ is positive f is said to have a *zero* along Y, and a *pole* if negative. The valuations allows to define divisors of rational functions in a natural way:

The divisor of f, denoted $(f) := \sum v_Y(f)Y$, sum taken over all prime divisors.
A divisor of this form, is called a *principal divisor.*

By the properties of valuations, we have a a homomorphism from the multiplicative group K^* to the additive group $DivX$,

$$\varphi : K^* \to DivX$$

sending a function f to its divisor (f), where $f.g \mapsto (f.g) = (f) + (g)$. $Im\varphi$ is the thus the subgroup of principal divisors in $DivX$.

We can now define an invariant of of such a scheme X,namely its *Divisor class group*, denoted $Cl(X) := DivX/Im\varphi$, We say that two divisors D and D' are *linearly equivalent* if $D - D'$ is a principal divisor.

1.2.2 Example. For a number field K and its ring of integers O_K, consider the affine scheme $SpecO_K$. (More generally for any $X = SpecA$ with A Dedekind.), $Cl(X)$ is nothing but the Ideal Class group as defined in algebraic number theory. Indeed, since each point of codimension 1 in a dimension 1 scheme is closed, we have an isomorphism given by associating to each Weil Divisor $D = \sum n_i x_i$, the fractional ideal $\prod \wp_i^{n_i}$, where $\wp_i$ is the maximal ideal associated to x_i.

1.2.3 Proposition. *A Noetherian ring A is a UFD* $\Leftrightarrow$ *$X = SpecA$ is normal and $Cl(X) = 0$.*

(Indeed, A UFD is integrally closed, so X is normal. And A is a UFD $\Leftrightarrow$ every prime ideal of height 1 is principal. So the statement above translates into proving that if A is an integral domain, then every prime ideal of height 1 is principal $\Leftrightarrow$ $Cl(X) = 0$, which follows from standard commutative algebra results.

Note that this generalizes the result from Algebraic number theory that A is a UFD $\Leftrightarrow$ its ideal class group is 0.

Since the divisor class group is not always easy to compute, lets look at at the particular example of projective n-space, where we can say something explicit about $Cl(X)$ Note that divisors are particularly important in studying embeddings of curves in projective space-A point P on a curve C in $\mathbb{P}^2_k$ can be recovered completely as a point of $\mathbb{P}^2_k$ itself, by a family of divisors parametrized by the set of lines passing through P.

1.2.4 Proposition. *Let X be projective space $\mathbb{P}^n_k$ over a field k and for $n \geq 1$. For a divisor $D = \sum n_i Y_i$, we define the degree of D to be $\sum n_i degY_i$ where $degY_i$ is the degree of the hypersurface (since by definition Y_i is of codimension 1). Let H be the hyperplane $x_0 = 0$. Then:*
(a) For any divisor of deg d, $D \sim dH$
(b) For any $f \in K^*$, deg f = 0
(c) $Cl(X) \cong \mathbb{Z}$

Proof : Any divisor D of deg d can be expressed as the difference of two effective divisors $D_1 - D_2$, of deg d_1 and d_2 respectively, where $d_1 - d_2 = d$. Now, any effective divisor is principal, since it can be written as (g) for a homogeneous polynomial g. Because every hypersurface corresponds to a prime ideal of height 1,and since the coordinate ring $k[x_0...x_n]$ is a UFD, by the above proposition, the prime ideal is principal. Thus, $D = (g_1) - (g_2) = (g_1/g_2)$ Thus $D - dH = (g_1/x_0^d g_2)$, which is the divisor of a rational function and hence principal. Thus $D \sim dH$. Any homogeneous polynomial g of deg d can be factored into irreducible polynomials $g_1^{n_i}...g_r^{n_r}$, so each of the g_i's defines a hypersurface Y_i of deg d_i and so we can define the divisor of g to be $\sum n_i Y_i$. Since a rational function f can be written as the quotient of two homogeneous polynomials (g/h), and $(g/h) = (g) - (h)$, we have that $degf = 0$. Let δ be the map from $Cl(X)$ to $\mathbb{Z}$ sending $D = \sum n_i Y_i$ to $degD = \sum n_i degY_i$. Now since $degH = 1$, $\delta([H]) = 1$ and since $\forall D, D \sim dH$, δ is surjective. Injectivity follows from (b). Thus the isomorphism.

1.2.5 Proposition. *Let X be a Noetherian, integral scheme, regular in codimension 1, and let Z be a proper closed subset of X. Set $U = X - Z$. Then:*
(a) There is a surjective homomorphism $\varphi : Cl(X) \to Cl(U)$

$$\sum n_i Y_i \mapsto \sum n_i (Y_i \cap U)$$

wherever $(Y_i \cap U) \neq \emptyset$
(b) If $codim(Z, X) \geq 2$, then φ is an isomorphism
(c) If Z is an irreducible subset of codimension 1, there exists an exact sequence:

$$\mathbb{Z} \to Cl(X) \to Cl(U) \to 0$$

where the first map sends 1 to $1.Z$.

Proof : (a) That such a homomorphism exists is easy to see since if Y is a prime divisor on X, then $Y \cap U$ is either empty or a prime divisor on U. For any $f \in K^*$ and $(f) = \sum n_i Y_i$, considering f as a rational function on U will give you $(f)_{|U} = \sum n_i (Y_i \cap U)$, making valid our homomorphism. Since every prime divisor on U is a restriction of it's closure on X, the homomorphism is indeed surjective. (b) Divisors are only defined on elements of codimension 1, so $DivX$ and $Cl(X)$ only depend on subsets of codimesion 1. Thus, removing a subset of codimension ≥ 2, means that U has the same divisors as X,giving us the isomorphism. (c) After (a), we already have the surjectivity of φ. Thus, in order to prove exactness, we only need to show that $ker\varphi = Imi$, where i is the first arrow of the sequence. $Ker\varphi$ just contains those divisors for which the n_is (which are $\neq 0$) associated to the Y_is are all contained in Z. So for Z irreducible, this is the subgroup generated by $1.Z$, which is nothing by Imi by definition.

Cartier Divisors

We now want a description of Divisors for arbitrary schemes, so we define something that locally looks like the divisor of a rational function. We begin by defining the sheaf of meromorphic functions, which generalizes the definition of rational functions to schemes that are not necessarily reduced. We do this by replacing the field of fractions of an integral domain by the total ring of fractions for any ring.

1.2.6 Definitions. Let A be a ring, and let $Frac(A)$ denote its complete ring of fractions. It is a ring, containing A as a subring. Any element $f \in A$, which is not a zero divisor, is called a regular element. Let $R(A)$ denote the multiplicative set of regular elements of A. Then $Frac(A)$ is nothing but the localization $R(A)^{-1}A$

The Sheaf of Stalks of Meromorphic functions: Let X be a scheme. $\forall U \in X$ open, define the sheaf $\mathcal{R}_X(U) := \{\ a \in O_X(U) | a_x \in R(O_{X,x}) \forall x \in X\}$, the set of elements with regular stalks.

1.2.7 Proposition. *Keeping the notation of above, we have the following:*
(a) $\mathcal{R}_X(U) = R(O_X(U))$ *if* U *is affine.*
(b) $\exists$ *a unique presheaf of algebras* $\mathcal{K}'_X$ *on* X, *containing* O_X *and verifying:*
(i) $\forall U \in X$ *open,* $\mathcal{K}'_X(U) = \Re_X(U)^{-1}O_X(U)$. *In particular,* $\mathcal{K}'_X(U) = Frac(O_X(U))$ *if* U *is affine.*
(ii) $\forall U \in X$ *open, the canonical homomorphism* $\mathcal{K}'_X(U) \to \prod \mathcal{K}'_{X,x}$ *is injective.*
(iii) If X *is locally Noetherian,* $\mathcal{K}'_{X,x} \simeq Frac(O_{X,x})$

1.2.8 Definition. Denote the sheaf associated to the presheaf $\mathcal{K}'_X$ by $\mathcal{K}_X$. This is called the *sheaf of stalks of meromorphic functions* on X. It contains O_X as a subsheaf. If X is locally Noetherian, then $\mathcal{K}'_{X,x} = \mathcal{K}_{X,x} = Frac(O_{X,x})$. An element of $\mathcal{K}_X(X)$ is called a *meromorphic function.* Note that this sheaf is analogous to the constant sheaf defined by the function field of an integral scheme, and in the case where X is integral, it is nothing but the constant sheaf defined by $K(X)$.
We denote the set of invertible elements of $\mathcal{K}_X$ by $\mathcal{K}^*{}_X$.

1.2.9 Definitions. A *Cartier Divisor* is a global section of the quotient sheaf $\mathcal{K}^*_X/O^*_X$. It is an element of the group $H^0(X, \mathcal{K}^*_X/O^*_X)$. A *principal Cartier divisor*, denoted $div(f)$, is an element of $H^0(X, \mathcal{K}^*_X/O^*_X)$, which is an image of an element $f \in H^0(X, \mathcal{K}^*_X)$. Two Cartier divisors, D_1 and D_2 are linearly equivalent if $D_1 - D_2$ is principal. An *effective* Cartier divisor is one which lies in the image of the canonical map $H^0(X, \mathcal{K}^*_X \cup O_X) \to H^0(X, \mathcal{K}^*_X/O^*_X)$. Since Cartier divisors are defined as sections of a sheaf, we have an obvious notion of restriction of a divisor to an open subset. By definition of quotient sheaves, we can describe a Cartier divisor D by giving an open cover $\{U_i\}$ of X and an $f_i \forall i$ which is the quotient of two regular elements of $O_X(U_i)^*$, where $f_{i|U_i \cup U_j} \in f_{j|U_i \cup U_j} O_X(U_i \cup U_j)^* \forall i, j$. Two such systems $\{(U_i, f_i)_i\}$ and $\{(V_j, g_j)_j\}$ represent the same Cartier divisor if f_i and g_j differ by a multiplicative factor in $O_X(U_i \cup V_j)^*$ The group of these isomorphism classes of Cartier divisors is denoted by CaCl(X).

Recall that an invertible sheaf $\mathcal{L}$ on a scheme X is defined to be a locally free O_X-module of rank 1. Isomorphism classes of Cartier divisors can be related to invertible sheaves in the following manner.
To every Cartier Divisor D, represented by the system $\{U_i, f_i\}$, we associate a subsheaf $O_X(D) \subset \mathcal{K}_X$, defined by

$$O_X(D)_{|U_i} = f_i^{-1} O_X(U_i)$$

It is infact an invertible sheaf and enables us to define a homomorphism from $CaCl(X)$ to $PicX$, the group of equivalence classes of invertible sheaves of $\mathcal{K}_X$.

1.2.10 Proposition. *Let $\rho : CaCl(X) \to PicX$ be the additive map defined by sending*

$$D \mapsto O_X(D).$$

($\rho(D_1 + D_2) = O_X(D_1)O_X(D_2) \simeq O_X(D_1) \otimes_{O_X} O_X(D_2)$.)
(a) ρ is an injective homomorphism
(b) The image of ρ corresponds to the invertible sheaves of $\mathcal{K}_X$.

Proof : (a) ρ sends a principal Cartier divisor to a free sheaf of rank 1, which is indeed an element of $PicX$. Thus ρ is infact a homomorphism from $CaCl(X)$ to $PicX$. A divisor $D \in ker\rho, \Rightarrow \exists f \in H^0(X, O_X(D)) \ni O_X(D) = fO_X$ By definition of $O_X(D), \Rightarrow D = div(f) \Rightarrow D$ is a principal divisor, thus in the equivalence class of 0, proving injectivity of ρ.
(b) For an invertible sheaf $\mathcal{L} \in \mathcal{K}_X$, we can choose an open covering $U_{i i}$ of X such that $\mathcal{L}_{|U_i}$ is free and generated by some $f_i \in \mathcal{K}'_X(U_i) \forall i$. Thus the system $U_i, f_{i i}$ is the desired representative for the Cartier divisor mapping to $\mathcal{L}$.

1.2.11 Corollary. *In the case where X is integral, ρ is an isomorphism.*

(Indeed,because here every invertible sheaf will be isomorphic to a subsheaf of $\mathcal{K}_X$.)

Note If D is an effective Cartier divisor, then $O_X(-D)$ is a sheaf of ideals of O_X. Consequently, D is naturally endowed with the closed subscheme structure $V(O_X(-D))$ of X.

For reasons that will become apparent later, it is important for us to be able to determine criteria for when the sheaf associated to a Cartier divisor is infact ample. We state without proof one very useful criterion.

1.2.12 Proposition. *Let X be a projective curve over a field k. Let $D \in CaCl(X)$ be a Cartier divisor. Then $O_X(D)$ is ample if and only if $degO_X(D)|_{X_i} > 0$ for every irreducible component X_i of X.*

Relating Weil and Cartier Divisors

In a particular setting, namely we can talk about both Cartier and Weil divisors on a scheme, and find a meaningful way to relate the two. More precisely, we have:

1.2.13 Proposition. *Let X be a Noetherian, integral, separated scheme for which all local rings are UFDs. Then, the group $Div(X)$ of Weil divisors and the group $\mathcal{K}^*/O_X^*$ of Cartier divisors are isomorphic, the principal divisors in each corresponding under this isomorphism.*

Since a UFD is integrally closed, X satisfies the property of being regular in codimension 1, ensuring that we can talk about Weil divisors in this setting. As we already know, for X integral, the sheaf $\mathcal{K}$ is nothing but the constant sheaf given by the function field K of X. Suppose U_i, f_i is a representative system for a Cartier divisor $\in \mathcal{K}^*/O_X^*$ such that the U_i form an open cover of X and each $f_i \in \mathcal{K}^*(U_i) = K^*$. We associate a Weil divisor D to this Cartier divisor by taking for each prime divisor Y, $v_Y(f_i)$ to be its coefficient, i being an index for which $Y \cap U_i \neq$, i.e $D = \sum v_Y(f_i)Y_i$. D is well defined, since if j were any other such index, then $f_i f_j$ is invertible on

$U_i \cap U_j \Rightarrow v_Y(f_i f_j) = 0 \Rightarrow v_Y(f_i) = v_Y(f_j)$.
Conversely we find a way to associate a Cartier divisor to each Weil divisor. Let D be a Weil divisor on X, and $x \in X$ any point. For each D, we then have a D_x, which is a Weil divisor on the local scheme $SpecO_{X,x}$, which is principal since $O_{X,x}$ is a UFD by assumption. Let $D_x = (f_x), f_x \in K$. Since $D_{|x} = D_x = (f_x)$, we can find an open neighbourhood $U_x \ni x$, such that $D_{|U_x} = (f_x)_{|U_x}$. Covering X with such U_x, the U_x, f_{xx} form a representative system for the Cartier divisor on X that we will associate to D. This is infact well-defined, since by proposition 0.1.4.3, we have that X is normal. Thus if f, f' give the same Weil divisor on some open $U \subseteq X$, then $f/f' \in O^*(U)$, thereby giving the same Cartier divisor.

Thus we have constructions which are inverse to one another, making the two groups isomorphic.

Morphisms to Projective Space

The importance of relating invertible sheaves to the isomorphism classes of Cartier divisors becomes apparent once we exploit the relationship between invertible sheaves and maps to projective space. Indeed, a morphism from a scheme X to some projective space can be determined entirely by giving an invertible sheaf $\mathcal{L}$ on X and a set of it's global sections. We show this formally, and outline a more explicit relationship between the two. Why this proves to be extremely useful to us will become clear once we study blow-ups and contractions, which are central to the problem of resolution of singularities.

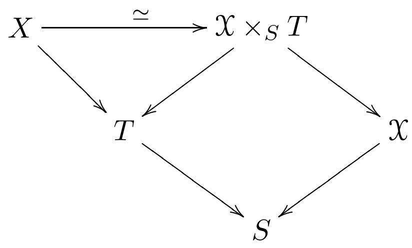
1.2.14 Theorem. *Let A be a ring, and let X be a scheme over A.*
If $\varphi : X \to \mathbb{P}^n_A$ is an A-morphism then the sheaf $\varphi^*(O(1))$ is an invertible sheaf on X generated by the global sections $s_i = \varphi^*(x_i),\ i = 0, 1, ...n$.
Conversely, if $\mathcal{L}$ is an invertible sheaf on X generated by $s_0, ..., s_n \in \mathcal{L}(X)$, then there exists a unique A-morphism $\varphi : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong \varphi^*(x_i)$ with $s_i = \varphi^*(x_i)$.

On $\mathbb{P}^n_A = Proj\ A[x_0, ...x_n]$ we naturally have the invertible sheaf $O(1)$ generated by the global sections given by the homogeneous coordinates $(x_0, ..., x_n)$. So for any scheme X and a morphism $\varphi : X \to \mathbb{P}^n_A$ we then have a sheaf on X given by $\mathcal{L} = \varphi^*(O(1))$ which is indeed invertible and which is generated by the global sections $s_i = \varphi^*(x_i) \in \mathcal{L}(X)$. Thus, given a morphism from X to projective space, we have found a corresponding invertible sheaf on X determined entirely by this morphism.
Now, suppose we are given an invertible sheaf $\mathcal{L}$ on X, generated by some global sections $s_0, ...s_n$. Define, for each i, the open subsets $X_i = \{P \in X | (s_i)_P \notin \mathbf{m}_P \mathcal{L}_P\}$ of X. Note that since the s_i generate $\mathcal{L}$, these X_i cover X. Now define morphisms from each X_i to the standard affine opens $U_i = D_+(x_i)$ of $\mathbb{P}^n_A$. (Note $U_i = SpecA[y_0, ...y_n]$ with $y_j = x_j/x_i$ and $y_i = 1$ omitted.) By the antiequivalence of categories we have the corresponding ring homomorphism from $A[y_0, ...y_n] \to O_{X_i}(X_i)$ sending $y_j \mapsto s_j/s_i$ which is a well defined element of $O_{X_i}(X_i)$ since $(s_i)_P \notin \mathbf{m}_P \mathcal{L}_P$ for every $P \in X_i$. So infact we have a morphism of schemes from $X_i \to U_i$ which glue together to give us our (unique) map to projective space $\varphi : X \to \mathbb{P}^n_A$ with $\mathcal{L} \cong \varphi^*(O(1))$. Thus we have our 1-1 correspondence between maps to projective space $\{X \to \mathbb{P}^n_A\}$ and invertible sheaves $\mathcal{L}$ on X generated by some global sections $s_0, ...s_n$ determined by the given map.

