MASTER THESIS

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Étale Cohomology

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Introduction

In this thesis we study the basics of étale cohomology . It is a vast and extremely rich area of mathematics, with plenty of applications . The theory is originally developed by Alexander Grothendieck and his numerous collaborators. Using this theory Deligne was able to prove the famous Weil Conjectures.

My main aim in this report has been to develop and study the basic theory. There are no new results here; all are known results, and I have produced them the way I understood them.

I begin with the study of étale morphisms which are basic kind of morphisms that we would always use. Next, I have developed the theory of Abelian Sheaves, mostly in the context of étale topology. The last chapter develops the cohomology. My study culminates, with a fundamental theorem relating Čech cohomology and derived functor cohomology (originally due to Michael Artin). But, the gates of the beautiful garden of étale cohomology remains wide open.

I would like to thank Professor Qing Liu, for numerous helpful discussions and I want to thank ALGANT for bringing me to this stage.

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Chapter 1

Étale Morphisms

All the schemes we consider in this report, would be locally noetherian and all commutative rings are assumed to be noetherian.

1.1 Étale Morphisms

This section deals with fundamental properties of *étale morphisms*. These are the maps which we would be basically concerned with in the whole of this report.

Definition 1.1.1. A morphism $f : X \to Y$, locally of finite-type is said to be unramified at x if $\mathfrak{m}_x = \mathfrak{m}_y \mathcal{O}_{X,x}$ and $k(y)$ is a finite separable extension of $k(x)$, where $y = f(x)$.

If f is unramified at all $x \in X$, then it is said to be $unramified$ morphism. The next propositon allows us an alternative definition of unramified, i.e in terms of differentials .

Theorem 1.1.2. *Let $f : X \to Y$ be locally finite-type morphism. The following conditions are equivalent :*

1. *f is unramified ;*
2. *the sheaf $\Omega^1_{X/Y}$ is zero.*

3. the diagonal morphism $\triangle_{X/Y} : X \to X \times_Y X$ is an open immersion.

Proof. (1) $\Rightarrow$ (2) The stalk $(\Omega^1_{X/Y})_x \simeq \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$. Let us assume that $\mathcal{O}_{Y,f(x)} = A$ and $\mathcal{O}_{X,x} = B$. We have a local morphism of local rings $A \to B$. By, condition (1) we know that $\mathfrak{m}_A B = \mathfrak{m}_B$ and $B/\mathfrak{m}_B$ is a finite separable extension of $A/\mathfrak{m}_A$. Denote, $\frac{B}{\mathfrak{m}_A B} = B \otimes_A \frac{A}{\mathfrak{m}_A}$ by B' and $A/\mathfrak{m}_A$ by A', then by the base change property of the differentials we know that

$$\Omega^1_{B'/A'} \simeq \Omega^1_{B/A} \otimes_B B' \simeq \frac{\Omega^1_{B/A}}{\mathfrak{m}_B \Omega^1_{B/A}}.$$

Since, B'/A' is a separble extension so $\Omega^1_{B'/A'} = 0$ and thus by Nakayama's lemma it follows that $\Omega^1_{B/A} = 0$
(2) $\Rightarrow$ (3). We know that the diagonal is locally a closed immersion ; there exists $U \subset X \times_Y X$, an open set such that $\triangle : X \to U$ is a closed immersion. Let $\mathcal{I}$ be the sheaf of ideals on U defining X. Then, we know that $\Omega^1_{X/Y} = \triangle^*(\mathcal{I}/\mathcal{I}^2)$. So, by condition (2)

$$\triangle^*(\mathcal{I}/\mathcal{I}^2)_x = (\mathcal{I}/\mathcal{I}^2)_x \otimes_{\mathcal{O}_{U,x}} \frac{\mathcal{O}_{U,x}}{\mathcal{I}_x} = 0$$

By, Nakayama's lemma therefore $\mathcal{I}_x = 0$ for all $x \in X$. So, $\mathcal{I}$ actually vanishes on some open subset V of U containing X. This means, from the defintion of $\mathcal{I}$, that V is isomorphic (as schemes) to X. Hence, (3) holds
(3)$\Rightarrow$ (1) Given any point $x \in X$ we shall show that f is unramified at x. Thus, first of all we might assume that $f^{-1}(y) = X = \operatorname{spec} A$ is affine, where $y = f(x)$. Then, in view of next proposition (see below) we can work on the geometric fibers, thus it is enough to show the result for $f : X' \to \operatorname{spec} k$, where $X' = \operatorname{spec}(A \otimes_{k(y)} k)$ with k being algebraic closure of $k(y)$.
Choose, some closed point x' in X', then as $k(x') = k$ (since k is algebraically closed) , we have a section $g : k \to X'$, with image of g being $\{x'\}$. Also, we have $g \circ f \circ g = g \circ Id_{X'}$. From, this it follows that $\phi \circ i = \triangle \circ i$ (just using the definition of fiber products and of $\triangle$), where $\phi := (Id_X, h_{x'}) : X' \to X' \times_k X'$, $i : x' \to X'$ and $h_{x'} : X' \to X'$ is the constant morphism mapping everything to x'. Now, note that $\phi(z) \in \triangle(X')$ implies that $z = x'$. So, $\phi^{-1}(\triangle(X')) = \{x'\}$, as diagonal is an open subset of X, this implies that $\{x'\}$ is open. Thus, every closed point of X is also open, which means that every prime ideal is maximal. So, X' is artinian, consisting of finite number of points.
We also get that $\operatorname{spec} \mathcal{O}_{X',x'} \to X'$ is an open immersion, as $\mathcal{O}_{X',x'}$ is local artinian. Now, the diagonal map restricted to this open set is still an open

immersion. So, $\mathcal{O}_{X',x'} \otimes_k \mathcal{O}_{X',x'} \longrightarrow \mathcal{O}_{X',x'}$ is isomorphism. This, means that $dim\,_k\mathcal{O}_{X',x'} = 1$ and thus $\mathcal{O}_{X',x'} = k$. So, $X' = \coprod \operatorname{spec} k$ a finite sum, hence using the next proposition (below) we get that $f : X' \to k$ is unramified. □

Proposition 1.1.3. Let $f : Y \to X$ be a morphism of locally finite-type. Then the following are equivalent

1. f is unramified ;
2. $X_y \longrightarrow \operatorname{spec} k(y)$ is unramified for all x ;
3. If we have spec $K \longrightarrow Y$ for K , some separably closed field, then $X \times_Y \operatorname{spec} K \longrightarrow \operatorname{spec} K$ is also unramified . This condition is referred as f has unramified geometric fibers ;
4. $\forall\, y \in Y\ X_y$ has an open covering by spectra of finite separable $k(y) - algebras$;
5. $\forall\, y \in Y$, X_y is a sum $\coprod \operatorname{spec} k_i$, where the k_i are finite separable field extensions of $k(y)$.(If f is of finite-type, then X_y itself is the spectrum of a finite separable $k(y) - algebra$ in (4), and X_y is a finite sum in (5); in particular f is quasi-finite);

Proof. : (1)⇔ (2) If we have $\phi : B \to A$ with $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$, where $\mathfrak{p} \in A$ and $\mathfrak{q} \in B$, are prime ideals . Then we have a canonical isomorphism

$$(A \otimes_B k(\mathfrak{q}))_{\mathfrak{p}} \approx A_{\mathfrak{p}} \otimes_{B_{\mathfrak{q}}} k(\mathfrak{q})$$

Thus, it follows from this that , $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x} \approx \mathcal{O}_{X_y,x}$. So, if f is unramified, then $\mathcal{O}_{X_y,x}$ is a finite separable extension of $k(y)$, and vice-versa.

(2) ⇒ (4) Choose an open affine subset $U = \operatorname{spec} A$ in X_y. Consider a prime ideal $\mathfrak{p} \subset A$ then, condition (2) implies that $A_{\mathfrak{p}}$ is a finite separable extension of $k(y)$. We, also have

$$k(\mathfrak{p}) \subset A/\mathfrak{p} \subset A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = A_{\mathfrak{p}}$$

Thus, $A/\mathfrak{p}$ should also be a field, and hence $\mathfrak{p}$ is a maximal ideal. Thus, A is a noetherian ring , with dimension 0, hence it is an Artin ring. Thus, $A = \prod A_{\mathfrak{p}}$,with $\mathfrak{p}$ running over the finite set spec A. Hence, we have (4)

Now, let us show that (3) implies (4).Denote $X \times_Y \operatorname{spec} K$ by Z. If $z \in Z$, then, as K is separably closed, so by condition (3) we have $K = \mathcal{O}_{Z,z}$. So, if spec K maps to y in spec $B \hookrightarrow Y$ and $f^{-1}(\operatorname{spec} B) = \cup \operatorname{spec} A_i$, then $K = (A_i \otimes_B K)_{\mathfrak{q}\otimes K}$, where $\mathfrak{q} \otimes K$ is a prime ideal in $A_i \otimes_B K$. From, this it follows that $(A_i \otimes_B k(y))_{\mathfrak{q}\otimes k(x)}$ is a separable extension of $k(x)$. Now, as $f^{-1}(y) \approx X_y$, so we can cover X_y by the affine open sets spec $(A_i \otimes_B k(y))$. Now, proceeding as in the previous proof, we get our required conclusion.
Now, (4)$\Rightarrow$ (5) $\Rightarrow$ (3) and (5)$\Rightarrow$ (2), follows from the next lemma and arguments very similar to (2)$\Rightarrow$ (4)

□

Lemma 1.1.4. Let k be a field and A ,an artinian $k-algebra.$ of finite type and $\bar{k}$ be the algebraic closure of k. If $A \otimes_k \bar{k}$ is reduced (no nilpotents), then A is a finite product of finite separable field extensions of k .

Proof. From the fact that an artinian ring has only finitely many prime ideals which are again maximal, we get that $A = \prod A_i$, where A_i are artinian local rings . Replacing A by A_i we may assume that A is local . Since the maximal ideal of A is nilpotent, it is zero and thus A is a field which is finite over k. Let α be an element of A and $f(T)$ its minimal polynomial over k. Then $k(\alpha) \cong k[T]/f(T)$; so, $k(\alpha) \otimes_k \bar{k} \cong \prod \bar{k}[T]/f_i(T)^{r_i}$ where the $f_i(T)$ are the distinct linear factors of $f(T)$. By hypothesis, $k(\alpha) \otimes_k \bar{k}$ is reduced. So, all $r_i = 1$; hence α is separable.

□

Definition 1.1.5. A morphism of schemes $f : X \to Y$ is said to be étale at $x \in X$, if it is $flat$ and $unramified$ at x (so it is locally finite,too). f is said to be an *étale morphism*, if it is *étale* at all the points .

Before, discussing étale morphisms in deatils, I would state (without proof) here some properties of *flat morphisms*, that we would use (often without explicit mention). For the proofs see for example [6]

- open immersions are flat , composition of flat morphism is flat, they are stable under base change
- A morphism spec $A \to$ spec B is flat, iff $B \to A$ is flat

- Again , let $A \to B$ is a ring homomorphism and M be a B-module. Denote spec B by X and spec A by Y, the $\mathcal{O}_X$-module $\tilde{M}$ is flat over Y if and only if M is flat over A.
- Let X be noetherian, and $\mathfrak{F}$ a coherent $\mathcal{O}_X$-module. Then $\mathfrak{F}$ is flat over X iff it is locally free.
- Any flat morphism which is locally of finite-type is open.
- Let $f : X \to Y$ is locally of finite-type. The set of points $x \in X$ such that $\mathcal{O}_x$ is flat over $\mathcal{O}_{f(x)}$ is open in X ; it is non-empty if X is integral

We have some common facts for étale morphisms , that we have for many other kinds of morphisms.

Proposition 1.1.6. 1. Any open immersion is *étale*

2. The compostion of two *étale* morphisms is *étale*

3. *étale* morphisms are stable under base change

Proof. Since we already know the corresponding facts for $flat$ morphisms, so we need to check that these are true, for unramified morphims. If $U \to X$ is an open immersion. Then, by definition, $\mathcal{O}_{U,x}/\mathfrak{m}_x\mathcal{O}_{U,x} = k(x)$, hence (1). Next, suppose $X \to Y \to Z$, and say
$x \mapsto y \mapsto z$, then note that by definition $\mathfrak{m}_z\mathcal{O}_{X,x} = (\mathfrak{m}_z\mathcal{O}_{Y,y}) \cdot \mathcal{O}_{X,x} = \mathfrak{m}_x$, hence $\mathcal{O}_{X,x}/\mathfrak{m}_z\mathcal{O}_{X,x}$ is a finite separble extension of $k(z)$. So, we have shown (2).
Now, suppose we make a base change by Z of an unramified morphism $X \to Y$. Also, suppose that we have spec $K \to Z \to X$, for some separably closed field K. Then to show that $X \times_Y Z \to Z$ is unramified , it is enough to show that $(X \times_Y Z) \times_Z \operatorname{spec} K \to \operatorname{spec} K$ is unramified because of proposition 1.1.3 . But $(X \times_Y Z) \times_Z \operatorname{spec} K \simeq X \times_Y \operatorname{spec} K$ and as $X \to Y$ is unramified, using propostion 1.1.3 (5) again , we get our required result (3). □

Finer results on local structure of étale morphisms can be described , if we use the '**main theorem**' of Zariski . We state here a following variant of that.

Theorem 1.1.7. *Let Y be locally noetherian scheme and let $f : X \to Y$ be a quasi-projective morphism (e.g a morphism of finite type with X,Y affine). Let X_y be a finite fiber. Then there exists an open neighborhood U of X_y such that $f|U : U \to Y$ factors into an open immersion $U \to Z$ followed by a finite morphism $Z \to Y$.*

Let us look at an example which is useful for us .

