Flasque resolutions of algebraic tori

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0 Introduction

Algebraic tori is a very special type of algebraic groups. Given a field k, it is well-known that the isomorphism classes of k-tori are classified by their character groups viewed as $\mathfrak{g}$-modules, where $\mathfrak{g}$ is the Galois group of the algebraic separable closure of k over k. In this article, we will study another birational invariant: the flaseque $\mathfrak{g}$-module associated to a k-torus $\mathbf{T}$, or more precisely, the "similarity class" of the flasque module associated to $\mathbf{T}$, and we denote it as $\rho(\mathbf{T})$. This invariant was introduced in Colliot-Thélène and Sansuc's paper [5], and also in Voskresenskiĭ's book [15].

Let $\mathbf{X}$ be a smooth k-compactification of $\mathbf{T}$. Note that by Brylinski and Künnemann's Theorem [4], for every k-torus $\mathbf{T}$, such an $\mathbf{X}$ always exists. We will show that $\rho(\mathbf{T})$ is just the similarity class of Pic$\overline{\mathbf{X}}$. Moreover, $\rho(\mathbf{T})$ characterizes the stably k-equivalent class of $\mathbf{T}$.

The birational invariant $\rho(\mathbf{T})$ is important on some arithmetical topics. Although I won't go into these topics in this article, I still want to mention a few. First, $\rho(\mathbf{T})$ can be used to compute the R-equivalence classes of $\mathbf{T}$. One can refer to Colliot-Thélène and Sansuc's paper [5] for a complete introduction of this topic. Second, $\rho(\mathbf{T})$ also has its application on weak approximation problem of k-tori. For this part, one can look up the details in Voskresenskiĭ's book [15], sec. 11.6.

Fix a profinite group G. In the first section, we introduce the definition and basic properties of flasque G-modules. At the end of section 1.1, we will see that actually we can reduce our case to G is finite. Next, we'll show that for each G-lattice M, we can associate a similarity class of flasque module to it. We will discuss some special properties of the semigroup formed by the similarity classes of flasque modules while We end the first section with a concrete example of flasque.

In the second section, we first recall Rosenlicht's Unit Theorem and some other ingredients, which together play important roles in defining $\rho(\mathbf{T})$. Then we define $\rho(\mathbf{T})$ precisely. From our definition, it will be easy to see $\rho(\mathbf{T})$ is a birational invariant. Furthermore, we will show it really characterizes the stably k-equivalence class of $\mathbf{T}$. After that, we provide two examples to demonstrate the link between algebraic and geometric point of view. We will see that those algebraic properties developed in the first section can really simplify the calculations in the geometric aspect.

In the last section, we write down in detail some interesting techniques which are used in the previous section.

The main idea and most proofs in this article are adapted from Colliot-Thélène and Sansuc's paper [5]. There are lots useful references on this topic. For algebraic techniques, one can refer to [1], [3], and [14]; for geometric background, see [9], [10]. Other references will be mentioned while we need them.

1 Flasque resolutions of G-modules

1.1 Permutation modules and flasque modules

Let $\mathfrak{g}$ be a profinite group, and $\mathcal{L}_{\mathfrak{g}}$ be the category of all free $\mathbb{Z}$-modules of finite rank equipped with a continuous left $\mathfrak{g}$-action. In this section, all the modules are assumed in $\mathcal{L}_{\mathfrak{g}}$. Let M, N be two modules in $\mathcal{L}_{\mathfrak{g}}$. Define $\mathrm{Hom}(\mathrm{M}, \mathrm{N})$ to be the $\mathfrak{g}$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathrm{N})$ with $\mathfrak{g}$-action defined as: $\sigma \circ f = \sigma f \sigma^{-1}$, for any $\sigma \in \mathfrak{g}$, $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathrm{N})$. Define $\mathrm{M} \otimes \mathrm{N}$ to be $\mathrm{M} \otimes_{\mathbb{Z}} \mathrm{N}$ with $\mathfrak{g}$-action defined as: $\sigma(m \otimes n) = \sigma(m) \otimes \sigma(n)$, for any $\sigma \in \mathfrak{g}, \mathrm{m} \otimes \mathrm{n} \in \mathrm{M} \otimes_{\mathbb{Z}} \mathrm{N}$. Define M° to be the dual $\mathfrak{g}$-module $\mathrm{Hom}(\mathrm{M}, \mathbb{Z})$. If $\mathfrak{h}$ is an open subgroup of $\mathfrak{g}$, then we define the augmentation $\varepsilon_{\mathfrak{g}/\mathfrak{h}} : \mathbb{Z}[\mathfrak{g}/\mathfrak{h}] \to \mathbb{Z}$ as $\varepsilon_{\mathfrak{g}/\mathfrak{h}}(\sum_{\mathfrak{g}/\mathfrak{h}} n_\sigma \sigma\mathfrak{h}) = \sum n_\sigma$, and the norm $\mathrm{Nr}_{\mathfrak{g}/\mathfrak{h}} : \mathbb{Z} \to \mathbb{Z}[\mathfrak{g}/\mathfrak{h}]$ as $\mathrm{Nr}_{\mathfrak{g}/\mathfrak{h}}(1) = \sum_{\mathfrak{g}/\mathfrak{h}} \sigma\mathfrak{h}$. We note the kernel of $\varepsilon_{\mathfrak{g}/\mathfrak{h}}$ as $\mathrm{I}_{\mathfrak{g}/\mathfrak{h}}$ and the cokernel of $\mathrm{Nr}_{\mathfrak{g}/\mathfrak{h}}$ as $\mathrm{J}_{\mathfrak{g}/\mathfrak{h}}$. If $\mathfrak{g}$ is finite, then let $\hat{\mathrm{H}}^i(\mathfrak{g}, \mathrm{M})$ denote the i-th Tate cohomology group, for any $i \in \mathbb{Z}$.
A $\mathfrak{g}$-module is called a *permutation module* if it has a $\mathbb{Z}$-base permutated by $\mathfrak{g}$. It is easy to verify that for a permutation $\mathfrak{g}$-module P, $\mathrm{P} \simeq \mathrm{P}^{\circ}$. Besides, if $\mathfrak{g}$ is finite, then a projective permutation $\mathfrak{g}$-module is $\mathbb{Z}[\mathfrak{g}]$-free. A $\mathfrak{g}$-module is said to be *invertible* if it is a direct summand of a permutation module.

Let $\mathfrak{g}$ be a profinite group, and take M, N in $\mathcal{L}_{\mathfrak{g}}$. Then M, N are said to be *similar* if there are permutation modules $\mathrm{P}_1, \mathrm{P}_2$ such that $\mathrm{M} \oplus \mathrm{P}_1 = \mathrm{N} \oplus \mathrm{P}_2$, and we note them as $\mathrm{M} \sim \mathrm{N}$. We then define $\mathrm{S}_{\mathfrak{g}} = \mathcal{L}_{\mathfrak{g}}/\sim$, and note the similarity class of M as [M]. A $\mathfrak{g}$-module M is said to be a *stably permutation module* if [M] = [0].

Remark 1.1. A stably permutation module is not necessarily a permutation module. We put an example in the end of section 1.

Let G be a finite group. Take i in $\mathbb{Z}$. A G-module M is said to be $\hat{\mathrm{H}}^i$-trivial if $\hat{\mathrm{H}}^i(\mathrm{G}', \mathrm{M}) = 0$ for any arbitrary subgroup $\mathrm{G}' \subseteq \mathrm{G}$, especially we call an $\hat{\mathrm{H}}^{-1}$-trivial module a *flasque* module and an $\hat{\mathrm{H}}^1$-trivial module is called a *coflasque*. A *flasque resolution* of a G-module M is an exact sequence of G-modules: $0 \to \mathrm{M} \to \mathrm{P} \to \mathrm{F} \to 0$, where P is a permutation module and F is a flasque module. A *coflasque resolution* of M is an exact sequence: $0 \to \mathrm{Q} \to \mathrm{R} \to \mathrm{M} \to 0$, where R is a permutation module and Q is a coflasque module. Let F_{G} (resp. $\mathrm{F}^{\circ}_{\mathrm{G}}$) denote the submonoid of S_{G}

consisting of all the similarity classes of flasque (resp. coflasque) modules, and $\mathrm{U_G}$ denote all the similarity classes of invertible modules in $\mathrm{S_G}$. Let $\mathrm{F^1_G} = \mathrm{F_G}/\mathrm{F_G} \cap \mathrm{F^\circ_G}$.