1.3 Models of Schemes over Discrete Valuation Rings

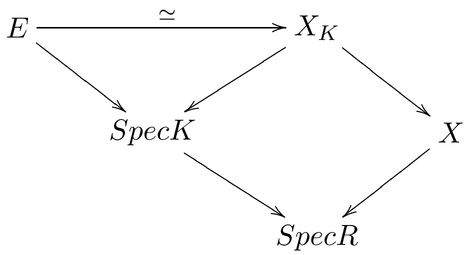
1.3.1 Definition. Let S be an arbitrary scheme, T a scheme over S and X a scheme over T. In full generality, we define a *model* of X to be an S-scheme, $\mathcal{X}$, such that

$X \simeq \mathcal{X} \times_S T$.



So X/T has a model over S if X can be defined over S. Our concern then becomes to find 'good' models of some kind- models that make 'geometric sense', perhaps preserve certain properties. For instance, if X/T is regular, flat, or smooth over S, then it seems sensible to ask for it's model $\mathcal{X}$ also to be regular,flat or smooth etc.

Our interest actually lies i'n defining models for schemes over DVRs. So what happens if we took S to be $SpecR$, where R is a DVR? We naturally have a scheme $SpecK$ over S where K is the field of fractions of R. And for a scheme E over $SpecK$ a model X would be an R-scheme with generic fiber X_K isomorphic to E.



Note that the nature and existence of such models will become clearer once we introduce the language and ideas of fibered surfaces and models of curves. We do not delve further into the subject of general models just yet. We only introduce an example, taken from [Liu] 8.3.54, which will recur through the course of this document, serving to illustrate the different ideas elaborated in each section. It is this example which motivates the title of the mémoire.

1.3.2 Example. Let K be the fraction field of a complete DVR R with uniformising parameter for the valuation π and algebraically closed residue field k. Consider the elliptic curve $E \to SpecK$ given by:

$$y^2 = \pi(x^3 + \pi^3)$$

Then homogenizing co-ordinates by setting $x = a/c, \ , y = b/c$, we get a projective model X over R defined by the homogeneous equation

$$b^2c = \pi(a^3 + c^3\pi^3)$$

Indeed E is nothing but the curve defined by the equations of X in the affine open $D_+(c)$ which is exactly the generic fiber X_K. Thus X is indeed a (projective) model of E.

1.4 Weierstrass Equations and Elliptic Curves

In this section,we take a slight detour and briefly look at Weierstrass equations, define elliptic curves and see how there exists a group law on elliptic curves giving them the structure of an abelian group. This will help us to define Weierstrass models in the next section, and much later to look at Neron models of Elliptic curves, which is the essential topic of this thesis.

Weierstrass Equations

An equation of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a^2X^2Z + a_4XZ^2 + a_6Z^3.$$

is called a Weierstrass equation. Note that we can view it as the locus of a cubic equation in $\mathbb{P}^2$, with a fixed *base point* $O = [0, 1, 0]$ being the only point on the line passing through ∞. We can re-write our equation in non-homogeneous co-ordinates by looking at the open affine where Z lives and setting $x = X/Z, y = Y/Z$. We thus get an equation of the form

$$y^2 + a_1xy + a^3y = x^3 + a_2x^2 + a_4x + a_6$$

By changing variables, $y \mapsto 2y + a_1x + a_3$, assuming the field characteristic $\neq 2$, the equation becomes:

$$y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6$$

where

$b_2 = a_1^2 + 4a_2$

$b_4 = 2a_4 + a_1a_3$

$b_6 = a_3^2 + 4a_6.$

Set $b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2,$

Then the discriminant is given by

$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6.$$

Singular Weierstrass Equations

Here we give a result that helps us classify the nature of a singularity (if it exists) on curves given by Weierstrass equations.

Consider a curve $f(x, y)$ given by the Weierstrass equation

$$f(x, y) = y^2 + a_1xy + a^3yx -^3 -a_2x^2 - a_4x - a_6 = 0$$

with a singular point $P = (x_0, y_0)$.

By the Jacobian Criterion, it follows that

$$\partial f/\partial x = \partial f/\partial y = 0$$

Using the Taylor series expansion for $f(x, y)$ at P, we get that $\exists\ \alpha, \beta \in k \ni$

$$f(x, y) - f(x_0, y_0) = (y - y_0) - \alpha(x - x_0)(y - y_0) - \beta(x - x_0) - (x - x_0)^3.$$

1.4.1 Definitions. With notation as above, the singular point P is called a *node* or *double point* if $\alpha \neq \beta$. In this case, the lines

$$y - y_0 = \alpha(x - x_0) \;\; and \;\; y - y_0 = \beta(x - x_0)$$

are the tangent lines at P . If $\alpha = \beta$, P is called a *cusp*, and the tangent line at P is given by

$$y - y_0 = \alpha(x - x_0).$$

1.4.2 Proposition. *The curve given by Weierstrass equation*

$$f(x, y) = y^2 + a_1xy + a^3yx -^3 -a_2x^2 - a_4x - a_6 = 0$$

satisfies the following:

(i) It is nonsingular $\Leftrightarrow \Delta \neq 0$.

(ii) It has a node $\Leftrightarrow \Delta = 0$ *and* $b_2^2 - 24b_4 \neq 0$*, where* b_2, b_4 *are as defined in the section above*

(iii) It has a cusp $\Leftrightarrow \Delta = (b_2^2 - 24b_4) = 0$*. In the latter two cases, there is only that one singularity.*

Proof : Let us first note that the point at ∞ is never singular. Indeed, consider the curve in $\mathbb{P}^2$ given by the homogeneous equation

$$F(X, Y, Z) = Y^2Z + a_1XYZ + a_3YZ^2 - X^3 - a^2X^2Z - a_4XZ^2 - a_6Z^3 = 0$$

and we see that the point $O = [0, 1, 0]$ is nonsingular since $\partial F/\partial Z(O) = 1 \neq 0$.

(i) Now suppose we have a singularity at $P = (x_0, y_0)$, the substitution $(x, y) \mapsto (x + x_0, y + y_0)$, translates our given singularity to the origin. Note that since this substitution leaves both Δ and $(b_2^2 - 24b_4)$ unchanged, we can assume, without loss of generality, that our singularity is at $(0, 0)$. Thus,

$$a_6 = f(0, 0) = 0, a_4 = \partial f/\partial x(0, 0) = 0, \;\; a_3 = \partial f/\partial y(0, 0) = 0.$$

So our equation becomes:

$$f(x, y) = y^2 + a_1xy - a^2x^2 - x^3 = 0.$$

Then $b_2 = a_1^2, b_4 = 0, b_6 = 0, b_8 = 0$ and thus we have that $\Delta = 0$.
Conversely, suppose nonsingular, and assume char(k) $\neq 2$. We can now use the Weierstrass form

$$y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6$$

The curve defined by this equation is singular $\Leftrightarrow \exists P = (x_0, y_0) \ni 2y_0 = 12x^2 + 2b_2x_0 + 2b_4 = 0$.
Hence all singular points will be of the form $(x_0, 0)$ where x_0 is a double root of $4x^3 + b_2x^2 + 2b_4x + b_6$. This polynomial has a double root $\Leftrightarrow$ its discriminant is zero. But its discriminant equals 16Δ. Thus if the curve is nonsingular, $\Delta \neq 0$.
So for both the following parts, that the curve has a node or a cusp $\Leftrightarrow \Delta = 0$ is clear. We only need to prove the respective second assertions.
(ii) By definition, the curve has a node at the origin if $y^2 + a_1xy - a_2x^2$ has distinct factors. This occurs $\Leftrightarrow a_1^2 + 4a_2 \neq 0$. This is nothing but b_2, so this proves (ii).
(iii) By definition, the curve has a cusp at the origin if $y^2 + a_1xy - a_2x^2$ has equal factors, which occurs $\Leftrightarrow a_1^2 + 4a_2 = 0$
$\Leftrightarrow (b_2^2 - 24b_4) = 0$ (Since we already have that $a_3 = a_4 = 0 \;\; and \;\; thus \;\; b_4 = 0$).
Finally, since a cubic equation can have atmost one double root, we have the curve has only one singularity.

1.4.3 Example. Given $S = Spec\,\mathbb{Z}$, consider the S-scheme $X = Proj\ \mathbb{Z}[x,y,z]/(y^2z + yz^2 - x^3 + xz^2)$, which is a curve sitting in $\mathbb{P}^2$. We want to locate its singular locus and examine the nature of the singularities if they exist.

We look at X more closely, by examining the fibers of the strutural morphism. Using the Jacobian Criterion, we find that X is smooth everywhere except at the fiber $p = 37$. Indeed, by Euler's formula for homogeneous polynomials $\sum x_i \partial f/\partial x_i = d.f$, where $d = degf$, we see that we can use the Jacobian Criterion directly for the homogeneous polynomial

$$y^2z + yz^2 - x^3 + xz^2$$

.

The curve is singular $\Leftrightarrow \exists P = (x,y,z)$ satisfying $F(x,y,z) \ni$ it satisfies the following three equations:

$$-3x^2 + z^2 = 0$$

$$2yz + z^2 = 0$$

$$y^2 + 2yz + 2xz = 0$$

Since we can't have $(x,y,z) = (0,0,0)$, we have that char(K) $\neq$ 2,3. This also allows us to equation (2) by z, giving us the following:

$$3x^2 = z^2$$

$$-2y = z \Rightarrow 4y^2 = 3x^2$$

$$-3y^2 - 4xy = 0 \Rightarrow -3y = 4x$$

Using (b) and (c), we have that $3x^2 = 4(16/9)x^2$, which is possible we have $p = 37$. Thus X is smooth everywhere except at the fiber $p = 37$.

In order to assess the nature of the singularity which in fact something local, we look at the the open affines $D_+(x), D_+(y),\ \ and\ \ D_+(z)$ which cover X and apply the J.C to each of these.

The following table runs through the process in each of the three affines:

$D_+(x)$	$D_+(y)$	$D_+(z)$
We set	We set	We set
$u = y/x, v = z/x$ to get	$a = x/y, b = z/y$ to get	$s = x/z, t = y/z$ to get
1.$F(u,v) = u^2v + uv^2 - 1 + v^2$	$F(a,b) = b + b^2 - a^3 + ab^2$	$F(s,t) = t^2 + t - s^3 + s$
2.$\partial F/\partial u = 2uv + v^2$	$\partial F/\partial a = -3a^2 + b^2$	$\partial F/\partial s = -3s^2 + 1$
3.$\partial F/\partial v = u^2 + +2uv + 2v$	$\partial F/\partial b = 1 + 2b + 2ab$	$\partial F/\partial t = 2t + 1$
There exists a singularity$P = (u,v)$	Following similar procedure	So clearly char(k) $\neq$ 2,3
$\Leftrightarrow$ all the above are satisfied.	as for $D_+(x)$	and we get
(1) ensures that $v \neq 0$	we get $16p^2/9 = 3p^2/4$	$8s^2/3 = 3s^2$
and so from (2) we have that	which is possible	which holds $\Leftrightarrow$
$2u = -v$ and $-3u^2 - 4u = 0$	$\Leftrightarrow$ we're working mod 37.	we're working mod 37.
so char(k) $\neq$ 2,3 and		Our singularity is thus at
from $F(u,v)$ we get		$(3/8, -1/2)$
$2u^3 + 4u^2 - 1 = 0$		
$\Leftrightarrow u \neq 0$		
and $3u + 4 = 0 \Leftrightarrow u = -4/3$		
which plugged in F holds		
$\Leftrightarrow$ we're working mod 37.		
Our singularity is thus $(-4/3, 8/3)$		

Notice that our original homogeneous equation is a Weierstrass equation with a base point $[0,1,0]$, the only one at ∞.
So we use our results on singular Weierstrass equations to assess the nature of the singularity on the curve.
We dehomogenize the equation to get something of the form:

$$y^2 + y = x^3 - x.$$

Comparing co-efficients with the general dehomogenized Weierstrass equation, and maintaining al notation from that section, we have the following associated quantities:

$$a_1 = 0, a_3 = 1, a_2 = 0, a_4 = -1, a_6 = 0$$

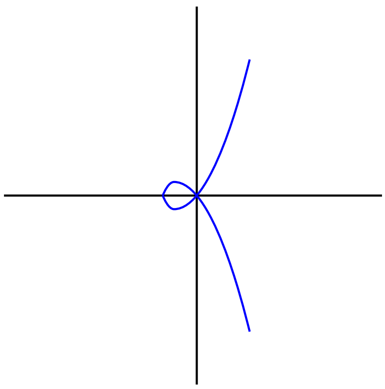
and $b_2 = 0, b_4 = -2, b_6 = 1$ *and* $b_8 = 1$ Thus $\Delta = b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 = 37 = 0(37)$ and $(b_2^2 - 24b_4) \neq 0(37)$. Thus proposition 1.4.2 tells us that the only singularity on the fibre X_{37} is a node or double point.
We found, while looking at $D_+(z)$ that our singular point was $P = (3/8, -1/2)$. We translate co-ordinates

$$(x, y) \mapsto (x - 3/8, y + 1/2)$$

in order to obtain our singularity at $(0,0)$. Our equation under this translation and mod 37 becomes:

$$17y^2 = x^3 + 9x^2$$

which would look something like this:



Using quadratic reciprocity we see that

$$(\frac{17}{37}) = (\frac{37}{17}) = (\frac{3}{17}) = (\frac{17}{3}) = (\frac{2}{3}) = -1$$

and thus 17 is not a square mod 37, implying that the tangents to the singularilty are not rational.

Elliptic Curves and the Group Law

Broadly speaking, elliptic curves are curves of genus 1 having a specified base point. More precisely, they are defined as follows:

1.4.4 Definition. An elliptic curve is a pair (E, O), where E is a nonsingular curve of genus one and the *base point* $O \in E$. The elliptic curve E is said to be *defined over* K, written E/K, if E is defined over K as a curve and $O \in E(K)$.

It can be shown (by using the Riemann Roch Theorem) that every such curve can be written as the locus in $\mathbb{P}^2$ of a cubic equation with only one point, the base point, on the line at ∞. After appropriate scaling, the curve can be written as a Weierstrass equation.

More precisely, we have:

1.4.5 Proposition. *Let E be an elliptic curve defined over K. There exist functions $x, y \in K(E)$ such that the map*

$$\varphi : E - \to \mathbb{P}^2, \quad \varphi = [x, y, 1],$$

gives an isomorphism of E/K onto a curve given by a Weierstrass equation

$$C : Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$$

with coefficients $a_1, ..., a_6 \in K$ and satisfying $\varphi(O) = [0, 1, 0]$. Conversely, every smooth cubic curve C given by a Weierstrass equation as above is an elliptic curve defined over K with base point $O = [0, 1, 0]$.

(ref [Si01], III.2.2)

The Group Law

Let E be an elliptic curve given, like above, by a Weierstrass equation. Then $E \subset \mathbb{P}^2$ consists of the points $P = (x, y)$ satisfying the Weierstrass equation, together with the point $O = [0, 1, 0]$ at infinity. Let $L \subset \mathbb{P}^2$ be a line. Then, since the equation has degree three, the line L intersects E at exactly three points, say P, Q, R (Counting multiplicites and applying Bezout's theorem). (ref Hartshorne)

We define a composition law $\oplus$ on E as follows:

Let $P, Q \in E$, let L be the line through P and Q (if $P = Q$, let L be the tangent line to E at P), and let R be the third point of $L \cap E$ Let L' be the line through R and O. Then L' intersects E at R, O and a third point. We denote that third point by $P \oplus Q$.

The composition law has the following properties:

1.4.6 Proposition. *(a) If a line L intersects E at the (not necessarily distinct) points $P,\ Q,\ R$, then*

$$(P \oplus Q) \oplus R = O.$$

(b) $P \oplus O = P$ for all $P \in E$
(c) $P \oplus Q = Q \oplus P$ for all $P,\ Q \in E$
(d) Let $P \in E$. There is a point of E, denoted by $\ominus P$, satisfying

$$P \oplus (\ominus P) = O.$$

(e) Let $P,\ Q,\ R \in E$. Then

$$(P \oplus Q) \oplus R = P \oplus (Q \oplus R).$$

In other words, the composition law makes E into an abelian group with identity element O.

1.5 Weierstrass Models

In this section,we define very particular models,given by explicit (Weierstrass) equations that help us in certain cases, to see what Neron models are like.

