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COMPUTATIONAL ALGEBRAIC GEOMETRY: APPLICATION TO EDDY CURRENTS

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Preface

Computing eddy currents (i.e. computing line integrals of a particular closed 1-form) on a surface involve solving some differential equations over it. Approximating the surface with a simplicial complex appears to be most convenient for computational purposes. There are many methods to address the problem, but most of them tend to be computationally demanding. A method is presented in article [5] which involves the use of the different definitions of the homology and cohomology of the surface together with duality theorems (Poincaré's and de Rham's duality theorems), to transform the problem of solving a differential equation into one of solving a system of linear equations. It turns out that, in order to efficiently compute the line integrals, Poincaré's duality theorem and de Rham's theory will play an important role.

The thesis aims to introduce the mathematical tools used on the modelling and to explain the mathematical aspects of the method proposed.

In chapter one, an introduction to homology and cohomology of a chain complex is presented.

In chapters two and three, the singular chain related to a topological space is introduced, which induces the homology and cohomology of a topological space. Simplicial complexes are introduced so that the homology and cohomology triangulable topological spaces can be handled with simplicial homology and cohomology.

Afterwards, CW complexes are introduced and the construction of the dual block complex of a simplicial complex is explained. Although dual block complex is not always a CW complex, the homology of the dual block complex is constructed in the same way as that of a CW complex, and hence it represents the homology of the topological space.

In chapter four, duality theorems are introduced as well. De Rham's complex is introduced allowing one to use differential forms to represent cochains. Poincaré's duality theorem is also introduced using the dual block complex, a complex with the same homology of that of a CW complex. This idea will be useful to explicitly compute the Poincaré dual of a simplex belonging to a simplicial complex.

At last, in chapter five, the problem with Eddy currents is exposed and the method presented in [5] is explained.

Contents

Chapter 1

Homology and Cohomology Groups

1.1 Chain Complexes

Definition The pair $K = \{K_n, \partial_n\}$ consisting of a sequence of abelian groups K_n and a sequence of group homomorphisms $\partial : K_n \to K_{n-1}$ $(n \in \mathbb{Z})$ satisfying the condition

$$\partial_{n-1}\,\partial_n = 0 \quad \forall n \in \mathbb{Z}$$

is called a **chain complex**. The elements of K_n are called *n-chains* and the homomorphisms ∂_n the **n-th boundary operator**.

The subgroups $\ker \partial_n$ and $\operatorname{im} \partial_{n+1}$ of K_n are denoted $Z_n(K)$ and $B_n(K)$ respectively and its elements are called **n-cycles** and **n-boundaries** respectively.

The quotient groups

$$H_n(K) := \frac{Z_n(K)}{B_n(K)}$$

are called the **n-th homology group of K**. Notice that they are well defined due to the condition over the boundary operators.

Definition Let $K = \{K_n, \partial_n\}$ and $K' = \{K'_n, \partial'_n\}$ be two chain complexes. A sequence of homomorphisms $f_n : K_n \to K'_n$ satisfying the commutative conditions

$$f_{n-1}\,\partial_n = \partial'_n\,f_n$$

is called a **chain map** and it s denoted as $f : K \to K'$.

Due to the commutative conditions, $f_n(Z_n(K)) \subset Z_n(K')$ and $f_n(B_n(K)) \subset B_n(K')$ are satisfied. Hence, a chain map induces an homomorphism

$$f_* : H_n(K) \to H_n(K')$$

given by

$$z_n + B_n(K) \mapsto f(z_n) + B_n(K')$$

Definition Let $f, g : K \to K'$ be two chain maps. A sequence of homomorphisms

$$D_n : K_n \to K'_{n+1}$$

satisfying

$$f_n - g_n = \partial'_{n+1} D_n + D_{n+1} \partial_n$$

for all n is called a **chain homotopy** between f and g and is denoted as $D : K \to K'$.

Two maps $f, g : K \to K'$ are said to be **chain homotopic** if a chain homotopy exists between them.

Proposition 1.1.1 *Let $f, g : K \to K'$ are chain homotopic then*

$$f_* = g_* : H_n(K) \to H_n(K')$$

for all n.

1.2 Homology Groups with Arbitrary Coefficients

Proposition 1.2.1 *Let $K = \{K_n, \partial_n\}, L$ be chain complexes, $f, g : K \to L$ chain maps, $D : K \to L$ a chain homotopy between f and g and G an abelian group. Then:*

The induced sequence

$$K \otimes G := \{K_n \otimes G, \partial \otimes 1_G\}$$

is a chain complex.

The induced sequence of homomorphisms

$$f_n \otimes 1_G : K_n \otimes G \to L_n \otimes G,$$

denoted as $f \otimes 1_G : K \otimes G \to L \otimes G$, *is a chain map.*

Then induced sequence of homomorphisms

$$D_n \otimes 1_G : K_n \otimes G \to L_{n+1} \otimes G$$

is a chain homotopy between $f \otimes 1_G$ *and* $g \otimes 1_G$.

Lemma 1.2.2 *If* K'' *is a chain complex of free abelian groups, then any short exact sequence* $0 \to K' \to K \to K'' \to 0$ *is split exact.*

Lemma 1.2.3 *Let* K *and* K' *be chain complexes of free abelian groups, and let* $f : K \to K'$ *be a chain map such that* $f_* : H_n(K) \to H_n(K')$ *is an isomorphism for all* n. *Then, for any abelian group* G, *the induced chain map* $f \otimes 1_G : K \otimes G \to K' \otimes G$ *also induces isomorphisms*

$$(f \otimes 1_G)_* : H_n(K \otimes G) \approx H_n(K' \otimes G)$$

for all n.

1.3 Cochain Complexes

Definition The pair of sequences $K = \{K^q, \delta_q\}$ of abelian groups K^q and homomorphisms $\delta_q : K^q \to K^{q+1}$ satisfying the condition

$$\delta_{q+1}\delta_q = 0$$

for all $q \in \mathbb{Z}$ is called a *cochain complex*. The homomorphisms δ_q are called **coboundary operators**.

The subgroups $\ker \delta_q$ and $\operatorname{im} \delta_{q-1}$ of K_q are denoted $Z^q(K)$ and $B^q(K)$ respectively and its elements are called **q-cocycles** and **q-coboundaries** respectively

The quotient groups

$$H^q(K) := \frac{Z^q(K)}{B^q(K)}$$

are called the **q-th cohomology group** of K.

Definition Let $K = \{K^q, \delta^q\}$ and $K' = \{K'^q, \delta'^q\}$ be two cochain complexes. A sequence of homomorphism $f^q : K^q \to K'^q$ satisfying the commutative conditions

$$f^{q+1}\delta^q = \delta'^q f^q$$

is called a **cochain map** and it is denoted as $f : K \to K'$.

Due to the commutative conditions, $f^q(Z^q(K)) \subset Z^q(K')$ and $f^q(B^q(K)) \subset B^q(K'^q)$ are satisfied. Hence, a chain map induces an homomorphism

$$f^* : H^q(K) \to H^q(K')$$

given by

$$z^q + B^q(K) \mapsto f(z^q) + B^q(K')$$

1.4 Cohomology Groups with Arbitrary Coefficients

Proposition 1.4.1 *Let $K = \{K_n, \partial_n\}, L$ be chain complexes, $f, g : K \to L$ chain maps, $D : K \to L$ a chain homotopy between f and g and G an abelian group. Then:*

The induced sequence

$$\mathrm{Hom}(K, G) := \{\mathrm{Hom}(K_n, G), \mathrm{Hom}(\partial_n, 1_G)\}$$

is a cochain complex, where $\mathrm{Hom}(\partial_n, 1_G)$ is the homomorphism mapping and element $f : K_n \to G$ to $1_G^{-1} \circ f \circ \partial_n : K_{n+1} \to G$.