For a profinite group $\mathfrak{g}$, and $\mathrm{M} \in \mathcal{L}_{\mathfrak{g}}$, we can extend the definition of $\hat{\mathrm{H}}^{-1}(\mathfrak{g}, \mathrm{M})$ as $\hat{\mathrm{H}}^{-1}(\mathfrak{g}, \mathrm{M}) = \varinjlim_{\mathfrak{h}} \hat{\mathrm{H}}^{-1}(\mathfrak{g}/\mathfrak{h}, \mathrm{M}^{\mathfrak{h}})$, where $\mathfrak{h}$ ranges over all open normal subgroups of $\mathfrak{g}$. Let $\mathfrak{h}$ be an open normal subgroup of $\mathfrak{g}$ acting trivially on M. Clearly, $\hat{\mathrm{H}}^{-1}(\mathfrak{g}, \mathrm{M}) = \hat{\mathrm{H}}^{-1}(\mathfrak{g}/\mathfrak{h}, \mathrm{M})$ by the above definition. We also modify the definitions of coflasque (resp. flasque) modules as $\hat{\mathrm{H}}^{-1}(\mathfrak{h}, \mathrm{M}) = 0$, (resp. $\hat{\mathrm{H}}^{1}(\mathfrak{h}, \mathrm{M}) = 0$) for all open subgroups $\mathfrak{h}$ of $\mathfrak{g}$.

Now let G be a profinite group. We will show that to find a coflasque (resp. flasque) resolution of $\mathrm{M} \in \mathcal{L}_{\mathrm{G}}$ can be reduced to the case for G is finite by the following three lemmas.

Lemma 1.2. *Take* $\mathrm{M} \in \mathcal{L}_{\mathrm{G}}$. *Then the following are equivalent:*
(1)M *is flasque.*
(2)M° *is coflasque.*
(3)$\hat{\mathrm{H}}^{-1}(\mathrm{G}', \mathrm{M}) = 0$, *for all open subgroups* $\mathrm{G}' \subseteq \mathrm{G}$.
(4)$\hat{\mathrm{H}}^{1}(\mathrm{G}', \mathrm{M}^\circ) = 0$, *for all open subgroups* $\mathrm{G}' \subseteq \mathrm{G}$.
(5)$\hat{\mathrm{H}}^{1}(\mathrm{G}, \mathrm{Hom}(\mathrm{M}, \mathrm{P})) = 0$, *for all permutation modules* P.
(6)$\hat{\mathrm{H}}^{1}(\mathrm{G}, \mathrm{M}^\circ \otimes \mathrm{P}) = 0$, *for all permutation modules* P.
(7) $\mathrm{Ext}^1_{\mathrm{G}}(\mathrm{M}, \mathrm{P}) = 0$, *for all permutation modules* P.
(8) $\mathrm{Ext}^1_{\mathrm{G}}(\mathrm{P}, \mathrm{M}^\circ) = 0$, *for all permutation modules* P.

Proof. First, let H be an open normal subgroup of G which fixes M. Then $\hat{\mathrm{H}}^{-i}(\mathrm{G}', \mathrm{M}) = \hat{\mathrm{H}}^{-i}(\mathrm{G}'/(\mathrm{G}' \cap \mathrm{H}), \mathrm{M})$. To see that (1),(2) are equivalent, we just note $\hat{\mathrm{H}}^{-i}(\mathrm{G}'/(\mathrm{G}' \cap \mathrm{H}), \mathrm{M})^* = \hat{\mathrm{H}}^{i}(\mathrm{G}'/(\mathrm{G}' \cap \mathrm{H}), \mathrm{M}^\circ)$, for all $i \in \mathbb{Z}$, where we denote $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) = A^*$ for a G-module A. So (1), (2), (3) and (4) are equivalent just by definition. To see (4) is equivalent to (5), it is enough to prove for $\mathrm{P} = \mathbb{Z}[\mathrm{G}/\mathrm{H}]$ for some arbitrary open subgroup $\mathrm{H} \subseteq \mathrm{G}$. Then $\mathrm{Hom}(\mathrm{M}, \mathrm{P}) \simeq \mathrm{Hom}(\mathrm{P}, \mathrm{M}^\circ) = \mathrm{Hom}(\mathbb{Z}[\mathrm{G}/\mathrm{H}], \mathrm{M}^\circ)) = \mathrm{Coind}^{\mathrm{H}}_{\mathrm{G}}\mathrm{M}^\circ$, and by Shapiro's Lemma, $\hat{\mathrm{H}}^{1}(\mathrm{G}, \mathrm{Hom}(\mathrm{M}, \mathrm{P})) = \hat{\mathrm{H}}^{1}(\mathrm{H}, \mathrm{M}^\circ)$, So (4), (5) are equivalent. Since $\mathrm{Hom}(\mathrm{M}, \mathrm{P}) \simeq \mathrm{M}^\circ \otimes \mathrm{P}$ as G-modules, it is clear that (5), (6) are equivalent. Since $\hat{\mathrm{H}}^{1}(\mathrm{G}, \mathrm{Hom}(\mathrm{M}, \mathrm{P})) = \mathrm{Ext}^1_{\mathrm{G}}(\mathbb{Z}, \mathrm{Hom}(\mathrm{M}, \mathrm{P})) = \mathrm{Ext}^1_{\mathrm{G}}(\mathbb{Z} \otimes \mathrm{M}, \mathrm{P})$, (5) and (7) are equivalent. For (5) is equivalent to (8), we just note $\mathrm{Hom}(\mathrm{M}, \mathrm{P}) = \mathrm{Hom}(\mathrm{P}, \mathrm{M}^\circ)$, for all permutation modules P. □

Remark 1.3. Note that we can replace the permutation modules in Lemma 1.2(5)-(8) by invertible modules.

Lemma 1.4. *Take a normal closed subgroup* H *of* G *and* $\mathrm{M} \in \mathcal{L}_{\mathrm{G}}$. *Then we have the following statements:*

(1) *If* M *is a permutataion module in* $\mathcal{L}_{\mathrm{G}}$*, then* M^{H} *is also a permutation module in* $\mathcal{L}_{\mathrm{G/H}}$.
(2) *If* M *is* G-*coflasque, then* M^{H} *is* G/H-*coflasque.*
(3) *If* M *has a coflasque resolution:* $0 \longrightarrow \mathrm{Q} \longrightarrow \mathrm{P} \longrightarrow \mathrm{M} \longrightarrow 0$ *, then* $0 \longrightarrow \mathrm{Q}^{\mathrm{H}} \longrightarrow \mathrm{P}^{\mathrm{H}} \longrightarrow \mathrm{M}^{\mathrm{H}} \longrightarrow 0$ *is a coflasque resolution of* M^{H} *in* $\mathcal{L}_{\mathrm{G/H}}$.

Proof. To prove (1), let's take $\mathrm{M} = \mathbb{Z}[\mathrm{G/H_0}]$, where $\mathrm{H_0}$ is an open subgroup. Since H is normal, $\mathrm{HH_0}$ is an open subgroup and $[\mathrm{HH_0} : \mathrm{H_0}] = [\mathrm{H} : \mathrm{H} \cap \mathrm{H_0}]$. Then, $\mathrm{G} = \bigcup s_i \mathrm{HH_0} = \bigcup s_i (\bigcup t_j \mathrm{H_0})$, and we have $\{s_i \sum t_j \mathrm{H_0}\}$ as a basis of M^{H} permuted by G/H. We can see (2) by the inflation map. And we easily get (3) by (1) and (2). □

Lemma 1.5. *Let* H, G *as defined above. Take* $\mathrm{M}, \mathrm{N} \in \mathcal{L}_{\mathrm{G/H}}$. *It is equivalent for* M, N *to possess properties* $\mathcal{P}$ *in* $\mathcal{L}_{\mathrm{G}}$ *and in* $\mathcal{L}_{\mathrm{G/H}}$ *for the following properties* $\mathcal{P}$*:*
(1) M *is a permutation module.*
(2) M *is a stably permutation module.*
(3) M *is invertible.*
(4) M, N *are similar.*
(5) M *is flasque.*
(6) M *is coflasque.*

Proof. From Lemma 1.4, it is clear that to possess properties (1), (2), (3), (4) in $\mathcal{L}_{\mathrm{G}}$ is equivalent to posses them in $\mathcal{L}_{\mathrm{G/H}}$. For the property (6), since $\mathrm{H}^1(\mathrm{H}, \mathrm{M}) = 0$, the inflation map is an isomorphism between $\mathrm{H}^1(\mathrm{G/H}, \mathrm{M})$ and $\mathrm{H}^1(\mathrm{G}, \mathrm{M})$, so it is equivalent to possess property (5) in $\mathcal{L}_{\mathrm{G}}$ and in $\mathcal{L}_{\mathrm{G/H}}$. And we can get (5) from (6) by duality. □

With Lemma 1.5, we can get the following corollary easily.