Example 1.1.8. Let A be a Noetherian ring, let $P(T) \in A[T]$ be a monic poylnomial, and let $P'(T)$ denote its derivative. Let $B = A[T, P'(T)^{-1}]/(P(T))$. Then B is flat over A, since its the localization of the flat A-algebra $A[T]/(P(T))$. Let $k(s)$ denote the residue field of a point $s \in \text{spec } A$; then :

$$B \otimes_A k(s) \;=\; k(s)[T, \bar{P}'(T)^{-1}]/(\bar{P}(T)),$$

here we have denoted by $\bar{P}$ as the image of P in $k(s)[T]$. Thus it is the direct sum of the decomposition fields of the simple roots of $\bar{P}(T)$. So, either spec B is empty, or spec $B \to$ spec A is étale. In the latter case, the morphism spec $B \to$ spec A is called a *standard étale morphism* .

Proposition 1.1.9. Let Y be locally noetherian $f : X \to Y$ be a morphism of finite type étale at a point x. Let $y = f(x)$. Then, if necessary by replacing X (respectively Y) by an open neighborhood of x (respectively y), there exists a standard étale morphism $h : Z \to Y$ and an open immersion $g : X \to Z$, such that $h \circ g = f$.

Proof. Since we are dealing with a local property, we may assume that $X =$ spec B and $Y =$ spec A are both affine. Also, as X is of finite type over y, we may as well assume B is local and y the closed point of Y. Now, by the preceding theorem we are able to assume that B is finite over A. The fiber X_y is the disjoint union of spec $k(x)$ and of an open subset U. As, $k(x)$ is separable over $k(y)$, there exists a $\bar{b} \in \mathcal{O}(X_y)$ such that $k(x) = k(y)[\bar{b}]$, $\bar{b} \neq 0$, and $\bar{b}|_U = 0$. Now, we can lift $\bar{b}$ to an element $b \in B$. Consider the subalgebra $C := A[b]$ of B. Let $\mathfrak{m}$ be the prime ideal of B corresponding to the point x, and $\mathfrak{q} = \mathfrak{m} \cap C$. We want to show that $C_\mathfrak{q} \to B \otimes_C C_\mathfrak{q}$ is actually an isomorphism. Firstly, note that $\mathfrak{m}$ is the unique prime ideal of B lying over $\mathfrak{q}$ because $\mathfrak{m}$ does not contain b while any other prime ideal of B lying over $\mathfrak{m}_y$ all contain b. Thus, $B \otimes_C C_\mathfrak{q}$ is a local ring . As $C \to C_\mathfrak{q}$ is flat, the morphism $C_\mathfrak{q} \to B \otimes_C C_\mathfrak{q}$ is finite and injective. Now, $(B \otimes_C C_\mathfrak{q}) \otimes_{C_\mathfrak{q}} k(\mathfrak{q}) = k(\mathfrak{q})$. From Nakyama's lemma now it follows that $C_\mathfrak{q} = B \otimes_C C_\mathfrak{q}$.
As B and C are finitely generated over A, the obtained morphism extends

to a neighborhood of x. We may thus replace B by C, and suppose that B is generated , as an A-algebra, by a single element b. Let $n = \dim_{k(y)} B \otimes_A k(y)$. Then, we check that $\{1, \bar{b}, ..., \bar{b}^{n-1}\}$ is a basis of $B \otimes_A k(y)$ as $k(y)$ vector-space. Again from Nakayama's lemma we obtain that $\{1, b, ..., b^{n-1}\}$ is a system of generators for the A-module B¿ Thus, there exists a monic polynomial $P(T) \in A[T]$ of degree n which vanishes at b, and we have a surjective homomorphism of A-algebras $A[T]/(P(T)) \to B$ which sends T to b. The image of $b' := P'(b)$ in $k(x)$ is non-zero beacuse $k(x)$ is separable over $k(y)$, and hence $x \in D(b')$. Now, by replacing X by the open subset $D(b')$ if necessary, we get a surjective homomorphism , $D := A[T, P'(T)^{-1}]/(P(T)) \to B$ Now, let $\mathfrak{n}$ be the inverse image of $\mathfrak{m}$ in D. We shall exhibit that $\psi : D_{\mathfrak{n}} \to B_{\mathfrak{m}}$ is an isomorphism. First of all we already have $psi \otimes k(y)$ as an isomorphism. let $I = \mathrm{Ker}\psi$. As, $B_{\mathfrak{p}}$ is flat over A, and ψ is surjective , we have $I \otimes_A k(\mathfrak{p}) = 0$, thus by Nakayama's lemma $I = 0$. Since, D is Noetherian, there exists an open neighborhood U of x such that the closed immersion spec $B \to \mathrm{spec} D$ is an isomorphism over U

□

1.2 Hensel Rings

Definition 1.2.1. A local ring A is said to be *Henselian* if the Hensel's Lemma holds for its residue field i.e if f is a monic polynomial with coefffficients in A, such that $\bar{f}$,the reduction modulo its maximal ideal, factors as $\bar{f} = g_0 h_0$ with g_0 and h_0 monic and coprime, then we can lift g_0 and h_0 to get g and h such that $f = gh$.

In what follows A shall be always local, with residue field $k := A/\mathfrak{m}A$

Theorem 1.2.2. *Let $X = spec A$, and x denote its closed point. The following are equivalent :*

1. *A is Henselian ;*
2. *any finite A-algebra B is a direct product of local rings $B = \prod B_i$ ($B_i \approx B_{\mathfrak{m}_i}$, where $\mathfrak{m}_i$ are the maximal ideals of B);*
3. *If $f : Y \to X$ is a quasi-finite and separated, then $Y = Y_0 \sqcup Y_1 \sqcup \cdots \sqcup Y_n$ where $f(Y_0)$ does not contain x and Y_i is finite over X and is the spectrum of a local ring, $i \geq 1$;*

4. *if $f : Y \to X$ is étale and there is a point $y \in Y$ such that $f(y) = x$ and $k(y) = k(x)$, then f has a section $s : X \to Y$*

5. *Consider the polynomial ring $A[T_1, ..., T_n]$ and let $f_1, ..., f_n$ be some elements there; if $\bar{f}_i$'s have a common zero $a = (a_1, ..., a_n)$ and the jacobian, $det((\partial \bar{f}_i/\partial T_j))(a) \neq 0$, then we can lift a to get a b in A such that $f_i(b) = 0$, $i = 1, ..., n$;*

Proof. (1)$\Rightarrow$ (2) Note first that B is local if and only if $\bar{B} := B/\mathfrak{m}B$, since any maximal ideal of B lies over $\mathfrak{m}$ (going-up theorem).
First we wrok out the case when $B = A[T]/(f)$, where $f(T)$ is a monic polynomial. If $\bar{f}$ is a power of an irreducible polynomial , then $\bar{B} = k[T]/(\bar{f})$ is local and so is B. If not, then from our hypothesis $B \approx A[T]/(g) \times A[T]/(h)$, an we repeat this procedure to get the desired decomposition.
Now, if B is any arbitrary finite A-algebra and is not local, then we could find a non-trivial idempotent $\bar{e}$ in B such that $\bar{e}$ is idempotent in $\bar{B}$. Let f be a monic polynomial such that $f(e) = 0$; and let $\phi : A[T]/(f) \to B$ be the homomorphism that send T to e. As, $A[T]/(f)$ is generated by a single element ($\bar{T}$). From the previous discussion, we get that there is an idempotent $\alpha \in A[t]/(f)$ such that $\bar{\phi}(\alpha) = \bar{e}$. So we have found a non-trivial idempotent $\phi(\alpha) = e'$; $B = Be' \times B(1 - e')$. We can continue this process to get the desired decomposition
(2)$\Rightarrow$ (3) From, the Zariski's main theorem f factors as $Y \xrightarrow{f'} Y' \xrightarrow{g} X$, with f' being open immersion and g is finite . Then from our hypothesis (2) , $Y' = \coprod \text{spec}\,(\mathcal{O}_{Y',y})$, where y runs over the finitely many closed points of Y'. let $Y_* = \coprod \text{spec}\,(\mathcal{O}_{Y',y})$, where y runs over the closed points of Y', which lie in Y. Y_* is contained in Y, and is open and closed as well, in Y, since it is so in Y'. Write $Y = Y_* \sqcup Y_0$. Then, $x \notin f(Y_0)$.

(3) $\Rightarrow$ (4) When translated into the laguage of rings this just means that A has no non-trivial étale "neighborhood " (this shall be cleared latter). More prcisely, from (3) we are in the situation that $A \to B$ is an étale local homomoprhism , with $k = K$, where K is the residue field of B. But B is a free A-module, and as $K = B \otimes_A k = k$, it must have rank 1, and so $A \approx B$.
(4)$\Rightarrow$ (5). Let $B = A[T_1, ..., T_n]/(f_1, ..., f_n) = A[t_1, ..., t_n]$ and let $J(T_1, ..., T_n) = \det(\partial f_i/\partial T_j)$. From the hypothesis of (3) , we get a prime ideal $\mathfrak{p}$ in B lying over $\mathfrak{m}$ such that the Jacobian , $J(t_1, ..., t_n)$ is an unit in $B_\mathfrak{p}$. It now follows that there exists a non-zero $b \in B$, such that $J(t_1, ..., t_n)$ is an unit in B_b. Writing B_b as $B[s]/sb-1$, and using the jacobian criterion, we get B_b is étale over A. Now, apply (4) to lift the solution in k^n to one in A^n.

(5)$\Longleftrightarrow$ (1) This a well known equivalent form of Hensel's Lemma in number theory. We skip the proof.

$\square$

Example 1.2.3. Any complete local ring is Henselian.

We list here a few more properties of *Hensel* rings . For proofs see [11]

- If A is Henselian, then so is any any finite local A-algebra B and any quotient ring A/I
- If A is Henselian, then the functor $B \mapsto B \otimes_A k$, is an equivalence between the category of finite étale A-algebra and the category of finite étale k-algebra.

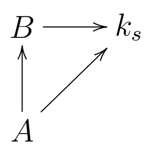
Now we define the notion of étale neighborhood of a local ring.

Definition 1.2.4. It is a pair $(B, \mathfrak{p})$ where B is an étale A-algebra and $\mathfrak{p}$ is a prime ideal of B lying over $\mathfrak{m}$, such that the induced homomorphism $k \to k(\mathfrak{p})$ is an isomorphism.

It turns out that the étale neighborhoods of A with connected spectra form a filtered direct system . Denote $(A^h, \mathfrak{m}^h) := \varinjlim(B, \mathfrak{p})$. Then, A^h is a local A-algebra with maximal ideal $\mathfrak{m}^h$, and $A^h/\mathfrak{m}^h = k$. Further, it satisfies the following universal property: if $\phi : A \to E$ is a local homomorphism, with E being Henselian, then ϕ factors through the canonical map $i : A \to A^h$.

Definition 1.2.5. A^h is called the *Henselization* of A.

A *strictly Henselian* ring is a local ring A, which is Henselian and its residue field is separably algebraically closed. There is also the notion of *strict henselization* of a ring (local) A. It is defined as $A^{sh} = \varinjlim B$, where B's are obtained from the following commutative diagrams :



where, $A \to B$ is étale and k_s denotes a fixed separable closure of k. A^{sh} has no finite étale extensions at all. It also satisfies the universal property , as in the case of A^h (just replace there *Henselian*, by *strict Henselian*) .

Now, let X be a scheme and $\bar{x} \to X$ be a geometric point of X , that is $\bar{x} := \operatorname{spec} K \to X$, maps to $x \in X$ where K is a separable closure of $k(x)$. An, *étale neighborhood* of $\bar{x}$ is a commutative diagram

$$\begin{array}{ccc} \bar{x} & \longrightarrow & U \\ & \searrow & \downarrow \\ & & X \end{array}$$

with $U \to X$ étale. Then, $(\mathcal{O}_{X,x})^{sh} = \varinjlim \Gamma(U, \mathcal{O}_U)$, where the limit is over all étale neighborhoods of $\bar{x}$. $(\mathcal{O}_{X,x})^{sh}$ is also denoted by $\mathcal{O}_{X,\bar{x}}$.

Chapter 2

Abelian Sheaves

2.1 Basics

Consider a class E of morphisms of schemes satisfying the following properties
:

- all isomorphisms are in E
- E is closed under composition
- any base change of a morphism in E is in E

Note that the last two conditions imply, that$E$ is also closed under fiber products .
A morphism in this class shall be referred to as E-morphism. The full subcategory of **Sch**$/X$ of X-schemes whose structure morphisms are in E shal be denoted by E/X.

Some examples of such classes are like : (Zar), the class of all open immersions , $(\acute{e}t)$ of all etale morphisms of finite-type,(fl) of all flat morphisms that are of locally of finite-type.

We would see that E-morphisms are to play role of the open subsets in an E-topology.

Now Fix a base scheme X , a class E as above , and a full subcategory $\mathbf{C}/X$ of $\mathbf{Sch}/X$ that is closed under fiber products and is such that, for any $Y \longrightarrow X$ in $\mathbf{C}/X$ and any E-morphism $U \longrightarrow Y$, the composite $U \longrightarrow X$ is in $\mathbf{C}/X$.

Definition 2.1.1. : $E-$ *covering* of an Object Y of $\mathbf{C}/X$ is a family $(U_i \xrightarrow{g_i} Y)_{i \in I}$ of E-morphisms such that $Y = \bigcup g_i(U_i)$. The class of all such coverings of all such in the $E-$ *topology* on $\mathbf{C}/X$. The category $\mathbf{C}/X$ together with E-topology is the $E-$ *site*. We shall write $(\mathbf{C}/X)_E$ or simply X_E to denote this. By an *étale site* X_{et} we shall mean $(ét/X)_{et}$, (this is also called as *small étale site*).