1.5.1 Definition. Let R be a discrete valuation ring with uniformizer π, field of fractions K and residue field k. Let E be an elliptic curve over given by the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

with $a_i \in R$.(such a Weierstrass equation is called integral)

Given an integral Weierstrass equation as above, we define a Weierstrass model for E over K to be the surface $W \to S = Spec\ R$, given by the closed subscheme of $\mathbb{P}^2_R$ defined by the equation,i.e W is the S-scheme

$$Proj\ R[x, y, z]/(y^2z + a_1xyz + a_3yz^2 - x^3 - a_2x^2z - a_4xz^2 - a_6).$$

Note that a Weierstrass model is indeed a model of a scheme over a DVR as described in section 1.3. What's more, the generic fiber of a Weierstrass model W is isomorphic to E, since it is precisely the curve over K given by the Weierstrass equation of E. Now the closed fibre W_k (unique, since we're working over a DVR) is a plane projective curve over k, given by a Weierstrass equation with coefficients $a_i(mod\ \pi R)$. Using results from the section on Weierstrass equations, we know exactly what this curve looks like: it can be an elliptic curve, or a cubic curve with one singularity. More precisely by means of proposition 1.4.2, we can distinguish the possibilities simply by calculating the discriminant. In particular, the special fiber of a Weierstrass model is smooth if and only if the discriminant Δ is in R^* .
The integral Weierstrass equation of an elliptic curve E over K, such that the valuation (associated to the DVR R) of the discriminant is minimal is called a *minimal Weierstrass equation*. Its discriminant modulo R^* is called the *minimal discriminant* and the resulting Weierstrass model is called *minimal Weierstrass model*.

1.5.2 Lemma.: *A Weierstrass model is integral and flat over R.*

Indeed, the dehomogenized polynomial in each affine open is irreducible, hence the model W is an integral scheme. By corollary 1.1.15 every non-constant morphism from an integral scheme to a Dedekind scheme is flat, thus our result.

2 Blowing-Up, Desingularization and Regular Models

Blow-ups are defined as specific morphisms associated to graded algebras. They are central to the study of desingularization. By 'blowing-up' singular schemes at certain points, it is possible to get rid of the singularities, or atleast make them 'nicer', there by making blowing-up an extremely useful tool in algebraic geometry. This section aims to develop the theory of blow-ups and show that the process of blowing-up is necessary in obtaining what we call regular models. There are also some notes on desigularization in general. The resolution of singularities, however, is a vast(and difficult!), widely studied area of algebraic geometry, but it is not our concern to delve into this here. Thus most results concerning desingularization will be stated without proof.

2.0.3 Definition. Let X be a reduced Noetherian scheme. Let $\xi_1, \xi_2, ...\xi_n$ be the its generic points. A morphism of finite tpe $f : Z \to X$ is called a *birational morphism* if Z admits exactly n generic points $\xi'_1, \xi'_2, ...\xi'_n$, if $f^{-1}(\xi_i) = \xi'_i$,and if $O_{X,\xi_i} \to O_{Z,\xi'_i}$ is an isomorphism $\forall i$.

2.0.4 Definition. Now let X be a scheme having only a finite number of irreducible components $X_1, X_2, ...X_n$ (endowed with the reduced closed sub-scheme structure.) The disjoint union $X' = \amalg_{1 \leq i \leq n} X'_i$ where each X'_i is the normalization of the integral scheme X_i as defined usually, is called the *Normalization of*X, and the canonical morphism $\pi : X' \to X$, the *Normalization morphism.* This forms an example of a birational morphism.

It will become clear once we define blow-ups, that what we call the *blowing-up morphism* is also a birational morphism.

2.1 Blowing-Up

Local Description:
Let A be a ring and I an ideal of A. Consider the graded A-algebra dependant on I:

$$\tilde{A} = \bigoplus_{d \geq 0} I^d, where I^0 = A$$

We can see this as sitting in $A \oplus A \oplus A \oplus A...$ and choose elements $1, t, t^2, t^3...$ in each A of the direct sum so as to explicitly define a map

$$\tilde{A} = \bigoplus_{d \geq 0} I^d \to A[t]$$

sending an element $i_1 i_2 ... i_d \mapsto i_1 i_2 ... i_d t^d$. In defining this explicitly, we are able to distinguish between elements of $A = \tilde{A}_0$ of degree 0 and the same element seen as an element of $I = \tilde{A}_1$ of degree 1. Now, let $(f_1, f_2, ...f_n)$ be a system of generators for I. Let t_i denote the element f_i seen as an element of $I = \tilde{A}_1$ of degree 1. We can now define a surjective map:

$$\varphi : A[T_1, T_2, ...T_n] \to \tilde{A}$$
$$T_i \mapsto t_i$$

The surjectivity of this map implies that $\tilde{A}$ is a homogeneous A-algebra. If $P(T)$ is a homogeneous polynomial with co-efficients in A, then $P(t_1, t_2, ...t_n) = 0 \Leftrightarrow P(f_1, f_2, ...f_n) = 0$.

2.1.1 Definition. Keeping the notation from above, let $\tilde{X} = Proj\tilde{A}$ where $X = SpecA$ is an affine Noetherian scheme. The canonical morphism $\pi : \tilde{X} \to X$ is called the *Blowing-up of X with centre (or along) V(I) (or I)*

Note: If I was generated by a regular element, $\tilde{A} \simeq A[T]$, the isomorphism being given by φ as defined above. This amounts to saying that $Proj\tilde{A} \to SpecA$ is an isomorphism.

2.1.2 Lemma. *Let A be a Noetherian ring, and I an ideal of A generated by $(f_1...f_n)$. Let φ be defined as above. Then we have the following two properties which help us characterise $Ker\varphi$ and thus sometimes, to compute the blow up of $X = SpecA$. (a) Let $S_i = T_i/T_1 \in O(D_+(T_1))$. Then $(Ker\varphi)_{(T_1)}$ is given by:*

$$J' = \{\ P(S) \in A[S_2...S_n] | \exists d \geq 0, f_1^d P \in (f_1 S_2 - f_2, ..., f_1 S_n - f_n)\}$$

(b) $J = (f_i T_j - f_j T_i)_{1 \leq i,j \leq n}$ is always contained in $Ker\varphi$.If the $\{f_i\}$ form a minimal set of generators for I, and $Z := V_+(J) \in \mathbb{P}_A^{n-1}$ is integral, then the closed immersion $Proj\tilde{A} \to Z$ is an isomorphism. Thus the blow-up $\tilde{X}$ is a union of affine schemes $SpecA_i$ where A_i is the sub A_{f_i}-algebra generated by the $f_j f_i^{-1} \in A_{f_i}, 1 \leq j \leq n$.

Proof : (a) By choosing a suitable $d \geq 0$, we can always write $f_1^d P(S)$ as $\sum Q_i(S)(f_1 S_i - f_i) + a, \quad a \in A$. For $P(S) \in (Ker\varphi)_{(T_1)}$, the image os a in $\tilde{A}_{(t_1)}$ is zero. $\Rightarrow \exists r \geq 0 \ni at_1^r = 0$
$\Rightarrow af_i^r = 0$
$\Rightarrow$ by replacing d by $d + r$ we can assume $a = 0$ and
$\Rightarrow P(S) \in J'$

Conversely, suppose that $P(S) \in J'$
$\Rightarrow \exists d \geq 0 \ni f_i^d P(S) \in (f_1 S_i - f_i)_{1 \leq i,j \leq n}$
$\Rightarrow$ if $P(S) = Q(T)/T_1^r$ with $Q(T)$ homogeneous of degree r, $\exists e \geq 0 \ni f_1^d T_1^e Q(T) \in (f_i T_j - f_j T_i)$
$\Rightarrow f_1^d t_1^e Q(T) = 0$
$\Rightarrow t_1^{d+e} Q(T) = 0$
$\Rightarrow P(S) \in (Ker\varphi)_{(T_1)}$

Note that $Ker\ \Psi$ where $\Psi : A[S_2...S_n] \to A_{f_1}$ automatically equals J_1 by the above. So we can identify $\tilde{A}_{(t_1)}$ with its image in A_{f_1}.

(b) That $J \subseteq Ker\varphi$ is clear since $P(S) \in Ker\varphi \Leftrightarrow P(f_1...f_n) = 0$
For the remaining, it suffices to show that the given map is an isomorphism on every non-empty principal open $U_i = D_+(T_i) \cap Z$. Consider the ideal generated by $\{f_j\}_{j \neq i}$. Since the set of generators was chosen to be minimal $\Rightarrow f_i$ is not in this ideal. So it is a non zero element of $O_Z(U_i)$ and thus not a zero divisor (Z integral $\Rightarrow O_Z(U_i)$ is an integral domain.)

2.1.3 Examples.

- Let's blow-up affine n-space $\mathbb{A}_k^n := Spec\ k[x_1....x_n]$ over a field k, at the origin $= (x_1 = 0.....x_n = 0)$. Here we are automatically in the "good" case, since our map φ behaves well and $Ker\varphi = J = (f_i T - j - f_j T_i)_{(0 \leq i,j \leq n)}$.
$\Rightarrow$ our blow-up $\tilde{X} = Proj A[T_1...T_n]/J \simeq V_+(J)$, which is the subscheme of $\mathbb{P}_A^{n-1} = \mathbb{A}_k^{n-1} \times_k \mathbb{P}_k^{n-1}$ given by J, where $A = k[x_1....x_n]$. For the subscheme of $\tilde{X}$ where x_i lives ($x_i \neq 0$), we have that $T_j = x_j T_i x_i^{-1}$. Thus $\tilde{X}_{(x_i)} \simeq X_{(x_i)}$. Thus the fibre of the origin, which is given by nothing more than the relation $x_i = 0$, is just projective $n-1$ space. So the blowing up of affine n-space at the origin, keeps the scheme intact everywhere except the point we blow-up, which is replaced by $\mathbb{P}_k^{n-1}$.
- Consider now, a scheme with a singularity-$SpecA$, $A := k[X,Y]/(X^2 - Y^3)$, which has a cusp at the origin. In order to 'resolve' this singularity we blow up the curve at the origin. Let x, y be such that $k[X,Y]/X^2 - Y^3 = k[x,y]$, so we blow-up along the ideal $I = xA + yA$. By the lemma above, we can cover $\tilde{X}$ with $SpecA_i$ for $i = 1, 2$ and $A_1 = k[x/y, y], A_2 = k[y/x, x]$ In A_1, $(x/y)^3 y = 1 \Rightarrow (x/y) \neq 0$, so we can infact write A_1 as $k[v, 1/v]$ where $v = x/y$. Thus $\tilde{X}$ is just $A_2 = k[y, x, x]$ where $(y/x)^2 = x$. Setting $u = y/x$ we have that A_2 is nothing but $k[u], u = y/x$ which is just the affine line over k, which is infact the normalization of X.

We now want a more general definition and procedure of blowing-up, for schemes that are only locally Noetherian.

2.1.4 Definition. Given a scheme X, a graded O_X-algebra is a quasi-coherent sheaf of O_X-algebras, together with a grading $\mathcal{B} = \oplus_{n \geq 0} \mathcal{B}_n$ where each $\mathcal{B}_n$ is a quasi-coherent sub-O_X-module. $\mathcal{B}$ is a homogeneous O_X-algebra, if $\mathcal{B}_1$ is finitely generated and $(\mathcal{B}_1)^n = \mathcal{B}_n, \forall n \geq 1$.

An immediate example of a homogeneous O_X-algebra is $\mathcal{B} := \oplus_{n\geq 0}\mathcal{I}^n$ where $\mathcal{I}$ is a finitely generated quasi-coherent sheaf of ideals over X.

2.1.5 Lemma. *For every scheme X and a graded O_X-algebra $\mathcal{B}$, $\exists$ a unique X-scheme $Proj\mathcal{B} \to X$, such that $\forall$ affine open sub-scheme $U \subseteq X$, we have an isomorphism of U-schemes $h_U : f^{-1}(U) \simeq Proj\mathcal{B}$ compatible with restrictions to every open affine $V \subset U$.*

Proof : Let us first assume X to be affine. The proof in the general case can be obtained by glueing. Set $Proj\mathcal{B}$ as $Proj\mathcal{B}(X)$. Then $\forall V \subseteq X$ open,

$$Proj\mathcal{B}(V) = Proj\mathcal{B}(X) \otimes_{O_X(X)} O_X)(V) \simeq Proj\mathcal{B}(X) \times_X V \simeq f^{-1}(V).$$

Note that if $Proj\mathcal{B}$ exits, then $Proj\mathcal{B}_{|W} = Proj\mathcal{B} \times_X W$ for every open subscheme W of X. Covering X by open affine subschemes X_i, we can glue together the $ProjX_i$s under the uniqueness of $Proj\mathcal{B}_{|X_i \cap X_j}$, and obtain $Proj\mathcal{B}$

2.1.6 Definition. Let $\mathcal{I}$ be a coherent sheaf of ideals on a locally Noetherian scheme X. Then the X-scheme

$$\tilde{X} := Proj(\oplus_{n\geq 0}\mathcal{I}^n) \to X$$

is called the *blowing-up of X with centre (or along) $V(\mathcal{I})(or\mathcal{I})$*. $\tilde{X}$ depends not only on $V(\mathcal{I})$ as a subset, but on the subscheme structure of $V(\mathcal{I})$. Note that the Zariski topology on schemes allows us to change $\mathcal{I}$ to $\mathcal{I}^n$ without changing the blow-up $\tilde{X}$. For X affine, the definition co-incides with that of the blowing-up of a Noetherian scheme.

2.1.7 Proposition. *Let X be a locally Noetherian scheme, $\mathcal{I}$ a quasi-coherent sheaf of ideals on X and $\pi : \tilde{X} \to X$, the blowing-up of X with centre $V(\mathcal{I})$. Then the following hold:*
(i) π is an isomorphism $\Leftrightarrow$ $\mathcal{I}$ is an invertible sheaf on X.
(ii) π is proper
(iii) $\pi^{-1}(X\ V(\mathcal{I})) \to X\ V(\mathcal{I})$ is an isomorphism and for $\mathcal{I} \neq 0$ and X integral, $\tilde{X}$ is integral and π is birational.
(iv) $\mathcal{I}O_{\tilde{X}} = O_{\tilde{X}}(1)$

Proof : Most of these results follow from the proposition proved in the case of blowing up a Noetherian scheme. The few things to note are that properness is a local property, that an invertible sheaf is locally free of rank 1, and that for $U := X\ V(\mathcal{I}), \tilde{U} = \tilde{X} \times_X U$, since $U \to X$ is an open immersion and thus flat. The only one that requires a proof perhaps is (iv):
It suffices to show the result for $X = SpecA$ affine. In this case, let $I = (f_1, ...f_n)$ be the ideal associated to $\mathcal{I}$. Let t_i denote f_i as seen as a homogeneous element of degree 1. Then $\mathcal{I}O_{\tilde{X}|D_+(t_i)}$ is generated by the f_i, $\forall i = 1, ...n$, since $f_j = f_i(t_j/t_i)$. Thus

$$\mathcal{I}O_{D_+(t_i)} = O_{D_+(t_i)}(1) \Rightarrow \mathcal{I}O_{\tilde{X}} = O_{\tilde{X}}(1)$$

Universal Property of Blowing-Up

For $\pi : Y \to X$, a morphism of locally Noetherian schemes, and for $\mathcal{I}$, a quasi-coherent sheaf of ideals on X, the canonical homomorphism $\pi^*\mathcal{I} \to \pi^*O_X = O_Y$, gives us a quasi-coherent sheaf, the image of $\pi^*\mathcal{I}$ in O_Y, called the inverse image sheaf and denoted $\mathcal{I}O_Y$.

2.1.8 Proposition. *Let $f : W \to X$ be a morphism of locally Noetherian schemes, and $\mathfrak{I}$, a quasi-coherent sheaf of ideals on X. Let $\mathfrak{J}$ be the inverse image sheaf $\mathfrak{I}O_Y$. Let $\pi : \tilde{X} \to X$ and $\rho : \tilde{W} \to W$ be the respective blow-ups along $\mathfrak{I}$ and $\mathfrak{J}$ respectively. Then $\exists$ a unique morphism $\tilde{f} : \tilde{W} \to \tilde{X}$ such that the follwing diagram commutes.*

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \pi} \\ W & \xrightarrow{f} & X \end{array}$$

Proof : We begin by showing the existence of $\tilde{f}$ in the case that X is affine, and then glue to get our result for the general case. Suppose $X = SpecA$ and $W = SpecB$. We have a homomorphism of graded algebras:

$$\oplus_{n \geq 0} I^n \to \oplus_{n \geq 0} (IB)^n, \quad I = \mathfrak{I}(X).$$

But $IB = \mathfrak{J}(W)$, so we have an induced morphism $\tilde{W} \to \tilde{X}$. For the general case where W is not affine, cover it with open affine subschemes W_i, and note that for any open affine subscheme U in $W_i \cap W_j$, we have that the induced morphisms constructed as above-$\tilde{W}_i \to \tilde{X}$ and $\tilde{W}_j \to \tilde{X}$, co-incide on U. So we glue together these f_is to get a morphism $f : \tilde{W} \to \tilde{X}$. We follow a similar procedure for X not affine. This proves the existence of $\tilde{f}$

Now for the uniqueness, in view of the assertion of uniquness itself, we may assume X is affine, say $X = SpecA$. Also we can simply show the existence and uniqueness of $g : \tilde{W} \to \tilde{X}$ such that the follwing diagram commutes:

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{g} & \tilde{X} \\ & \searrow^{f.\rho} & \downarrow{\scriptstyle \pi} \\ & & X \end{array}$$

Then taking $W = \tilde{W}$ we would have $\mathfrak{J}$ is an invertible sheaf.