Corollary 1.6. *Take a closed normal subgroup* H *of* G. *Then* $\mathrm{S_{G/H}}$ *is a submonoid of* $\mathrm{S_G}$ *and* $\mathrm{U_{G/H}} = \mathrm{U_G} \cap \mathrm{S_{G/H}}$*;* $\mathrm{F_{G/H}} = \mathrm{F_G} \cap \mathrm{S_{G/H}}$*;* $\mathrm{F^{\circ}_{G/H}} = \mathrm{F_G} \cap \mathrm{S_{G/H}}$

Instead to find a coflasque (resp. flasque) resolution of M in $\mathcal{L}_{\mathrm{G}}$, Lemma 1.5 allows us just to find a coflasque (resp. flasque) resolution of M in $\mathcal{L}_{\mathrm{G/H}}$, where H is an open normal subgroup of G acting trivially on M. So actually, we only need to know how to find a coflasque (resp. flasque) resolution in $\mathcal{L}_{\mathrm{G}}$ under the assumption that G is finite.

Remark 1.7. Let H be a closed subgroup of G and consider the restriction map $\mathrm{Res}^{\mathrm{H}}_{\mathrm{G}}$: $\mathcal{L}_{\mathrm{G}} \to \mathcal{L}_{\mathrm{H}}$. Then a G-module M satisfying the properties in

Lemma 1.5 or the property $\mathcal{I}$: M is a $\hat{\mathrm{H}}^i$-trivial module in $\mathcal{L}_{\mathrm{G}}$ also satisfies those properties in $\mathcal{L}_{\mathrm{H}}$. (Here $i \in \mathbb{Z}$ for G is finite and $i \geq -1$ for G is profinite.) On the other hand, suppose H is of finite index in G. Then we have the coinduced map $\mathrm{Coind}_{\mathrm{G}}^{\mathrm{H}}$: $\mathcal{L}_{\mathrm{H}} \to \mathcal{L}_{\mathrm{G}}$. If an H-module N satisfies properties in Lemma 1.5, then $\mathrm{Coind}_{\mathrm{G}}^{\mathrm{H}}\mathrm{N}$ also satisfies those properties in $\mathcal{L}_{\mathrm{G}}$. Moreover, by Shapiro's Lemma, if N satisfies property $\mathcal{I}$, then $\mathrm{Coind}_{\mathrm{G}}^{\mathrm{H}}$ also satisfies properties $\mathcal{I}$ in $\mathcal{L}_{\mathrm{G}}$. Since N is a direct summand in $\mathrm{Res}_{\mathrm{G}}^{\mathrm{H}}\mathrm{Coind}_{\mathrm{G}}^{\mathrm{H}}(\mathrm{N})$, if $\mathrm{Coind}_{\mathrm{G}}^{\mathrm{H}}(\mathrm{N})$ is a flasque (resp. coflasque) G-module, then N is a flasque (resp. coflasque) H-module.

1.2 Some properties of flasque resolutions

We begin this section by proving the existence of flasque resolutions in $\mathcal{L}_{\mathrm{G}}$.

Lemma 1.8. *Every* G*-module admits a flasque resolution and a coflasque resolution.*

Proof. Since we can find a flasque resolution of M by taking a dual sequence of a coflasque resolution of M°, it is enough to show that every G-module admits a coflasque resolution. Moreover, by Lemma 1.5, it is enough to find a coflasque resolution of M in $\mathcal{L}_{\mathrm{G/H}}$, where H is an open normal subgroup acting trivially on M, so we can assume G is finite. If we can find a permutation module P with a surjective G-homomorphism j onto M such that $\mathrm{P}^{\mathrm{G}'}$ is still surjective to $\mathrm{M}^{\mathrm{G}'}$ for all subgroups $\mathrm{G}' \subseteq \mathrm{G}$, then the kernel of j is coflasque, and we get a coflasque resolution of M. So we only need to construct such a P and j. Let's take $\mathrm{P} = \oplus_{\mathrm{G}'}\mathbb{Z}[\mathrm{G}/\mathrm{G}'] \otimes \mathrm{M}^{\mathrm{G}'}$, where G' ranges over all the subgroups of G and we define the G-action on $\mathbb{Z}[\mathrm{G}/\mathrm{G}'] \otimes \mathrm{M}^{\mathrm{G}'}$ as $\sigma(g\mathrm{G}' \otimes m) = \sigma(g\mathrm{G}') \otimes m$, for all g, $\sigma \in \mathrm{G}$, $m \in \mathrm{M}^{\mathrm{G}'}$, and we define j as $j(g\mathrm{G}' \otimes m) = gm$. Then it is easy check such a P, j satisfying our requirement. □

Lemma 1.9. *Take a flasque resolution of a* G*-module* M*:* $0 \longrightarrow \mathrm{M} \xrightarrow{i} \mathrm{P} \longrightarrow \mathrm{F} \longrightarrow 0$ *, and a coflasque resolution of* M*:* $0 \longrightarrow \mathrm{Q} \longrightarrow \mathrm{R} \xrightarrow{j} \mathrm{M} \longrightarrow 0$ *. Let* P_0 *be a permutation* G*-module. Then every morphism* α*:* $\mathrm{M} \to \mathrm{P}_0$ *factorizes through* i*, and every morphism* β*:* $\mathrm{P}_0 \to \mathrm{M}$ *factorizes through* j*.*

Proof. Let's prove the statement for a coflasque resolution. Let $\mathrm{P}_0 \times_{\mathrm{M}} \mathrm{R}$ be the pullback of P_0, R over M, i.e. $\mathrm{P}_0 \times_{\mathrm{M}} \mathrm{R} := \{(a, b) \in \mathrm{P}_0 \times \mathrm{R} | \ \beta(a) = j(b)\}$. Then we have the following exact sequence:

$$0 \to \mathrm{Q} \to \mathrm{P}_0 \times_{\mathrm{M}} \mathrm{R} \to \mathrm{P}_0 \to 0 \,.$$

Since Q is coflasque, by Lemma 1.2(8), $\mathrm{P}_0 \times_{\mathrm{M}} \mathrm{R} \simeq \mathrm{Q} \oplus \mathrm{P}_0$. Hence β factorizes through j. For the case for flasque resolutions, we just take the dual sequences, and then the results follow. □

Lemma 1.10. *Take a flasque resolution of a* G*-module* M*:* $0 \longrightarrow M \longrightarrow P \longrightarrow F \longrightarrow 0$ *, and a coflasque resolution:* $0 \longrightarrow Q \longrightarrow R \longrightarrow M \longrightarrow 0$ *. The similarity classes of* F *and* Q *don't depend on the resolutions of* M*, so we denote them as* $\rho(M)$*,* $\varsigma(M)$ *respectively. Moreover, it is well-defined over the similarity class of* M*.*

Proof. It is enough to prove the coflasque case. Let $0 \longrightarrow Q' \longrightarrow R' \longrightarrow M \longrightarrow 0$ be another coflasque resolution. Take $N = R \times_M R'$. Then since Q, Q' are coflasque, we have $Q \oplus R' \simeq N \simeq Q' \oplus R$. So [Q] is independent of the coflasque resolution. And clearly, $0 \longrightarrow Q \longrightarrow R \oplus R' \longrightarrow M \oplus R' \longrightarrow 0$ is a coflasque resolution of $M \oplus R'$, for any permutation module R', so $\varsigma(M)$ is invariant over the similarity class of M. □