Definition 2.1.2. : A *presheaf* of abelian groups on $(\mathbf{C}/X)_E$ is a contravariant functor $(\mathbf{C}/X)^o \rightarrow \mathbf{Ab}$. $P(f) : P(U) \rightarrow P'(U)$ shall often be denoted by $res_{U',U}$
A *morphism* $\phi : P \longrightarrow P'$ of presheaves on $(\mathbf{C}/X)_E$ is just a natural transformation of the functors P and P'.

The presheaves and presheaf morphisms over $(\mathbf{C}/X)_E$ form an additive category $\mathbf{P}(X_E)$.

Example 2.1.3. :

1. Given any abelian group M, the constant presheaf P_M is defined to be the $P_M(U) = M$, $\forall\ U \in (\mathbf{C}/X)_E$, and $P_M(f) = 1_M$ for all f and $P_M(\varnothing) = 0$ ($\varnothing$ denotes the empty scheme). We denote by $\mathbb{Z}$, the constant sheaf defined by the ring of integers.

2. The presheaf $\mathbb{G}_a$ is defind to be $\mathbb{G}_a = \Gamma(U, \mathcal{O}_U)$ regarded as an additive group forall U; for any morphism $f : U \longrightarrow U'$, $\mathbb{G}_a(f) : \Gamma(U, \mathcal{O}_U) \longrightarrow \Gamma(U', \mathcal{O}_{U'})$ is the map induced by f.

3. The presheaf $\mathbb{G}_m$ has $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^*$ for all U, and the obvious restriction maps

Definition 2.1.4. : A presheaf P on X_E is a *sheaf* if it satisfies :
(S_1) if $s \in P(U)$ and there is an $E-$ *covering* $(U_i \rightarrow U)$ of U such that $res_{U_i,U}(s) = 0$ for all i, then $s = 0$;
(S_2) if $(U_i \rightarrow U)_{i \in I}$ is a covering and the family $(s_i)_{i \in I}$, $s_i \in P(U_i)$ is such that

$$res_{U_i \times_U U_j, U_i}(s_i) = res_{U_i \times_U U_j, U_j}(s_j)$$

for all i and j , then there exists an $s \in P(U)$ such that $res_{U_i,U}(s) = s_i$ for all i.

In other symbols , P is a sheaf if the sequence

$$(S) \qquad P(U) \to \prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_U U_j)$$

is exact for all U in X_E and for all coverings $(U_i \to U)$.

Note that in the case when the class E contains all open immersions, for example the class $(\acute{e}t)$, then any open covering $U = \bigcup U_i$ in the zariski topology is also a covering in the E-topology. Thus a sheaf F on X_E defines by restriction, sheaves in the usual sense on all schemes U in $\mathbf{C}/X$.

Following is an criterion which makes it easy to check if a presheaf is a sheaf.

Proposition 2.1.5. Let P be presheaf for the etale site on X. Then P is a sheaf if and only if it satisfies the following conditions :
(a) For any U in $\mathbf{C}/X$, the restriction of P to the usual Zariski topology on U is a sheaf;
(b) For any covering $(U' \to U)$ with U and U' both affine ,
$P(U) \to P(U') \rightrightarrows P(U' \times_U U')$ is exact.

Proof. The necessity of the above conditions are obvious from the defintion of sheaf.
For proving the sufficiency note that (a) implies that if a scheme V is a sum $V = \coprod V_i$ of subschemes V_i, then $P(V) = \prod P(V_i)$ (as we have in the case of Zariski topology). Thus we have that the sequence (S) arising from a covering $(U_i \to U)$, is isomorphic to that coming from a single morphism $(U' \to U)$, $U' = \coprod U_i$, which follows from the relation

$$(\coprod U_i) \times_U (\coprod U_i) = \coprod (U_i \times_U U_j)$$

.

So, from (b) we get that (S) is exact for coverings $(U_i \to U)_{i \in I}$, in which the indexing set I is finite and each U_i is affine , since then $\coprod U_i$ is affine. Now, the rest of the proposition is just diagaram chasing, so we skip it .

□

Let us have a closer look at the étale site of a field. A presheaf P on $(\text{spec } k)_{et}$ can be regarded as a covariant functor $\mathbf{Et}/k \to \mathbf{Ab}$, where $\mathbf{Et}/k$ deontes the category of étale k algebras . So, P would be a sheaf iff $P(\prod A_i) = \oplus P(A_i)$ for every finite family (A_i) of étale k-algebras and $P(k') \xrightarrow{\approx} P(K)^{Gal(K/k')}$, for every finite Galois extension K/k' of fields k' of finite degree over k.

Choose a separable closure k^{sep} of k, and let $G = Gal(k^{sep}/k)$. Given a sheaf P on $(\text{spec } k)_{et}$, define

$$M_P = \varinjlim P(k')$$

Where k' runs through the subfields k' of k^{sep}, that finite and Galois over k.. Then M_P is a discrete G-module. Conversely, if M is a discrete G-module, we define

$$P_M(A) \;=\; \text{Hom}_G(F(A), M)$$

where $F(A)$ is the G-set $\text{Hom}_{k-alg}(A, k^{sep})$. Then P_M turns out to be sheaf on spec k. The functors $P \mapsto M_P$ and $M \mapsto P_M$ defines an equivalence between the category of sheaves on $(\text{spec } k)_{et}$ and the category of discrete G-modules.

2.2 Sheaves on Sites

Definition 2.2.1. : Given two sites $(\mathbf{C}'/X')_{E'}$ and $(\mathbf{C}/X)_E$. A morphism $\pi : X' \to X$ of schemes defines a *morphism of sites* $(\mathbf{C}'/X')_{E'} \to (\mathbf{C}/X)_E$ if :
(a) for any Y in $\mathbf{C}/X$, $Y_{(X')}$ is in $\mathbf{C}'/X'$;
(b) for any E-morphism $U \to Y$ in $\mathbf{C}/X$, $U_{(X')} \to Y_{(X')}$ is an E'-morphism

Now, as we know that the base change of a surjective family of morphism is again surjective , hence once we have π we also get a functor

$$\pi^\circ = \;(Y \mapsto Y_{(X')}) : \mathbf{C}/X \to \mathbf{C}'/X'$$

that take coverings to coverings. We shall often refer simply π, as a continuous morphism $\pi : X'_{E'} \to X_E$.

Let $\pi : X'_{E'} \to X_E$ be continuous . We denote the *direct image* of a presheaf P' on $X'_{E'}$, by $\pi_p(P') := P' \circ \pi^\circ$. $\pi_p(P')(U) = P'(U_{(X')}$. Clearly, π_p is a functor from $\mathbf{P}(X_E) \to \mathbf{P}(X'_E)$. Now, we have the following result of category theory (we state it here without proof)

Proposition 2.2.2. Let C and C' be two small categories, and let p be a functor $C \to C'$. Let A be a category that admits direct limits, denote $[C, A]$ and $[C', A]$ for the category of funtors from $C \to A$ and $C' \to A$. Then, the functor

$$(f \mapsto f \circ p) : [C', A] \to [C, A]$$

admits a left adjoint.

Proof: see [7]

Immediately from this proposition, we get the *inverse image* functor π^p : $\mathbf{P}(X_E) \to \mathbf{P}(X'_{E'})$, this is the left adjoint of π_p.
More, explicitly $(\pi^p P)(U') = \varinjlim P(U)$, which comes from the following commutative diagram :

$$\begin{array}{ccc} U' & \xrightarrow{g} & U \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\pi} & X \end{array}$$

where the limit is over all $U \to X$ in $(\mathbf{C}/X)_E$. Note that this is dependent on g (so that the U' 's are not necessarily distinct).

Let us consider an example where the description of π^p is particularly simple . Suppose that $\pi : X' \to X$ is in $(\mathbf{C}/X)_E$ and $\mathbf{C}'/X' = ((\mathbf{C}/X)_E)/X'$. In this case we have an initial object in the category over which we took the limit (above), it is with $g = id_{U'}$. Thus, we get $\Gamma(U', \pi^p P) = \Gamma(U', P)$, and so in this π^p simply restricts the functor P to the category $\mathbf{C}'/X'$.

Proposition 2.2.3. The functor π_p is exact and π^p is right exact . π^p is left exact , if we have finite inverse limits in $(\mathbf{C}/X)_E$, for example in X_{et} or X_{zar}, the étale site or the zariski site.

Proof. : The first statement is clear from the definition. π^p is right exact beacuse arbitrary direct limits are right exact in **Ab** (the category of abelian groups). Now, if finite inverse limits exists in $(\mathbf{C}/X)_E$, then it turns out that $(\pi^p P)(U') = \varinjlim P(U)$, is actually a cofiltered limit, and such limits are exact in **Ab**. Hence, our result. □

Proposition 2.2.4. If F is a sheaf, then $\pi_p F$ is a sheaf.

Proof. : Let π be a morphism $(\mathbf{C}'/X')_E \to ((\mathbf{C}/X)_E)$. For any U in $\mathbf{C}/X$ we write $U' = U \times_X X'$. If $(U_i \to U)$ is a covering, then so also is $(U_i' \to U')$, and so

$$F(U') \to \prod_i F(U_i') \rightrightarrows \prod_{i,j} F(U_i' \times_{U'} U_j')$$

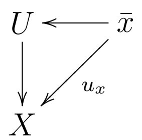
is exact. But $U_i' \times_{U'} U_j' \approx (U_i \times_U U_j)'$, and so this sequence is isomorphic to the sequence

$$(\pi_p F)(U) \to \prod_i \pi_p F(U_i) \rightrightarrows \prod_{i,j} \pi_p F(U_i \times_U U_j),$$

this shows that $\pi_p F$ is a sheaf. $\square$

Now, recall the notion of geometric point which we defined at the end of the 1 st chapter. Given a point x in a scheme X, henceforth $\bar{x}$ would be used to denote spec $k(\bar{x})$, where $k(\bar{x})$ is any separable closure of $k(x)$ and $u_x : \bar{x} \to X$, is the map induced by the inclusion $k(x) \hookrightarrow k(\bar{x})$.

Definition 2.2.5. If P is a presheaf on X_{et} . We define the *stalk* of P at $\bar{x}$ as the abelian group, $P_{\bar{x}} := (u_x^p P)(\bar{x}) = \varinjlim P(U)$, where the limit is over all the étale neighborhoods of $\bar{x}$ in X, i.e the U comes from the diagrams:



Before proceeding, further I would like to recall a basic defintion from abelian category. In an abelian category $\mathbf{A}$, we say that a sequence

$$0 \to M' \to\to M \xrightarrow{\alpha} M''$$

is exact if the sequence of abelian groups

$$0 \to \text{Hom}\,(N, M') \to \text{Hom}\,(N, M) \to \text{Hom}\,(N, M'')$$

is exact for all objects N in $\mathbf{A}$. In this case we say M' is the *kernel* of α. Similarly, a sequence

$$M' \xrightarrow{\beta} M \to M'' \to 0$$

is said to be exact if

$$0 \to \text{Hom}\,(M'', N) \to \text{Hom}\,(M, N)\text{Hom}\,(M', N)$$

is exact for all N. In this case we say that M'' is the cokernel of β

The next theorem is the étale topology analogue of sheafification, in ordinary sheaf theory.

Theorem 2.2.6. *For any presheaf P on X_E there is an associated sheaf aP on X_E, and a morphism $\phi : P \to aP$, such that any morphism from P to another sheaf F factors uniquely, through aP, i.e in terms of category theory , the inclusion functor $i : \boldsymbol{S}(X_E) \to \boldsymbol{P}(X_E)$ has a left adjoint $a : \boldsymbol{P}(X_E) \to \boldsymbol{S}(X_E)$. Further, the functor a is exact and preserves direct limits, while i preserves inverse limits.*

Proof : See [9] for a proof

Thus, now we define the category $\mathbf{S}(X_{et})$ as the full subcategory of $\mathbf{P}(X_{et})$, whose objects are sheaves of abelian groups. Now, by a morphism of sheaves we shall always mean a natural transformation of functors.

Given an abelian group Θ, there is a particular kind of *skyscraper sheaf*, associated which shall be useful to us .We discuss it in brief. For an étale map $\phi : U \to X$, define

$$\Theta^x(U) \;=\; \oplus_{u\in\phi^{-1}(x)}\Theta.$$

Thus, $\Theta^x(U) \;=\; 0$ unless $x \in \phi(U)$, in which case it is a sum of copies of Θ indexed by points of U mapping to x. Θ^x is a sheaf, and its stalks are zero except at $\bar{x}$ (assuming, that x is a closed point of X), where the stalk is Θ. If F is some sheaf on X and $F_{\bar{x}} \to \Theta$ is a homomorphism of abelian groups , then by choosing an $u \in \phi^{-1}(x)$ such that (U, u) is an étale neighborhood of x, we obtain a map $F(U) \to F_{\bar{x}} \to \Theta$. Thus, combining it with the maps for $u \in \phi^{-1}(x)$, we get :

$$F(U) \to \Theta^x(U) := \oplus_u\Theta$$

These maps are compatible with restrictions, hence we have a morphism of sheaves $F \to \Theta^x$. And, it turns out that the following holds :

$$\text{Hom}(F, \Theta^x) \simeq \text{Hom}(F_{\bar{x}}, \Theta)$$

A *locally surjective* morphism $\phi : F \to F'$ between sheaves ,is one, staisfying the property that for every U and $s' \in F'(U)$, there is a covering $(U_i \to U)_i$, such that $s|U_i$ is in the image of $F(U_i) \to F'(U_i)$, for each i.