Before we begin, we admit the following result about invertible sheaves:

For a scheme X and any invertible sheaf $\mathfrak{L}$ on X generated by n elements $s_1, ...s_n$, $\exists$ a morphism $f : X \to \mathbb{P}^n - 1 = ProjA[T_1, ...T_n]$ such that $\mathfrak{L} \simeq f^*O_Y(1)$ and $f^*T_i = s_i$ under this isomorphism.

Now, let $I := \mathfrak{I}(X)$ be generated by $f_1, ...f_n$. Let $s_1, ...s_n$ be the respective canonical images of the f_i in $\mathfrak{J}(W)$. Then $\mathfrak{J}$ is an invertible sheaf generated by $s_1 ... s_n$. Let $g : W \to \tilde{X}, \ni \pi \circ g = f$. Then:

$$\mathfrak{J} = g^{-1}(\mathfrak{I}O_{\tilde{X}})O_W = g^{-1}(O_{\tilde{X}}(1))$$

$\Leftrightarrow g^*O_{\tilde{X}}(1) \to \mathfrak{J}$ is an isomorphism. Let $i : \tilde{X} \to \mathbb{P}^{n-1}$ be the canonical closed immersion associated to our original homomorphism φ. And let $h := i \circ g$. Then $h^*O_{\mathbb{P}_X^{n-1}}(1) \simeq \mathfrak{J}$ and such an h is unique by the admitted result. Thus since i is a closed immersion, h completely determines g, and thus we have our result regarding uniqueness.

2.1.9 Corollary. *As an obvious corollary to the above, we have the actual statement for the Universal Property of Blowing-Up: Let $\pi : \tilde{X} \to X$ be the blowing up of a locally Noetherian scheme X with centre $\mathfrak{I}$. Then π has the following Universal property: For any morphism $f : W \to X$ such that $(f^{-1}\mathfrak{I})O_W$ is an invertible sheaf of ideals on W, $\exists$ a unique morphism $g : W \to \tilde{X}$ making the following diagram commute.*

$$\begin{array}{ccc} W & \xrightarrow{g} & \tilde{X} \\ & \searrow^{f} & \downarrow{\scriptstyle \pi} \\ & & X \end{array}$$

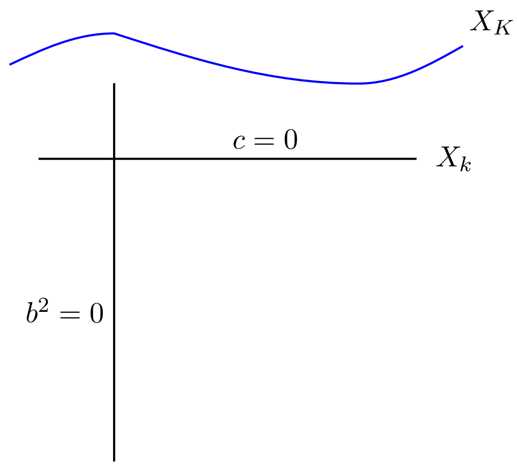
2.1.10 Example. Let's go back to example 1.3.2, locate the singularities on the model X, if any, and blow them up. We had an elliptic curve $E \to Spec K$ given by:

$$\pi^{-1}y^2 = (x^3 + \pi^3)$$

It can be seen as the smooth plane projective curve over K with homogeneous equation $\pi^{-1}b^2c = a^3 + \pi^3c^3$. We found a model of E given by

$$X = Proj R[a, b, c]/(b^2c - \pi(a^3 + \pi^3c^3))$$

Let's look at X more closely. The generic fibre X_K is nothing but E as we saw and is thus smooth, with its points regular in X. So we don't have to worry about $X_K \simeq E$. The special fibre X_k (unique since R is a DVR), occurs at $\pi = 0$ and is thus given by $b^2c = 0$, which is the union of the projective (reduced) line $c = 0$, and the non-reduced projective line given by $b^2 = 0$. We define the *multiplicity* of an irreducible component to be $d := length\ O_{X,\eta}$ where η is the generic point. Under this, the line $b^2 = 0$ has multiplicity 2. The two components meet at a single point where $b = c = 0$. And so we have something that looks like this:



Now, we cover X with the standard open affines, and just as we did in example 1.4.3, we use the Jacobian criterion to check for singularities. It turns out that the only singularity that exists is in the open $D_+(c)$, and it lies on the special fibre X_k. Indeed $D_+(c) = Spec\ R[x, y]/(y^2 - \pi(x^3 + \pi^3))$ where $x = a/c, y = b/c$. Taking partial derivatives and applying the J.C, we find a singularity at the point $p = (0 : 0 : 1) \in X$ which corresponds to the maximal ideal $\mathbf{m} = (x, y, \pi)$. To check, we look at the local ring $O_{X,p}$ which is the quotiented regular local ring $R[x, y]_{\mathbf{m}}/(y^2 - \pi(x^3 + \pi^3))$. Now $y^2 - \pi(x^3 + \pi^3) \notin \mathbf{m}^2$, and so by proposition 1.1.20, $O_{X,p}$ is not regular.

So we now have our singular locus. It is simply the point p corresponding to $\mathbf{m} = (x, y, \pi)$.

Let's blow-up X with centre p. By the definition of blowing-up we know that $X\backslash\{p\}$ remains unchanged. So we can work locally on $U = D_+(c)$. Applying all that we learnt in the section on Blowing ups, this is what we have:
Let $\phi : A := Spec\ R[x, y]/(y^2 - \pi(x^3 + \pi^3))$. Then we have the maps:

$$A[u, v, w] \to \oplus\mathbf{m}^n$$

$$u \mapsto x$$

$$v \mapsto y$$

$$w \mapsto \pi$$

The kernel of ϕ is rather easy to calculate: It consists of the three obvious (proposition 2.1.2) candidates:

$$yu - xv, \pi u - xw, \pi v - yw$$

together with the candidate coming from the relation that determines $D_+(c)$, namely

$$v^2 - \pi(xu^2 + \pi w^2)$$

So

$$\oplus \mathbf{m}^n \simeq A[u, v, w]/(yu - xv, \pi u - xw, \pi v - yw, v^2 - \pi(xu^2 + \pi w^2))$$

Thus our blow-up U' of U with centre p is given by

$$U' = Proj\ A[u, v, w]/(yu - xv, \pi u - xw, \pi v - yw, v^2 - \pi(xu^2 + \pi w^2))$$

2.2 Desingularization

Fibred surfaces

2.2.1 Definition. Let S be a Dedekind scheme. We define a *Fibered Surface* X to be an integral, projective, flat S-scheme of dimension 2, with structural morphism

$$\pi : X \to S$$

Let η be the generic point of S. Then $X_\eta = X \times_S k(\eta)$ is called the generic fibre of X, and for a closed point $s \in S$, $X_s = X \times_S k(s)$ is called a closed fibre.

Note: The flatness of π is equivalent to it's surjectivity: That surjective $\Rightarrow$ flat is clear to see from Proposition 0.1.1.14 and the Corollary that follows . The converse arises from the fact that dimension of fibres is preserved by flatness and that $X_s \simeq \pi^{-1}(s)$

A *morphism* of fibered surfaces is a morphism of schemes that is complatible with the S-scheme structure.
Fibered surfaces can be broadly classified into two kinds-One where dim $S = 0$, which is the "Geometric case". Here X is an integral, projective, algebraic surface over a field. And the other where dim $S = 1$, the "Arithmetic case". This is where we call X a *Relative curve* over S.

Note: From now on, we assume all fibred surfaces to be over a Dedekind scheme of dimension 1 unless specified otherwise. Although most results would also hold for the *dim* $S = 0$ case, for our purposes (which is eventually to work over DVRs) we choose to stick to the case of *dim* $S = 1$.

2.2.2 Remark. For a fibred (respectively normal fibred) surface X (i.e-over a Dedekind scheme S of dimension 1), in general, the generic fibre X_η is an integral (respectively normal integral) curve over $K(S)$, and $\forall s \in S$, the fibres X_s are projecive curves over $k(s)$.

2.2.3 Example. Let's go back to example 1.4.3.

$$X = Proj\mathbb{Z}[x, y, z]/(y^2z + yz^2 - x^3 + xz^2)$$

It turns out that it is indeed an example of a normal fibered surface. That it is a projective, integral (the dehomogenized polynomials in each of the standard affines is irreducible) surface is clear, and the structural morphism is flat since flat is equivalent to torsion free over a PID. The only thing to verify then, is that X is normal. Since we saw that X is smooth everywhere except for X_{37}, we have that its generic fibre is normal. Further, since X_{37} is reduced, we have a lemma ([Liu] 4.1.18) that says such a scheme is indeed normal, thereby making it a normal fibred surface as we wanted.

As something of an application of the valuative criterion of properness to fibered surfaces, we have the following proposition, which proves to be insightful and raises some questions that concern regular models of curves, and even more specifically reveals how the properties of regularity and properness are used in constructing the Neron model.

2.2.4 Proposition. *Although this could be proved for a more arbitrarily chosen fibered surface, it suits our purposes to choose the following setting, as will become clear later on. Let R be a complete DVR with algebraically closed residue field k. Let X be a normal fibered suface over R, and X_K be its generic fibre.*
(a) If X is proper over R. Then

$$X_K(K) = X(R)$$

(b) If X is regular, and if $X^0 \in X$ is that largest subscheme of X such that $X^0 \to SpecR$ is smooth, i.e, X^0 is the smooth part of X, then

$$X(R) = X^0(R)$$

(c) In particular, if X is both regular and proper over R, then

$$X_K(K) = X(R) = X^0(R)$$

Proof : (a) This is infact just a special case of the valuative criterion of properness. We have a canonical map $X(R) \to X_K(K)$. That it is injective follows from the fact that $SpecK$ is dense in $SpecR$ and the fact that if two morphisms $SpecR \to X$ agree on a dense open set, then they are the same. (Ref Liu3.3.25,3.3.11) Surjectivity follows from the valuative criterion of properness which says that since X is proper over R, for every point $P \in X_K(K)$, there is a morphism $\rho_P : SpecR \to X$ making the following diagram commute:

$$\begin{array}{ccc} X_K & \longrightarrow & X \\ \uparrow{\scriptstyle P} & & \uparrow{\scriptstyle \rho_P} \\ SpecK & \longrightarrow & SpecR \end{array}$$

Thus every point in $X_K(K)$ comes from a point in $X(R)$, implying $X(R) = X_K(K)$.
(b) It can be proved that for a fibred surface as above, if it is regular, then every R-valued point, i.e a point belonging to $X(R)$, intersects each fibre at a non-singular point. In other words, if $x \in X_k$, the special fibre, lies in the image of an R-valued point $P \in X(R)$, then X_k is non-singular at x. Since X^0 is precisely the complement of the singular points of X, this amounts to saying that the natural inclusion $X^0(R) \hookrightarrow X(R)$ is infact a bijection.
(c) It follows directly from the above two that

$$X_K(K) = X(R) = X^0(R)$$

What this proposition actually tells us is that the smooth part of a regular, proper fibred surface over a DVR as above, is large enough to contain all the rational points on the generic fibre. So every K-valued point on the generic fibre extends to an R-valued point on the smooth part of the fibred surface X. It compels us to ask some questions about (and also gives insight into the use of) models of curves:

- Given a DVR R with field of fractions K, a curve E over K, can we find a proper, regular fibred surface X over R such that its generic fibre X_K is isomorphic to E?
- If so, is there a minimum such model?

These questions lie at the heart of this memoire, and form the base for finding and studying Neron models.

Before we formally define models of curves, we explore the question of regularity, i.e the existence of such a regular fibred surface X as above. This forces us to turn to the theory of the resolution of singularities, which helps us obtain regular fibred surfaces from those with singularities in an explicit way. The theory of the resolution of singularities, or disingularization as it is sometimes called, is vast and complex, and has not been treated here in great detail. Oscar Zariski and his students S.Abhayankar, J.Lipman and H.Hironaka have contributed greatly to this field in the years 1939-1965.

2.2.5 Definition. Let X be a reduced, locally Noetherian scheme. A proper birational morphism $\pi : Z \to X$ with Z regular is called a *desingularization of* X (or a *resolution of singularities of* X. If π is an isomorphism above all regular points of X, it is called a *desingularization in the strong sense.*

2.2.6 Example. For a reduced curve X over a field k, the normalization morphism $X' \to X$ is a desingularization of X.

For any (non-reduced) curve this may not be the case. But since the notions of normal and regular co-incide on curves over fields, the problem of disingularization is rather easy in this case. Thus, our problem essentially becomes one of higher dimensions. However, to get an intuitive understanding of desingularization, we first take the example of a (singular) curve over a field and show how we can obtain an explicit desingularization: Construct a sequence of blow-ups of an integral projective curve X over k as follows: Let us suppose X is singular. Let $X_1 \to X_0 = X$ be the blowing up of X_0 along it's singular locus endowed with it's closed subscheme structure. i.e along $S_0 = X_0 \backslash Reg(X_0)$, where $Reg(X_0)$ is the set of regular points of X_0. If X_1 is still singular, define another blowing-up morphism $X_2 \to X_1$ in the same manner, and for X_2 singular, $X_3 \to X_2$ and so on, blowing up the X_is along $S_i = X_i \backslash Reg(X_i)$.

2.2.7 Proposition. *With the notation above, the sequence*

$$... \to X_n \to X_{n-1} \to ... \to X_0 = X$$

is finite. This is equivalent to saying that given an integral projecive curve X that is singular, we can desingularize X in a finite number of steps with regular centres.

Proof : Notice first the claim that the singular locus of X is closed. Indeed we saw earlier that the generic fibre X_η s an integral curve over the field $K(S)$ (Remark 2.2.2). Thus $Reg(X_\eta)$ is a non-empty open set, since for a curve the set of regular points is equal to the set of normal points, and the latter is open for an integral curve over a field. Since X_η is open in X, we have that $Reg(X)$ contains a non-empty open subset of X. Let Y be a proper closed subscheme of X. Then $dim\ Y \leq 1$. If $Y \to S$ is dominant, $Reg(Y)$ contains the open subset Y_η. If $Y \in X_s$, then Y is an integral curve over a field and thus $Reg(Y)$ is open just as we saw in the case of X_η. Thus for any integral closed subscheme Y of X, $Reg(Y)$ contains a non-empty open subset. If this is the case, it can be shown that this implies $Reg(X)$ is open.

Now, lets go back to the sequence. Let $\pi_n : X_n \to X_{n-1}$ be the blowing-up morphism along S_{n-1}. We know that π_n is a proper birational morphism of integral curves, and therefore finite. We thus have the following exact sequence:

$$0 \to O_{X_{n-1}} \to \pi_{n*}O_{X_n} \to \mathcal{F}_n \to 0$$

where $\mathcal{F}_n$ is a skyscraper sheaf with support in the singular locus of X_{n-1}. This sequence allows us to obtain a relation between an invariant (namely the arithmetic genus) of the schemes in the sequence, which is infact strictly decreasing and bounded from below, and thus stationary. What this means is that since $\mathcal{F}_n$ is a skyscraper sheaf, $\mathcal{F}_n = 0$. Thus the second arrow of the exact sequence is an isomorphism which in tuen implies that π_n is an isomorphism since it is finite. Thus the singular locus is defined by an invertible ideal by proposition 2.1.7, which is not possible and thus the singular locus is empty, implying that X_{n-1} is regular. Thus in a finite number of blow-ups we have desingularized X.

The following is the main theorem of Disingularization (rephrased to suit our purposes):

2.2.8 Theorem. *Let S be a Dedekind scheme of dimension 1. Let $\pi : X \to S$ be a fibred surface with smooth generic fibre. Then X admits a desingularization in the strong sense.*

The key ingredient of the theorem is a result by Lipman and it's corollary which we admit:

2.2.9 Theorem. *Let X be an excellent, reduced, Noetherian scheme of dimension 2. Consider the sequence:*

$$... \to X_{n+1} \to X_n \to ... \to X_1 \to X$$

such that $X_1 \to X$ is the normalization morphism and $\forall i \geq 1$, the maps $X_{i+1} \to X_i$ are the composition of the blowing up morphism $X_i' \to X$ along the singular locus $S_i = X_i \backslash Reg(X_i)$ (which is closed because X is excellent-see definition below.) and the normalization morphism $X_{i+1} \to X_i'$. Then the sequence is finite and stops at n when X_n is regular. In other words, such an X admits a desingularization in the strong sense.

2.2.10 Corollary. *Let S be an excellent Dedekind scheme. Let $X \to S$ be a fibred surface. Then X admits a desingularization in the strong sense.*

We must first define an excellent scheme:

2.2.11 Definition. A ring A is said to be *excellent* if it satifies the following properties: (i) $SpecA$ is *universally catenary*, i.e every finitely generated A-algebra satifies the following-for any triplet of prime ideals $\mathbf{q} \subseteq \wp \subseteq \mathbf{m}$ we have the equality of heights $ht(\mathbf{m}/q) = ht(\mathbf{m}/\wp) + ht(\wp/\mathbf{q})$
(ii) $\forall \wp \in SpecA$, the formal fibres of $A_\wp$ are geometrically regular
(iii) $\forall$ finitely generated A-algebra B, the set of regular points in $SpecB$ is open in $SpecB$

A locally Noetherian scheme X is said to be *excellent* if $\exists$ a covering $\{U_i\}$ of X, such that $O_X(U_i)$ is excellent $\forall i$.