Lemma 1.11. *The applications of* ρ *and* ς *are additive and dual to each other in the following sense:* $\rho(M)^\circ = \varsigma(M^\circ)$*. The application of* ρ *(resp.* ς*) is a surjective map from* S_G *to* F_G *(resp.* F_G°*). The restriction of* ρ *on* F_G° *induces an isomorphism to* F_G *while* ς *restricted on* F_G *is its inverse. Besides,* ρ *and* ς *coincide on* U_G *with* ρ, ς*:* $M \to -M$*.*

Proof. Let F be a flasque module, and $0 \to Q \to P \to F \to 0$ be its coflasque resolution. Then $\rho(Q) = F$, and $\varsigma\rho(Q) = Q$. So ρ is a surjective map onto F_G. By taking flasque resolutions of coflasque modules, we can also show that ς is surjective onto F_G°, and clearly, ρ restricted to F_G° and ς restricted to F_G are inverse to each other. The rest part of this Lemma is obvious, so we omit the proof. □

Lemma 1.12. (1) *Let* $0 \longrightarrow M \longrightarrow N \longrightarrow Q \longrightarrow 0$ *, where* Q *is an invertible module. Then* $\rho(N) = \rho(M) + \rho(Q)$*. In particular, if* Q *is a permutation module, then* $\rho(N) = \rho(M)$*.*
(2) *Let* $0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0$ *be an exact sequence, where* P *is a permutation module. Then* $\rho(M) = \rho(\varsigma(N))$ *and* $\varsigma(N) = \varsigma(\rho(M))$*.*

Proof. Let the exact sequence of G-modules: $0 \to N \to P \to F \to 0$ be a flasque resolution of N. Then we have the following sequence: $0 \to N/M \to P/M \to F \to 0$. Note that $N/M \simeq Q$ is an invertible module. By Lemma 1.2(7), $P/M \simeq Q \oplus F$ and is clearly a flasque module. So $\rho(M) = [P/M] = [Q] \oplus [F] = [Q] \oplus \rho[N]$. This establishes (1).

For (2), let $0 \to Q \to R \to N \to 0$ be a coflasque resolution of N. Then we have the following two exact sequences:

$$0 \to Q \to P \times_N R \to P \to 0$$
$$0 \to M \to P \times_N R \to R \to 0.$$

So by (1), we have $\rho(M) = \rho(P \times_N R) = \rho(Q) = \rho(\varsigma(N))$. By taking the dual sequence and Lemma 1.11, we get the other equality. □

Lemma 1.13. *Take two* G*-modules* M *and* N*. The following conditions are equivalent:*
(1) $\rho(\mathrm{M}) = \rho(\mathrm{N})$*;*
(2) *There are two exact sequences:* $0 \to \mathrm{M} \to \mathrm{E} \to \mathrm{P} \to 0$ *and* $0 \to \mathrm{N} \to \mathrm{E} \to \mathrm{R} \to 0$ *with permutation modules* P, R.

Proof. First, we show (1) implies (2). Since $\rho(\mathrm{M}) = \rho(\mathrm{N})$, we find flasque resolutions: $0 \to \mathrm{M} \to \mathrm{P}_1 \to \mathrm{F} \to 0$, and $0 \to \mathrm{N} \to \mathrm{P}_2 \to \mathrm{F} \to 0$. Let $\mathrm{E} = \mathrm{P}_1 \times_{\mathrm{F}} \mathrm{P}_2$. Then we get (2). Next, if (2) is true, then from Lemma 1.12(1), we have $\rho(\mathrm{M}) = \rho(\mathrm{E}) = \rho(\mathrm{N})$. □

1.3 Flasque G-modules for G is finite

In this subsection, we will give some good properties of F_{G} for G is finite, especially while G is metacyclic, i.e. every Sylow p-subgroup of G is cyclic. These properties will provide a simpler way to calculate the geometric invariants which we will introduce in Section 2. Here we let C_p denote the cyclic group of order p.

Lemma 1.14. *(Lenstra) Take a finite group* G*, and* $\mathrm{M} \in \mathcal{L}_{\mathrm{G}}$*. The following conditions are equivalent:*
(1) M *is invertible;*
(2) M *is an invertible* G_{p}*-module for all Sylow subgroups* G_{p}*;*
(3) $\mathrm{Ext}^1_{\mathrm{G}}(\mathrm{M}, \mathrm{Q}) = 0$*, for all coflasque* G*-module* Q*;*
(4) $\mathrm{Ext}^1_{\mathrm{G}}(\mathrm{F}, \mathrm{M}) = 0$*, for all flasque* G*-module* F*.*

Proof. Obviously, (1) implies (2). For (2) implies (3), we just note $\mathrm{Ext}^1_{\mathrm{G}}(\mathrm{M}, \mathrm{Q})$ is injective into $\oplus_p \mathrm{Ext}^1_{\mathrm{G}_p}(\mathrm{M}, \mathrm{Q})$, where p ranges over distinct primes dividing $|\mathrm{G}|$. Then by Lemma 1.2(8), we obtain (3). We can also obtain (4) from (2) by Lemma 1.2(7). If M satisfied (3), then take a coflasque resolution of M, and we conclude (1). For (4) implies (1), we just take a flasque resolution instead, and conclude (1). □

Proposition 1.15. *Let* G *be a finite group and* ε *be the augmentation* $\mathbb{Z}[\mathrm{G}] \to \mathbb{Z}$*. Then we have the following:*
(1) *For all exact sequences of* G*-modules:* $0 \to \mathrm{Q} \to \mathrm{L} \xrightarrow{\omega} \mathbb{Z}[\mathrm{G}] \xrightarrow{\varepsilon} \mathbb{Z} \to 0\ (*)$*, where* L *is a free* G*-module, we have* $\varsigma(\mathrm{I}_{\mathrm{G}}) = [\mathrm{Q}]$ *and* $\rho(\mathrm{J}_{\mathrm{G}}) = [\mathrm{Q}^\circ]$*.*
(2) *In particular, for all subgroups* G' *of* G*:* $\mathrm{H}^1(\mathrm{G}', \rho(\mathrm{J}_{\mathrm{G}})) = \mathrm{H}^3(\mathrm{G}', \mathbb{Z})$*.*

Proof. First, we note that there is always an exact sequence like $(*)$. For example, we can let $\mathrm{L} = \mathbb{Z}[\mathrm{G}] \times \mathbb{Z}[\mathrm{G}]$ and $\omega(g_0, g_1) = g_0 - g_1$.

We can split $(*)$ into two exact sequences: $0 \to \mathrm{Q} \to \mathrm{L} \to \mathrm{I}_{\mathrm{G}} \to 0$, and $0 \to \mathrm{I}_{\mathrm{G}} \to \mathbb{Z}[\mathrm{G}] \to \mathbb{Z} \to 0$. Since $\hat{\mathrm{H}}^i(\mathrm{G}, \mathrm{L}) = 0$, for all free $\mathbb{Z}[\mathrm{G}]$-modules L, for all $i \in \mathbb{Z}$. We have $\hat{\mathrm{H}}^{i+1}(\mathrm{G},\mathrm{Q}) \simeq \hat{\mathrm{H}}^{i}(\mathrm{G},\mathrm{I_G})$, and $\hat{\mathrm{H}}^{i}(\mathrm{G},\mathrm{I_G}) \simeq \hat{\mathrm{H}}^{i-1}(\mathrm{G},\mathbb{Z})$. Therefore, $\hat{\mathrm{H}}^{i}(\mathrm{G},\mathrm{Q}) \simeq \hat{\mathrm{H}}^{i-2}(\mathrm{G},\mathbb{Z})$. Since $\hat{\mathrm{H}}^{-1}(\mathrm{G},\mathbb{Z}) = 0$, Q is coflasque and $\varsigma(\mathrm{I_G}) = [\mathrm{Q}]$. Since $\mathrm{J_G} = \mathrm{I}_G^{\circ}$, by Lemma 1.11, $\rho(\mathrm{J_G}) = [\mathrm{Q}^{\circ}]$.