Proposition 2.2.7. Let $\phi : F \to F'$ be morphism of sheaves on X_{et}, then the following conditions are equivalent :

1. $F \xrightarrow{\phi} F' \to 0$ is exact, i.e ϕ is epimorphism.
2. ϕ is locally surjective .
3. for every geometric point $\bar{x}$ of X, $\phi_{\bar{x}} : F_{\bar{x}} \to F'_{\bar{x}}$ is surjecive.

Proof : (2)⇒ (1) Let $\psi : F' \to G$ be a map of sheaves such that $\psi \circ \phi = 0$, we want to show that then $\psi = 0$.
Let $s' \in F'(U)$ for some étale $U \to X$. By assumption, there exists a covering $(U_i \to U)_{i\in I}$ and $s_i \in F'(U)$ such that $\phi(s_i) = s'|U_i$. We have :

$$\psi(s')|U_i = \psi(s'|U_i) = \psi \circ \phi(s_i) = 0, \quad \forall\, i$$

As, G is a sheaf, this gives us $\psi(s') = 0$
(1)⇒ (3). We prove this by contradiction. So, suppose that $\phi_{\bar{x}}$ is not surjective for some geometric point $\bar{x}$ of X, and let $\Theta \neq 0$ be the cokernel of $F_{\bar{x}} \to F'_{\bar{x}}$. Let Θ^x be the sheaf defined as above, i.e. we have $\mathrm{Hom}(G, \Theta^x) = \mathrm{Hom}(G_{\bar{x}}, \Theta)$, for any sheaf G on X_{et}. The map $F'_{\bar{x}} \to \Theta$ defines a non-zero morphism $F' \to \Theta^x$, whose composite with $F \to F'$ is 0 (since it corresponds to the composition $F_{\bar{x}} \to F'_{\bar{x}} \to \Theta$). This means that $F \xrightarrow{\phi} F' \to 0$ is not exact.
(3)⇒ (2). Again let $U \to X$ be an étale, and $\bar{u}$ be a geometric point of U. The composite $\bar{u} \to U \to X$ determines a geometric point $\bar{x}$ of X. So by our choice of $\bar{x}$, it is clear that if F is a sheaf on X_{et} then $F_{\bar{u}} \simeq F_{\bar{x}}$. Thus from our hypothesis it follows that $F_{\bar{u}} \to F'_{\bar{u}}$ is surjective for evry geometric point $\bar{u}$ of U. Let $s \in F'(U)$ be given. So now, there exists an étale map $V \to U$ whose image contains u, and which is such that $s|V$ is in the image of $F(V) \to F'(V)$. Continuing this process for sufficiently many $u \in U$, we obtain a desired covering.

Proposition 2.2.8. Let

$$0 \to F' \to F \to F''$$

be a sequence of sheaves on X_{et}. The following are equivalent :

1. the sequence is exact in the category of sheaves ;
2. the sequence
$$0 \to F'(U) \to F(U) \to F''(U)$$
is exact for all étale $U \to X$;
3. the sequence
$$0 \to F'_{\bar{x}} \to F_{\bar{x}} \to F''_{\bar{x}}$$
is exact for all geometric points

Proof: From theorem 2.2.6, we know that i is left exact, this shows the equivalence of (1) and (2).
Now, it is a standard fact from homological algebra that direct limits of exact sequences of abelian groups are exact, and thus (2)⇒ (3). Using the fact that $s \in F(U)$ is zero if and only if $s_{\bar{u}} = 0$ for all geometric points $\bar{u}$ of U, a similar argument as in the last proposition proves (3) ⇒ (2).

From the preceding theorem, we see that a sequence $0 \to F' \to F \to F''$ is exact in $\mathbf{S}(X_{et})$ iff it is so in $\mathbf{P}(X_E)$. Also, from the propositions thereafter , we deduce that, to form arbitrary inverse limits (for example, kernels, products) in $\mathbf{S}(X_{et})$ form the inverse limit in $\mathbf{P}(X_E)$, and then the resulting presheaf is a sheaf and is the inverse limit in $\mathbf{S}(X_{et})$. To form arbitrary direct limit (for example, cokernels, sums) in $\mathbf{S}(X_{et})$ form the direct limit in $\mathbf{P}(X_E)$, and then the associated sheaf is the direct limit in $\mathbf{S}(X_{et})$. All, this lets us deduce that.

Proposition 2.2.9. The category of sheaves of abelian groups on X_{et}, $\mathbf{S}(X_{et})$, is abelian.

Proof : We have shown the existence of kernels and cokernels for every morphism , so we are just reduced to show that for any morphism $\phi : F \to F'$ in $\mathbf{S}(X_{et})$, the induced morphism $\bar{\phi} : coim(\phi) \to im(\phi)$ is an isomorphism . But, as they are isomorphic at stalks, hence at the level of morphism.

Example 2.2.10. The group scheme, $\mathbb{G}_m$, which also defines a sheaf for the étale topology, has a subsheaf μ_n , it is defined as $\mu_n(U)$ = the group of n-th roots of unity in $\Gamma(U, \mathcal{O}_U)$. Actually, it is the sheaf defined by the group scheme, $\mu_n = \operatorname{spec} \mathbb{Z}[T]/(T^n - 1)$. Now, consider the *Kummer sequence*
$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \to 0$$

Here,$(\cdot)^n : \mathbb{G}(U) \to \mathbb{G}(U)$ is the n power map. Clearly, $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m$ is exact in $\mathbf{P}(X_E)$, and so in $\mathbf{S}(X_E)$. But, $(\cdot)^n : \mathbb{G}(U) \to \mathbb{G}(U)$, is not necessarily surjective. But, if we consider a strict local ring A, where n is invertible, then by Hensel's lemma we have

$$0 \to \mu_n(A) \to A^* \xrightarrow{(\cdot)^n} A^* \to 0$$

is exact, since $A[T]/(T^n - a)$ is an étale algebra over A, $\forall\ a \in A^*$, so from proposition 2.2 we have that :

$$0 \to \mu_n \to \mathbb{G})_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \to 0$$

is an exact sequence in $\mathbf{S}(X_E)$ (of course, assuming that characteristic of $k(x)$ does not divide n for any $x \in X$)

2.3 Direct and Inverse Images of Sheaves

Suppose we have a morphism of sites $(\mathbf{C}'/X')_{E'} \to (\mathbf{C}/X)_E$, defined by $\pi : X' \to X$

Definition 2.3.1. : The *direct image* of a sheaf F' on $X'_{E'}$ is $\pi_* F' = \pi_p F'$ and the *inverse image* of a sheaf F on X_E is defined as $\pi^* F = a(\pi^p F)$.

Note that $\pi_* F'$ is a sheaf as we have seen earlier. Now , we also have the following canonical isomorphisms

$$Hom_{\mathbf{S}(X_E)}(F, \pi_* F') \approx Hom_{\mathbf{P}(X'_{E'})}(\pi^p F, F') \approx Hom_{\mathbf{S}(X'_{E'})}(\pi^* F, F')$$

which we get from the fact that π^p is left adjoint of π_p and the inclusion functor $i : \mathbf{S}(X_E) \to \mathbf{P}(X_E)$ has a as left adjoint. In particular, we get that π_* and π^* are adjoint functors ;$\mathbf{S}(X'_{E'}) \leftrightarrows \mathbf{S}(X_E)$. Thus π_* is left exact and commutes with inverse limits and π^* is right exact and commutes with direct limits.

Proposition 2.3.2. If π^p is exact then so is π^*.

Proof. By definition we have

$$\pi^* := \mathbf{S}(X'_{E'}) \hookrightarrow \mathbf{P}(X'_{E'}) \xrightarrow{\pi^p} \mathbf{P}(X_E) \xrightarrow{a} \mathbf{S}(X_E)$$

Thus, in this case it right as well as left exact . □

Remark : Let $\pi : X' \to X$ be a morphism in $(\mathbf{C}/X)_E$, then

$$\pi^* : \mathbf{S}(X_{et}) \to \mathbf{S}(X'_{E'})$$

is just the restriction functor;

If we have continuous morphisms $X''_{E''} \to X'_{E'} \to X_E$. We see that $(\pi'\pi)_* = \pi'_*\pi_*$ and $\pi^*\pi'^*$ is adjoint to $\pi'^*\pi^*$ which implies that $\pi'^*\pi^* = (\pi'\pi)*$.

Now let us examine the behaviour of π_* and π^* at the stalks, for the étale topology.

Theorem 2.3.3. *Let $\pi : X' \to X$ be a morphism (étale) of schemes. Then we have the following:*
(a) For any sheaf F on X_{et} and any $x' \in X'$, $(\pi^*F)_{\bar{x'}} \approx F_{\pi(\bar{x'})}$ that is to say that the stalk of π^*F at $\bar{x'}$ is isomorphic to the stalk of F at $\pi\bar{x'}$. So that in particular , if π is the canonical morphism spec $\mathcal{O}_{X,\bar{x}} \to X$, then we have

$$F_{\bar{x}} = (\pi^*F)_{\bar{x}}$$

(b) Let us suppose that π is quasi-compact. For $x \in X$ and a fixed geometric point $\bar{x}$consider the canonical morphism f from $\tilde{X} :=$ spec $\mathcal{O}^{sh}_{X,x} \to X$, and write $\tilde{X'} = X' \times_X \tilde{X}$:

$$\begin{array}{ccc} \tilde{X'} & \xrightarrow{f'} & X' \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \tilde{X} & \xrightarrow[f]{} & X \end{array}$$

Then $(\pi_*F)_{\bar{x}} = \Gamma(\tilde{X'}, \tilde{F})$ for any sheaf F on X' and its restriction $\tilde{F} = f'^*F$ to $\tilde{X'}$.

Proof. (a) Denote $\pi(x')$ by x. Assume $\bar{x} = \bar{x'}$ (we can do this , since by defintion $\bar{x}$ is just spec $k(x)_{sep}$) . Thus, we get a commutative diagram (remember that $\mathfrak{u}_x$ map is just , the map induced by the inclusion $k(x) \hookrightarrow k(x)_{sep}$ followed by the canonical map spec $k(x) \to X$) :

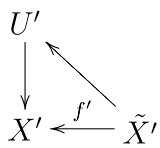
$$\begin{array}{ccc} X' & \xleftarrow{\mathfrak{u}_{x'}} & \bar{x'} = \bar{x} \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \mathfrak{u}_x} & \\ X & & \end{array}$$

Now by defintion of stalk, $(\pi^* F)_{\bar{x'}} \;=\; (u'^*_x \pi^* F)(\bar{x'})$
but $\mathfrak{u}^*_{x'}\pi^* \;= (\;\pi\mathfrak{u}_{x'})*$ (by remark 3.2) and $\mathfrak{u}_{x'}\pi \;= \mathfrak{u}_x$
hence we get that $(\pi^* F)_{\bar{x'}} = F_{\bar{x'}}$.

(b) To prove this first note that , f' being base change of an étale morphism is étale" and f'^p here takes sheaves to sheaves. Thus, we have here,

$$(f'^* F)(\tilde{X}') \;=\; \lim_{\rightarrow} F(U')$$

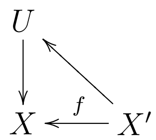
where the limit runs over all commutative diagrams with $U' \to X'$ *é*tale morphism . But $\tilde{X}$ is a limit, $\tilde{X}$



On the other hand from the definitions, we have

$$(\pi_* F)_{\bar{x}} \;=\; \lim_{\rightarrow} \pi_p F(U) \;=\; \lim_{\rightarrow} F(U_{(X')})$$

. Here the limit is over all U which comes from base extensions the diagrams of the following kind :



Thus, in order to show that above two limits are same it is sufficient to show that any morphism $\tilde{X}' \to U'$ factors through $\tilde{X}' \to U_{(X')}$ for some étale $U \to X$, beacuse it would show that the second set of diagrams that we mentioned are cofinal in the first, hence the limits being equal. Now, $\tilde{X} = \lim_{\leftarrow} U$, where U is affine and étale over X. As fiber products and inverse limits commutes and π is also quasi-compact, we have $\tilde{X}' \;=\; X' \times (\lim_{\leftarrow} U) \;=\; \lim_{\leftarrow} U_{(X')}$, where $U_{(X')}$ is quasi-compact and the transition morphisms are all affine . Now, if we use the next lemma , then as U' is of finite-type over X', so $\tilde{X}' \to U'$ factors through some $U_{(X')}$. □

Lemma 2.3.4. Let X be a scheme and let $Y = \lim_{\leftarrow} Y_i$, where (Y_i) is a filtered inverse system of X-schemes such that the transition morphisms $Y_i \to Y_j$ are affine . Assume that the Y_i are quasi-compact and let Z be a scheme that is locally of finite-type over X . Then any X-morphism $Y \to Z$ factors through $Y \to Y_i$ for some i. In, particular we have, $\mathrm{Hom}_X(Y, Z) = \varinjlim \mathrm{Hom}_X(Y_i, Z)$.

Proof. We do not prove the general case. The affine case is trivial , $X =$ spec A, $y_i =$ spec B_i, and $Z =$ spec C, just says that if $B = \varinjlim B_i$ and C is finitely generated over A, then any A-homomorphism $C \to B$ factors through some B_i. For the general case see [5] ☐

Corollary 2.3.5.
1. Let $\pi : V \to X$ be an open immersion, and F a sheaf on V_{et} . If $x \in V$, and say $\pi(z) = x$. Then, $(\pi_* F)_{\bar{x}} = F_{\bar{z}}$
2. If $\imath : Z \to X$ is a closed immersion, and F is a sheaf on Z_{et}, then $(i_* F)_{\bar{x}} = 0$, if $x \notin i(Z)$ and is $F_{\bar{z}}$ otherwise, where $i(z) = x$

Proof.
1. Note that we can find étale neighborhoods $\phi : U \to X$ of $\bar{x}$ such that $\phi(U) \subset V$ and we have $U = \phi^{-1}(V) = U \times_X V$. Thus, the neighbourhoods of the form U_V are cofinal in the set of étale neighborhoods of $\bar{z}$. Hence, our result.