For the purpose of this memoire, it suffices to consider fibred surface X over a scheme $S = SpecR$ where R is a complete DVR, with an algebraically closed residue field k. Thus from this point on, a fibred (respectively normal,regular etc) surface is an integral projective, flat, (with respective additional conditions of normal, regular etc) R-scheme of dimension 2. Note that such an S is always excellent.

Thus we rephrase theorem 2.2.6 as:

2.2.12 Theorem. *Let $S = SpecR$ with R a complete DVR with algebraically closed residue field k. Let $\pi : X \to S$ be a fibred surface. Then X admits a desingularization in the strong sense.*

Proof : The proof then directly follows from the corollary to Lipmans result since our S is an excellent Dedekind scheme.

2.3 Models of Curves and The Regular Model

Having inroduced the language of fibred surfaces and raised the relavent questions (see remarks following proposition 2.2.3), we now formally introduce *models of curves.* They are models in the sense of section 1.3, but have a rather strict definition. We will notice later, when we introduce the concept of Weierstrass and Neron models of curves, that the latter need not necessarily be models of curves as we define them below.

We continue to work over $S = SpecR$, with R a complete DVR with algebraically closed residue field k and field of fractions K.

2.3.1 Definitions. Let E be a normal, projective connected curve over K. A normal fibred surface $X \to S$ is called a *model* of E over S or over R if the generic fibre X_K of X is isomorphic to E. We say that it is a *regular* model if X is regular.
If X , X' are two models of E over R, the identification of the generic fibres gives us a birational map between X and X'.

2.3.2 Proposition. *Weierstrass models of elliptic curves are models of curves.*

That a Weierstrass model for an elliptic curve is projective, integral and flat of dimension 2 is clear. The only thing to verify then is normality. And this follows directly from the argument used in showing that example 1.4.3 was indeed an example of a normal fibered surface, since the generic fibre, which is isomorphic to the elliptic curve, is regular (and thus normal) and the (unique, as it was defined over a DVR) special fibre is reduced.

2.3.3 Example. We return to our main example 1.3.2. We first note that the model X we found for the elliptic curve E is indeed a model of a curve in the sense of definition 2.3.1. First note that E satisfies the conditions for us to be able to define a model in the sense of definition 2.3.1. Indeed it is a regular connected projective curve over K and since normal is equivalent to regular, we can define a model of the curve E. Now, we already have that the generic fibre X_K of X is isomorphic to E. Thus, we must show only that X is a normal fibered surface. That it is projective and of dimension 2 is obvious. Since the polynomial that defines it, when dehomogenized, is irreducible in all three standard affine opens, we have that it is integral. That it is flat follows from the fact that every non-constant morphism from an integral scheme to a Dedekind scheme is flat. The only thing that remains to be verified is that it is normal. This requires admitting a criterion for normality due to Serre that says for a scheme such as X, (which happens to be one that we call Cohen-McCaulay; In particular it verifies the property that for any point $x \in X$, we have $depth\ O_{X,x} \geq inf\{2,\ dim\ O_{X,x}\}$. As a counter example consider the scheme $X = SpecR[x, y]/(x^2 - \pi, xy)$ over our DVR R. This scheme contains an embedded point x (an associated point that is not generic) corresponding to the ideal $(x - \pi, y)$, which will not verify property above (indeed, $depth\ O_{X,x} = 0$) and is thus not Cohen-McCaulay.) we have the following criterion for normality: X is normal if and only if it is normal at the points of codimension 1. (see [Liu] 8.2.23 for details). Admitting this, we must only check that X is normal at its points of codimension 1. These points are either closed points of the generic fibre $X_K \simeq E$ or the generic points of the special fibre X_k. Since we already know that E is regular and thus normal, we needn't worry about its closed points. We turn our attention to X_k. A generic point η of X_k corresponds to the prime ideal generated by π together with a polynomial in $R[x,y]$ who's image in $k[x,y]$ is irreducible. We look at this locally, in each of the stadard open affines covering X. We have the equations:

$$y^2 - \pi(x^3 + \pi^3);\ x = a/c,\ y = b/c$$

$$u^2v - \pi(1 + \pi^3 v^3);\ u = b/a,\ v = c/a$$

$$t - \pi(s^3 + \pi^3 t^3);\ s = a/b,\ t = c/b$$

In each, we look at what a generic point of X_k would look like: and see that it corresponds to (π, y^2), (π, u^2v), *and* (π, t) respectively. Now, by proposition 1.1.20, X would be normal at each of these points respectively if and only if $(x^3 + \pi^3)$, $(1 + \pi^3v^3)$, $(s^3 + \pi^3t^3)$ did not belong to $(\pi, y^2)^2$, $(\pi, u^2v)^2$, *and* $(\pi, t)^2$ respectively. And since this holds true, X is indeed normal.

Thus we have that X is a normal fibered surface over R with generic fiber isomorphic to E,which makes X a model of E in the sense of definition 2.3.1.

The Regular Model

We return to the question posed after proposition 2.2.3 which asked if given a curve E over K, is it possible to find a regular model X over R for this curve? The disingularization theorem helps us to answer this. What the following proposition says, is that, for a curve E over K as above, given any model X over R for E, it is possible to obtain the regular model. What's more, by applying the Lipman procedure of repeated normalizations and blow-ups, the X_n obtained in theorem 2.2.8 is precisely the regular model.

2.3.4 Proposition. *If X is a model of a regular connected projective curve E, then X_n as in proposition 2.2.8 is a regular model of E.*

Proof : Since we have maintained our particular setting of R a complete DVR with algebraically closed residue field k, we know from theorem 2.2.11 that X does infact admit a desingularization in the strong sense. What this means by definition is that we have a proper birational morphism $\pi : X_n \to X$ such that π is an isomorphism above all regular points of X. Now by the definition of a model, we have that the generic fibre X_K of X is isomorphic to E, which is given to be regular. Thus we have that π is an isomorphism above $E(\cong X_K)$. This gives us an isomorphism between the generic fibre of X_n and E. Now X_n is regular and projective. (Projectivity follows from the fact that the composition of two projective maps is projective-Thus since X_n is projective over X and X is projective over R we have that X_n is projective over R.) By 2.1.7(iii), X_n is also integral, and thus flat. (non-constant morphism from an inegral scheme to a Dedekind scheme). It is normal since it is regular and irreducible (regular implies any connected component is normal). Since proper birational morphisms preserve dimension, (see Liu 8.2.7), we have that $dim X_n = 2$. Thus X_n is a regular, normal fibred surface of dimension 2 over R whose generic fibre is isomophic to E. In other words, X_n is indeed a regular model of E.

2.3.5 Example. By the above proposition, we have that for our E given by $\pi^{-1}y^2 = (x^3 + \pi^3)$, and its model $X = Proj R[x,y]/(b^2c - \pi(a^3 + \pi^3c^3))$ there exists a regular model X'' which can be arrived at by following Lipman's procedure of repeated blow-ups and normalizations. So we proceed now to do this in order to find a regular model X'' of E. We have already blown-up the singular locus of X to get a regular scheme. If this is normal, then we have found a regular model for E. But by looking at the relations defining the affine open $D_+(w)$ of U', we see that the scheme contains the open $Spec\ R[x_1, y_1]/(y_1^2 - \pi^2(x_1^3 + 1))$ which is indeed not normal by proposition 1.1.20 and the argument used to show X was normal. Thus we must now normalize. If the resulting scheme is regular, we will have our regular model. Consider the affine opens covering X'

$$D_+(u) = Spec\ A[v_1, w_1]/(y - xv_1, \pi - xw_1, \pi v_1 - xw_1, v_1^2 - \pi(x + \pi w_1^2));\ v_1 = v/u,\ w_1 = w/u$$

$$D_+(v) = Spec\ A[u_2, w_2]/(yu_2 - x, \pi u_2 - xw_2, \pi - yw_2, 1 - \pi(u_2^2 + \pi w_2^2));\ u_2 = u/v,\ w_2 = w/v$$

$$D_+(w) = Spec\ A[u_3, v_3]/(yu_3 - xv_3, \pi u_3 - x, \pi v_3 - y, v_3^2 - \pi(xu_3^2 + \pi));\ u_3 = u/w,\ v_3 = v/w$$

Begin with $D_+(u)$: $\pi = xw_1$. Substituting in the last relation we have:

$$v_1^2 = x^2 w_1 + x^2 w_1^4 = x^2 w_1(1 + w_1^3)$$

$$\Rightarrow (v_1/x)^2 = w_1(1 + w_1^3)$$

So we found an element $v_2 := (v_1/x)$ that is integral over function ring of $D_+(u)$. If the overring it generates is normal, then it is clearly the normalization of $D_+(u)$. Let us look at the overring: $A[v_2, w_1]/(y - v_2 x^2, \pi - xw, \pi v_2 x - yw_1, v_2^2 - w_1(1 + w_1^3)) \simeq R[v_2, w_1, x]/(\pi - xw_1, v_2^2 - w_1(1 + w_1^3))$ Since it is integral and thus irreducible, if it is regular it is normal. Thus we check for normality by checking for regularity. Once again we use the Jacobian criterion to eliminate the possibilities and we see that the only point that causes any suspicion at all is the one corresponding to the maximal ideal $\mathbf{m}' = (v_2, w_1, x, \pi)$. But by proposition 1.1.20,we only need to look at the local ring at this point, and this is simply the regular local ring $R[x, w_1]/(\pi - xw_1)$ quotiented by $v_2 - w_1(1 + w_1^3)$ which does not belong to $\mathbf{m}'^2$, and is thus regular. Thus the overring described above is indeed the normalization $D_+(u)''$.

Now consider $D_+(v)$. The relation $1 = \pi(u_2^2 + \pi w_2^2)$ implies the π is invertible in $D_+(v)$ which in turn implies that $D_+(v)$ is contained in $X_K \simeq E$ which is in any case regular in X, so we needn't worry about $D_+(v)$ any further.

Lastly we have $D_+(w)$ which we already know is not normal. We follow what we did for $D_+(u)$ and find an integral element not in $D_+(w)$ and find its overring. We have $x = \pi u_3$. Substitute this in the last relation to get $v_3^2 = \pi(\pi u_3 + \pi) \Rightarrow (v_3/\pi)^2 = u_3^2 + 1$. So we have an element $v_4 := (v_3/\pi)$ integral over the function ring of $D_+(w)$. The overring it generates is quite obviously regular and thus we have our desired normalization $D_+(w)''$: Indeed the overring is $A[u_3, v_4]/(yu_3 - xv_4, \pi^2 v_4 - y, v_4^2 - (u_3^3 + 1)) \simeq R[u_3, v_4]/(v_4^2 - (1 + u_3^3))$, so applying the Jacobian criterion it is instantly seen to be regular since $char\ k \neq 2, 3$.

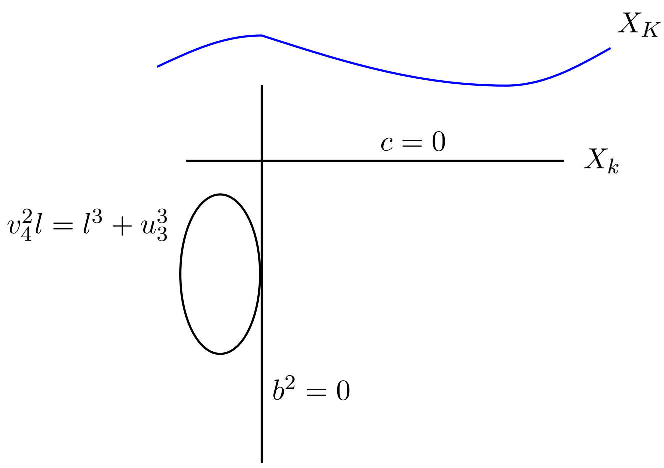
Thus we have now obtained U'', the normalization of our blow-up U', which is regular and defines our regular model. Since we knew the precise location and nature of our singularity, we were able to work locally on $U = D_+(c)$. By this we now have a regular normal fibered surface X'' over R with generic fiber isomorphic to E. Thus X'' is a regular model of E. In order to see what X'' looks like, lets look at its fibers. We already know exactly what its generic fiber looks like. Its special fiber X_k'' is made up of the reduced projective line we saw earlier that remains unchanged ($c = 0$) and what we made by resolving the singularity, leaves us with a projective double line and an elliptic curve. Indeed, in $D_+(u)''$, the special fibre is given by the equations

$$xw_1 = 0, v_2^2 = w_1(1 + w_1^3).....(1)$$

and also by $v_1^2 = 0$, which gives us the double projective line mentioned above. In $D_+(w)''$, the special fiber is given by the equation

$$v_4^2 = 1 + u_3^3....(2)$$

Now, the relation $u_3 = 1/w_1$ when substituted in (2) gives us $v_4 = v_2 u_3^2$, which enables us to glue the equations to get a single elliptic curve, which is the third component of the special fiber of U''. Ou figure will now look something like this:



3 Contraction and The Existence of a Minimum Regular Model

3.1 Contraction: Definition and Existence

3.1.1 Definitions. Let $\pi : X \to S$ be a fibred surface over a Dedekind scheme S. Then an irreducible Weil divisor D on X is called *horizontal* if $\pi_{|D} : D \to S$ is surjective and thus finite. If $\pi(D)$ is reduced to a point, we call D *vertical.* An arbitrary Weil divisor is called *horizontal* (respectively *vertical*), if each of its irreducible components are horizontal (respectively vertical).

As a complement to the process of blowing-up, one would imagine it natural to define something that reverses the process, something that perhaps *blows down.* This would somewhat amount to characterising 'exceptional' divisors on the blow-up of a regular scheme. More precisely we define the following:

3.1.2 Definition. Let $X \to S$ be a regular fibred surface over a Dedekind scheme of dimension 1. A prime divisor E on X is called an *exceptional divisor* or *(-1)-curve* if there exist a regular fibered surface $Y \to S$ and a morphism $f : X \to Y$ of S-schemes such that $f(E)$ is reduced to a point, and that

$$f : X \backslash E \to Y \backslash f(E)$$

is an isomorphism.

We now define a *contraction* which is essentially the morphism f above, defined for the exceptional locus.

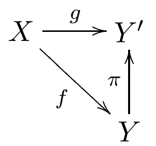
3.1.3 Definition. Let X be a normal fibred surface. Let ε be a strict subset of the irreducible components of the special fibre X_k. Let $f : X \to Y$ be a morphism from X to another normal fibred surface Y, such that $f(E)$ is reduced to a point $\forall E \in \varepsilon$ and f induces an isomorphism

$$f : X \backslash \cup E \to Y \backslash \cup f(E)$$

Such an f is called a contraction morphism.

Note that the contraction morphism is also a blow-up of Y along the closed point $f(E)$. This seems intuitively clear, however we will be able to give an easy neat proof once we have the result of the factorization theorem, which we will see soon. One of the main objects of this section is to prove the existence of the contraction morphism.

However,keeping notation from above, if such a contraction f exists, then it is unique upto unique isomorphism. Indeed if $g : X \to Y'$ was another contraction, we have by by Stein factorization that g factors as follows:



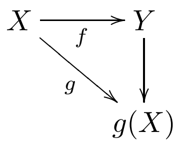
with $g = \pi \circ f$ such that f is unique with geometrically connected fibres.

In order to understand what the morphism f actually looks like, and why it exists, we must now admit results from the section on divisors and invertible sheaves. We first show that the existence of the contraction morphism is equivalent to the existence of an effective Cartier divisor that meets the closed fibres in a certain way. Then we show that in our setting of a complete DVR with algebraically closed residue field (infact the result is for Henselian rings) such a divisor exists, and thus so does the contraction morphism.

3.1.4 Proposition. *Let $X \to S$ be a normal fibered surface with dim $S = 1$. Let ε be a strict subset of integral projective vertical curves on X. Then the following conditions are equivalent:*
(a) A contraction $f : X \to Y$ of the $E \in \varepsilon$ exists.
(b) There exists a Cartier divisor D on X such that deg$(D|_{X_K}) > 0$, $O_X(D)$ is generated by its global sections and for any vertical curve E, $O_X(D)|_E \cong O_E$ if and only if $E \in \varepsilon$.