The induced sequence of homomorphisms

$$\mathrm{Hom}(f_n, 1_G) : \mathrm{Hom}(K_n, G) \to \mathrm{Hom}(L_n, G),$$

denoted as $\mathrm{Hom}(f, 1_G) : \mathrm{Hom}(K, G) \to \mathrm{Hom}(L, G)$, is a cochain map.

Then induced sequence of homomorphisms

$$\mathrm{Hom}(D_n, 1_G) : \mathrm{Hom}(K_{n+1}, G) \to \mathrm{Hom}(L_n, G)$$

is a cochain homotopy between $\mathrm{Hom}(f, 1_G)$ and $\mathrm{Hom}(g, 1_G)$.

Chapter 2

Singular Homology and Cohomology of a Topological Space

2.1 Simplexes on $\mathbb{R}^n$

Definition A subset A of $\mathbb{R}^n$ is called an **affine space** if for every pair of points x, y contained in A the line determined by them is contained in A.

An **affine combination** of points $p_0, p_1, \ldots, p_m$ in R^n is a point

$$x = t_0 p_0 + t_1 p_1 + \ldots + t_m p_m$$

for some coefficients $t_0, t_1, \ldots, t_m$ such that $t_0 + t_1 + \ldots + t_m = 1$.

A subset C of $\mathbb{R}^n$ is called a **convex set** if for every par of points x, y contained in A the segment determined by them is contained in C.

A **convex combination** of points $p_0, p_1, \ldots, p_m$ in R^n is a point

$$x = t_0 p_0 + t_1 p_1 + \ldots + t_m p_m$$

for some coefficients $t_0, t_1, \ldots, t_m$ such that $t_0 + t_1 + \ldots + t_m = 1$ and $0 \leq t_i \leq 1$.

Let $X \subset \mathbb{R}^n$. The affine space (resp. convex set) spanned by X is the smallest affine space (resp. convex set) containing X. The convex set spanned by X is denoted $[X]$.

Theorem 2.1.1 *Let $p_0, p_1, \ldots, p_n \in \mathbb{R}^n$. Then $[p_0, p_1, \ldots, p_n]$ is exactly the set of all convex combinations of $p_0, p_1, \ldots, p_n$.*

Definition An ordered set of points $\{p_0, p_1, \ldots, p_m\} \subset \mathbb{R}^n$ is said to be **affine independent** if $\{p_1 - p_0, p_2 - p_0, \ldots, p_m - p_0\}$ is a linearly independent subset of $\mathbb{R}^n$.

Theorem 2.1.2 *Let $p_0, p_1, \ldots, p_m$ be and affine independent subset of $\mathbb{R}^n$ and A the affine set spanned by it. Then, $\forall x \in A$ there exists a unique (m+1)-tuple $(t_0, t_1, \ldots, t_m)$ satisfying $\sum t_i = 1$ and*

$$x = \sum_{i=1}^{m} t_i p_i$$

*This (m+1)-tuple is called the **barycentric coordinates** of x.*

Definition Let $\{p_0, p_1, \ldots, p_m\}$ be and affine independent subset of $\mathbb{R}^n$. The convex set spanned by it is called the **m-simplex** with vertices $p_0, p_1, \ldots, p_m$. It is denoted by $[p_0, p_1, \ldots, p_m]$.

The barycentric coordinates of elements of $[p_0, p_1, \ldots, p_m]$ are all non-negative. The element with barycentric coordinates $(1/(m+1), \ldots, (1/(m+1))$ is called the **barycentre**.

For each vertex p_i, the $(m-1)$-simplex $[p_0, \ldots, \hat{p}_i, \ldots, p_m]$ is called the **face opposite** p_i. The union of all its faces is called the boundary of $[p_0, p_1, \ldots, p_m]$.

Definition The n-simplex $\Delta^n = [e_0, e_1, \ldots, e_n]$ spanned by the origin and the elements of the canonical basis of $\mathbb{R}^n$ is called the **standard n-simplex**.

An ordering $e_{\pi 0}, e_{\pi 1} \ldots, e_{\pi n}$ of the vertices $\{e_0, e_1, \ldots, e_n\}$ is called an *orientation* on Δ^n. Two orientations $e_{\pi 0}, e_{\pi 1} \ldots, e_{\pi n}$ and $e_{\rho 0}, e_{\rho 1} \ldots, e_{\rho n}$ are said to be equivalent if $\rho^{-1}\pi$ is an even permutation.

2.2 The Singular Complex of a Topological Space

Definition Let X be a topological space. A continuous map $\sigma : \Delta^n \to X$ is called a **(singular) n-simplex** in X.

Definition Let X be a topological space. For each $n \geq 0$, let $S_n(X)$ be the free abelian group with basis all singular n-simplexes in X. Let $S_{-1}(X) = 0$. The elements of $S_n(X)$ are called **n-chains** in X.

Definition The map $\varepsilon_i^n : \Delta^{n-1} \to \Delta^n$ given by

$$\varepsilon_i^n(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_{n-1})$$

in barycentric coordinates, is called the **i-th face map**.

Let $\sigma : \Delta^n \to X$ be a n-simplex and denote $p_i = \sigma(e_i)$.

The $n-1$-simplexes $\sigma\varepsilon_i^n$ are called the **face opposite** p_i.

The $n-1$ chain given by

$$\begin{aligned} &\partial_n \sigma = \textstyle\sum_{i=0}^n (-1)^i \sigma\varepsilon_i^n, \quad n > 0 \\ &\partial_0 \sigma = 0. \end{aligned}$$

is called the **boundary** of σ.

Theorem 2.2.1 *For each $n \geq 0$, there exist a unique homomorphism $\partial : S_n(X) \to S_{n-1}(X)$ extending the boundary definition over the n-simplexes above.*

Theorem 2.2.2 *For all $n \geq 0$, $\partial_n \partial_{n+1} = 0$.*

Definition The pair of sequences $(S_n(X), \partial_n)$ form a chain complex, called the **singular chain complex** of X. The group of n-cycles is denoted $Z_n(X)$ and the group of boundaries $B_n(X)$. Its homology group is called the **singular homology** and denoted $H_n(X)$.

The pair of sequences $(\mathrm{Hom}(S_n(X), \mathbb{Z}), \mathrm{Hom}(\partial_n, 1_Z))$ form a cochain complex, called the **singular cochain complex** of X. The group of n-cocycles is denoted $Z^n(X)$ and the group of boundaries $B^n(X)$. Its cohomology group is called the **singular cohomology** and denoted $H^n(X)$.

2.3 Universal Coefficients Theorems

Theorem 2.3.1 (Universal Coefficients Theorem for Homology)

- *For every space X and every abelian group G, there are exact sequences for all $n \geq 0$*

$$0 \longrightarrow H_n(X) \otimes G \xrightarrow{\alpha} H_n(X;G) \longrightarrow \mathrm{Tor}(H_{n-1}(X), G) \longrightarrow 0$$

  where $\alpha : (clsz) \otimes g \mapsto cls(z \otimes g)$ and Tor *represents the torsion part.*

- *This sequence splits; that is,*

$$H_n(X;G) \approx H_n(X) \otimes G \oplus \mathrm{Tor}(H_{n-1}(X), G)$$

See [1] Theorem 9.32.