2. From, the notation in the theorem $\tilde{Z}$ is empty if $x \notin i(Z)$ and so from part (2) of the theorem we get the result. If $i(z) = x$, then assuming X is affine (we can, since we are working locally, at stalks), we get $\tilde{Z} =$ spec $(\mathcal{O}_{X,\bar{x}}/\mathfrak{I}\mathcal{O}_{X,\bar{x}}) =$ spec $\mathcal{O}_{Z,\bar{z}}$, where $\mathfrak{I}$ is the sheaf of ideals defining Z. Now, from part (1) of the theorem we get our result.

☐

Relations with Subschemes

Let X be a scheme and consider the situation

$$Z \xrightarrow{i} X \xleftarrow{j} U$$

where U is an open subscheme of X and Z it's complement. Given a sheaf F on X_{et} we have correspondingly, sheaves $F_1 = i^* F$ and $F_2 = j^* F$ on Z and U , respectively. Now, $Hom(j^*F, j^*F) \approx Hom(F, j_* j^* F)$, so we have a canonical morphism $F \to j_* j^* F$ corresponding to the identity morphism of j^*F. By, applying i^* to this, we get a canonical morphism $\phi_F : F_1 \to i^* j_* F_2$. Thus. with every sheaf $F \in \mathbf{S}(X_{et})$ we have associated a triple (F_1, F_2, ϕ_F), it shall be shown that there is an equivalence of categories here.

We define a new category $\mathbf{T}(X)$ whose objects are triples (F_1, F_2, ϕ) with $F_1 \in \mathbf{S}(Z_{et})$, $F_2 \in \mathbf{S}(U_{et})$, and ϕ a morphism $F_1 \to i^*j_*F_2$. Morphisms in this category are defined as : $(F_1, F_2, \phi) \xrightarrow{(\psi_1,\psi_2)} (F_1', F_2'\phi')$, with ψ_1 being a morphism from $F_1 \to F_1'$ and ψ_2 is a morphism $F_2 \to F_2'$ and we should have the following commutativity :

$$\begin{array}{ccc} F_1 & \xrightarrow{\phi} & i^*j_*F_2 \\ {\scriptstyle \psi_1}\downarrow & & \downarrow{\scriptstyle i^*j_*(\psi_2)} \\ F_1' & \xrightarrow[\phi']{} & i^*j_*F_2' \end{array}$$

Theorem 2.3.6. *The categories $\boldsymbol{S}(X_{et})$ and $\boldsymbol{T}(X)$ are equivalent, under the natural transformation $F \mapsto (i^*F, j^*F, \phi_F)$*

Proof Consider $\psi in \mathrm{Hom}_S(F, F')$. Then we get a morphism in $\mathbf{T}$; $(i^*(\psi), j^*(\psi))$ from

$$(i^*F, j^*F, \phi_F) \to (i^*F', j^*F', \phi_{F'}).$$

So, we get a canonical functor $t : \mathbf{S}(X_{et}) \to \mathbf{T}(X)$.
Now, if $(F_1, F_2, \phi) \in \mathrm{Ob}(\mathbf{T}(X))$, define $s(F_1, F_2, \phi)$ to be the categorical fiber product of i_*F_1 and j_*F_2 over $i_*i^*j_*F_2$, so the diagram

$$\begin{array}{ccc} s(F_1, F_2, \phi) & \longrightarrow & j_*F_2 \\ \downarrow & & \downarrow \\ i_*F_1 & \xrightarrow{i_*(\phi)} & i_*i^*j_*F_2 \end{array}$$

is cartesian. Now, from a given morphism

$$(\psi_1, \psi_2) : (F_1, F_2, \phi) \to (F_1', F_2', \phi')$$

we obtain a canonical morphism from the universal property of fiber products,

$$s(\psi_1, \psi_2) : s(F_1, F_2, \phi) \to s(F_1', F_2', \phi')$$

it is induced by the maps, $s(F_1, F_2, \phi) \to j_*F_2 \xrightarrow{j_*\psi_2} j_*F_2'$ and $s(F_1, F_2, \phi) \to i_*F_1 \xrightarrow{i_*\psi_1} i_*F_1'$. Thus, we get a functor $s : \mathbf{T}(X) \to \mathbf{S}(X_{et})$.
For any $F \in \mathbf{S}(X_{et})$ the canonical maps $F \to i_*i^*F$ and $F \to j_*j^*F$ induce a

map $F \to st(F)$. To show that F is actually isomorphic to the fiber product in $\mathbf{S}(X_{et})$, it suffices to show that the diagram :

$$\begin{array}{ccc} F & \longrightarrow & j_*j^*F \\ \downarrow & & \downarrow \\ i_*i^*F & \longrightarrow & i_*i^*j_*j^*F \end{array}$$

is cartesian. Since, we are in the étale topology, this is enough that this is true at stalks. Now , if $x \in U$, then by corollary 3.4 ,the diagram we have in our hand is :

$$\begin{array}{ccc} F_{\bar{x}} & \longrightarrow & F_{\bar{x}} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

This is clearly cartesian. If $x \in Z$, then the diagram we have is :

$$\begin{array}{ccc} F_{\bar{x}} & \longrightarrow & (j_*j^*F)_{\bar{x}} \\ \downarrow & & \downarrow \\ F_{\bar{x}} & \longrightarrow & (j_*j^*F)_{\bar{x}} \end{array}$$

This is cartesian too.
Now, again taking the stalks we see that , s and t induce inverse maps on morphisms. So, we have shown that the two categories in question are actually equivalent.

■

If Y is a subscheme of X and F is a sheaf on X_{et}, we say that F has its support on Y if for any geometric point, $F_{\bar{x}} = 0$ where $x \notin Y$

Corollary 2.3.7. If $i :\to X$ is a closed immersion, then the functor i_* : $\mathbf{S}(Z_{et}) \to \mathbf{S}(X_{et})$ induces an equivalence between $\mathbf{S}(Z_{et})$ and the full subcategory of $\mathbf{S}(X_{et})$ consisiting of the sheaves with support on $i(Z)$

Proof Note, that F has support in Z if and only if $t(F)$ has the form $(F_1, 0, 0)$. Then, by applying the theorem, we get our result.

Now, there are six functors which relates the categories $\mathbf{S}(U_{et})$, $\mathbf{S}(Z_{et})$ and $\mathbf{S}(X_{et})$:

$$\begin{array}{ccccc} & \xleftarrow[i^*]{\qquad} & & \xleftarrow[j_!]{\qquad} & \\ \mathbf{S}(Z_{et}) & \xrightarrow{i_*} & \mathbf{S}(X_{et}) & \xrightarrow{i^*} & \mathbf{S}(U_{et}) \\ & \xleftarrow[i^!]{\qquad} & & \xleftarrow[j_*]{\qquad} & \end{array}$$

By the identification of $\mathbf{S}(X_{et})$ with $\mathbf{T}(X)$, these functors are defined as

$$i^* : (F_1, F_2, \phi) \mapsto F_1 \qquad j_! : F_2 \mapsto (0, F_2, 0)$$

$$i_* : F_1 \mapsto (F_1, 0, 0) \qquad j^* : (F_1, F_2, \phi) \longmapsto F_2$$

$$i^! : (F_1, F_2, \phi) \mapsto ker\phi \qquad j_* : F_2 \mapsto (i^* j_* F_2, F_2, 1)$$

Following is a list of basic properties of these functors

- Any given functor functor is left adjoint to the one below it.
- i^*, i_*, j^*, $j_!$ are exact, j_*, $i^!$ are left exact.
- $i^* j_! = i^! j_! = i^! j_* = j^* i_* = 0$
- i_*, j_* are fully faithful, and $F \in \mathbf{S}(X_{et})$ has support in Z if and only if $F \approx i_* F_1$ for some $F_1 \in \mathbf{S}(Z_{et})$.
- The functors j_*,j^*, $i^!$, i_* maps injectives to injectives

Note that for the functor s that was defined earlier and a geometric point $\bar{x}$, we have $s(F_1, F_2, \phi)_{\bar{x}} = (F_1)_{\bar{x}}$ if $x \in Z$ and is equal to $(F_2)_{\bar{x}}$ otherwise. In view of this and theorem 3.4 , a sequence

$$(F_1', F_2', \phi') \to (F_1, F_2, \phi) \to (F_1'', F_2'', \phi'')$$

in $\mathbf{T}(X)$ is exact if and only if the sequences

$$F_1' \to F_1 \to F_1'', \quad F_2' \to F_2 \to F_2''$$

exact (just take stalks).

Thus we get a canonical exact sequence :

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0$$

in $\mathbf{S}(X_{et})$, which comes from the following exact sequence in $\mathbf{T}(X)$:

$$0 \to (0, j^* F, 0) \to (i^* F, j^* F, \phi_F) \to (i^* F, 0, 0) \to 0$$

For more details of these discussions, see [9]

Remark Let $j : U \to X$ be an object in $\mathbf{C}/X$ for some arbitrary site $(\mathbf{C}/X)_E$. We shall exhibit that there is a left adjoint functor for j^*.
Let p be the functor $\mathbf{C}/U \to \mathbf{C}/X$, $p(Y \xrightarrow{g} U) = (Y \xrightarrow{jg} X)$. Then the functor $j^p : \mathbf{P}(X) \to \mathbf{P}(U)$, that we had defined earlier is idnetical to the functor

$$(f \mapsto f \circ p) : [\mathbf{C}/X, \mathbf{Ab}] \to [\mathbf{C}/U, \mathbf{Ab}]$$

. According to proposition 2.1 , we know that this has a left adjoint,written as $j_! : \mathbf{P}(U) \to \mathbf{P}(X)$. This is the *extension by zero* functor . Explicitly, if $P \in \mathbf{P}(U)$ and $V \in \mathbf{C}/X$, then $j_! P(V) = \lim_{\leftarrow} P(V')$, where V' comes from the following kind of commutative diagrams,

$$\begin{array}{ccc} V' & \longleftarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

The limit breaks up into

$$(j_! P)(V) = \bigoplus_{\phi \in Hom_X(V,U)} \varinjlim_{S(\phi)} P(V')$$

where $S(\phi)$ is the set of squares with $(V \to V' \to U) = \phi$. Since $S(\phi)$ contains a final object (the square with $V' = V$), we see that

$$(j_! P)(V) = \bigoplus_{\phi \in Hom_X(V,U)} P(V_\phi)$$

where V_ϕ is the object $V \xrightarrow{\phi} U$ of $\mathbf{C}/U$. Thus $j_!$ is exact. If j is an open immersion, then $(j_! P)(V) = P(V)$ if $V \to X$ factors through U and is zero

otherwise.
On, sheaves $j_!$ is defined to be the composite

$$\mathbf{S}(X) \hookrightarrow \mathbf{P}(X) \xrightarrow{j_!} \mathbf{P}(U) \xrightarrow{a} \mathbf{S}(U).$$

This is clearly adjoint to, j^*, the restriction functor. Being a left adjoint, it is automatically right exact, and it is left exact because it is a composite of left exact functors. If, we wrok on the étale site, and j is an open immersion, then we are back to our old extension by zero, defined before.

Chapter 3

Cohomology

3.1 Preliminaries on homological algebra and cohomology

All the categories in this section are assumed to be abelian and the functors are assumed to be additive.

In an abelian category $\mathbf{A}$ an object I is said to be *injective*, if the functor $M \mapsto \mathrm{Hom}_{\mathbf{A}}(M, I)$ is an exact functor. $\mathbf{A}$ is said to have *enough injectives* if, for every object M in $\mathbf{A}$ has a monomorphism into an injective object . Now, I'm listing here some propeties of the *right derived functors* $R^i f : \mathbf{A} \to \mathbf{B}$, $i \geq 0$ for a given left exact functor $f : \mathbf{A} \to \mathbf{B}$ between two abelian categories, with $\mathbf{A}$ having enough injectives. (the proofs are skipped)

1. $R^0 f \; = \; f$
2. $R^i f(I) \; = \; 0$ for $i\, 0$, if I is an injective object
3. Then we have the notion of connecting morphisms. For any exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathbf{A}$, there are morphisms $\delta^i : R^i f(M'') \to R^(i+1) f(M')$, $i \geq 0$, such that the sequence

$$\cdots \to R^i f(M) \to R^i f(M'') \xrightarrow{\delta^i} R^{i+1} f(M') \to R^{i+1} f(M) \to \cdot\cdot$$

4. The association in (3)of the long exact sequence to the short exact sequence is functorial.

An object M in $\mathbf{A}$ is said to be $f-acyclic$ if $R^i f(M) = 0$ for all i 0. Further if we have an injective resolution of M

$$0 \to M \to N^0 \to N^1 \to N^2 \to \cdot\cdot$$

With N^i being $f-acyclic$ objects then the objects $R^i f(M)$ are canonically isomorphic to the cohomology objects of the complex

$$0 \to fN^0 \to fN^1 \to fN^2 \to \cdot\cdot$$

To know more about the derived functors see for example [7].