Proof : Suppose the contraction morphism f as above exists. Then $f(E)$ is a point fo every $E \in \varepsilon$ and f induces an isomorphism outside $\cup_{E\in\varepsilon} E$. Note that ε and thus $f(\varepsilon)$ are finite as sets. We look at Y as sitting in projective N-space over S, i.e embed $Y \hookrightarrow \mathbb{P}^N_S$. Now we know from properties of projective morpshisms that there exists a hypersurface $V_+(F)$ of $\mathbb{P}^N_S$ such that it doesn't meet the image of $f(\cup E)$ in $\mathbb{P}^N_S$ or the generic point of Y. Consider this F that defines the hypersurface and let D_0 be the Cartier divisor associated to it, let it have degree d. Then $O_{\mathbb{P}^N_S}(D_0) \simeq O_{\mathbb{P}^N_S}(d)$ and $Supp\ D_0 = V_+(F)$. Thus, by definition of $V_+(F)$, $SuppD_0 \cap (\cup_{E\in\varepsilon} f(E)) = \emptyset$ and so $Y \nsubseteq Supp\ D_0$. Let $D_1 := D_0|_Y$, the restriction of D_0 to Y, which is also a Cartier divisor, such that $Supp\ D_1 \cap (\cup_{E\in\varepsilon} f(E)) = \emptyset$ Since we have the closed immersion $Y \hookrightarrow \mathbb{P}^N_S$, $O_Y(D_1)$ is very ample, and what's more, we can suppose it to be generated by its global sections, since we can replace D_1 by a multiple if necessary. Now consider $D := f^*(D_1)$. It turns out that this is the divisor we are looking for. We have that D is effective, $O_X(D) \simeq f^*(O_X(D_1))$ and is generated by its global sections (pullback of a very ample sheaf generated by its global sections). Further, $Supp\ D \cap E = \emptyset\ \forall E \in \varepsilon$ and thus $O_X(D)|_E \simeq O_E$. If $E \notin \varepsilon$, then $f(E)$ is a vertical curve on Y and we have a finite birational morphism $g : E \to f(E)$ and $O_X(D)|_E = g^*(O_Y(D_1))|_f(E)$. Since the pullback of an ample sheaf under a finite morphism is itself ample, we have that $O_X(D)|_E$ is ample, and thus $O_X(D)|_E \ncong O_E$, proving the if and only if condition of (b). We thus have our divisor D as desired, constructed such that the associated sheaf is the pullback of an ample sheaf on Y.

Now suppose we have a Cartier divisor D on X as defined in part (b). We want to show that the contraction f exists. We first construct f for the affine case $S = SpecA$ and then the general case follows from uniqueness of f. Now for such a D, we have the associated sheaf $O_X(D)$. We know from the assumptions of our setting and remark 2.2.2 that the generic fibre X_K is an integral projecive curve over K. Thus by proposition 1.2.12, the restriction $O_X(D)|_{X_K}$ is an ample sheaf on X_K. Further, it can be assumed to be very ample (substituting D by a multiple if necessary). Thus, if $s_0, ..., s_n$ are the sections which generate $O_X(D)|_{X_K}$, then we can construct a morphism $g : X \to \mathbb{P}^n_S$ associated to these sections. Now $g(X)$ can be endowed with its closed integral subscheme structure to make $g : X \to g(X)$ a dominant morphism which identifies the respective generic fibres and is thus birational and projective. Under a projective morphism, we have that g_*O_X will be a coherent sheaf on $g(X)$. Since X is normal, the canonical morphism $Y := Spec\ g_*O_X \to g(X)$ is the same as the normalization morphism $X \to g(X)$. So $X \to g(X)$ factors as follows:



where $f : X \to Y$ is the contraction morphism we desire. In order to check this we admit the following lemma:

3.1.5 Lemma. *Let X be a scheme that is locally of finite type over a Noetherian ring A. Let $\mathfrak{L}$ be an invertible sheaf on X generated by global sections $s_0, ...s_n$. Associate the morphism $f : X \to \mathbb{P}^n_S$ to these sections. Then for $s \in Spec\ A$ such that X_s is projective over $k(s)$, and Z a connected closed subscheme of X_s, we have that $f(Z)$ is reduced to a point if and only if $\mathfrak{L}|_Z \simeq O_Z$.*

It follows directly from the lemma above that in proposition 3.1.4, second part of the proof, that for any integral vertical curve E on X, $g(E)$ is reduced to a point if and only if $O_X(D)|_E \simeq O_E$ and since $Y \to g(X)$ is finite, $f(E)$ is reduced to a point if and only if $g(E)$ is reduced to a point. Thus the result.

Before we proceed to show that the above equivalent conditions are satisfied in our case of a normal fibred surface X over a complete DVR R with algebraically closed residue field k, we need the following lemma, which, given an effective Cartier divisor D and the associated sheaf $O_X(D)$, assures us of being able to find an n large enough for the sheaf $O_X(nD)$ to be generated by its global sections.

3.1.6 Lemma. *Let X be a fibred surface over $S = SpecA$ Dedekind. Let D be an effective Cartier divisor on X and $O_X(D)$ the associated sheaf. Then there exists n_0 such that $O_X(nD)$ is generated by its global sections for all $n \geq n_0$.*

Proof : By proposition 1.2.12, we have that $O_X(D)|_{X_K}$ is an ample sheaf. Since K is flat over A, it follows from the commutativity of cohomology and flatness that $H^1(X, O_X(nD)) \otimes_A K = H^1(X_K, O_X(nD)|_{X_K})$ for every $n \geq 0$. But for a fixed $n_0 \geq 0$, the right hand side of the equality will vanish for all $n \geq n_0$, implying that $H^1(X, O_X(nD))$ is torsion and thus of finite length (as it is finitely generated). Now, based on the remark following 1.2.12, we endow D with the closed subscheme structure $V(O_X(-D))$ and consider the exact sequence

$$0 \to O_X(-D) \to O_X \to O_D \to 0$$

Tensor with $O_X((n+1)D)$ and take the cohomology to get the sequence:

$$H^0(X, O_X((n+1)D)) \to H^0(D, O_X((n+1)D|_D) \to H^1(X, O_X(nD))$$

$$\to H^1(X, O_X((n+1)D)) \to H^1(D, O_X((n+1)D|_D) = 0$$

The sequence tells us that the length of the cohomology groups $H^1(X, O_X(nD)$ decreases as $n \geq n_0$ increases, and thus becomes stationary at some rank m_0. Then $H^1(X, O_X((m)D)) \simeq H^1(X, O_X(m+1)D)$ and $H^0(X, O_X(mD)) \to H^0(D, O_X(mD|_D))$ is surjective for every $m > m_0$. Now $O_X(mD)|_D$ is generated by its global sections. But by Nakayama's lemma, the homomorphism $H^0(X, O_X(mD)) \otimes_{O_X(X)} O_{X,x} \to O_X(mD)_x$ is surjective for every $x \in Supp\ D$ but also for all other x, since if $x \notin Supp\ D$ then $O_X(mD)_x = O_{X,x}$ by definition. So the above homomorphism remains surjective. Thus we have that $O_X(mD)$ is generated by its global sections as claimed.

Note that this lemma together with proposition 3.1.4 says that, in order for a contraction morphism to exist, we must be able to find a Cartier divisor that meets the fibres in a desired way. It turns out that in our setting, of a normal fibered surface over $S = Spec\ R$, where R is a complete DVR with algebraically closed residue field k, we are able to find such a divisor. The result holds true for all fibred surfaces over the spectrum of a Henselian ring, but since our chosen DVR is indeed Henselian, this is something we do not worry about. We simply present the result in the particular chosen setting.

3.1.7 Proposition. *Let R be a complete DVR with algebraically closed residue field. Let $X \to S = Spec\ R$ be a normal fibered surface. Then for any proper subset ε of the irreducible components of the closed fibre X_k, the contraction morphism exists for each of the $E \in \varepsilon$.*

Proof :

The key to be able to choose the desired Cartier divisor that ensures the existence of the contraction morphism lies in the fact that for a scheme $S = SpecR$ such as ours, any scheme that is finite over such an S is a disjoint union of local schemes. So for an $x \in X_k$, there exists an effective horizontal Cartier divisor D such that $Supp\ D \cap X_k = x$. Indeed we can choose a regular element f such that $V(f)$ doesn't contain any irreducible component of $Spec\ O_{X_k,x}$. Then there exists an open neighbourhood $U \ni x$ and an element $g \in O_X(U)$ such that $g_x = f$ and $V(g) \cap U_x = x$. Now consider $V(g)$ as, we can regard it as an effective divisor on U. Consider $V(g)$ with its endowed closed subscheme structure. Then $V(g) \to S$ is both quasi-finite and quasi projective. As a consequence of Zariskis main theorem (see [Liu] Cor.4.4.6) we know there exists an affine open neighbourhood V of x such that $V(g) \cap V$ is an open subscheme of a scheme Z that is finite over S. Let D be a connected component of Z, then it is closed in Z and thus in X,making it the Cartier divisor we desire as it satisfies the condition claimed above. Now that we know we can choose such a divisor given a fibered surface such as ours, let ε be a subset of the irreducible components of X_k. Let $Z_1, ..., Z_n$ be the irreducible components of X_k not in ε. Let D_i be a Cartier divisor such that $Supp\ D_i \cap Z_i$ is a single point in Z_i obviously not lying in any component belonging to ε. Let $D = \sum_{0 \leq i \leq n} D_i$. By Lemma 3.1.6 we have that (replacing D by a multiple if necessay), $O_X(D)$ is generated by its global sections. Furthur, for $E \in \varepsilon$, $E \cap Supp\ D = \emptyset$. Thus $O_X(D)|_E \simeq O_E$. And if $E \notin \varepsilon$, $E = Z_i$ for some i, and $deg\ O_X(D)|_E > deg\ O_X(D_i)|_{Z_i}$, so $O_X(D)|_E \not\simeq O_E$. Thus by proposition 3.1.4 the contraction morphism exists for all $E \in \varepsilon$.

3.1.8 Remark. To see more clearly what the morphsim itself looks like, consider this. What we have shown above is that in our setting, the contraction morphism exists for any proper subset ε of irreducible components of X_k. And its existence is is equivalent to be able to choose an effective relative Cartier divisor that meets exactly the components of X_k not in ε. Since we are able to do this in our setting, we can explicitly state that the following morphism is a contraction morphism:

$$X \to Y := Proj\ (\oplus_{n \geq 0}(H^0(X, O_X(nD)))$$

3.1.9 Example. Before we go on to looking at intersection theory, let's return to our example, in order to apply the results of this section- i.e in order to contract an irreducible component of the special fiber X_k. Let's say we want to contract E_1, the reduced projective line given by $c = 0$. By proposition 3.1.4, we see that we want to find an effective horizontal divisor D that meets only the components of X_k other than E_1. One such divisor is the closed subscheme given by $D := V_+(b^2, a + \pi c)$, which is of degree 2 and clearly disjoint from E_1. Also by proposition 3.1.4 we know that we want to look at $O_X(D)$, the associated invertible (locally free) sheaf and its global sections. We can do so by covering X by open affines on which $O_X(D)$ is trivial. By the definition of global sections we should look for a set of sections on the open sets of the cover such that they coincide on intersections. But since X is integral we have the advantage of being able to determine any section of an invertible sheaf by its restriction to any open set. So we can instead compute a global section by taking any section on one of the opens and making sure it extends to the whole scheme.

Keeping the above comment in mind, we choose the following opens to cover X:

$$U_1 = D_+(c),\ U_2 = D_+(a(a + \pi c)),\ U_3 = D_+(b)$$

Then we have

$$O_X(U_1) = R[u, v]/(v^2 - \pi(u^3 + \pi^3);\ u = a/c,\ v = b/c.$$
$$O_X(U_2) = R[p, q, 1/(1 + \pi q)]/(p^2 s - \pi(1 + \pi^3 s^3));\ p = b/a,\ q = c/a.$$
$$O_X(U_3) = R[s, t]/(t - \pi(s^3 + \pi^3 u^3));\ s = a/b,\ t = c/b.$$

Note that we have the relations:

$$q = 1/u,\ t = 1/v = q/p,\ s = 1/p = u/v,$$

Following proposition 3.1.4 and remark 3.1.8, we know we want to look at $H^0(X, O_X(nD))$ for some $n \geq 0$ in order to be able to define the contraction morphism for E_1. We already know by proposition 1.1.12 and the proof of proposition 3.1.4 that for a suitable $n \geq 0$, $O_X(nD)|_{X_K} = O_{X_K}(nD_K)$ is very ample and is generated by its global sections, which define a closed immersion into projective space (under the morphism we called g in the proof of proposition 3.1.4). Moreover, we have the following result which tells us explicitly what our n needs to be in order for $O_{X_K}(nD_K)$ to be very ample: For a smooth, geometrically connected projective curve over a field, with genus g, an invertible sheaf on the curve is very ample as soon as it's degree is $\geq 2g + 1$. (see [Liu] prop.7.4.4) Since we know X_K to satisfy the above conditions with genus 1, we have that $deg\ O_{X_K}(nD_K)$ should be atleast 3. Given that $deg\ D_K = 2$, $n = 2$ will give us what we want. We also have that $dim\ H^0(X_K, O_{X_K}(nD_K) = (deg\ D_K)n = 2n$, by the Riemann Roch Theorem, an important result in algebraic geometry, which is neither stated or treated here, but can be referred to in [Liu] prop. 7.3.33 and [Ha] chapter IV. Thus what we want is to find a basis $H^0(X_K O_{X_K}(2D_K)$ which defines the closed immersion $X_K \to \mathbb{P}^3_K$ (see theorem 1.2.14). Moreover, we want this basis to generate $O_X(D)$.

What we do now is to choose a global section, see what it looks like on U_1, make sure it extends to all of X, i.e to U_1 and U_2 as well, and obtain a description of $H^0(X, O_X(2D))$.

Let $f \in H^0(X, O_X(2D))$ be a global section of $O_X(2D)$. In U_1, every element can be written as an expression of the form $P(u) + vQ(u)$ (given the relation that defines it) which is easy to work with so we look at the restriction of f to U_1.

$$f|_{U_1} = F[u, v] = 1/(u + \pi)^2(P(u) + wQ(u))$$

Using the relations $q = 1/u,\ p = 1/s = u/v$ we have:

$$F(1/q, p/q) = (q/(1+\pi q))^2(P(1/q)+(p/q)Q(1/q)) = 1/(1+\pi q)^2(q^2 P(1/q)+pqQ(1/q))$$

If this were to be defined on all of U_2, this means that $F(1/q, p/q)$ should have no pole at $q = 0$, which in turn means that $deg\ P(u) \leq 2$ and $degQ(u) \leq 1$. Now using relations $s = u/v,\ t = 1/v$ we have:

$$F(s/t, 1/t) = (t/(s + \pi t))^2(P(s/t) + (1/t)Q(s/t)) = 1(s + \pi t)^2(t^2P(s/t) + tQ(s/t))$$

If this were to be defined on all of U_3, this means that $F(s/t, 1/t)$ must have no pole at $s + \pi t = 0$. In U_3 we have $t = \pi(s^3 + \pi^3t^3) = \pi(s + \pi t)(s^2 - st\pi + \pi^2t^2) = 0\ mod(s + \pi t)$ This implies $s = (s+\pi t) - \pi t = 0\ mod(s+\pi t)$, $t = 0\ mod(s+\pi t)^2$, and $s \neq 0\ mod(s+\pi t)^2$.

Given these relations, we must see what it means for $F(s/t, 1/t)$ to have no pole at $s + \pi t = 0$. $t^2P(s/t)$ is an expression with terms of the form t^is^{2-i} each of which, by above, is $0\ mod(s + \pi t)^2$. $tQ(s/t)$ is an expression with terms of the form $s^it^{1-i};\ i \geq 1$, each of which by above are $0\ mod(s + \pi t)^2$, together with the term a_1t which will be $0\ mod(s + \pi t)^2$ if and only if $a_1 = 0$. Thus what the condition becomes for $F(s/t, 1/t)$ to have no pole at $s + \pi t = 0$ is that $deg\ Q(s/t) = 0$.

So given the conditions, we have that $H^0(X, O_X(D))$ is the set of $\{F(u, w) = 1/(u+\pi)^2(P(u) + wC);\ deg\ P(u) \leq 2\}$. with basis:

$$s_0 = 1/(u + \pi)^2,\ s_1 = u/(u + \pi)^2,\ s_2 = u^2/(u + \pi)^2,\ s_3 = v/(u + \pi)^2$$

. This defines the morphism:

$$\phi : X \to \mathbb{P}^3_R$$

$$(u : v : 1) \mapsto (1 : u : u^2 : v)$$

which in homogeneous coordinates is:

$$(a : b : c) \mapsto (c^2 : ac : a^2 : bc)$$

The equation defining X tells us $b^2c = \pi(a^3 + \pi^3c^3)$. So we have $\pi a^3 = c(b^2 - \pi^3a^3)$, and can thus cover X by $D_+(a), D_+(c)\ and\ D_+(b^2 - \pi^3a^3)$.

Since we have the relation $c = \pi a^3/(b^2 - \pi^3a^3)$ in the last open, our map sends

$$(a : b : c) \mapsto ((\pi a^3/(b^2 - \pi^3a^3))^2a^4 : (\pi a^3/(b^2 - \pi^3a^3))a^2 : 1 : (\pi a^3/(b^2 - \pi^3a^3))ab)$$

So any point on the component E_1 (which by virtue of being on $E_1 \subset X_k$ satisfies $c = \pi = 0$)

$$x = (a : b : c) \mapsto (0 : 0 : 1 : 0)$$

Our map ϕ is thus indeed a contraction.

What we want now is to define the contracted scheme $Y := \phi(X) \subset \mathbb{P}^3_R$.

If we set the homogeneous coordinates of Y to be $(l : m : n : o)$; $l = c^2,\ m = ac,\ n = a^2,\ o = bc$, we have the relations $ln = m^2$ and also from the equation of X, $b^2c = \pi(a^3 + \pi^3c^3)$ we have $o^2/c = \pi(an + \pi^3lc) \Rightarrow o^2 = \pi(mn + \pi^3l^2)$.

So $Z \subset D_+(l) \cup D_+(n) \subset \mathbb{P}^3_R$ is given by the equations $ln = m^2\ and\ o^2 = \pi(mn + \pi^3l^2)$. On $D_+(l)$ and $D_+(n)$ we dehomogenize the equations of Z to see that following the argument in example 2.3.3 to show that X was normal (Serre's criterion together with proposition 1.1.20), we find that Z is indeed normal, but not regular.