Theorem 2.3.2 (Universal Coefficients Theorem for Cohomology)

- *For every space X and every abelian group G, there are exact sequences for all $n \geq 0$*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}, G) \longrightarrow H^n(X;G) \xrightarrow{\beta} \mathrm{Hom}(H_n(X), G) \longrightarrow 0$$

  where β is given by $\beta(cls(\varphi) = \phi'$ and $\phi'(z_n + B_n) = \phi(z_n)$.

- *This sequence splits; that is, there are isomorphisms for all $n \geq 0$,*

$$H^n(\mathrm{Hom}(C_*, G)) \approx \mathrm{Hom}(H_n(C_*), G) \oplus \mathrm{Ext}(H_{n-1}(C_*), G)$$

See [1] Theorem 12.11.

2.4 Relative Homology

Definition Let (X, A) be a pair of topological spaces such that $A \subseteq X$ and it has the subspace topology, and $S_*(X), S_*(A)$ their respective singular chain complexes. The groups $\frac{S_n(X)}{S_n(A)}$ form a chain complex. Its homology groups $H_n(S_*(X)/S_*(A))$ are called the **relative homology groups** and denoted $H_n(X, A)$.

Theorem 2.4.1 *If A is a subspace of X, there exists a long exact sequence*

$$\ldots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \ldots$$

The proof is based on applying the snake lemma to the short exact sequences $0 \to H_n(A) \to H_n(X) \to H_n(X, A) \to 0$.

Definition The group of **relative n-cycles** mod A is defined as

$$Z_n(X, A) = \{c \in S_n(X) : \partial\, c \in S_{n-1}(A)\}$$

The group of **relative n-boundaries** mod A is defined as

$$B_n(X, A) = \{c \in S_n(X) : c = c' + \partial\, d, \quad \text{for some} \quad c' \in S_n(A) d \in C_{n+1}(X)\}$$

Theorem 2.4.2 *For all $n \geq 0$,*

$$H_n(X, A) \approx Z_n(X, A)/B_n(X, A)$$

Chapter 3

Simplicial Homology and Cohomology

3.1 Simplicial Complexes

Definition Let $s = [v_0, \ldots, v_q]$ be a q-simplex in $\mathbb{R}^n$. Its **vertex set** is defined by

$$\mathrm{Vert}(s) = \{v_0, \ldots, v_q\}$$

A simplex s' satisfying $\mathrm{Vert}(s') \subseteq \mathrm{Vert}(s)$ is called a **face** of s and is denoted as $s' \leq s$. If the inclusion is proper, then s' is called a **proper face** of s and is denoted as $s' < s$.

Definition A **simplicial complex** K is defined as a finite collection of simplexes in $\mathbb{R}^n$ satisfying:

- if $s \in K$, then every face of s belongs to K
- if $s, t \in K$, then $s \cap t$ is either empty or a common face of both s and t

The collection of 0-simplexes in K is called the **vertex set** and is denoted $\mathrm{Vert}(K)$.

The subspace $|K|$ of the ambient space defined as

$$|K| = \bigcup_{s \in K} s$$

is called the **underlying space** of K.

Definition A topological space X is called a **polyhedron** if there exists simplicial complex K and an homeomorphism $h : |K| \to X$. The ordered pair (K, h) is called a **triangulation** of X.

By abuse of notation, the identification $X = |K|$ might be done.

Definition The **dimension** of a simplicial complex K is defined by

$$\dim K := \sup_{s \in K}\{\dim s\}$$

where a $q - simplex$ is assumed to have dimension q.

Definition A simplicial complex K equipped with a partial order on $\mathrm{Vert}(K)$ whose restriction to any simplex in K is a total order is called an **oriented simplicial complex**.

Definition If K is an oriented simplicial complex and $q \geq 0$, let $C_q(K)$ be the abelian group having the following presentation:

Generators: $(q+1)$-tuples $(p_0, \ldots, p_q)$ with $p_i \in \mathrm{Vert}(K)$ such that $\{p_0, \ldots, p_k\}$ spans a simplex in K.

Relations:

1. $(p_0, \ldots, p_q) = 0$ if some vertex is repeated
2. $(p_0, \ldots, p_q) = (sgn\pi)(p_{\pi 0}, \ldots, p_{\pi q})$, where π is a permutation of $\{0, 1, \ldots, q\}$.

One denotes the element of $C_q(K)$ corresponding to $(p_0, \ldots, p_q)$ by $< p_0, \ldots, p_q >$.

Lemma 3.1.1 *Let K be an oriented simplicial complex of dimension m. Then,*

1. *$C_q(K)$ is a free abelian group with basis of symbols $< p_0, \ldots, p_q >$, where $\{p_0, \ldots, p_q\}$ spans a q-simplex in K and $p_0 < p_1 < \ldots < p_q$. Moreover, $< p_{\pi 0}, \ldots, p_{\pi q} >= (sgn\pi) < p_0, \ldots, p_q >$.*
2. *$C_q(K) = 0$ for all $q > m$.*

Definition Define a boundary operator $\partial : C_q(K) \to C_{q-1}(K)$ by setting

$$\partial_q(< p_0, \ldots, p_q >) \sum_{i=0}^{q} (-1)^i < p_0, \ldots, \hat{p}_i, \ldots, p_q >$$

and extending it by linearity.

Theorem 3.1.2 *Let K be and oriented simplicial complex of dimension m. Then,*

$$0 \to C_m(K) \to \ldots \to C_1(K) \to C_0(K) \to 0$$

is a chain complex.

Definition Let K be an oriented complex. Then, define

$$\begin{array}{ll} Z_q(K) = \ker \partial_q, & \text{the group of } \textbf{simplicial q-cycles}, \\ B_q(K) = \operatorname{im} \partial_{q+1}, & \text{the group of } \textbf{simplicial q-boundaries}, \\ H_q(K) = \frac{Z_q(K)}{B_q(K)}, & \text{the } \textbf{q-th simplicial homology group}. \end{array}$$

Theorem 3.1.3 *For each oriented simplicial complex K, there exists a chain map $j : C_*(K) \to S_*(|K|)$ where each $j_q : C_q(K) \to S_q(K)$ is injective. Furthermore it induces isomorphisms,*

$$H_q(K) \approx H_q(|K|)$$

where $S_q(K)$ refers to the q-th singular complex of $|K|$ and $H_q(|K|)$ its homology group.

See [1] Theorem 7.22.

The Barycentric Decomposition of a Simplicial Complex

Definition Let K be a simplicial complex. If $s \in K$ is a simplex, let b^s denote its barycentre. The **barycentric subdivision** of K, denoted $Sd(K)$, is the simplicial complex given by

$$Vert(Sd(K)) = \{b^s : s \in K\}$$

and simplexes $[b^{s_0}, \ldots, b^{s_q}]$, where s_i are simplexes in K and s_i is a face of s_{i+1}

If the simplicial complex is oriented, an orientation can be induced on $Sd(K)$ by setting $b^\tau \leq b^\sigma$ whenever σ is a face of τ.

3.2 CW Complexes

Definition Let $D^n \in \mathbb{R}^n$ be the closed unit disc $D^n = \{(x_1, \dots, x_n) | x_1^2 + \dots + x_n^2 \leq 1\}$, $\mathring{D}^n$ its interior and $\dot{D}$ its topological boundary. A topological space X is said to be a **cell** of dimension m if it isomporphic to D^m.