Definition 3.1.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two abelian categories. A (covaraint) $\delta - functor$ from $\mathfrak{A}$ to $\mathfrak{B}$ is a collection of functors $T = (T^i)_{i \geqslant 0}$, together with a morphism $\delta^i : T^i(A'') \to T^{i+1}(A')$ for each short exact sequence $0 \to A' \to A'' \to 0$, and each $i \geqslant 0$, such that :

1. For each short exact sequence as above , there is a long exact sequence

$$0 \to T^0(A') \to T^0(A) \to T^0(A'') \xrightarrow{\delta^0} T^1(A') \to \cdots$$

$$\cdots \to T^i(A) \to T^i(A'') \xrightarrow{\delta^i} T^{i+1}(A') \to T^{i+1}(A) \to \cdots$$

2. For each morphism of one short exact sequence (as above) into another $0 \to B' \to B \to B'' \to 0$, we have the following commutative diagram

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\delta^i} & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B'') & \xrightarrow[\delta^i]{} & T^{i+1}(B') \end{array}$$

Definition 3.1.2. The δ-functor $T = (T^i) : \mathfrak{A} \to \mathfrak{B}$ is said to be *universal* if, given any other δ-functor $T' = (T'^i) : \mathfrak{A} \to \mathfrak{B}$, and given any morphism of functors $f^0 : T^0 \to T'^0$, there exists an unique sequence of morphisms $f^i : T^i \to T'^i$ for each $i \geqslant 0$, starting with given f^0, which commutes with the δ^i for each short exact sequence.

Definition 3.1.3. An additive functor $F : \mathfrak{A} \to \mathfrak{B}$ is said to be $effaceable$ if for each obeject A of $\mathfrak{A}$, there is a monomorphism $u : A \to M$, such that $F(u) = 0$. It is called $coeffaceable$ if for each A there exists an epimorphism $u : P \to A$ such that $F(u) = 0$.

Theorem 3.1.4. *Let $T = (T^i)_{i \geqslant 0}$ be a covariant $\delta - functor$ from $\mathfrak{A}$ to $\mathfrak{B}$. If T^i effaceable for each $i > 0$, then T is universal.*

Proof. For proof see [1]

Corollary 3.1.5. Assume that $\mathfrak{A}$ has enough injectives. Then for any left exact functor $F : \mathfrak{A} \to \mathfrak{B}$, the derived functors $(R^iF)_{i \geqslant 0}$ form a universal δ-functor with $F \cong R^0F$. Coversely, if $T = (T^i)_{i \geqslant 0}$ is any universal δ-functor, then T^0 is left exact, and the T^i are isomorphic to R^iT^0 for each $i \geqslant 0$.

Proof. If F is left exact functor, then the $(R^iF)_{i \geq 0}$ form a δ-functor . For some object M in , let $u : M \to I$ be a monomorphism into an injective object. Then $R^iF(I) = 0$ for $i > 0$, so $R^iF(u) = 0$ (see above). Thus R^iF is effaceable for each $i > 0$. So, from the theorem it follows that (R^iF) is universal.
Conversely, if we are given an universal δ-functor T, we have from the definitions T^0 as a left exact functor. As $\mathfrak{A}$ has enough injectives, the derived functors R^iT^0 exists. Now, as (R^iT^0) is also universal and $R^0T^0 = T^0$, we obtain that $R^iT^0 \cong T^i$ for each i □

We need the following lemma from category theory .

Lemma 3.1.6. (a) Product of two injective objects is injective
(b) If $f : \mathbf{A} \to \mathbf{B}$ has an exact left adjoint $g : \mathbf{B} \to \mathbf{A}$, then f preserves injectives.
Thus, if π is a continuous morphism of sites, then π_* preserves injectives, when π^* is exact.

Proof. (a) Using the fact that functor $\mathrm{Hom}(M, \cdot)$ commutes with products and that, for a given functors (additive) F and G, $X \mapsto F(X) \oplus G(X)$ is exact if and only if F and G are exact.
(b) Consider an injective object I in $\mathbf{A}$. As g is the left adjoint of f so by defintion, we have $M \mapsto \mathrm{Hom}_{\mathbf{B}}(M, fI)$ is same as the functor $M \mapsto \mathrm{Hom}_{\mathbf{A}}(gM, I)$. But, by hypothesis $\mathrm{Hom}_{\mathbf{A}}(\mathbf{.}, I)$ and g are both exact and hence their composite. □

Proposition 3.1.7. $\mathbf{S}(X_{et})$ has enough injectives .

Proof. Let $i_x : \bar{x} \to X$ be a geometric point of X. Now, the category of sheaves on $(\bar{x}_{et})$ is ismorphic to **Ab**, (see chpater 2 after proposition 1.5 , $F \mapsto F(\bar{x})$ gives an equivalence). Thus, it has got enough injectives. Let $F \in \mathbf{S}(X_{et})$, for each $x \in X$ consider an embedding $i_x^* \to I'_x$, where I' is an injective object in $\mathbf{S}(\bar{x})$. Now, define $F^* = \prod_{x \in X} i_{x*} i_x^* F$, then we have a canonical injection $F \hookrightarrow F^*$, and from the definitions of direct and inverse images, we also get the injection $F^* \hookrightarrow \prod_{x \in X} i_{x*} I_x$. By the previous lemma $i_{x*}F$ and $\prod_{x \in X} i_{x*} I_x$ are both injective, thus the composition of these two monomorphisms gives us the required embedding in the category $\mathbf{S}(X_{et})$ □

Spectral Sequences

Let M be an object of an abelian category $\mathbf{A}$. A (decreasaing) *filtration* of M is a family $(F^p(M))_{p \in \mathbb{Z}}$ of subobjects $F^p(M)$ of M, such that $F^{p+1}(M) \subset F^p(M)$, for all p. We define $gr_p(A) = F^p(M)/F^{p+1}(M)$. Given objects M and N with filtrations, a morphism $u : M \to N$ is said to be compatible with the filtrations if $u(F^p(M)) \subset F^p(B)\ \forall\, p \in \mathbb{Z}$.

Definition 3.1.8. : A *spectral sequence* in the abelian category $\mathbf{A}$ is a system

$$E = (E_r^{p,q}, E^n)$$

consisting of the following data

1. $E_r^{p,q}$ are objects of $\mathbf{A}$, for all $p, q\ \in \mathbb{Z}$ and $r \geq 2$

2. morphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ in $\mathbf{A}$ such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$.

3. There are isomorphisms $\alpha_r^{p,q} : \ker(d_r^{p,q})/\mathrm{im}(d_r^{p-1,q-r+1}) \xrightarrow{\approx} E_{r+1}^{p,q}$,

4. E^n are filtered (decreasing) objects of $\mathbf{A}$. And we may assume that for every fixed pair (p, q), the morphisms $d_r^{p,q}$ and $d_r^{p-r,q-r+1}$ vanish for sufficiently large r. By the previous condition it follows that $E_r^{p,q}$ are independent of r, for sufficiently large r, so we denote this object by $E_\infty^{p,q}$. Another assumption that we consider is that for every $n \in \mathbb{Z}$,

$F^p(E^n) = E^n$ for sufficiently small p and $F^p(E^n) = 0$ for sufficiently large p.

5. there are isomorphisms $\beta^{p,q} : E_\infty^{p,q} \xrightarrow{\approx} gr_p(E^{p+q})$.

We shall denote such a spectral sequence by $E_2^{p,q} \Rightarrow E^{p+q}$. If $E^{p,q} = 0$ for $p < 0$ and $q < 0$, then it is said to be *cohomological spectral sequence*. Following are some important propeerties of such a spectral sequence .

Proposition 3.1.9. 1. There exists morphisms $E_2^{n,0} \to E^n$ and $E^n \to E_2^{0,n}$, that are functorial. They are called *edge morphisms*

2. The sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to E^2$$

is exact

3. If the terms $E_2^{p,q}$ vanish for $0 < q < n$. Then $E_2^{m,0} \cong E^m$ for $m < n$ and the following sequence is exact

$$0 \to E_2^{n,0} \to E^n \to E_2^{0,n} \to E_2^{n+1,0} \to E_2^{n+1}$$

Thus, if $E_2^{p,q} = 0$ for all $q > 0$, we have

$$E_2^{n,0} \cong E^n$$

for all n. The spectral sequence is then said to be *trivial*

Proof. For proofs of these see for example [3]. □

Theorem 3.1.10. *Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be abelian categories, where $\boldsymbol{A}$ and $\boldsymbol{B}$ have enough injectives . Let $f : \boldsymbol{A} \to \boldsymbol{B}$ and $g : \boldsymbol{B} \to \boldsymbol{C}$ be left exact functors. If f maps injective objects to g-acyclics, then there is a cohomological spectral sequence :*

$$(R^p g)(R^q f)(A) \Rightarrow R^{p+q}(gf)(A)$$

for any object A of $\boldsymbol{A}$. In particular, there is an exact sequence

$$0 \to R^1 g(fA) \to g(R^1 f)A \to R^2 g(fA) \to \cdots.$$

Proof . [7]

Definition 3.1.11. : We, know that the functor $\Gamma(X,-) : S(X_E) \to \mathbf{Ab}$ with $\Gamma(X,F) = F(X)$, is left exact and so we can consider its right derived functors:

$$R^i\Gamma(X,-) = H^i(X,-) = H^i(X_E,-)$$

The group $H^i(X_E,F)$ is called the i^{th} *cohomology group* of X_E with values in F.
We have a few more important functors here.

1. For any fixed sheaf F_0 on X_E, the functor $F \mapsto \mathrm{Hom}_{\mathbf{S}}(F_0,F)$ is left exact , hence has right derived functors $R^iHom_{\mathbf{S}}(F_0,-) = \mathrm{Ext}^i_{\mathbf{S}}(F_0,-)$.

2. For any $U \to X$ in $\mathbf{C}/X$, the right derived functors for $F \mapsto F(U) : \mathbf{S}(X_E) \to \mathbf{Ab}$ is denoted by $H^i(U,F)$. We shall see that it is same as $H^i(U,F|U)$.

3. From ...we know that $i : \mathbf{S}(X_E) \to \mathbf{P}(X_E)$ is left exact. Its right derived functors are denoted as $\underline{H}^i(F)$.

4. Given any continuous morphism $\pi : X'_{E'} \to X_E$, we can define the right derived functors $R^i\pi_*$ of the functor $\pi_* : \mathbf{S}(X'_{E'}) \to \mathbf{S}(X_E)$ are defined. The sheaves $R^i\pi_*F$ are called *higher direct images* of F

5. If we consider $\pi^* : \mathbf{S}(X_E) \to \mathbf{S}(X'_{E'})$ corresponding to some continuous morphism $\pi : X'_{E'} \to X_E$, then the canonical map $H^0(X,F) \to H^0(X',\pi^*F)$ induces , maps on the higher cohomology groups by the universality of the derived functors.

A natural question that might arise now, is what is the relation between étale cohomology theory and Zariski Cohomology (i.e the ordinary sheaf cohomology for schemes). A partial answer to this, is the following :

Theorem 3.1.12. *(Leray spectral seqence) The continuous morphism $\pi : X_{et} \to X_{Zar}$, of the étale site on X into the Zariski site, gives rise to the spectral sequence :*

$$H^p(X_{Zar}, R^q\pi_*F) \Rightarrow H^{p+q}(X_{et},F)$$

where F is a sheaf on X_{et}.

Proof. We want to apply theorem 3.1.10 here to the functors $\Gamma(X_{Zar}, -)$ and π_*. As, from the definition of right derived functors, we know that injective sheaves on X_{Zar} are $\Gamma(X_{Zar}, -)$ acyclic, so we are just reduced to show that π_* maps injective sheaves to injectives. But, note that π^* is exact, here as we are working on the Zariski site (see 2.3.2 and 2.2.3).That means π_* has an exact left adjoint and hence it preserves injectives.

□

In general we would have $R^q\pi^* \neq 0$, so that the spectral sequence is non-trivial.

Cohomology with support

Consider the following

$$Z \xrightarrow{i} X \xleftarrow{j} U \;=\; X - Z$$

Here, i is a closed immersion and j is an open immersion. The group

$$\Gamma(X, i_* i^! F) = \Gamma(Z, i^! F) = Ker(F(X) \to F(U))$$

is called the group of sections of F with support on Z. The functor $F \mapsto \Gamma(Z, i^! F)$ is left exact, and its right derived functors are called the *cohomolgy groups of F with support on Z.*

Proposition 3.1.13. For any sheaf on X_{et} there is a long exact sequence,

$$0 \to (i^! F)(Z) \to F(X) \to F(U) \to \cdots H^p(X, F) \to H^p(U, F) \to H^{p+1}_Z(X, F) \to \cdots$$

Proof. Given any sheaf F on X_{et} we have the following exact sequence

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0$$

If we consider the constant sheaf $\mathbb{Z}$ then we have the exact sequence

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Z \to 0$$

here we have denoted $j_! j^* \mathbb{Z}$ by $\mathbb{Z}_U$ and $i_* i^* \mathbb{Z}$ by $\mathbb{Z}_Z$. From this we obtain a long exact sequence for the functor $\mathrm{Ext}^i(-, F)$

$$\cdots \to \mathrm{Ext}^p(\mathbb{Z}, F) \to \mathrm{Ext}^p(\mathbb{Z}_U, F) \to \mathrm{Ext}^{p+1}(\mathbb{Z}_Z, F) \to \cdots.$$

(Note, this comes from a general statement in homological algebra, i.e if we are given a short exact sequence $B' \rightarrowtail B \twoheadrightarrow B''$, then for any object A we take an injective resolution $I^{\cdot}$, then we get an exact sequence

$$0 \to \text{Hom}\,(B'', I^{\cdot}) \to \text{Hom}\,(B, I^{\cdot}) \to \text{Hom}(B', I^{\cdot}) \to 0$$

, From this we obtain a long exact sequence for the *Ext*- functor in second variable.)
Now, the functor $\Gamma(X, -) : \mathbf{S}(X_E) \to Ab$ is representable by the constant sheaf $\mathbb{Z}$ which implies that $\text{Ext}^p(\mathbb{Z}, F) = H^p(X, F)$. We also know that $\text{Hom}_{S(X)}(j_! j^* \mathbb{Z}, F) \approx \text{Hom}_{S(U)}(\mathbb{Z}, j^* F)$. Thus,by the definition of Ext functor, we get that $\text{Ext}^p(\mathbb{Z}_U, F)$ is actually the p^{th} right derived functor for

$$F \mapsto \text{Hom}_{\mathbf{S}(U)}(\mathbb{Z}, j^* F) \;=\; \Gamma(U, F|U).$$

Since, j^* sends injectives to injectiives and is also exact (see, chapter 2)

, we get that $\text{Ext}^p(\mathbb{Z}_U, F) = H^p(U, F|U)$. There are also canonical ismorphisms

$$\text{Hom}_{\mathbf{S}(X)}(\mathbb{Z}_Z, F)_{\mathbf{S}(Z)}(\mathbb{Z}, i^! F) \approx H^0_Z(X, F).$$

Which means that $\text{Ext}^p(\mathbb{Z}_Z, F) \approx H^p_Z(X, F)$. Thus from the exact sequence for the Ext we get our required exact sequence for the cohomology groups with support on Z. □

Now, we shall discuss excision in the setting of étale site.