Note that a free $\mathbb{Z}[\mathrm{G}]$-module is also a free $\mathbb{Z}[\mathrm{G}']$-module, for all subgroups G' of G. So take the $\mathbb{Z}$-dual sequence of $(*)$, then we get $\hat{\mathrm{H}}^{i}(\mathrm{G}',\mathrm{Q}^{\circ}) \simeq \hat{\mathrm{H}}^{i+2}(\mathrm{G}',\mathbb{Z})$, and (2) follows. □

Example 1.16. Let G be a finite group of order n. Consider the exact sequence: $0 \to \mathrm{I_G}\otimes\mathrm{I_G} \to \mathbb{Z}[\mathrm{G}]\otimes\mathrm{I_G} \to \mathbb{Z}\otimes\mathrm{I_G} \to 0$. Since $\mathbb{Z}[\mathrm{G}]\otimes\mathrm{I_G} \simeq \mathbb{Z}[\mathrm{G}]^{n-1}$ (see [11] p. 59), by Proposition 1.15, $\rho(\mathrm{J_G}) = [(\mathrm{I_G}\otimes\mathrm{I_G})^{\circ}] = [\mathrm{J_G}\otimes\mathrm{J_G}]$.

Corollary 1.17. *Let* G *be a finite group. The following conditions are equivalent:*
(1) *All the Sylow subgroups of* G *are cyclic or generalized quaternion;*
(2) *All the abelian subgroups of* G *are cyclic;*
(3) $\mathrm{H}^3(\mathrm{G}',\mathbb{Z}) = 0$*, for all subgroups* G' *of* G*;*
(4) $\rho(\mathrm{J_G})$ *is coflasque.*

(For the definition of generalized quaternion, see [1] p. 98.)

Proof. The equivalence of (3) and (4) is just Proposition 1.15. For (3) implies (2), we just note $\mathrm{H}^3(\mathrm{C}_p\times\mathrm{C}_p,\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. For (2) ⇒ (1) ⇒ (3), see [3], Chap. XII. □

Corollary 1.18. *Let* G *be a finite group. The following conditions are equivalent:*
(1) G *is metacyclic;*
(2) $\mathrm{F}^1_{\mathrm{G}} = 0$*, i.e. All the flasque* G*-modules are coflasque.*

Proof. If G is cyclic, then G has periodic cohomology with period 2, so all flasque G-modules are coflasque. For G is metacyclic, we just note that $\hat{\mathrm{H}}^{i}(\mathrm{G},\mathrm{M}) \hookrightarrow \oplus_p\hat{\mathrm{H}}^{i}(\mathrm{G}_p,\mathrm{M})$, where p ranges over distinct primes divide $|\mathrm{G}|$, and then conclude a flasque G-module is also coflasque. If all the flasque G-modules are coflasque, in particular, $\rho(\mathrm{J_G})$ is coflasque, then by corollary 1.15, we know that G_p is cyclic for $p \neq 2$. Suppose G_2 is a generalized quaternion which is not cyclic. Then $\mathrm{G}_2/<\pm1>$ contains $\mathrm{V}_4(\simeq \mathrm{C}_2\times\mathrm{C}_2)$ as a subgroup. By Lemma 1.5 and remark 1.7, $\mathrm{F}^1_{\mathrm{V}_4} \hookrightarrow \mathrm{F}^1_{\mathrm{G}_2/<\pm1>} \hookrightarrow \mathrm{F}^1_{\mathrm{G}_2} \hookrightarrow \mathrm{F}^1_{\mathrm{G}}$. Again by corollary 1.17, we have $\mathrm{F}^1_{\mathrm{V}_4} \neq 0$ which contradicts to $\mathrm{F}^1_{\mathrm{G}} = 0$. So G_2 is also cyclic. □

Proposition 1.19. *Take a finite group* G*. The following conditions are equivalent:*
(1) G *is metacyclic;*
(2) $\mathrm{F_G}$ *is a group, i.e. all the flasque modules are invertible;*
(3) $\rho(\mathrm{J}_G) \in \mathrm{U_G}$.

Proof. We first show (1) implies (2). By Lemma 1.14, it is enough to consider the case for G is a p-group. Assume G is a p-group with order p^k and M is a flasque G-module. By corollary 1.18, M is also a coflasque module. We want to prove that M is invertible by induction on $k \geqslant 0$. Assume the Proposition is true for $k < n$. Now, for $k = n$, let $f(x) = (x^{p^n} - 1)/(x^{p^{n-1}} - 1)$ and take a generator $s \in \mathrm{G}$. Consider the map $f(s)$: $\mathrm{M} \xrightarrow{\times f(s)} \mathrm{M}$. Let M' and M'' be the kernel and image respectively. Then M' is a submodule of M annihilated by $f(s)$. Since $f(x)$ and $x^{p^{n-1}} - 1$ are coprime in $\mathbb{Q}[x]$, the elements fixed by $s^{p^{n-1}}$ in M' are $\mathbb{Z}$-torsion elements, but M' is a submodule of M and hence $\mathbb{Z}$-torsion free. So the only element in M' fixed by $s^{p^{n-1}}$ is 0. So the long exact sequence derived from: $0 \longrightarrow \mathrm{M}' \longrightarrow \mathrm{M} \longrightarrow \mathrm{M}'' \longrightarrow 0$ shows M'' is also a flasque G-module. Since M'' is a $\mathrm{G}/\langle s^{p^{n-1}} \rangle$-module, by Lemma 1.5(5), (3) and the induction hypothesis, M'' is an invertible G-module.

Since M is a coflasque module, to prove M is invertible is equivalent to prove $\rho(\mathrm{M})$ is invertible. By Lemma 1.12, $\rho(\mathrm{M}) = \rho(\mathrm{M}') + \rho(\mathrm{M}'')$, since M'' is invertible, it is equivalent to prove $\rho(\mathrm{M}')$ is invertible. If we identify $\mathbb{Z}[\mathrm{G}]$ with $\mathbb{Z}[x]/(x^{p^n} - 1)$, then as M' is annihilated by $f(s)$, M' can be regarded as a $\mathbb{Z}[\lambda]$-module, where λ is a primitive p^n-th root. Since M' is $\mathbb{Z}$-torsion free, it is also $\mathbb{Z}[\lambda]$-torsion free. Since $\mathbb{Z}[\lambda]$ is a Dedekind domain, M' is a projective $\mathbb{Z}[\lambda]$-module. So it is enough to prove $\rho(\mathbb{Z}[\lambda])$ is an invertible G-module. Then

$$0 \longrightarrow \mathbb{Z}[\lambda] \simeq \mathbb{Z}[x]/f(x) \xrightarrow{\times x^{p^{n-1}} - 1} \mathbb{Z}[x]/(x^{p^n} - 1) \xrightarrow{\times f(x)} \mathbb{Z}[x]/(x^{p^{n-1}} - 1) \longrightarrow 0$$

is a flasque resolution of $\mathbb{Z}[\lambda]$ and $\mathbb{Z}[x]/(x^{p^{n-1}} - 1)$ is invertible by induction hypothesis. So $\rho(\mathbb{Z}[\lambda])$ is invertible and we complete the proof of (1) $\Rightarrow$ (2). (2) imply (3) is trivial.

For (3) implied (1), let's take an exact sequence as in Proposition 1.19(1). Assume $\rho(\mathrm{J_G})$ is invertible and let $\mathrm{N} \in \mathcal{L}_\mathrm{G}$ such that $\mathrm{Q} \oplus \mathrm{N} = \mathrm{P}$ for some permutation module P. As we mentioned in the proof of Proposition 1.19, $\hat{\mathrm{H}}^2(\mathrm{G}', \mathrm{Q}) \simeq \hat{\mathrm{H}}^0(\mathrm{G}', \mathbb{Z}) \simeq \mathbb{Z}/|\mathrm{G}'|\mathbb{Z}$, for all subgroups $\mathrm{G}' \subseteq \mathrm{G}$. Let G' be a p-Sylow subgroup of G. Since $\hat{\mathrm{H}}^2(\mathrm{G}', \mathrm{Q})$ is a subgroup of $\hat{\mathrm{H}}^2(\mathrm{G}', \mathrm{P})$, there is an element of order $|\mathrm{G}'|$ in $\hat{\mathrm{H}}^2(\mathrm{G}', \mathrm{P})$. Since P is a permutation module, $\mathrm{P} \simeq \oplus \mathbb{Z}[\mathrm{G}'/\mathrm{H}_i]$ as a G'-module for some subgroups H_i in G'. By Shapiro's Lemma $\hat{\mathrm{H}}^2(\mathrm{G}', \mathbb{Z}[\mathrm{G}'/\mathrm{H}_i]) = \hat{\mathrm{H}}^2(\mathrm{H}_i, \mathbb{Z})$ which is annilated by $|\mathrm{H}_i|$, so $\mathbb{Z}$ must be a direct summand of P and $\hat{\mathrm{H}}^2(\mathrm{G}', \mathbb{Z})$ has an element of order $|\mathrm{G}'|$. So G' is a cyclic group and G is metacyclic. □

In the following Proposition, we will give an even more concrete description of $\mathrm{U_G}$ while G is a cyclic group of prime order. From this proposition, we can find a concrete example of an invertible G-module which is not a permutation module.