Definition A par (X, E) consisting of a topological space X and a family of cells $E = \{e_\alpha\}$ which are pairwise disjoint and cover X is said to be a **CW complex** if:

- X is a Hausdorff space
- For each open m-cell e_α, there exists an isomorphism $f_\alpha : D^m \to X$ whose restriction to $\mathring{D}^m$ is an homeomorphism into e_α and $f_\alpha(\dot{D})$ is a union of cells of dimension lower than m.
- A subset A in X is closed iff $A \cap \bar{e}_\alpha$ is closed in $\bar{e}_\alpha$ for each α. It is said that X has the **weak topology**.

The maps f_α related to e_α are called **characteristic maps**.

If the family of cell e_α is finite, then X is said to be a **finite CW complex**. We will only deal with finite CW complexes.

The subspaces $X^p = \cup e_\alpha$, where the union goes over all cells of dimension lower or equal to p are called the p**-skeleton** of X.

A CW complex (Y, F) is called a **CW subcomplex** of (X, E), if $X \subseteq Y$ and $F \subseteq E$.

Proposition 3.2.1 *Let (X, E) be a CW complex. If a topological subset $Y \subseteq X$ can be expressed as union of cells for some $F \subseteq E$, where $\bar{e}_\alpha \subseteq Y$ for all $e_\alpha \in F$, then (Y, F) is a CW subcomplex of (X, E).*

In particular the p-skeleton X^p of a CW complex (X, E) equipped with the family cells of E of dimension lower or equal to p is a CW subcomplex.

Definition Let X be a CW complex. Define,

$$D_p(X) = H_p(X^p, X^{p-1})$$

together with $\partial_p : D_p(X) \to D_p(X)$ as the composition of

$$H_p(X^p, X^{p-1}) \to H_p(X^{p-1}) \to H_p(X^{p-1}, X^{p-2})$$

where the first homomorphism is the boundary operator on relative homology and the second one the homomorphism induced by the inclusion $j : (X^{p-1}, \emptyset) \subseteq (X^{p-1}, X^{p-2})$.

It follows that $\partial_{p-1}\,\partial_p$ for all p and hence $\mathcal{D} = \{D_p, \partial_p\}$ is a chain complex. It is called the **cellullar chain complex** of X.

Theorem 3.2.2 *Let (X, E) be a CW complex. The groups $H_i(X^p, X^{p-1})$ vanishes if $i \neq p$, and is free abelian if $i = p$. If γ generates $H_p(D^p, S^{p-1})$, then $H_p(X^p, X^{p-1})$ is generated by elements $(f_\alpha)_*(\gamma)$, where f_α are the characteristic maps of the cells e_α of dimension p.*

Theorem 3.2.3 *Let X be filtered by the subspaces $X_0 \subset X_1 \subset \ldots$. Assume that $H_i(X_p, X_{p-1}) = 0$ whenever $i \neq 0$, and that for any compact $C \subset X$, there exists X_p such that $C \in X_p$. Let $\mathcal{D}(X)$ be the chain complex defined by setting $D_p = H_p(X_p, X_{p-1})$ and boundary operator δ_* defined as composition between the boundary operator on relative homology of and inclusion, as before.*

Then there exists an isomorphism

$$\lambda : H_p(\mathcal{D}(X)) \to H_p(X)$$

between the homology of the chain complex above and the singular homology of X.

In particular, the homology of a CW complex is isomorphic to the singular homology.

See [4] theorem 39.4.

3.3 The Dual Block Complex to a Simplicial Complex

In the next section, a complex related to simplicial complexes will be introduced. In general, it will not have a CW-complex structure, because the elements are not necessarily cells. However, it admits a chain complex constructed as the cellular chain complex of a CW-complex which determines its homology. This will allow to compute the cohomology of the underlying topological space of the simplicial complex in a different way. Besides, when the original simplicial complex is a triangulation of some manifold, the blocks will actually be cells, and hence they'll constitute a CW-complex. This structure will prove useful to prove Poincaré's duality, and it will give an explicit way to compute the isomorphism.

Definition Let X be a polyhedron with triangulation (K, h), where K is a finite oriented simplicial complex, and let $Sd(K)$ be its barycentric subdivision with induced orientation. Let $\sigma \in K$ be a simplex and b^σ its barycentre. The **dual block** to σ, denoted $D(\sigma)$ is defined as the union of all open simplexes for which b^σ is the final vertex, i.e.,

$$\hat{\sigma} = \bigcup \tau$$

where the union runs over all simplexes τ such that

$$b^\sigma \in \mathrm{Vert}(\tau) \quad \text{and} \quad v \leq b^\sigma, \forall v \in \mathrm{Vert}(\tau)$$

Its topological closure $\bar{D}(\sigma)$ is called the **closed dual block** to *sigma*. Let $\dot{D}(\sigma)$ denote the topological boundary.

Theorem 3.3.1 *Let X be the underlying topological space of a simplicial complex K of dimension n consisting entirely of n-simpices and their faces. Let σ be a k-simplex. Then,*

1. *The dual blocks are disjoint and their union is X.*
2. *$\bar{D}(\sigma)$ is the polytope of a subcomplex of $Sd(K)$ of dimension $n-k$.*
3. *$\dot{D}(\sigma)$ is the union of all blocks $D(\tau)$ for which τ has σ as a proper face. These blocks have dimensions lower than $n-k$.*
4. *If $H_i(X, X - b^(\sigma)) \approx \mathbb{Z}$ for $i = n$ and vanishes otherwise, then $(\bar{D(}\sigma), \partial(D(\sigma))$ has the homology of an $n-k$ cell modulo its boundary.*

Definition Let X be the underlying topological space of a finite simplicial complex K. The collection of dual blocks $\{D(\sigma) : \sigma \in K\}$ is called the **dual block decomposition** of X. The union of blocks of dimension at most p are denoted by X_p and called the **dual p-skeleton** of X. The **dual chain complex** $\mathcal{D}(\mathcal{X})$ is defined by

$$D_p(X) = H_p(X_p, X_{p-1})$$

with boundary operator δ_* given as in the definition of a CW complex.

Theorem 3.3.2 *Let X be the underlying topological space of a finite simplicial complex K, X_p the dual p-skeleton and $\mathcal{D}(X)$ the dual chain complex. Then,*

1. *The group $H_i(X_p, X_{p-1})$ vanishes for $i \neq p$ and is a free abelian group for $i = p$. A basis for it is obtained by choosing generators for the groups $(\bar{D}(\sigma), \dot{D}(\sigma))$, as $D(\sigma)$ ranges over all p-blocks of X, and taking their images in $H_p(X_p, X_{p-1})$.*
2. *$\mathcal{D}(X)$ can be used to compute the homology of X. In fact, $D_p(X)$ equals to a subgroup of $C_p(Sd(X))$ consisting of those chains in X_p such that their boundary is on X_{p-1} The inclusion map $D_p(X) \to C_p(Sd(X))$ induces a homology isomorphism.*

A remark could be done on the fact that this blocks are very similar to simplexes. By how they are constructed, the dual block to a simplex σ^k is the union of some open simplexes of dimensions lower or equal to $n-k$. These are the same as closed simplexes of dimension exactly $n-k$ of the previous union. Hence, they can be viewed as chains in $Sd(K)$. Hence, they admit an alternative definition, and handling them as chains of simplexes is often convenient.

Definition Let K be an oriented simplicial complex of dimension n, $\mathrm{Sd}(K)$ its barycentric subdivision and $i : K \to Sd(K)$ the inclusion. Let σ be a simplex in K and $i(\sigma)$ its image on $\mathrm{Sd}(K)$.