Proposition 3.1.14. Let $Z \subset X$ and $Z' \subset X'$ be closed subschemes, and let $\pi : X' \to X$ be an étale morphism, such that restriction of π to Z' is an isomorphism onto Z and $\pi(Z'^C) \subset Z^C$. Then, the canonical map $H^p_Z(X, F) \to H^p_{Z'}(X', \pi^* F)$ is an isomoprphism for all $p \geq 0$ and all sheaves F on X_{et}.

Proof. In order to make the required conclusion we need to show two things; firstly show that the functors $H^0_Z(X, -)$ is isomorphic to $H^0_{Z'}(X', \pi^* -)$ and secondly $H^p_{Z'}(X', \pi^* -)$ is an universal δ-functor .
Since, here $\pi \in (\mathbf{C}/X)_E$, so π^* is exact and thus composing it with the δ-functor $H^p_{Z'}(X', -)$ we again get a δ-functor. Further, here we also know that π^* preserves injectives , hence $H^p_{Z'}(X', \pi^* I) = 0$ if I is an injective sheaf . This means that $H^p_{Z'}(X', \pi^* -)$ is $effaceable$, thus in the light of theorem 3.1.4, we see that $H^p_{Z'}(X', \pi^* -)$ is an universal δ-functor.

Now , we exhibit the first claim
From the hypothesis of the proposition, we have :

$$\begin{array}{ccccc} U' & \xrightarrow{j'} & X' & \xleftarrow{i'} & Z' \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \approx} \\ U & \xrightarrow{j} & X & \xleftarrow[i]{} & Z \end{array}$$

Here we have denoted Z^C by U and Z'^C by U'. Here i, j, i' and j' are the canonical immersions. Now, $\pi : X' \to X$ is an étale morphism, so the functor $\pi^* F$ is just the restricted sheaf $F|X'$. Thus, we get a commutative diagram .

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0_Z(X, F) & \longrightarrow & \Gamma(X, F) & \longrightarrow & \Gamma(U, F) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0_{Z'}(X', F|X') & \longrightarrow & \Gamma(X', F) & \longrightarrow & \Gamma(U', F) \end{array}$$

Note that the first arrow comes from the the fact that $i^! F(Z) = \mathrm{Ker}(F(X) \to F(U))$. Now, we do a little diagram chasing. Suppose $t \in H^0_Z(X, F)$ maps to zero in $H^0_{Z'}(X', F|X')$. Then by exactness at $\Gamma(X, F)$ we get that $t|U = 0$ (here we regard t as an element of $\Gamma(X, F)$) and by the commutativity of the first square we get $t|X' = 0$. Now, note that in our étale site, $U \to X$ and $X' \to X$ is a covering of X . Thus, by property of sheaf, $t = 0$. This proves the required injectivity.

Now, let $t' \in H^0_{Z'}(X', F|X')$ and look at it as an element of $\Gamma(X', F)$. Note that by the commutativity of the second square, $t' \in \Gamma(X', F)$ and $0 \in \Gamma(U, F)$ agree on $X' \times_X U = \pi^{-1}(U) = U'$, thus we are just reduced to verify that restrictions of t', under the two maps $X' \times_X X' \rightrightarrows X'$ actually coincide. For a point in $U' \times_X U'$, both the restrictions are zero. The two maps , $Z' \times_X Z' \rightrightarrows Z'$ are same , hence restrictions coincide. But, $X' \times_X X'$ is a disjoint union of $U' \times_X U'$ and $Z' \times_X Z'$. Thus, by property of sheaf t' and 0 come from a section $t \in \Gamma(X, F)$, which again should lie in $H^0_Z(X, F)$ because of the exactness at $\Gamma(X, F)$. Thus, the required surjectivity.

□

3.2 Čech Cohomology

Let $\mathfrak{U} = (U_i \xrightarrow{\phi_i} X)_{i \in I}$ be covering for X in the E-topology on X. For any $(p+1)$-tuple $(i_0, ..., i_p)$ with the i_j in I, denote $U_{i_0} \times_X .... \times_X U_{i_p} = U_{i_0....i_p}$. The projection map

$$U_{i_0...i_p} \to U_{i_0..\hat{i_j}..i_p} = U_{i_0} \times \cdots U_{i_{j-1}} \times U_{i_{j+1}} \times \cdots \times U_{i_p}$$

induces restriction morphism

$$P(U_{i_0..\hat{i_j}..i_p}) \to P(U_{i_0...i_p})$$

We shall denote this by res_j. Now, we define a complex $C^{\cdot}(\mathfrak{U}, P)$ of abelian groups as follows. For each $p \geqslant 0$, let

$$C^p(\mathfrak{U}, P) = \prod_{i_0 < ... < i_p} P(U_{i_0,...i_p})$$

Thus an element $\alpha \in C^p(\mathfrak{U}, P)$ is determined by giving an element $\alpha_{i_0,...i_p} \in P(U_{i_0,...,i_p})$ for each $(p+1)$-tuple $i_0 < .. < i_p$ of elemnts of I. We define the coboundary maps $d^p : C^p(\mathfrak{U}, P) \to C^{P+1}(\mathfrak{U}, P)$ by setting

$$(d^p \alpha)_{i_0,...i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j res_j(\alpha_{i_0..\hat{i_j}..i_p}).$$

As usual we have $d^{p+1} \circ d^p = 0$, Thus, we have a complex and the cohomology groups of this complex are called the *Čech cohomology groups*, $\check{H}^p(\mathfrak{U}, P)$, of P with respect to the covering $\mathfrak{U}$ of X.

Lemma 3.2.1. Given X ,$\mathfrak{U}$ and P as above, we have $\check{H}^0(\mathfrak{U}, P) \cong \Gamma(X, P)$.

Proof. By defintion $\check{H}^0(\mathfrak{U}, P) = Ker(\prod P(U) \xrightarrow{d^0} \prod P(U_{ij}))$. If $t \in \prod P(U_i)$, then for each $i < j$, $(dt)_{ij} = t_i - t_j$, where t_i, t_j denotes the i^{th} and j^{th} co-ordinates of t. Thus, $dt = 0$ implies that t_i and t_j agree on $U_i \times U_j$. So, from the sheaf axioms it follows that $\check{H}^0(\mathfrak{U}, P) = \Gamma(X, P)$. □

If we consider any other covering, say $\mathfrak{V} = (V_j \xrightarrow{\psi_j} X)_{j \in J}$, then we say that it is a *refinement* of $\mathfrak{U}$, if we have maps $\tau : J \to I$ such that for each j, ψ_j factors through $\phi_{\tau j}$ that is we have $\psi_j = \phi_{\tau j}\eta_j$ for some $\eta_j : V_j \to U_{\tau j}$. These induces a cochain map between the two complexes, $\sigma : C^{\cdot}(\mathfrak{U}, P) \to C^{\cdot}(\mathfrak{V}, P)$. For, $t = (t_{i_0 \ldots i_p}) \in C^p(\mathfrak{U}, P)$, it is given by :

$$(\sigma^p t)_{j_0 \ldots . j_p} = res_{\eta j_0 \times \eta j_1 \times \ldots . \times \eta j_p}(t_{\tau j_0 \ldots . \tau j_p}).$$

Taking cohomology, we get a map

$$\rho(\mathfrak{V}, \mathfrak{U}, \tau) : \check{H}^p(\mathfrak{U}, P) \to \check{H}^p(\mathfrak{V}, P).$$

For another similar map τ' and a family (η'_j) from $J \to I$, we get a $\rho'(\mathfrak{V}, \mathfrak{U}, \tau')$. But, this map is actually same as $\rho(\mathfrak{V}, \mathfrak{U}, \tau)$ because there exists a homotopy $\Delta : C^{\cdot}(\mathfrak{U}, P) \to C^{\cdot}(\mathfrak{V}, P)$, which is given on each term by

$$(\Delta^p t)_{j_0 \ldots . j_{p-1}} = \sum (-1)^r res_{\eta j_0 \times \ldots \times (\eta j_r, \eta' j_r) \times \ldots \times \eta' j_{p-1}}(t_{\tau j_0 \ldots . \tau j_r \tau' j_r \ldots \tau' j_{p-1}})$$

Note here also that if we are given any third covering $\mathfrak{W}$, which is a refinement of $\mathfrak{V}$, then we have $\rho(\mathfrak{W}, \mathfrak{U}) = \rho(\mathfrak{W}, \mathfrak{V})\rho(\mathfrak{V}, \mathfrak{U})$. This allows us to define *Čech cohomology groups* of P over X, as $\check{H}^p(X_E, P) = \varinjlim \check{H}^p(\mathfrak{U}, P)$, the limit being taken over all coverings $\mathfrak{U}$ of X.

Remark: The category of coverings of X need not be filtered in general. So, we can actually consider the direct limit over the category of coverings modulo an equivalence relation: $\mathfrak{U} \equiv \mathfrak{V}$, if each is a refinement of the other. Call this category as $\mathcal{I}_X$, then $\mathcal{I}$ is actually filtered because any two coverings $\mathfrak{U} = (U_i)$ and $\mathfrak{V} = (V_i)$ has a common refinement $(U_i \times V_i)$. According to our discussion above the functor $\mathfrak{U} \mapsto \check{H}^p(\mathfrak{U}, P)$, then factors through $\mathcal{I}$, and so we take limit over this category.

If $U \to X$ is in $\mathbf{C}/X$ and P is a presheaf on $(\mathbf{C}/X)_E$, then analogusly we may define the cohomology groups $\check{H}^p(\mathfrak{U}/U, P)$ and $\check{H}^p(U, P) = \varinjlim \check{H}^p(\mathfrak{U}/U, P)$, where $\mathfrak{U}$ now denotes a covering of U. From the definitions we also have $\check{H}^p(U, P)$ isomorphic to $\check{H}^p(U, P|U)$

We also have a canonical long eaxct sequence of Čech cohomology groups corresponding to any given short exact sequence of presheaves , $0 \to P' \to P \to P'' \to 0$ and a covering $\mathfrak{U}$ of X. The sequence

$$0 \to C^p(\mathfrak{U}, P') \to C^p(\mathfrak{U}, P) \to C^p(\mathfrak{U}, P'') \to 0$$

is exact for all p, being a product of exact sequence of abelian groups. This gives us a long exact sequence of complexes

$$0 \to C^{\cdot}(\mathfrak{U}, P') \to C^{\cdot}(\mathfrak{U}, P) \to C^{\cdot}(\mathfrak{U}, P'') \to 0,$$

This gives rise to the connecting homomorphisms $\delta : \check{H}^p(\mathfrak{U}, P) \to \check{H}^{p+1}$ that satisfy the following properties

- The δ-s gives rise to a long exact sequence of cohomology groups

$$0 \to \check{H}^0(\mathfrak{U}, P') \to \cdots \to \check{H}^p(\mathfrak{U}, P) \to \check{H}^p(\mathfrak{U}, P'') \xrightarrow{\delta^i} \check{H}^{p+1}(\mathfrak{U}, P') \to \cdots.$$

- For a morphism of the short exact sequence of presheaves into another, $0 \to Q' \to Q \to Q'' \to 0$, we have the following :

$$\begin{array}{ccc} \check{H}^p(\mathfrak{U}, P'') & \xrightarrow{\delta} & \check{H}^{p+1}(\mathfrak{U}, P') \\ \downarrow & & \downarrow \\ \check{H}^p(\mathfrak{U}, Q'') & \xrightarrow{\delta} & \check{H}^{p+1}(\mathfrak{U}, Q') \end{array}$$

This, shows that $\check{H}^p(\mathfrak{U}, -)$ is actually an exact δ-functor. Similarly $\check{H}^p(\mathfrak{U}/U, -)$ is also a δ-functor.