Proposition 1.20. *Let* $\mathrm{G} = \mathrm{C}_p$*, where* p *is a prime number. Then* $\mathrm{F_G} = \mathrm{U_G} \simeq \mathrm{Cl}(\mathbb{Z}[\zeta_p])$*, where* ζ_p *is a primitive* p*-th root of unity and* $\mathrm{Cl}(\mathbb{Z}[\zeta_p])$ *is the ideal class group of* $\mathbb{Z}[\zeta_p]$*.*

Proof. First, we note that for a Dedekind domain A, we has an isomorphism $\mathrm{Cl}(A) \simeq \overline{\mathrm{K}}_0(A)$, where $\overline{\mathrm{K}}_0(A)$ is the reduced K_0 group of A (see [12] p. 20). Now we want to construct an isomorphism between $\mathrm{U_G}$ and $\overline{\mathrm{K}}_0(\mathbb{Z}[\zeta_p])$. Let σ be a generator of G and ϕ_p be the primitive polynomial. For $\mathrm{M} \in \mathrm{U_G}$, we define

$$\mathrm{M}^{\phi_p} = \{m \in \mathrm{M} | \phi_p(\sigma)m = 0\}.$$

As we mentioned in the proof of Proposition 1.19, M^{ϕ_p} is a projective $\mathbb{Z}[\zeta_p]$-module. Hence we define $\Phi : \mathrm{U_G} \to \overline{\mathrm{K}}_0(\mathbb{Z}[\zeta_p])$ as $\Phi([\mathrm{M}]) = [\mathrm{M}^{\phi_p}]$. For a $\mathbb{Z}[\zeta_p]$-module M, we define a left G-action on it as $\sigma(m) = \zeta_p(m)$, for all $m \in \mathrm{M}$. We denote by $\widetilde{\mathrm{M}}$ the corresponding G-module defined by M. Clearly, under the G-action defined above, $\widetilde{\mathbb{Z}[\zeta_p]}$ is a permutation G-module, and for any projective $\mathbb{Z}[\zeta_p]$-module M, $\widetilde{\mathrm{M}}$ is a invertible G-module. Moreover, for a projective $\mathbb{Z}[\zeta_p]$-module which comes from M^{ϕ_p}, the G-action we define coincides with the original G-action on M^{ϕ_p}. So we define $\Psi : \overline{\mathrm{K}}_0(\mathbb{Z}[\zeta_p]) \to \mathrm{U_G}$ as $\Psi([\mathrm{M}]) = [\widetilde{\mathrm{M}}]$. To check Φ and Ψ are inverse of each other, first, we check $[\mathrm{M}] = [\mathrm{M}^{\phi_p}]$ in $\mathrm{U_G}$. Consider the following exact sequence of G-modules:

$$0 \to \mathrm{M}^{\phi_p} \to \mathrm{M} \to \phi_p(\sigma)\mathrm{M} \to 0.$$

Note that $\phi_p(\sigma)\mathrm{M}$ is a trivial G-module, since $(1-\sigma)\phi_p(\sigma) = 0$. By Lemma 1.12, we have $\rho([\mathrm{M}]) = \rho([\mathrm{M}^{\phi_p}])$. Since M, M^{ϕ_p} are invertible, we conclude $[\mathrm{M}] = [\mathrm{M}^{\phi_p}]$ in $\mathrm{U_G}$ and hence $\Psi \circ \Phi$ is the identity map on $\mathrm{U_G}$. Now, take a projective $\mathbb{Z}[\zeta_p]$-module N. Then under our definition, $\phi_p(\sigma)\widetilde{\mathrm{N}} = \phi_p(\zeta_p)\mathrm{N} = 0$. So, $\Phi \circ \Psi(\mathrm{N}) = \mathrm{N}$, and we conclude this proposition. ☐

Example 1.21. ([16] p. 7) Let $K = \mathbb{Q}[\sqrt{-23}]$, $L = \mathbb{Q}[\zeta_{23}]$, G be defined as in Corollary 1.20, and $p = 23$. It is known that $K \subseteq L$. Let $\theta = \frac{1+\sqrt{-23}}{2}$, and $\mathcal{P} = (2, \theta)$ which is a prime ideal in $\mathcal{O}_K$ lying over (2). Then it is easy to check that $\mathcal{P}$ is not principal, and $\mathcal{P}^3 = (\theta - 2)$, so $\mathcal{P}$ is a nontrivial element in $\mathrm{Cl}(\mathcal{O}_K)$ with order 3. Let $\mathfrak{P} = \mathcal{P}\mathcal{O}_L$ which is still a prime ideal in $\mathcal{O}_L$. Then we claim $\mathfrak{P}$ is a nontrivial element in $\mathrm{Cl}(\mathcal{O}_L)$. Suppose $\mathfrak{P} = (a)$, for some $a \in \mathcal{O}_L$. Then since $[\mathcal{O}_L/\mathfrak{P} : \mathcal{O}_K/\mathcal{P}] = 11$, we have $\mathcal{P}^{11} = (\mathrm{Nr}_{L/K}(a))$ which contradicts to the order of $\mathcal{P}$. So $\mathfrak{P}$ is a nontrivial element of $\mathrm{Cl}(\mathcal{O}_L)$, and by Corollary 1.20, $\widetilde{\mathfrak{P}}$ is invertible but not a permutation G-module.

Next, we will give an example, as we promised in Remark 1.1, to show there are stably permutation G-modules which are not permutation modules.

Example 1.22. ([5] R 1.) Let G be a finite group generated by s, t, where s, t satisfy the following: $s^2 = t^3 = 1$, and $sts = t^{-1}$. Let's consider the exact sequence of G-modules:

$$(*) \quad 0 \to \mathrm{M} \to \mathbb{Z}[\mathrm{G}] \xrightarrow{\omega} \mathbb{Z}[\mathrm{G}/\langle s\rangle] \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\omega(1) = 1 - \bar{t}$, and $\{1, \bar{t}, \bar{t}^2\}$ is a permutation basis of $\mathbb{Z}[\mathrm{G}/\langle s\rangle]$. Let I be the kernel of ε as usual. We will show M is not a permutation module. Suppose M is a permutation module. Since $\hat{\mathrm{H}}^{-1}(\mathrm{G}, \mathrm{I}) = \hat{\mathrm{H}}^0(\mathrm{G}, \mathbb{Z}[\mathrm{G}]) = 0$, $\hat{\mathrm{H}}^0(\mathrm{G}, \mathrm{M}) = 0$. Hence, to be a permutation module, M must be a free $\mathbb{Z}[\mathrm{G}]$-module. However, $\hat{\mathrm{H}}^2(\mathrm{G}, \mathrm{M}) = \hat{\mathrm{H}}^1(\mathrm{G}, \mathrm{I}) = \mathbb{Z}/3\mathbb{Z}$, which contradicts to M is a free $\mathbb{Z}[\mathrm{G}]$-module. So M is not a permutation module.

Now let's show M is a stably permutation module. Consider the exact sequence of G-modules:

$$\mathbb{Z}[\mathrm{G}] \oplus \mathbb{Z} \xrightarrow{\omega'} \mathbb{Z}[\mathrm{G}/\langle s\rangle] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\omega' = \omega \oplus 0$. Clearly, $\mathrm{M} \oplus \mathbb{Z}$ is the kernel of ω'. Also, we have the following exact sequence:

$$0 \to \mathbb{Z}[\mathrm{G}/\langle ts\rangle] \oplus \mathbb{Z}[\mathrm{G}/\langle t\rangle] \xrightarrow{\iota} \mathbb{Z}[\mathrm{G}] \oplus \mathbb{Z} \xrightarrow{\omega'} \mathbb{Z}[\mathrm{G}/\langle s\rangle],$$

where $\iota(1, 0) = (1 + ts, 1)$, and $\iota(0, 1) = (1 + t + t^2, 1)$.
So $\mathrm{M} \oplus \mathbb{Z} \simeq \mathbb{Z}[\mathrm{G}/\langle ts\rangle] \oplus \mathbb{Z}[\mathrm{G}/\langle t\rangle]$, and M is a stably permutation module.