For each 0-simplex $\sigma^0 \in K$, define

$$\hat{\sigma}^0 = \sum \tau^n \in C_n(\mathrm{Sd}(K))$$

where the sum runs over all n simplexes τ^n in $Sd(K)$, which have $i(\sigma^0)$ as a face and are oriented positively with respect to the orientation on K.

For each q-simplex $\sigma^q = [v_0, \ldots, v_k] \in K$, define

$$\hat{\sigma}^k = \sum \tau^{n-q} \in C_{n-q}(\mathrm{Sd}(K))$$

where the sum runs over all $n-q$ simplexes $\tau^{n-q} \in \mathrm{Sd}(K)$, which are common faces of $[\hat{v_k}]$ for each $k = 0, \ldots, q$, and are oriented positively with respect to $[\hat{v_0}]$.

The elements $\hat{\sigma}^k$ are called **dual block** to σ^k. The elements of

$$\hat{K}_q := \{\Delta_q : \Delta_q = \hat{\sigma}^{n-q} \quad \text{for some} \quad \sigma^{n-q} \in K_{n-q}\} \subset C_q(Sd(K))$$

are called **dual q-blocks**. The set $\hat{K} = \bigcup_{q=0}^{n} K_q$ is called the **dual block complex** to K.

Proposition 3.3.3 *Let K be an oriented simplicial complex and $\hat{K}$ its dual block complex. Let $C_q(\hat{K})$ be the free abelian group with basis K_q and $j_q : K_q \to C_q(Sd(K))$ the inclusion. Then,*

1. *j_q can be extended to an injective group homomorphism $j : C_q(\hat{K}) \to C_q(Sd(K))$.*
2. *There exists unique set of homomorphisms $\hat{j_k} : C_k(\hat{K}) \to C_{k-1}(\hat{K})$, $k = 0, \ldots, n$ such that*

$$\partial_q \, j_q = j_{q-1}\hat{\delta_q}, \quad \textit{for each} \quad q = 0, \ldots, n$$

where δ is the boundary operator on $C_*(\mathrm{Sd}(K))$. In other words, $\partial_q \, j(\Delta_q) \in C_{q-1}(Sd(K))$ for all $\Delta_q \in \hat{K}_q$.

3. *$\hat{\partial}_{q-1}\hat{\partial}_q = 0$ for all q.*

As a result, $(C_*(\hat{K}), \hat{\partial})$ can be seen as a subchain complex of $(C_*(Sd(K), \partial)$.

Chapter 4

Homology and Cohomology of a Manifold and Duality Theorems

4.1 De Rham's Duality Theorem

4.1.1 De Rham's Cochain Complex of a Topological Space

Definition Let A be a ring and M an A-module. The abelian group defined as the quotient of $F := A \times M \times \ldots M$ (p copies of M) by the subgroup S generated by:

$$\begin{aligned}
&(a, m_1, \ldots, m_i + m_i', \ldots, m_p) - (a, m_1, \ldots, m_i, \ldots, m_p) - (a, m_1, \ldots, m_i', \ldots, m_p)\\
&(a + a', m_1, \ldots, m_p) - (a, m_1, \ldots, m_p) - (a', m_1, \ldots, m_p)\\
&(a, m_1, \ldots, m_p) - (1, m_1, \ldots, am_i, \ldots, m_p)\\
&(a, m_1, \ldots, m_p) \quad \text{if} \quad m_i = m_j \quad \text{for} \quad i \neq j
\end{aligned}$$

for all possible values of $a, a', m_i, m'i$ and indexes i, j, is called **p-th exterior power** of M and is denoted $\bigwedge^p M$.

The cosets $(a, m_1, \ldots, m_p) + S$ are referenced as $a \wedge m_1 \wedge \ldots \wedge m_p$ and satisfy relations induced by the definition of S.

Proposition 4.1.1 *If $M = A^n$, it is a free A-module with basis $e_1, \ldots, e_n$. Then, $\bigwedge^p M$ has basis $\{e_{i_1} \wedge e_{i_p} | 1 \leq i_1 < \ldots < i_p \leq n\}$, and each element $\omega \in \bigwedge^p M$ admits a unique decomposition*

$$\omega = \sum a_{i_1 \ldots i_p} e_{i_1} \wedge \ldots \wedge e_{i_p}$$

Definition Let M be a $\mathbb{R}$-manifold and $C^\infty(M)$ the ring of differentiable functions $f : M \to \mathbb{R}$. The $C^\infty(M)$-module $\Omega_p(X) := \bigwedge^p C^\infty(M)^n$ is called the space of **differential p-forms** on M. The elements of the basis of $C^\infty(M)^n$ are denoted $\mathrm{d}\,x_i$, and hence the elements of $\Omega^p(M)$ admit decomposition

$$\omega = \sum \alpha_{i_1 \ldots i_p} \mathrm{d}\,x_{i_1} \wedge \ldots \wedge \mathrm{d}\,x_{i_p}$$

for $\alpha_{i_1 \ldots i_p} \in C^\infty(M)$.

The $C^\infty(M)$-module homomorphism denoted $\mathrm{d}^p : \Omega^p(M) \to \Omega^{p+1}(M)$ given by:

$$\begin{aligned} \mathrm{d}^0(\omega) &= \sum_{j=1}^{n} \left(\frac{\partial \alpha}{\partial x_j} \right) \mathrm{d}\,x_j, && \text{if} \quad \omega = \alpha \in C^\infty(M) \\ \mathrm{d}^p(\omega) &= \sum \mathrm{d}^0(\alpha_{i_1 \ldots i_p}) \wedge \mathrm{d}\,x_{i_1} \wedge \ldots \wedge \mathrm{d}\,x_{i_p}, && \text{if} \quad p \geq 1, \omega = \alpha_{i_1 \ldots i_p} \wedge \mathrm{d}\,x_{i_1} \wedge \ldots \wedge \mathrm{d}\,x_{i_p} \end{aligned}$$

is called the **exterior derivative**.

$(\Omega^*(M), \mathrm{d})$ is a cochain complex calle the **de Rham's cochain complex** of M. The k-forms ω satisfying $\mathrm{d}\,\omega = 0$ are called the **closed differential k-forms** and the ones who admit and expression $\omega = \mathrm{d}\,\omega'$ for some $k-1$-for ω' are called **exact differential k-forms**. Its cohomology groups, consisting of closed forms modulo exact ones, are called the **de Rham's homology** of M.

A singular p-simplex $\sigma : \Delta^p \to M$ determines n coordinate functions σ_i. Given a p-form $\omega = \sum \alpha_{i_1,\ldots,i_p} \mathrm{d}\,x_{i_1} \wedge \ldots \wedge \mathrm{d}\,x_{i_p}$ and a singular p-simplex $\sigma : \Delta^p \to M$, define

$$\int_\sigma \omega = \int_{\Delta^p} \sigma_\# \omega$$

where $\sigma_\#\omega = \sum \sum \alpha_{i_1,\ldots,i_p} J \,\mathrm{d}\,x_{i_1}$, with $J = \det(\partial \sigma_{i_j} / \partial x_{i_k})$ and $\mathrm{d}\,x_{i_k}$ are the differentials on $\mathbb{R}^n$.