Since we are working with abelian groups, we can also pass on to the direct limit over all coverings over X (strictly speaking, over $\mathcal{I}_X$). The exactness is preserved there and so we also have another long exact sequence :

$$0 \to \check{H}^0(U, P') \to \cdots \check{H}^p(U, P) \to \check{H}^p(U, P'') \to \check{H}^{p+1}(U, P') \to \cdots$$

Theorem 3.2.2. *The functors $\check{H}^p(\mathfrak{U}/U, -)$ ($p > 0$), are effaceable .*

Proof. . We want to show here that $\check{H}^p(\mathfrak{U}/U, P) = 0$ for any injective presheaf P. So, we wish to show that the cochain complex

$$\prod P(U_i) \xrightarrow{d^1} \prod P(U_{i_0 i_1}) \xrightarrow{d^2} \cdots$$

is exact. For a given $W \to U$ in $\mathbf{C}/X$. Consider the presheaf $\mathbb{Z}_W$ on X which we have defined previously. (see remark at the end of chapter 2, our $\mathbb{Z}_W$ here corresponds to $j_!\mathbb{Z}$ there.)
Recall, that it satisfies $\mathrm{Hom}(\mathbb{Z}_W, P') = P'(W)$ for any presheaf $P' \in \mathbf{P}(X)$, and $\mathbb{Z}_W(V) = \bigoplus_{\mathrm{Hom}_X(V,W)} \mathbb{Z}$. Thus, we shall show that

$$\prod Hom(\mathbb{Z}_{U_i}, P) \to \prod Hom(\mathbb{Z}_{U_{i_0 i_1}, P}) \to \cdots$$

=

$$Hom(\bigoplus \mathbb{Z}_{U_i}, P) \to Hom(\bigoplus \mathbb{Z}_{U_{i_0 i_1}}, P) \to \cdots \text{is exact}$$

As, P is an injective presheaf, so it is enough to show that

$$\bigoplus \mathbb{Z}_{U_i} \leftarrow \bigoplus \mathbb{Z}_{U_{i_0 i_1}} \leftarrow \cdots$$

is an exact sequence of presheaves (note here that the maps are determined by the maps of the previous line), i.e. we must show that for each $V \in \mathbf{C}/X$.

$$(*) \qquad \bigoplus \mathbb{Z}_{U_i}(V) \leftarrow \bigoplus \mathbb{Z}_{U_{i_0 i_1}}(v) \leftarrow \cdots$$

.

Further we have that

$$Hom_X(V, U_{i_0 \dots i_p}) = \bigsqcup_{\phi \in Hom_X(V,U)} (Hom_\phi(V, U_{i_0}) \times \dots\dots \times Hom_\phi(V, U_{i_0 \dots i_p}))$$

Denote $S(\phi) = \sqcup_i Hom_X(V, U_i)$. Then, we get

$$\bigsqcup_{i_0 \dots i_p} Hom_X(V, U_{i_0 \dots i_p}) = \bigsqcup_{\phi \in Hom_X(V,U)} (S(\phi) \times \dots \times S(\phi))$$

.

So, from the definitions, $\bigoplus \mathbb{Z}_{U_{i_0 \dots i_p}}(V)$ is the free abelian group on $\bigsqcup_{\phi \in Hom_X(V,U)}(S(\phi) \times \dots \times S(\phi))$. So, (*) can now be written as

$$\bigoplus_{\phi \in Hom_X(V,U)} (\bigoplus_{S(\phi)} \mathbb{Z} \leftarrow \bigoplus_{S(\phi) \times S(\phi)} \mathbb{Z} \leftarrow \dots)$$

.

But, the complex inside the bracket is homotopically trivial, i.e. we have a contracting homotopy given by

$$n_{i_0 \dots i_p} \mapsto n_{s, i_0 \dots i_p}$$

Where s is any fixed element of $S(\phi)$ and $n_{i_0 \dots i_p}$ means n in the $(i_0 \dots i_p)$-th component of $\bigoplus_{S(\phi)^{p+1}} \mathbb{Z}$. Presence of contracting homotopy implies the sequence is exact. □

Corollary 3.2.3. The functors $\check{H}^p(\mathfrak{U}/U, -)$ are the derived functors for H^0 .

Proof. We have already seen that $\check{H}^p(\mathfrak{U}/U,-)$ $(p \geq 0)$ form a δ-functor, thus in the light of corollary 3.1.5 , we get our result. □

Corollary 3.2.4. In the case of sheaves the functors $\check{H}^p(X,-)$ coincides with $H^p(X,-)$, iff for any short exact sequence of sheaves there is a functorially associated long exact sequence of Čech cohomology groups. For example if for any surjection of sheaves $F \to F''$, the map $\varinjlim_{\mathfrak{U}} \prod F(U_{i_0 \dots i_p}) \to \varinjlim_{\mathfrak{U}} \prod F'(U_{i_0 \dots i_p})$ is surjective (limit is over all coverings $\mathfrak{U}$ of X), then our hypothesis is satisfied.

Proof. . $(\Rightarrow)$ This just follows from the definition of $H^p(X,-)$.
$(\Leftarrow)$ Let F be an injective sheaf . Take a short exact sequence of presheaves

$$0 \to M' \to M \to M'' \to 0$$

As $a : \mathbf{P}(X_{et}) \to \mathbf{S}(X_{et})$ is an exact functor,so

$$0 \to aM' \to aM \to aM'' \to 0 \qquad \text{is exact}$$

Now, from the isomorphism $\mathrm{Hom}(P, iF) \approx \mathrm{Hom}(aP, F)$ we have

$$0 \to Hom(M'', iF) \to Hom(M, iF) \to Hom(M', iF) \to 0$$

is exact . Thus, an injective object in the category $\mathbf{S}(X_{et})$ is also injective in $\mathbf{P}(X_{et})$.
Now we already know that $H^0(X,-) = \check{H}^0(X,-)$. Thus the hypothesis of the theorem holds in this case for the category $\mathbf{S}(X_{et})$. Hence $H^p(X,-) = \check{H}^p(X,-)$.
For the last statement , if we are given a short exact sequence of sheaves $0 \to F' \to F \to F'' \to 0$, by the given condition we have

$$0 \to C^p(\mathfrak{U}, P') \to C^p(\mathfrak{U}, P) \to C^p(\mathfrak{U}, P'') \to 0$$

is exact. Then we consider the exact sequence for the corresponding complex and get a long exact sequence of Čech cohomology groups. Passing onto the direct limit, we see that our hypothesis is satisfied. □

Corollary 3.2.5. Let $U \to X$ be in $\mathbf{C}/X$; let $\mathfrak{U}$ be a covering of U, and let F be a sheaf on X_E. There are spectral sequences

$$\check{H}^p(\mathfrak{U}/U, \underline{H}^q(F)) \Rightarrow H^{p+q}(U, F),$$

$$\check{H}^p(U, \underline{H}^q(F)) \Rightarrow H^{p+q}(U, F).$$

Proof. : Here we want to use theorem 3.1.10,to the pair of functors $\check{H}^0(\mathfrak{U}/U, -)$, $i : \mathbf{S}(X_E) \to \mathbf{P}(X_E)$ and $\check{H}^0(U, -)$, i. Now, we know that $\check{H}^0(\mathfrak{U}/U, P) \cong P(U)$ if P is any presheaf. So, from the definitions we have

$$\check{H}^0(\mathfrak{U}/U, iF) \ = \ H^0(U, F) \ = \ \check{H}^0(U, iF)$$

As i preserves injectives we want to check if injective presheaves are acyclic for $\check{H}^0(\mathfrak{U}/U, -)$ and $\check{H}^0(U, -)$. This follows form the theorem.

□

Now we will examine a case where čech cohomology group and étale cohomolgy groups coincide. In what follows we shall assume that X is a quasi-compact scheme, such that every finite subset of X is contained in an open affine subset . For, example we might consider a quasi-projective scheme over an affine scheme.

We need the following lemma

Lemma 3.2.6. Let A be a ring, $\mathfrak{p}_1, ..., \mathfrak{p}_r$ be prime ideals of A, and $A^{sh}_{\mathfrak{p}_1}, ..., A^{sh}_{\mathfrak{p}_r}$ denote the strict heneselizations of the local rings $A_{\mathfrak{p}_1}, ..., A_{\mathfrak{p}_r}$. Then, every map $A^{sh}_{\mathfrak{p}_1} \otimes_A .... \otimes_A A^{sh}_{\mathfrak{p}_r} \longrightarrow B$, which is faithfully flat and étale has a section , $B \longrightarrow A^{sh}_{\mathfrak{p}_1} \otimes_A .... \otimes_A A^{sh}_{\mathfrak{p}_r}$.

Proof : [10],

Theorem 3.2.7. *Let X be as above and let F be a sheaf on X_{et}. Then, we have canonical isomorphisms $\check{H}^p(X_{et}, F) \cong H^p(X_{et}, F)$ for all p.*

To prove this we need to prove another lemma .But, first we fix some notations. If Y is a $X - scheme$ then Y^n will denote the $n - th$ fibered power over X. Given an r tuple of geometric points $(p) = (p_1, ...p_r)$ of X (not necessarily distinct), we will denote by $X_{(p)}$ the strict localizations X_{p_i} of X at p_i.

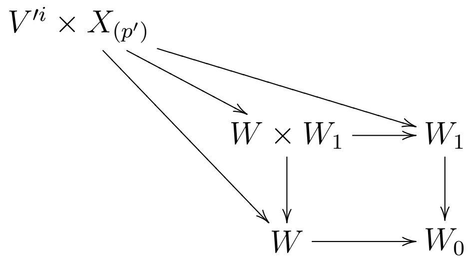
Lemma 3.2.8. Let X be as above in the theorem. Let $U \to X$ be étale and of finte type, and let $p_1, ..., p_r$ be geometric points of X. Let $W \to U^n \times X_(p)$ be an étale cover. Then there exists an étale surjective map $U' \to U$, such that the canonical $X - morphism$

$$U'^n \times X_{(p)} \to U^n \times X_{(p)}$$

factors through W.

Proof. For $n \geq 1$ we proceed by induction. Let $V \to U$ be étale and of finite type, but not necessarily surjective, and let the map $V^n \times X_{(p)} \to U^n \times X_{(p)}$ factor through W, (we might take V to be empty for instance). Let q be a geometric point of U which is not covered by V, note that $X_q \approx U_q$ (since, $U \to X$ is étale). Now, $(V \sqcup X_q)^n$ is a sum of schemes of the form $V^i \times X_q^j$ $(i + j = n)$. Now, we have a canonical map $V^i \times X_q^j \times X_{(p)} \to U^i \times U^j \times X_{(p)} = U^n \times X_{(p)}$ (since we have canonically $X_q^j = U_q^j \to U^j$). Deonte $V^i \times X_q^j \times X_{(p)}$ by W_1 and $U^n \times X_{(p)}$ by W_0. Then the pullback $W \times_{W_0} W_1$, of W to $V^i \times X_q^j \times X_{(p)}$ satisfies the original hypothesis on W. So, by induction we see that for $i < n$, we have V' an étale cover of V such that , $V'^i \times X_{(p')} \to V^i \times X_{(p')}$, factors through $W \times_{W_0} W_1$, where (p') denotes the $r + j$-tuple $(q, ...q, p_1, ..., p_r)$.

so we have the following diagram :



This means that for $i < n$ if we replace V by this V' then , $W_1 \to W_0$ factors through W. For $i = n$, the map factors by assumption. Thus we may assume that $(V \sqcup X_q)^n \times X_{(p)} \to U^n \times X(p)$ factors through W.
Now, by definition $U_q = \text{Spec } \varinjlim \Gamma(Y, \mathcal{O}_Y)$, where the limit is over all étale neighborhoods Y of q. Now, here the affine étale neighborhoods are clearly cofinal in the set of all étale neighborhoods. So, that we may write $U_q = \varprojlim \text{Spec } A_i$ where $Y_i := \text{Spec } A_i$ are affine étale neighborhoods of q. Thus X_q is a limit of schemes Y_i étale over U, then by lemma 2.3.4 and the fact that inverse limits commute with fiber products, we get some Y_i, such that the map $(V \sqcup Y_i)^n \times X_{(p)} \to U^n \times X_{(p)}$ also factors through W. By proceeding in this way we obtain a sequence of maps $V_1 \to U$, $V_2 \to U$,.... whose images in U form a strictly increasing sequence of open subsets of U. As U is a noetherian topological space ,after finitely many steps we obtain a

surjective map.

For the case $n = 0$ we proceed as follows. The images of $p_1, ...p_r$ are contained in some open affine subset $specA$ of X, thus we have

$$X_{(p)} = Spec\,(A^{sh}_{p_1} \otimes_A \cdots A^{sh}_{p_r})$$

. So, by the previous lemma $W \to X_{(p)}$ has a section . □

Now, we have all the ingredients to prove the theorem 2.7. We want to use the spectral sequence of proposition 3.2.5 , i.e $\check{H}^p(X, \underline{H}^q(F)) \Rightarrow H^{p+q}(U, F)$. Now, it follows from proposition 3.1.9 that to show our required result , it enough to show that $E_2^{p,q} := \check{H}^p(X, \underline{H}^q(F)) = 0$, for all $q > 0$. But to show this , it suffices to prove that for every étale cover U of X, and every class $\alpha \in H^q(U^n, F)$, there is a cover $U' \to U$ such that $\alpha \mapsto 0$ in $H^q(U'^n, F)$. Since cohomology is locally trivial, there is a cover $W \to U^n$ for which α is zero in $H^q(W, F)$. Then, apply the theorem with no points .

Alternatively, we could use corollary 3.2.4. Let $F \to F''$ be a surjective map of sheaves in $\mathbf{S}(X_{et})$ and let Y be in X_{et}. Note that since X and Y are both quasi-compact so the coverings $U \to Y$ consisting of single morphism is cofinal in the set of all coverings of Y. Let $s'' \in F''(U^n)$. By the surjectivity of sheaves, we have an étale covering $W \to U^n$ and a $s \in F(W)$ such that $s \mapsto s''|W$. Thus, according to our previous lemma , we have a covering $U' \to U$ such that $U'^n \to U^n$ factors through W. So, $s|U'^n \mapsto s''|U'^n$, then from corollary 2.4. our result follows.

■

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