For more examples, one can refer to [7].

2 Flasque resolution of algebraic tori

In this section, all the k-varieties are assumed to be separated, geometrically integral k-schemes of finite type. For a field k, we denote its algebraic separable closure as k_s and $\mathrm{Gal}(k_s/k) = \mathfrak{g}$. For a k-variety $\mathbf{X}$ and a field extension K over k, we define $\mathbf{X}_K = \mathbf{X} \times_k K$, and $\overline{\mathbf{X}} = \mathbf{X} \times_k k_s$. Let $K[\mathbf{X}]$ be the regular functions on $\mathbf{X}_K$, and $K(\mathbf{X})$ be the function field of $\mathbf{X}_K$. Let $\mathrm{Div}\mathbf{X}$ be the group generated by Cartier divisors on $\mathbf{X}$; $\mathbf{Z}^1\mathbf{X}$ be the group generated by Weil divisors on $\mathbf{X}$; and $\mathrm{Cl}\mathbf{X}$ be the Weil divisor class group.

Let $\mathbf{X}_1$, $\mathbf{X}_2$ be two k-varieties. We say that $\mathbf{X}_1$ is *stably k-equivalent* to $\mathbf{X}_2$ if there exist m, n such that $\mathbf{X}_1 \times \mathbb{A}^m$ is k-birational isomorphic to $\mathbf{X}_2 \times \mathbb{A}^n$. For two k-variety $\mathbf{X}$ and $\mathbf{Y}$, we say $\mathbf{X}$ is a k-compactification of $\mathbf{Y}$ if $\mathbf{X}$ is proper and contains $\mathbf{Y}$ as an open subvariety.

Let $\mathcal{T}_k$ be the category of all algebraic k-tori, and $\mathcal{T}_{K/k}$ be the category of all algebraic k-tori splitting over K. For $\mathbf{T} \in \mathcal{T}$, let $\hat{\mathbf{T}}$ be the character group of $\mathbf{T}$, i.e. $\hat{\mathbf{T}} = \mathrm{Hom}_{grp}(\overline{\mathbf{T}}, \mathbf{G}_{m,k_s})$. Note that $\hat{\mathbf{T}}$ is a free $\mathbb{Z}$-module of finite rank. In this section, we want to associate a flasque $\mathfrak{g}$-module $\rho(\mathbf{T})$ to each $\mathbf{T} \in \mathcal{T}_k$ in a geometrical way, which will be a stably k-equivalent invariant. Besides, we will show under our definition, $\rho(\mathbf{T})$ is just $\rho(\hat{\mathbf{T}})$ which is defined in the first section. We start this section with a useful Theorem: Rosenlicht's unit Theorem.

2.1 Rosenlicht's Theorem

Theorem 2.1. *(Rosenlicht) Let* $\mathbf{X}$ *be a* k*-variety and define* $\mathrm{U}_k(\mathbf{X})$=$k[\mathbf{X}]^*/k^*$. *Then* $\mathrm{U}_k(\mathbf{X})$ *is a free* $\mathbb{Z}$*-module of finite rank and the contravariant functor* $\mathbf{X} \to \mathrm{U}_k(\mathbf{X})$ *is an additive functor over the category of all* k*-varieties in the following sense:* $\mathrm{U}_k(\mathbf{X}) \oplus \mathrm{U}_k(\mathbf{Y}) \simeq \mathrm{U}_k(\mathbf{X} \times_k \mathbf{Y})$.

Proof. Let $\mathbf{Y}$ be an open affine subset of $\mathbf{X}$, and $\mathbf{Y}'$ be the normalization of $\mathbf{Y}$. Since $\mathbf{X}$ is integral, $\mathrm{U}_k(\mathbf{X}) \hookrightarrow \mathrm{U}_k(\mathbf{Y}')$. So we can reduce our case to a normal affine k-variety $\mathbf{X}$. Since $\mathbf{X}$ is affine and normal, we can find a normal projective k-compactification $\mathbf{X}'$ of $\mathbf{X}$. Then since $\mathbf{X}'$ is normal, we have the exact sequence: $1 \to \mathrm{U}_k(\mathbf{X}') \to \mathrm{U}_k(\mathbf{X}) \to \mathbf{Z}^1_{\mathbf{X}'\setminus\mathbf{X}}\mathbf{X}'$. The variety $\mathbf{X}'$ is geometrically integral and proper, so $\mathrm{U}_k(\mathbf{X}') = 1$, which implies $\mathrm{U}_k(\mathbf{X})$ injects into $\mathbf{Z}^1_{\mathbf{X}'\setminus\mathbf{X}}$. Hence the first claim in this theorem follows.

Now the second part of this Theorem is equivalent to prove the exactness of the sequence:

$$(1) \quad 0 \to k^* \xrightarrow{\delta} k[\mathbf{X}]^* \times k[\mathbf{Y}]^* \to k[\mathbf{X} \times_k \mathbf{Y}]^* \to 0,$$

where δ is defined as $\delta(c) = (c, c^{-1})$ for all $c \in k^*$.

First, assume there are $x_0 \in \mathbf{X}(k)$ and $y_0 \in \mathbf{Y}(k)$, so we have the map:

$$k[\mathbf{X}] \simeq k[\mathbf{X} \times_k \{y_0\}] \xleftarrow{i_x^*} k[\mathbf{X} \times_k \mathbf{Y}] \xleftarrow{p_x^*} k[\mathbf{X}].$$

Pick $u \in k[\mathbf{X}\times_k\mathbf{Y}]^*$ and note that the surjectivity of the above exact sequence is equivalent to the equation: $u(x, y) = u(x, y_0)u(x_0, y)u(x_0, y_0)^{-1}$, where $u(x, y_0)$ (resp. $u(x_0, y)$) is $p_x^* i_x^* u(x, y)$ (resp. $p_y^* i_y^* u(x, y)$), and $u(x_0, y_0)$ is the valuation of u at (x_0, y_0). Let's assume $k = k_s$. Since $\mathbf{X}, \mathbf{Y}$ are geometrically reduced, $\mathbf{X}(k), \mathbf{Y}(k)$ are nonempty. Let $v(x, y)$ denote $u(x, y_0)u(x_0, y)u(x_0, y_0)^{-1}$. Since $\mathbf{X}, \mathbf{Y}$ are geometrically integral, if we pick normal affine open subsets $\mathbf{U}, \mathbf{V}$ of $\mathbf{X}, \mathbf{Y}$ respectively, then $k[\mathbf{X} \times_k \mathbf{Y}] \hookrightarrow k[\mathbf{U} \times_k \mathbf{V}]$. Therefore, if $u = v$ on $\mathbf{U} \times_k \mathbf{V}$, then $u = v$ on $\mathbf{X} \times_k \mathbf{Y}$. Hence we can suppose $\mathbf{X}, \mathbf{Y}$ are normal and affine, and there are open immersions $\mathbf{X} \hookrightarrow \widetilde{\mathbf{X}}$, $\mathbf{Y} \hookrightarrow \widetilde{\mathbf{X}}$, where $\overline{\mathbf{X}}, \overline{\mathbf{Y}}$ are projective and normal.