More generally, given $c = \sum k_i \sigma_i \in S_p(M)$, then

$$\int_c \omega = \sum k_i \int_{\sigma_i} \omega$$

Theorem 4.1.2 (Stoke's theorem) *If c is a $(p+1)$-chain and ω a differential p-form, then*

$$\int_c \mathrm{d}\,\omega = \int_{\partial c} \omega$$

4.1.2 De Rham's Cochain Complex of a Simplicial Complex: Piecewise Linear Differential Forms

An approach in [3] allows working with differential forms that are piecewise linear, which can be used to represent differential forms on a simplicial complex.

Let K be a simplicial complex and $\sigma^n \in K$ a simplex. σ^n can be seen as elements $\{(t_0, t_1, \ldots, t_n) \in R^{n+1} : 0 \leq t_i \leq 1, \sum_{i=0}^n t_0 = 1\}$ given by the barycentric coordinates. Let Ω^* be the de Rham cochain complex of $\mathbb{R}^{n+1}$. Consider the restriction to σ^n of the forms in Ω^* of the form

$$\sum \varphi_{i_1,\ldots,i_j} \,\mathrm{d}\, t_{i_1} \wedge \ldots \wedge \mathrm{d}\, t_{i_j}$$

where $\varphi_{i_1,\ldots,i_j}$ are polynomials on $\mathbb{Q}$. $\sum_{i=1}^n \mathrm{d}\, t_i = 0$ is met due to $\sum t_i = 1$. The group of all p-forms of the above form is denoted $A^p(\sigma)$.

Definition Let K be a simplicial complex. Define,

$$A^p(K) = \{(\omega_\sigma^p)_{\sigma \in K} : \omega_\sigma^p \in A^p(\sigma) \quad \text{and} \quad \omega_{\sigma|\tau}^p = \omega_\tau^p, \forall \tau < \sigma\}$$

together with $\mathrm{d}_p : A^p(K) \to A^{p+1}(K)$ given by $\mathrm{d}_p((\omega_\sigma^p)) = (\mathrm{d}_p\, \omega_\sigma^p)$ and wedge product $(\omega_\sigma^p) \wedge (\omega_\sigma^q) = (\omega_\sigma^p \wedge \omega_\sigma^q)$.

It is a cochain complex, called the **de Rham's cochain complex** of K. It's cohomology group is called the **de Rham's homology** of K. Its elements can be understood as forms in each simplex of K that are compatible in their common faces.

4.1.3 De Rham's Duality Theorem for Simplicial Complexes

Theorem 4.1.3 (Piecewise Linear De Rham's Duality Theorem) *Let K be a simplicial complex and $A^*(K)$ its de Rham's cochain complex. The product defined as*

$$< (\omega_\tau)_{\tau \in K}, \sigma^n >:= \int_{\sigma^n} \omega_\sigma$$

induces a cochain homomorphism $A^*(K) \to C^*(K; \mathbb{Q})$.

See [3] P.L. deRham Theorem.

4.2 Poincaré's Duality Theorem and Intersection

Poincaré's duality theorem is introduced following [4], which uses the dual block complex of a simplicial complex to compute its dual block, and find the associated isomorphism between the k-th homology and the $n-k$-th cohomology.

Definition Let M be a compact manifold of dimension n that admits triangulation K. We say that M is **orientable** if K admits an orientation such that $\sum_{\sigma^n \in K_n} \sigma^n = 0$.

Theorem 4.2.1 *let M be a compact manifold of dimension n that admits a triangulation K. Then,*

- *If M is orientable, there exists an isomorphism*

  $$H^p(X; G) \approx H_{n-p}(X; G)$$

  for each $p = 0, \ldots, n$, G abelian group.

- *If M is non orientable, there exists an isomorphism*

  $$H^p(X; \mathbb{Z}/2\mathbb{Z}) \approx H_{n-p}(X; \mathbb{Z}/2\mathbb{Z})$$

  for each $p = 0, \ldots, n$.

Proof. Let $\hat{K}$ be the dual block complex of K. One can use the simplicial cochain complex $C^*(K)$ to compute the cohomology on M and the chain complex of the dual block complex $\mathcal{D}_*(\hat{K})$ to compute the homology on M.

By construction of the dual block complex, there exists a one-to-one correspondence between p-simplexes on K and $n-p$ blocks on $\hat{K}$. Hence, the free abelian groups $C^p(K)$ and $D_{n-p}(\hat{K})$ are isomorphic, with isomorphism φ mapping σ^*, algebraic dual to an oriented simplex σ, to some generator of $H_{n-p}(\bar{D}(\sigma), \dot{D}(\sigma))$. If M is orientable, then φ can be defined so that the diagram

$$\begin{array}{ccc} C^{p-1}(K) & \xrightarrow{\varphi} & D_{n-p+1}(\hat{K}) \\ \delta \downarrow & & \downarrow \partial \\ C^p(K) & \xrightarrow{\varphi} & D_{n-p}(\hat{K}) \end{array}$$

commutes.

φ is defined inductively on the dimension of the cohomology group:

- For each $s \in K_n$, $D(s)$ is the barycentre b^s of s which is a generator for $H_0(b^s$. Thus an isomorphism
$$\varphi_n C^n(K) \to D_0(K)$$
has been defined.

- For each $s \in K_{n-1}$ there are exactly 2 n-simplexes σ_0, σ_1 such that s is a face of them. Choose them so that s has positive relative orientation with respect to σ_1 and negative relative orientation with respect to σ_0. Then $\delta\, s^* = \sigma_1^* - \sigma_0^*$. Thus, $\varphi(s^*) = \varphi(\sigma_1^*) - \varphi(\sigma_0^*) = b^{\sigma_1} - b^{\sigma_0}$.

  As a result,
$$\varphi(s^*) = [b^{\sigma_0}, b^s] + [b^s, b^{\sigma_1}]$$
is a fundamental cycle for $(\bar{D}(s), \dot{D}(s))$ and $\partial\, \varphi(s^*) = b^{\sigma_1} - b^{\sigma_0} = \varphi(\delta\, s^*)$ as expected.

- Let $s \in K_p$, $p < n-1$, and assume that φ has been defined for simplexes of dimension greater than p. $\varphi(s^*)$ must be defined such that $\partial\, \varphi(s^*) = \varphi(\delta\, s^*)$.

  $\delta\, s^* = \sum_{s<\sigma_i} \epsilon_i \sigma_i^*$ can be expressed, where σ_i are the $p+1$ simplexes of whom s is a face, and ϵ_i is 1 if they have same relative orientation and -1 if they have opposite relative orientation. By hypothesis, $\varphi(\sigma_i^*)$ is a fundamenta cycle for $\bar{D}(\sigma_i)$ as σ_i is a $p+1$ simplex. Because s is a face of σ_i, $\bar{D}(\sigma_i) \subseteq \dot{D}(s)$. Hence $\varphi(\delta\, s^*)$ is carried by $\dot{D}(S)$.

  $\varphi(\delta\, s^*)$ is a cycle, because $\partial\, \varphi(\delta\, s^*) = \varphi(\delta\, \delta\, s^*) = 0$. Hence, $\varphi(\delta s^*)$ generates $H_{n-p-1}(\dot{D}(s))$.