Now that we have overcome the problem of existence of contraction, our major concern is what becomes of the irreducible components that you contract. As we saw in the example above, we obtained an arithmetic surface on contraction that wasn't regular. Being in a natural way the inverse of the process of blowing up, we want to be able to regulate the singularities that result. We have aleady raised the question of a minimal regular model in the note following proposition 2.2.3. In order to obtain such a minimal model from a regular model and ensure that it continues to be regular, we will have to choose the components we contract wisely. In order to recognise these exceptional divisors in some way, we have to be able to define an intersection theory for divisors on regular fibred surfaces, which we develop below.

3.2 Intersection Theory on a Regular fibred Surface

General intersection theory is defined between arbitrary divisors. However, since a regular fibred surface is not complete, we cannot do the same here. It turns out however, for $X \to S$, a regular fibred surface, over a complete DVR with algebraically closed residue field, we can define an intersection between a vertical divisor and an effective divisor. Thus we have a mapping

$$Div_k X \times Div X \to \mathbb{Z}$$

where $Div_k X$ is the subgroup of $Div X$ generated by the vertical divisors, i.e the subgroup generated by the irreducible components of the special fibre X_k. More precisely, for a prime vertical divsor E on X equipped with the closed immersion $i : E \hookrightarrow X$ and an arbitrary effective divisor F on X, equipped with the closed immersion $j : F \hookrightarrow X$,we can define an intersection number $E.F$ as follows: Since X is given to be regular, we have an equality of Weil and Cartier divisors on X, and thus, the ideal sheaf $\mathfrak{I}$ associated to F is invertible and $i^{-1}(\mathfrak{I})$ is the induced invertible sheaf on E. Since E is a curve over k, we can define the intersection number $E.F := deg\ (i^{-1}(\mathfrak{I}))$ If $E \neq F$ then $E.F$ is atleast the number of points in $Supp\ E \cap F$ and thus it is positive. If E and F intersect each other transversally at all points, then $E.F$ equals exactly the number of points in $Supp\ E \cap F$. So under the mapping defined above:

$$Div_k X \times Div X \to \mathbb{Z}$$

we send

$$(E, F) \mapsto E.F$$

.

We state, wihout proof, some properties of this map.

3.2.1 Proposition. *Let X be a regular fibred surface over a complete DVR R with algebraically closed residue field k. Let E, F be divisors on X with E vertical. Then:*

- *If F is vertical, then $E.F = F.E$, i.e to say that the restriction of the bilinear form to $Div_k X \times Div_k X \to \mathbb{Z}$ is symmetric.*
- *If F is principal, then $E.F = 0$.*
- *If E is prime, then $E.F = deg_E(O(F) \otimes O_E)$*

3.2.2 Theorem. *Let X be a regular fibered surface as above and let $E_1, ..., E_r$ be the irreducible components of X_k. Then:*

- *(i) For all vertical divisors F, $X_k.F = 0$*

- *(ii)* $E_i.E_j \geq 0$ *for* $i \neq j$ *and* $E_i^2 \leq 0$

Proof : (i) This is easy to see since X_k is nothing but the pullback of the closed point in $SpecR$ which is a principal Cartier divisor and therefore X_k is a principal Cartier divisor in X. Thus by part two of above proposition, the result follows. (ii) For $i \neq j$, $E_i.E_j \geq \sharp(E_i \cap E_j) \geq 0)$. And by (i), $E_i^2 = (E_i \backslash X_k).E_i = -\sum_{j \neq i} E_j.E_i \leq 0$ by above. Thus E_i^2 is minus the intersection number of E_i with the special fibre X_k

Note that for $E,\ F \in Div_k X \times Div_k X$, if E and F intersect at finitely many points $P_1, ..., P_n$, then

$$E.F = \sum_{1 \leq i \leq n} dim_k\ (O_i/(e_i, f_i))$$

where O_i is the local ring at P_i and e_i and f_i are the local equations for E and F at P_i respectively. The proof of this equality requires some very standard commutative algebra and can be found in [Lic] (prop.1.6).

3.3 Castelnuovo's Criterion and The Minimal Regular Model

In this section, equipped with our knowledge of contraction and the arithmetic information from intersection theory for fibered surfaces, we finally address the question of the existence of a minimal regular model. We first state the Factorization theorem, which describes projective birational morphisms between regular fibered surfaces as a sequence of blow-ups of closed points. We do not give a proof, only an outline. We then state Castelnuovo's criterion for recognising exceptional divsors and finally, after having formally defined what a minimal regular model is, we prove the minimal models theorem.

Factorization Theorem

3.3.1 Theorem. *Keeping to our notation in the previous sections, let* $f : X \to Y$ *be a birational morphism of regular fibered surfaces over* $S = Spec\ R$. *Then* f *is made up of a finite sequence of blow-ups along closed points.*

We admit the following lemma:

3.3.2 Lemma. *Let* $f : X \to Y$ *be a birational morphism of regular fibered surfaces over* $S = Spec\ R$. *Let* $y \in Y$ *be a closed point such that dim* $X_y \geq 1$. *Then* f *factors as follows:*

$$X \xrightarrow{g} \tilde{Y} \xrightarrow{\pi} Y$$

where π *is the blowing-up morphism of* Y *with centre* y.

Given this, let ε be the exceptional locus of f, i.e $\varepsilon = X \backslash f^{-1}(W)$ where W is the union of open subsets U such that $f^{-1}(U) \to U$ is an isomorphism. Suppose it is non empty, and let $y \in f(\varepsilon)$. Then $dim\ X_y \geq 1$. Let $\pi : \tilde{Y} \to Y$ be the blowing up of Y with centre y. Then by the lemma above, f factors into $X \xrightarrow{g} \tilde{Y} \xrightarrow{\pi} Y$. Consider ε', the exceptional locus of g. Then we know that we have a strict containment $\varepsilon' \subset \varepsilon$. Indeed, the irreducible component of ε whose image in $\tilde{Y}$ is $\pi^{-1}(y)$ will obviously not be contained in ε'. Performing an induction on the number of irreducible components of ε will give us the result.

As a direct corollary to the factorisation theorem, we have the following, which we claimed to be true while defining the contraction morphism.

3.3.3 Corollary. *Let $f : X \to Y$ be a birational morphism of regular fibered surfaces over $S = SpecR$. Suppose the exceptional locus of f is irreducible. Then f is the blowing-up of Y along a closed point y. In particular if $f : X \to Y$ is the contraction morphism of an exceptional divisor E, then it is also the blow-up of Y along the closed point $f(E)$.*

Intersection properties of exceptional divisors

Exceptional divisors are central to understanding contractions. In fact it is precisely these divisors that enable us to use contractions in ways that are useful to us-namely in regulating the singularities that result once we contract, and in turn in finding minimal regular models. By applying the results of intersection theory to exceptional divisors, we attempt to get a more intuitive understanding of these divisors and their properties. One of the main results of this subsection is Castelnuovo's criterion, which gives us an explicit way to recognise an exceptional divisor. Although much of the following is necessary to prove Castelnuovo's criterion, I do not give a proof of the criterion itself. However the brief treatment below will perhaps serve to intuitively see why the criterion should hold true.

3.3.4 Definitions. Let $f : X \to Y$ be a proper birational morphism between regular fibered surfaces. Let D be a prime divisor on Y, and $\mathcal{L}$ the associated invertible sheaf of ideals. We define the total transform of D to be $f^{-1}(D)$, the divisor associated to the sheaf $f^*(\mathcal{L})$. Let y be the generic point of D. We define the proper transform of D to be $f^{-1}[D] := \overline{f^{-1}(y)}$, the Zariski closure of $f^{-1}(y)$. Note that $f^{-1}[D]$ is a divisor on X.

We state, without proof the following properties of divisors and proper transforms.

3.3.5 Proposition. *Let $f : X \to Y$ be as above. Then the total transform from divisors on Y to divisors on X preserves linear equivalence. If C, D are two prime divisors on Y such that for a field k, the intersection number $C.D$ is well defined, then $f^{-1}(C).f^{-1}(D)$ is well defined and equal to $C.D$.*

We fix an exceptional divisor E on X (Definition 3.1.2), so we have that $f(E)$ is reduced to a point on Y and f is an isomorphism off E.

3.3.6 Lemma. *Let D be a divisor on Y. Then over $k = k(f(E))$, we have $E.f^{-1}(D) = 0$.*

Indeed we can find a divisor D' that is linearly equivalent to D such that it does not contain $f(E)$. Then $f^{-1}(D')$ and E are disjoint,so $E.f^{-1}(D') = 0$, but by the proposition above $E.f^{-1}(D') = E.f^{-1}(D)$, hence the result.

3.3.7 Lemma. *Let D be a prime divisor of Y passing through $f(E)$. Then D has a regular point at $f(E)$ if and only if $f^{-1}(D) = f^{-1}[D] + E$ and $E.f^{-1}[D] = 1$. If D is regualar at $f(E)$ then f induces an isomorphism between $f^{-1}[D]$ and D.*

From the above two lemmas we have:

3.3.8 Proposition. *E as above is isomorphic to the projective line over $H^0(E, O_E) = k$, $H^1(E, O_E) = 0$ and the self intersection number $E^{(2)} := E.E = -1$*

Proof :

By the factorization theorem we saw that f is also the blow-up of Y with centre $f(E)$. Now, by the construction of blowing-up, $E = f^{-1}(f(E))$ is a complete curve over $H^0(E, O_E) = k$ covered by two affine lines over $H^0(E, O_E) = k$, and thus $E \simeq \mathbb{P}^1_k$. Choose a divisor D having a regular point at $f(E)$. By the two lemmas above, we have

$$0 = E.f^{-1}(D) = E.(f^{-1}[D] + E) = 1 + E.E$$

Thus, $E^{(2)} = -1$.

3.3.9 Proposition. *Let C be a complete integral curve on X over $k' = H^0(C, O_C)$. Suppose $C \neq E$. Then let $D := f(C)$ be the prime divisor on Y and let $k = H^0(D, O_D)$. Then*
(a) If D does not contain $f(E)$ then C and D are isomorphic and $C^{(2)} = D^{(2)}$
(b) If it does, then

$$D^{(2)} \geq [k : k'](C^{(2)} + 1)$$

with equality if and only if C is isomorphic to D via f and D has a regular point at $f(E)$.

Proof : (a) Since f is proper and an isomorphism off E, we have by the integrality of C that D is an irreducible curve in Y overcome k. Now since f is of finite type, $[k : k']$ is finite. By definition of f and E, it is clear that if D does not contain $f(E)$, then C and D are isomorphic. Thus be proposition 3.3.5, $C^{(2)} = D^{(2)}$.

(b) Now, if D does contain $f(E)$, then $f^{-1}(D) = C + nE$ for some integer $n \geq 1$. Thus

$$D^{(2)} = D.D = f^{-1}(D).f^{-1}(D) = (C + nE).(C + nE) = (C + nE).C$$

, the fourth equality coming from lemma 3.3.6. Now $(C + nE).C = C.C + nE.C = [k : k'](C._k C) + n(E.C)$ (intersection number computed over k). Now since C is a curve over k' $E.C = C.E$ and since C and E intersect properly non trivially,$E.C \geq 1$. Thus,

$$D^{(2)} \geq [k : k'](C^{(2)} + 1).$$

Equality holds if and only if $n = 1$ and $E.C = 1$ which by lemma 3.3.7 is true if and only if D has a regular point at $f(E)$.

Before we state Castelnuovo's Criterion, we state the following result:

3.3.10 Proposition. *Let X be a regular fibered surface and E a prime divisor on X which is proper over $H^0(E, O_E)$. If $H^1(E, O_E) = 0$ and $E^{(2)} = -1$, then $E \simeq \mathbb{P}^1_{H^0(E,O_E)}$ (see [Chin] prop.5.3)*

Castelnuovo's Criterion

3.3.11 Theorem. *Let X be a regular fibered surface over $S = SpecR$, where R is a complete DVR with algebraically closed residue field k. Let E be prime divisor on X. Then E is an exceptional divisor (Definition 3.1.2) if and only if all the following hold: $E \subset X_k$, $H^1(O_E, E) = 0$ and $E^{(2)} = -1$. In this case $E \simeq \mathbb{P}^1_{H^0(E,O_E)}$.*

What we are saying in other words is that for a vertical prime divisor E on an X such as above, there exists a contraction of E if and only if $E \simeq \mathbb{P}^1_{H^0(E,O_E)}$ and $E^{(2)} = -1$

Minimal Models

3.3.12 Definition. A minimal (regular) model of a normal, projective, connected curve C over K is a (regular) model X of C over $S = Spec\ R$, in the sense of Definition 2.3.1, (and in keeping with the setting from Section 2.3) such that, for every (regular) model

Y over S , the natural birational map $f : Y \dashrightarrow X$ is a morphism. A relative (regular) minimal model of C is a (regular) model of C such that it contains no exceptional divisors. Clearly a minimal regular model is relatively minimal. (Indeed, if it did contain exceptional divisors, by Castelnuovo's criterion, these can be blown down. Since the blow-down map is not an isomorphism, we have a contradiction to minimality)

The aim of this section is to prove that given a curve C over K as above, a minimal regular model X over S for C exists.

3.3.13 Theorem. *Let X be a regular fibered surface over $S = Spec\ R$. Consider the following sequence of contractions of exceptional divisors:*

$$X = X_0 \to X_1 \to \dots \to X_i \to \dots$$

Then the sequence is necessarily finite and ends at some $Y := X_n$ for some n with a birational morphism $X \to Y$. If X is a regular model for some C over K as above, then the last term in the sequence $Y = X_n$ is a relatively minimal regular model of X. Thus, given a regular model for a curve C over K, we can obtain a relatively minimal regular model for the same.

Proof : If B_i is the set of exceptional divisors on X_i, then certainly $B_{i+1} \subset B_i$, (since even if you only contract some of the exceptional divisors on X_i, the ones you do not contract are sent to isomorphic exceptional divisors on X_{i+1} by the definition of contraction.) Thus as i increases $|B_i|$ is strictly decreasing, implying that the sequence is finite. By the definition of contraction, $X \to Y$ is birational.

Now if X over S is a regular model for some C over K, then by the birational morphism $X \to Y$ we can identify generic fibres, making Y over S a regular model for C. Since Y contains no exceptional divisors by above, it is indeed a relatively minimal model for C.

3.3.14 Theorem. *Suppose that C is a smooth, geometrically connected curve over K of genus $g \geq 1$, then C has a minimal regular model, unique upto unique isomorphism.*

By theorem 2.3.3,we know that C has a regular model X' over S. By the theorem above, we can construct a sequence of contractions to obtain a relatively minimal regular model X over S of C. To show that there exists a minimal regular model. we only have to show that any two relatively minimal regular models are isomorphic. Let us suppose X' is another such model. Then by [Lic] prop 4.2 or [Chin] prop 2.2, we have a regular model Y with birational morphisms $Y \to X$ and $Y \to X'$ which by the Factorization theorem, factor into a finite sequence of blow-ups as follows:

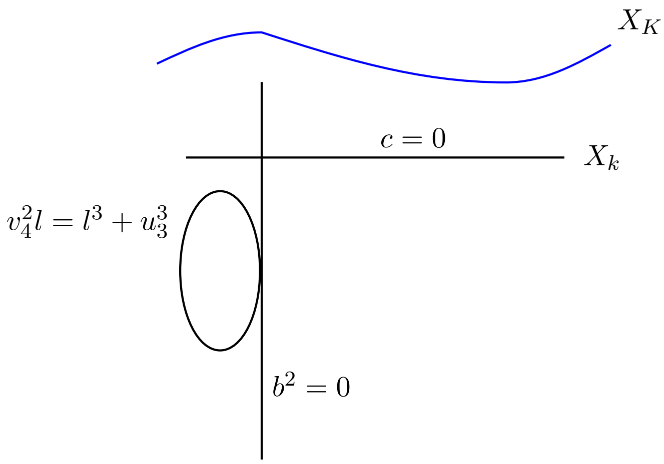
$$Y = X_n \to X_{n-1} \to \dots \to X_0 = X$$

$$Y = X'_m \to X'_{m-1} \to \dots \to X'_0 = X'$$

We may choose Y such that $m + n$ is minimal. If $m > 0$, then $Y \to X_{m-1}$ has an exceptional divisor E. Since X' has no exceptional divisors, the image of E in X' is not an exceptional divisor. Thus there is a t for the image of E in X'_t would be an exceptional divisor for $X'_t \to X'_{t-1}$. (E is either blown down to a point or sent to an isomorphic exceptional curve. (Note: More precisely we have that for a birational morphism $f : X \to Y$ which is a contraction of an exceptional divisor E on X, If X_K has genus > 0 and if C is another exceptional curve on X, then either $C = E$ or $f(C)$ is an exceptional curve on Y not containing the point $f(E)$. It is here that we use the fact of positive genus. The proof of this statement is by contradiction, where one supposes

$f(C)$ contains the point $f(E)$ and then concludes that genus of $X_K < 0$, a contradiction to our assumption.) For this same reason, for all $i = t, ..., m-1$, the image of E in X'_i does not contain the centre of the blow-up $X'_{i+1} \to X'_i$. This allows us then to rearrange the sequence of blow-ups in such a way so as to assume that E is the exceptional divisor of $Y \to X'_{m-1}$. But then $X'_{m-1} \simeq X_{n-1}$, contradicting the minimality of $m+n$. Thus $m = 0$. This implies $X \simeq X'$ by the definition of relatively minimal. Thus we have that a unique minimal regular model exists.