  By considering the exact sequence
$$0 \to H_{n-p}(\bar{D}(s), \dot{D}(s)) \to H_{n-p-1}(\dot{D}(s)) \to 0$$
once can define $\varphi(s^*)$ as the fundamental cycle for $(\bar{D}(s), \dot{D}(s))$ for which
$$\partial(\varphi(s)) = \varphi(\delta\, s^*)$$

In conclusion, $\varphi : C^*(K) \to \mathcal{D}_*(\hat{K}$ has been defined, and is a group isomorphism at each link of the chain. It induces isomorphism,
$$\mathrm{Hom}(C_p(M), \mathbb{Z}) \to D_{n-p}(M).$$

By the universal coefficient theorem, an isomorphism

$$\mathrm{Hom}(C_p(M), G) \approx \mathrm{Hom}(C_p(M), \mathbb{Z}) \otimes G \to D_{n-p}(M) \otimes G$$

can be induced.

For the non orientable case, the proof is analogue. It cannot be assumed that for $n-1$ simplexes, the two simplexes of whom they are faces will be oriented one with same relative orientation and the other one with opposite. This difficulty is overcome by taking coefficients in $\mathbb{Z}/2\mathbb{Z}$, taking away the importance of the relative orientation. Thus, φ is directly defined as,

$$\mathrm{Hom}(C_p(M), \mathbb{Z}/2\mathbb{Z}) \to D_{n-p}(M) \otimes \mathbb{Z}/2\mathbb{Z}$$

□

4.2.1 Intersection number

Notice that the proof of Poincaré's duality provides a way to orientate dual blocks, and also induces a monomorphism $D_*(\hat{K}) \to C_*(Sd(K))$, by regarding $\bar{D}(s) = \varphi(s^*)$ as the unique oriented chain in $Sd(K)$ whose boundary is $\varphi(\delta s^*)$ and whose underlying topological space is $\bar{D}(s)$. This allows for a definition of intersection number between primal and dual chains, which is useful in many ways.

Definition The intersection number between a p-simplex σ and a dual $n-p$-simplex $\hat{\tau}$ is defined as

$$I(\sigma, \hat{\tau}) = \begin{cases} +1, & \text{if} \quad \hat{\tau} = \bar{D}(s) \\ -1, & \text{if} \quad \hat{\tau} = -\bar{D}(s) \\ \;\;0, & \text{otherwise} \end{cases}$$

it can be extended to a map $I : C_p(K) \times D_{n-p}(\hat{K})$ by

$$I\left(\sum_i a_i \sigma_i, \sum_j b_j \hat{\tau}_j\right) = \sum_{i,j} a_i b_j I(\sigma_i, \hat{\tau}_j)$$

Chapter 5

Eddy Currents on a manifold

Let Γ be a compact, connected and orientable surface. For simplicity, assume it has no boundary. Any such surface admits a finite oriented simplicial complex K that is a triangulation for Γ and let $\hat{K}$ its dual block complex (reference to previous part).

For simplicity, 0-simplexes in K will be called vertices and denoted v_i, 1-simplexes in K will be called edges and denoted e_i and 2-simplexes will be called faces and denoted f_i. σ will be used when its dimension is irrelevant.

The number of vertices, edges and faces in K will be denoted n_v, n_e, n_f respectively.

The corresponding elements in the dual block complex will be addressed as dual vertices, dual edges and dual faces. The corresponding element to $\sigma \in K$ will be denoted $\hat{\sigma} \in \hat{K}$.

The density of an eddy current over Γ can be represented as a 1-form on Γ which is closed, and hence, be seen as an element $\omega \in Z^1_{dR}(\Gamma)$. It is computed from given differential equations. The goal of the method presented consists on using the tools of simplicial homology and cohomology to convert the problem into solving a system of linear equations.

From previous chapters, there exists an isomorphism

$$\begin{array}{cccc} \varphi : & C_1(K) & \to & C^1(\hat{K}) \\ & c = \sum_i c_i e_i & \mapsto & \varphi(c) = \sum_i c_i \hat{e}_i^* \end{array}$$

By construction, given $a = \sum_i a_i \in C_1(K)$ the element $\varphi(a)$ is the linear map given by

$$\varphi(a)(\hat{b}) = \sum_i a_i \hat{e}_i^* \left(\sum_j b_j \hat{e}_j \right) = \sum_k a_k b_k = I(a, \hat{b})$$

Its algebraic dual is the isomorphism

$$\begin{array}{rccc} \psi = \varphi^* : & C_1(\hat{K}) & \to & C^1(K) \\ & \hat{c} = \sum_i c_i \hat{e}_i & \mapsto & \psi(\hat{c}) = \sum_i c_i e_i^* \end{array}$$

which also induces the same Poincaré's isomorphism $H_1(\Gamma) \to H^1(\Gamma)$. In fact, given $\hat{b} = \sum_i c_i \hat{e}_i \in C_1(\hat{K})$, the element $\psi(\hat{b})$ is the linear map given by

$$\psi(\hat{b})(a) = \sum_j b_j \hat{e}_j^* \left(\sum_i a_i e_i \right) = \sum_k b_k a_k = I(a, \hat{b})$$

In order to describe the problem in a simple way, a basis for the first homology of K is needed. The algorithm proposed in [6] offers a way to compute 1-cycles $h_i \in Z_1(K)$, whose classes $[h_i] \in H_1(K)$ will form a basis of the first homology. This basis has the property of being the "shortest", which is optimal for computational purposes.

A basis for $H_1(\hat{K})$ can be induced in the following way: for each vertex v_j that is a the starting point of some edge e_j of h_i, consider the edges e_{j_k} on K with v_j as its ending point. Select those that lie on the correct side of e_j (see the explanation of the algorithm to see how this is ensured). Then, the dual blocks of this edges sum into a 1-cycle $\hat{h}_i' = \sum_j \sum_k \hat{e}_{j_k}$ on $\hat{K}$. Notice that h_i and $\hat{h}_i'$ bound a region on $\mathrm{Sd}(K)$, and hence have the same homology class on Γ.

The isomorphism $\psi_* : H_1(\hat{K}) \to H^1(K)$ induced by ψ gives rise to a first cohomology basis with elements $[\psi(\hat{h}_i')] \in H^1(K)$, who have representatives $\psi(\hat{h}_i') = h_i' \in Z^1(K)$.

Because K is a triangulation for Γ,

$$H_1(K) \approx H_1(\Gamma) \quad \text{and} \quad H^1(K) = H^1(\Gamma)$$

as described in chapter 3. As a consequence, elements h_i can be assumed to belong to $H_1(\Gamma)$ and elements $h_i'^*$ to $H^1(\Gamma)$.

Due to de Rhams duality, $H^1(\Gamma)$ is isomorphic to the de Rham cohomology space $H^1_{dR}(\Gamma)$. This means that for each cochain $h_i'^*$ there exist a 1-form $\omega_i \in Z^1_{dR}(\Gamma)$ such that

$$h_i'(c) = \psi(\hat{h}_i')(c) = \int_c \omega_i$$

Notice that the classes $[\omega_i]$ form a basis for $H^1_{dR}(\Gamma)$.

Let $[\omega_i]$ be the previously given basis of $H^1_{dR}(\Gamma)$ and let $\omega \in Z^1(K)$ be the eddy current density. Then, the class of the eddy current density can be decomposed as $[\omega] = \sum_{i=1}^n \alpha_i[\omega_i]$. At coset level can be expressed as $\omega = \omega_0 + \sum_{i=1}^n \alpha_i\omega_i$ for an appropriate $\omega_0 \in B^1_{dR}(\Gamma)$. Since ω_0 is exact, it can also be written as $\mathrm{d}\, f$ for some $f \in Z^0_{dR}(\Gamma) \subseteq C^\infty(\Gamma)$ and hence,

$$\omega = \mathrm{d}\, f + \sum_{i=1}^n \alpha_i\omega_i$$

As a result, given $c \in C_1(\Gamma)$ the integral $\int_c \omega$ can be decomposed as:

$$\int_c \omega = \int_c \left(\mathrm{d}\, f + \sum_{k=1}^n \alpha_k\omega_k\right) = \int_{\partial c} f + \sum_{k=1}^n \alpha_k \int_c \omega_k$$

By restricting to K and $\hat{K}$, we can assume that $c \in C_1(K)$ and $\int_c \omega_k = I(c, \hat{h}'_k)$. Since $C_1(K)$ has basis consisting of edges e_i on K, knowing the integrals over the edges equals to knowing the integrals on all chains.