3.3.15 Example. We are now in a position to judge whether our regular model X'' of X, which looks something like this:



is infact a minimal regular model, and if not, we should be able to find one. Thus we first find the possible exceptional divisors, which will occur, if at all, as irreducible components of the special fiber. The three components of the special fiber are the reduced projective line E_1 given by $c = 0$, the projective double line E_2 given by $v_1^2 = 0$ and the elliptic curve E_3 described in example 2.3.5. As the figure shows E_1 meets E_2 in a single point and E_2 meets E_3 in a single point. We know from Castelnuovo's criterion that the only component that might infact be an exceptional divisor is E_1. So we check by computing it's self intersection. Let us first calculate $E_1.E_2$. By the note following theorem 3.2.2 we have that $E_1.E_2 = dim_k k[b,c]/(b^2,c]$. But the later equals 2, and so we have $E_1.E_2 = 2$. Now, also by theorem 3.2.2 we have $E_1.X_k = 0$. This implies

$$E_1(E_1 + E_2) = 0 \Rightarrow E_1^2 + 2 = 0 \Rightarrow E_1^2 = -2$$

Thus E is not an exceptional divisor, implying that X'' contains no exceptional divisors. Implying that it is indeed a minimal surface. Since our original elliptic curve E is smooth and geometrically connected, by theorem 3.3.14, it is infact *the* minimal regular model of E.

4 Neron Models

Neron models were invented by French mathematician André Néron in the early 1960s, with the intention of being able to study abelian varieties over number fields. The novelty of Neron models lies in the fact that Neron are not always projective. However, despite relaxing the condition of properness, Neron found that one could preserve the point extension property by emphasising smoothness and group-scheme structure. As the authors of [BLR] would have us believe, "it came as a surprise for arithmeticians and algebraic geometers" that such models exist in a canonical way. Soon after the discovery, the work of Néron was put in appropriate context using the recently developed "revolutionary" language of schemes devised by A. Grothendieck. Here, we define Neron models and give a brief exposeé of group schemes, before moving on to focus our attention o nnerNeron models of elliptic curves.

4.0.16 Definition. Let R be a Dedekind scheme of dimension 1 with field of fractions K. And $S = Spec\ R$. Let $X \to K$ be a smooth and separated K-scheme of finite type. Then $\mathcal{N}$, a smooth and separated S-scheme of finite type is called a Neron Model of X if X is isomorphic to the generic fibre $\mathcal{N}_K$ of $\mathcal{N}$ and which satisfies the following universal property, called the Neron mapping property:

For every smooth S-scheme Y with generic fibre Y_K, the K-morphism $\varphi_K : Y_K \to \mathcal{N}_K = X$ extends uniquely to an S-morphism $\varphi : Y \to \mathcal{N}$. In other words, the map $\mathcal{N}(Y) \to X(Y_K)$ is a bijection.

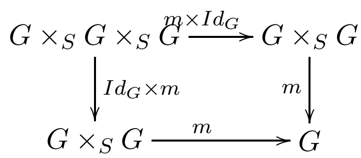
Note that Neron Models do not always exist. Note also that if X were taken to be a smooth and separated curve over K, its Neron model, if it exists may not be a model of a curve in the sense of Definition 2.3.1.

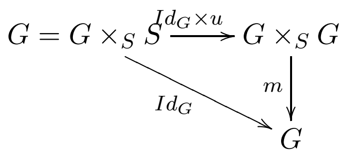
4.1 Group Schemes

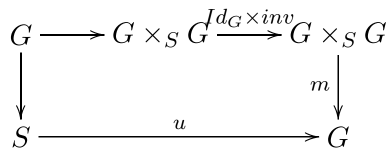
4.1.1 Definition. Let S be a scheme. A group scheme over S is an S-scheme G that is endowed with the following morphisms:

$$m : G \times_S G \to G$$
$$u : S \to G$$
$$inv : G \to G$$

such that the following diagrams commute:

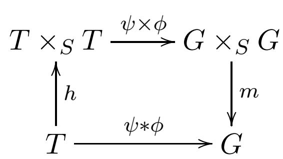






where the diagrams establish associativity, right identity and right inverse respectively.

For an arbitrary S-scheme T, the morphisms m and inv induce maps $m(T) : G(T) \times_S G(T) \to G(T)$ and $inv(T) : G(T) \to G(T)$. The group scheme axioms above then make $G(T)$ into a group. Indeed for any two element ψ and $\phi \in G(T)$, we define a new element $\psi * \phi$ by the commutativity of the following diagram



where we have $\psi * \phi = m \circ (\psi \times \phi) \circ h \in G(T)$ gives $G(T)$ the structure of a group. If $\pi_T : T \to S$ is the canonical structural morphism, then $G(T)$ is a group with identity element $u_T := u \circ \pi_T$.

We say that the group scheme G is a *commutative group scheme* if $G(T)$ is commutative for every S-scheme T. A *subgroup scheme* of G is a closed subscheme H of G such that $H(T)$ is a subgroup of $G(T)$ for every S-scheme T. A group scheme G is called a *group variety* or *algebraic group* if G is an algebraic variety over a field k.

Note that, since G is not a group but a family of groups parametrized by points of S, we cannot define maps given by translations of points. Instead, we define the following:

4.1.2 Definition. Let $G \to S$ be a group scheme. Let $\sigma \in G(S)$ be an S-valued point. Then a *translation by σ map* is the S-morphism $\tau_\sigma : G \to G$ defined as follows:

$$G = G \times_S S \xrightarrow{Id \times \sigma} G \times_S G \xrightarrow{m} G$$

Now an S-valued point σ is a map $S \to G$,and so for every point $s \in S$, we get a point $\sigma(s) \in G_s$, where G_s, the fibre of G at s is a group variety over the resifue field $k(s)$. A group variety is indeed a group and so τ_σ restricted to the fibre G_s is nothing but the translation by $\sigma(s)$. Thus τ_σ can be viewed as a family of translations of the fibres of G.

4.2 Neron Models of Elliptic Curves

We focus our attention on Neron models of Elliptic curves. We once again fix a complete DVR R with algebraically closed residue field k and fraction field K. And we fix an elliptic curve E over K. An important property of Ellipic curves is that they have a group law given by a morphism $E \times E \to E$. If E is given by a Weierstrass equation, and we use this to define it's Weierstrass model W as in section 1.5.1, then this morphism extends in a natural way to a map $W \times_R W \to W$, which may not be a morphism. It turns out however, that if we took the smooth part of the model W, i.e discarded all singular points, then $W^0 \times_R W^0 \to W^0$ is a morphism. However, we would have lost the point extension property guaranteed to us by properness, namely that every point in $E(K)$ extends to a given point in $W(R)$. (see Proposition 2.2.3). A Neron model of E is one that preserves both these properties. The aim of this section is to prove that such a model exists. In the particular case of Elliptic curves, if $\mathcal{C}$ is its unique minimal regular model (It exists by theorem 3.3.14), then it turns out that the Neron model for E can be seen as sitting inside $\mathcal{C}$ i.e, the open set of smooth points of $\mathcal{C}$ is precisely the Neron model of E. For abelian varieties of higher dimension, Neron models are not naturally embedded into a known scheme attached to the variety. Here one has to use a group law defined birationally (called the normal law) but this is not of our concern. The main results in this section are due to Weil.

4.2.1 Proposition. *The group law on E makes it a commutative group variety (abelian variety)*

Indeed you can define morphisms

$$\phi : E \times E \to E, \qquad \psi : E \to E,$$

and

$$(P_1, P_2) \mapsto P_1 + P_2, \qquad P \mapsto -P.$$

Now the algebraic group law on E would be as follows:

$$m : E \times_K E \to E$$

such that $m(x, y) + o \sim x + y$, where o is the fixed base point of E, acting as the unit element of $E(K)$ which is a commutative group under the map m.

4.2.2 Theorem. *Let E over K be an elliptic curve as above. Let $\mathcal{C}$ be its minimal regular model. Let $\mathcal{N}$ be the largest subscheme of $\mathcal{C}$ that is smooth over R. Then the group law extends to $\mathcal{N}$ and defines a smooth group scheme structure on it.*

Proof : First let us note that $\mathcal{N}(R) = \mathcal{C}(R) = E(K)$. Since $\mathcal{N}$ is the largest smooth subscheme of X, this follows directly from Proposition 2.2.3. (Note the minimal regular model $\mathcal{C}$ is projective and thus proper over R.) As for the first equality, since $\mathcal{C}$ is regular, the sections of $\mathcal{C}(R)$ have their image in the smooth locus of $\mathcal{C}$, namely $\mathcal{N}$. Thus $\mathcal{N}(R) \to \mathcal{C}(R)$ is surjective and thus an equality as it is naturally an injection too. Thus $\mathcal{N}(R) = \mathcal{C}(R)$.

Now let $m : E \times_K E \to E$ be the algebraic group law as defined above. Let $(m, q) : E \times_K E \to E \times_K E$ be the automorphism such that q is the second projection $E \times_K E \to E$. We claim that it extends to an automorphism $t : \mathcal{C} \times_S \mathcal{N} \to \mathcal{C} \times_S \mathcal{N}$.

Consider the special fibre $\mathcal{N}_k$. Let η be a generic point. And let $T := Spec\ O_{\mathcal{N},\eta}$. T is regular and of codimension 1 and $\mathcal{C} \times T$ is smooth over $\mathcal{C}$ and thus regular. (smooth morphism to a regular scheme). It can be shown then that $\mathcal{C} \times T$ is a minimal surface. (We admit this point here, see Liu 9.3.30 for more details). Thus (m, q) extends to an automorphism of $\mathcal{C} \times T \to \mathcal{C} \times T$. Indeed, since $(m, q) : E \times_K E \to E$ is an automorphism, by definition of minimality there exists a birational morphism from $\mathcal{C} \times T \to \mathcal{C} \times T$. Applying the same logic to $(m, q)^{-1}$, we see that we get an automorphism. So what we now have is that t is defined everywhere on $\mathcal{C} \times U \to \mathcal{C} \times U$ for some open set U of η containing E. What we do now is show that t is indeed defined everywhere, by performing translations on U. Choose an arbitrary section $\sigma \in \mathcal{N}(S)$. And let t_σ be the associated translation by σ map. Then consider $t' = (t_\sigma \times t_\sigma) \circ t \circ (Id_{\mathcal{C}} \times t_\sigma^{-1})$. It is an automorphism on $\mathcal{C} \times t_\sigma(U)$ which coincides with t on $Spec\ K$ since E is commutative. Thus they coincide on all of $\mathcal{C} \times U \cap t_\sigma(U)$ implying that t is defined on $\mathcal{C} \times t_\sigma(U)$ but since σ was chosen arbitrarily, we have that t is defined on $\mathcal{C} \times \cup_\sigma t_\sigma(U)$ where σ runs through the sections of $\mathcal{N}(S)$. What we want to show is that $\cup_\sigma t_\sigma(U) = \mathcal{N}$ Let $y_s \in U_s$ and $z_s \in \mathcal{N}_s$ be closed points. They lift to sections $y, z \in \mathcal{N}(S)$. If $\sigma = t_y^{-1}(z)$, then $t_\sigma(y) = z$. Thus $z_s = t_\sigma(y_s) \in t_\sigma(U)$. Thus $\mathcal{N} = \cup_\sigma t_\sigma(U)$ as desired. We now have that t is defined everywhere. To see that it is infact an automorphism, note that by the uniquness of the minimal regular model, the automorphism on $inv : E \to E$ extends to an automorphism $inv_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$. (By same argument used earlier in the proof.) One can then verify that $(Id_{\mathcal{C}} \times inv_{\mathcal{N}}) \circ t \circ (Id_{\mathcal{C}} \times inv_{\mathcal{N}})$ is the inverse of t, ensuring that it is indeed an automorphism.

Now when we restrict t to $\mathcal{N} \times_S \mathcal{N}$, its image lies in the smooth locus of $\mathcal{C} \times_S \mathcal{N}$. which is nothing but $\mathcal{N} \times_S \mathcal{N}$ itself. So we have an automorphism $\tau : \mathcal{N} \times_S \mathcal{N} \to \mathcal{N} \times_S \mathcal{N}$ induced by t. Let $p : \mathcal{N} \times_S \mathcal{N} \to \mathcal{N}$ be the first projection. Then the algebraic group law on the generic fibre, namely E, is induced precisely by the composition $p \circ \tau : \mathcal{N} \times_S \mathcal{N} \to \mathcal{N}$ and $inv_{\mathcal{N}}$. What this means is that the commutative diagrams satisfied by a group scheme are verified by E, the generic fibre of $\mathcal{N}$. Since they are commutative on the generic fibre, they are indeed commutative over S, thereby making $\mathcal{N} \to S$ a group scheme. (defined by $p \circ \tau$). (Indeed two S-morphisms agreeing on a dense open subset are infact everywhere equal).

Thus we have that the group law on E extends to make $\mathcal{N} \to S$ a group scheme.

4.2.3 Theorem. *$\mathcal{N}$ as above is the Neron model of E.*

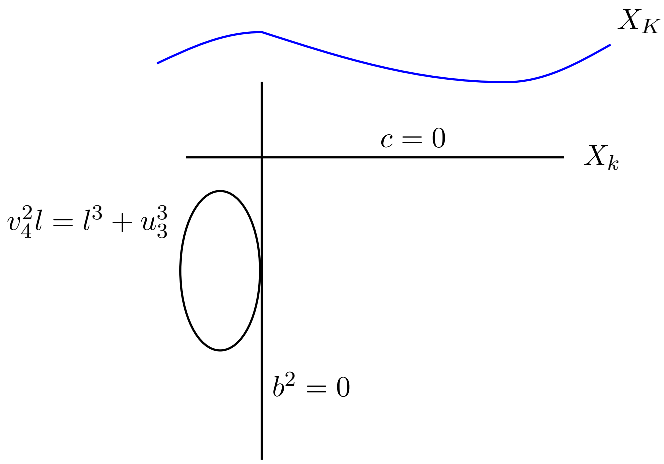
By the theorem above, we have that $\mathcal{N}$ is a smooth group scheme, separated and of finite type over S with generic fibre isomorphic to E. Thus the only thing that we need to verify is the Neron mapping property.

In order to do this, we need the following result by Weil, which we admit:

4.2.4 Theorem. *Let $G \to S$ be a smooth and separated group scheme with S normal and Noetherian. Let X be an arbitrary smooth S-scheme, and let $f : X \to S$ be an S-rational map. If f is defined in codimension ≤ 1, f is defined everywhere.*

Proof : (Theorem 4.2.3) Let X be a smooth S-scheme with generic point η, and let $f : X_\eta \to E$ be a morphism considered as a rational map $X \to \mathcal{N}$. After the result of Weil, it suffices to prove that f is defined at every point of codimension 1 in X. So let $x \in X$ be a point of codimension 1. Let $T := Spec\ O_{X,x}$. By the same argument as in the proof of theorem 4.2.3, we have that $\mathcal{C} \times_S T$ is a minimal regular surface with smooth locus $\mathcal{N} \times_S T$. So we have that $\mathcal{N}_T(T) \to E_{K(T)}K(T)$ is bijective. But $\mathcal{N}(T) = \mathcal{N}_T(T)$ and $E_{K(T)}K(T) = E(K(X))$. So f is defined at x. Since x was an arbitrarily chosen point of codimension 1, it holds for all $x \in X$ of codimension 1, and we are done.

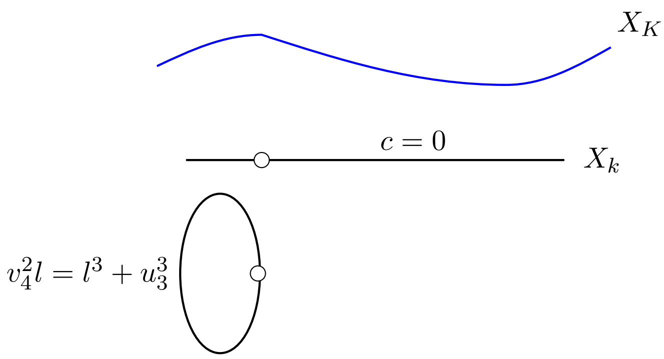
4.2.5 Example. We are now in a position to describe the Neron model for our original elliptic curve $E \to K$, given by $y^2 = \pi(x^3 + \pi^3)$. Indeed we have its minimal regular model that looks like this:



So by theorem 4.2.2 and 4.2.3, the Neron model $\mathcal{N}$ of E is nothing but the smooth part of the minimal regular model X''.

Thus $\mathcal{N} = X'' \backslash \{E_2\}$ where E_2 is the irreducible component which is the (non-reduced) double projective line given by $b^2 = 0$.

The Neron model would thus look something like this:



An interesting and important invariant of the Neron model of an elliptic curve is the order of the group of components of the elliptic curve. The elliptic curve, being a group variety, has an associated quotient group E/E^0, where E is the curve with its group structure and E^0 is the connected component containing the identity element (base point) of E. It can be shown that it is equal to the determinant of the incidence matrix obtained from the intersetion numbers of the irreducible components of the special fibre or equivalently, it is equal to the number of multiplicity 1 components of the special fibre.

Here in our example, $\mathbb{Z}/2\mathbb{Z}$ is the group of components of E. And the number 2, an interesting invariant of our Neron model.

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