Let i be the column vector consisting of values $\int_{e_i} \omega$, i_0 be the column vector with values $f(v_l)$ and i_t be the column vector with values α_k. Due to the equation above, there exist matrices $G \in M_{n_e,n_v}(Z)$ and $Q_{n_e,n}(Z)$ satisfying

$$i = Gi_0 + Qi_t$$

where n is the first Betti number.

Matrices G and Q are called node-to-edge and hloop-to-edge incidence matrices respectively due to the content they hold: (the term hloop being used for loops representing the homology classes).

The elements $G_{i,l}$ on row i and column l of G have values

$$G_{i,l} = \begin{cases} 1, & \text{if } v_l \text{ is the ending vertex of } e_i, \\ -1, & \text{if } v_l \text{ is the starting vertex of } e_i, \\ 0, & \text{otherwise.} \end{cases}$$

and the elements $Q_{i,k}$ on row i and column k of Q have values

$$Q_{i,k} = I(e_i, \hat{h}'_j) = \begin{cases} 1, & \text{if } \hat{h}'_j \text{ intersects } e_i \text{ positively}, \\ -1, & \text{if } \hat{h}'_j \text{ intersects } e_i \text{ negatively}, \\ 0, & \text{if } \hat{h}'_j \text{ and } e_i \text{ do not intersect.} \end{cases}$$

Notice in this decomposition, that G and Q are independent from ω, they only depend on the choice of K. As a result, computing G and Q makes knowing ω equivalent to knowing the vectors i_0, i_t, which will become the unknowns of the linear equations.

As mentioned above, the matrix Q has elements $Q_i^j = I(\sigma_i^1, \hat{h}_j')$, i.e. the intersection between the i-th 1-simplex in K and the cycle in $\hat{K}$ homotopic to a representative of the j-th element of the shortest homology basis of K. By following the previous steps, this intersections can be calculated by the following algorithm:

See Intersection Algorithm

The algorithm works the following way:

It loops through each pair (h_j, e_{j_k}) where h_j is a cycle representative of the homology basis and e_{j_k} and edge appearing in h_j. Sets $e = e_{j_k}$.

Afterwards, finds the faces of K that have e contained, selects the one with same relative orientation to e (call it Δ) and picks the edge (call it e') with vertices v_0, the vertex of Δ not in e_{j_k}, and v_1 the vertex in which e ends. Then, the value of Q at row representing e' and column representing $\hat{h}_j'$ (as described above), i.e., the intersection number between e' and $\hat{h}_j'$ will be 1 if e' starts at v_0 and ends at v_1 and -1 if it starts at v_1 and ends at v_0. This value is the incidence between e' and v_1 and hence it appears at matrix G at row representing e' and column representing v_1.

The last step is repeated for edge e' just obtained until this edge lies in h_j.

This process ensures that the dual blocks of the edges considered form a cycle in $\hat{K}$, namely h_j'.

Because $\mathrm{d}\,\omega = 0$, the edge-to-face incidence matrix $C \in M_{n_f,n_e}(\mathbb{Z})$, which has elements

$$C_{i,j} = \begin{cases} 1, & \text{if } e_j \text{ lies in } \Delta_i \text{ with same relative orientation,} \\ -1, & \text{if } e_j \text{ lies in } \Delta_i \text{ with opposite relative orientation,} \\ 0, & \text{otherwise.} \end{cases}$$

will satisfy the equation $Ci = 0$.

By construction, matrix G represents the coboundary operator $\delta_0 : C^0(K) \to C^1(K)$ while C represents the coboundary operator $\delta_1 : C^1(K) \to C^2(K)$. As a result, the kernel of C are the coordinates of the 1-cocycles in K and the image of G the coordinates of the 2-coboundaries.

1 Intersection algorithm

```
for r = 1 : #hloops do
    hloop ← hloops(r)                          ▷ select r-th hloop h_r
    last ← hloop(#hloop)
    N ← dim(hloop)                             ▷ check number of edges in h_r
    for s = 1 : N do
        vert ← hloop(s)
        next ← hloop(s mod N + 1)
        lastedg ← {last, vert}                 ▷ select previous edge e_{r_{k-1}}
        nextedg ← {vert, next}                 ▷ select current edge e_{r_k}
        other ← vert
        edge ← lastedg
        while other ≠ next do                  ▷ cycle through all edges e' associated to e_{r_k}
            cells ← neight(edge)               ▷ find faces that have e_{r_k} contained
            if cells[0] > 0 then               ▷ select the one with same relative orientation
                cell ← cells[0]
            else
                cell ← cells[1]
            end if
            other ← cell \ edge
            edge ← {other, vert}               ▷ find intersecting edge
            Q[iedge, r] ← G[iedge, ivert]      ▷ write incidence between e' and v_1 to Q
        end while
        last ← vert                            ▷ new hloop node
    end for
end for
```

Matrix Q has in its columns the coordinates of the cocycles that span the cohomology. As a result, the kernel of C is spanned by G and Q, and the problem is well posed for surfaces with non-trivial homology.

Introducing notations $G_m = [G; Q]$ (Q appended to G) and $i_m = [i_0; i_t]$, condition $i = Gi_0 + Qi_t$ can be rewritten as $i = G_m i_m$.

The equation $i = G_m i_m$ for currents on K can be complemented by an equation for electro-magnetic fields on $\hat{K}$, by Faraday's law $G^T e + j\alpha b_0 = 0$, where e and b_0 are arrays of induced electro-magnetic fields on dual edges and magnetic fluxes on dual faces, j is the imaginary unit, and α an angular frequency of the magnetic fields. Additional constraints lead to equation:

$$G_m^T e + j\alpha b_m = 0$$

where $b_m = [b_0; b_t]$.

The relationship between the newly given data and the intensity of the eddy currents i given before is given by physical laws $e = Ri$ and the pair $a = Li$, $b_m = G_m^T a + G_m^T a_s$.

Edge elements w_i who interpolate the eddy current density form ω can be used to approximate 1 cochains in Γ. Then, the resistance matrix R will have elements

$$R_{i,j} = \sum_k \int_{\sigma_k} \rho w_i(x) \cdot w_j(x) \, \mathrm{d}\, x$$

where ρ the resistivity. The inductancce matrix L has elements

$$L_{i,j} = \frac{\mu}{4\pi} \sum_{r,s} \int_{\sigma_r} \int_{\sigma_s} \frac{w_i^r(x) \cdot w_j^s(y)}{\|x - y\|} \, \mathrm{d}\, x \, \mathrm{d}\, y$$

where μ is the magnetic permeability. a_s is the array of line integrals of the source magnetic vector potential.

By merging it all the relations together,

$$(G_m^T Z G_m) i_m = -j\alpha G_m^T a_s$$

is obtained, where $Z = R + j\alpha L$.

It allows to compute approximated values of i_m which can be used to retrieve a value for i, thus computing the eddy current.

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