Let $\{\mathbf{X}_i\}_{i\in I}$, $\{\mathbf{Y}_j\}_{j\in J}$ be those irreducible codimension-1 components complementary to $\mathbf{X}, \mathbf{Y}$ in $\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}$ respectively. Since $\widetilde{\mathbf{X}} \times \widetilde{\mathbf{Y}}$ are projective,

$k[\widetilde{\mathbf{X}} \times \widetilde{\mathbf{Y}}] = k$. So by normality, we have the following exact sequence: $1 \to k^* \to k[\mathbf{X} \times_k \mathbf{Y}]^* \xrightarrow{ord} \mathbb{Z}^I \oplus \mathbb{Z}^J$, where $ord(f)$ is the order of f along $\mathbf{X}_i \times \widetilde{\mathbf{Y}}$'s and $\widetilde{\mathbf{X}} \times \mathbf{Y}_j$'s. Then we use the following lemma (see [2]) :

Lemma 2.2. *Choose a nonzero $f \in k[\mathbf{X} \times_k \mathbf{Y}]$. There is a Zariski-dense open $\mathbf{U} \subseteq \mathbf{Y}$ such that for all $y \in \mathbf{U}(k)$ we have $f|_{\mathbf{X}\times\{y\}} \neq 0$ and $ord_{\mathbf{X}_i \times \widetilde{\mathbf{Y}}}(f) = ord_{\mathbf{X}_i}(f|_{\mathbf{X}\times\{y\}})$, for all i's.*

Now, for $u(x, y) \in k[\mathbf{X} \times_k \mathbf{Y}]^* \hookrightarrow k(\widetilde{\mathbf{X}} \times \widetilde{\mathbf{Y}})^*$, by Lemma 2.2, we can find $\mathbf{U} \subseteq \mathbf{X}, \mathbf{V} \subseteq \mathbf{Y}$ such that $ord_{\mathbf{X}_i \times \widetilde{\mathbf{Y}}}(u) = ord_{\mathbf{X}_i}(u|_{\mathbf{X}\times\{y\}})$, $ord_{\widetilde{\mathbf{X}} \times \mathbf{Y}_i}(u) = ord_{\mathbf{Y}_i}(u|_{\{x\}\times\mathbf{Y}})$, for all $x \in \mathbf{U}(k)$, $y \in \mathbf{V}(k)$. Pick $x_0 \in \mathbf{U}(k)$, $y_0 \in \mathbf{V}(k)$. Then $u(x, y_0)$ has the same order with u along $\mathbf{X}_i \times \widetilde{\mathbf{Y}}$ and has order zero along $\widetilde{\mathbf{X}} \times \mathbf{Y}_i$; while $u(x_0, y)$ has the same order with u along $\widetilde{\mathbf{X}} \times \mathbf{Y}_j$ and has order zero along $\mathbf{X}_i \times \widetilde{\mathbf{Y}}$. So $u(x, y)u(x, y_0)^{-1}u(x_0, y)^{-1}$ is a global section on $\widetilde{\mathbf{X}} \times \widetilde{\mathbf{Y}}$. Hence $u(x, y)u(x, y_0)^{-1}u(x_0, y)^{-1}$ is a constant, and we denote this constant as c. So $u(x, y) = cu(x, y_0)u(x_0, y)$. Evaluating u at (x_0, y_0), we get $c = u(x_0, y_0)^{-1}$. This proves the surjectivity of (1) for $k = k_s$.

Now let's consider the injectivity. Still, we assume that $k = k_s$. Let u_x, u_y be the units on $\mathbf{X}$, $\mathbf{Y}$ respectively. Suppose $p_x^* u_x = p_y^* u_y$. Then we want to show $u_x{=}u_y \in k$. Pick $y_0 \in \mathbf{Y}(k)$. Then $u_x = i_x^* p_x^* u_x = i_x^* p_y^* u_y \in k_s$, so $u_x{=}u_y \in k$. In general case, we have $k[\overline{\mathbf{X}}] = k[\mathbf{X}] \otimes_k k_s$. So by the exact sequence: $0 \to k_s^* \xrightarrow{\delta} k_s[\overline{\mathbf{X}}]^* \times k_s[\overline{\mathbf{Y}}]^* \to k_s[\overline{\mathbf{X}} \times_{k_s} \overline{\mathbf{Y}}]^* \to 0$, and Hilbert Theorem 90, we prove (1) is exact for general k. □

Remark 2.3. Actually, Rosenlicht's Theorem can be proved in a more general setting. See [2].

Corollary 2.4. *Let $\mathbf{G}$ be a smooth connected algebraic group scheme over k. Then $\hat{\mathbf{G}}(k) \simeq \mathrm{U}_k(\mathbf{G})$.*

Proof. Let e be the identity element in the group scheme $\mathbf{G}$, and $\mathbf{G}_m = \operatorname{Spec} k[t, t^{-1}]$. Note e is a k-rational point in $\mathbf{G}$, and $\mathbf{G}$ is geometrically integral. It suffice to prove that given a unit u on $\mathbf{G}$ satisfying $u(e) = 1$, we can define a group morphism $f \in \hat{\mathbf{G}}(k)$ by sending t to u. Clearly, $f \in \mathbf{G}_m(\mathbf{G})$. Let $\Delta : \mathbf{G} \times_k \mathbf{G} \to \mathbf{G}$ be the multiplication map on $\mathbf{G}$. Let u_Δ be the unit on $\mathbf{G} \times \mathbf{G}$ corresponding to the map $f \circ \Delta \in \mathbf{G}_m(\mathbf{G} \times \mathbf{G})$. By Theorem 2.1, $u_\Delta = u_1 u_2$ where u_1, u_2 are units on $\mathbf{G}$. If we extend the scalars to k_s, then for all g_1, $g_2 \in \mathbf{G}(k_s)$, $u_\Delta(g_1, g_2) = u_1(g_1)u_2(g_2)$. Substitute g_1, g_2 by e. Then we get three identities: $u(g_2) = u_\Delta(e, g_2) = u_1(e)u_2(g_2)$, $u(g_1) = u_\Delta(g_1, e) = u_1(g_1)u_2(e)$, and $1 = u(e) = u_1(e)u_2(e)$. Combining the three identities above, we conclude $u_\Delta(g_1, g_2) = u(g_1)u(g_2)$. So f is a group morphism. □

2.2 Definitions and Properties of Flasque resolutions of algebraic tori

Lemma 2.5. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two smooth* k*-varieties. If* $\mathbf{Y}$ *is* k*-rational, then the canonical morphism* $\mathrm{Pic}\mathbf{X} \oplus \mathrm{Pic}\mathbf{Y} \to \mathrm{Pic}(\mathbf{X} \times_k \mathbf{Y})$ *is an isomorphism.*

Proof. Let $p_{\mathbf{X}}$ and $p_{\mathbf{Y}}$ be the natural projections from $\mathbf{X} \times_k \mathbf{Y}$ to $\mathbf{X}$ and $\mathbf{Y}$ respectively. First, we show the injectivity. Suppose $\mathbf{X}$ and $\mathbf{Y}$ have k-rational points x and y respectively. Then we consider the maps:

$$\mathbf{Y} \simeq \{x\} \times_k \mathbf{Y} \xrightarrow{i} \mathbf{X} \times_k \mathbf{Y} \xrightarrow{p_{\mathbf{Y}}} \mathbf{Y} \ .$$

Take $\mathcal{L}$ and $\mathcal{N}$ in $\mathrm{Pic}\mathbf{X}$ and $\mathrm{Pic}\mathbf{Y}$ respectively. If $\mathcal{M} = p_{\mathbf{X}}^*\mathcal{L} \otimes p_{\mathbf{Y}}^*\mathcal{N}$ is trivial on $\mathbf{X} \times_k \mathbf{Y}$, then $\mathcal{N} \simeq i^*\mathcal{M}$ is also trivial on $\mathbf{Y}$. By the same argument, we also have $\mathcal{L}$ is trivial on $\mathbf{X}$. This proves the injectivity when $\mathbf{X}$ and $\mathbf{Y}$ have k-rational points. In general case, we consider the following exact sequence derived from Theorem 2.1:

$$1 \longrightarrow {k_s}^* \longrightarrow k_s[\mathbf{X}]^* \times k_s[\mathbf{Y}]^* \longrightarrow k_s[\mathbf{X} \times_k \mathbf{Y}]^* \longrightarrow 1 \ ,$$

where we can derive the long exact sequence, and by Theorem 90, we finally get $0 \longrightarrow \mathrm{H}^1(\mathfrak{g}, k_s[\mathbf{X}]^* \times k_s[\mathbf{Y}]^*) \longrightarrow \mathrm{H}^1(\mathfrak{g}, k_s[\mathbf{X} \times_k \mathbf{Y}]^*)$ Next, by Lemma 3.1 in our appendix, we have the following exact sequence:

$$0 \longrightarrow \mathrm{H}^1(\mathfrak{g}, k_s[\mathbf{X}]^*) \longrightarrow \mathrm{Pic}\mathbf{X} \longrightarrow \mathrm{Pic}\overline{\mathbf{X}} \ .$$

Combining the two exact sequences above, we have the following diagram:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{H}^1(\mathfrak{g}, k_s[\mathbf{X}]^*) \oplus \mathrm{H}^1(\mathfrak{g}, k_s[\mathbf{Y}]^*) & \longrightarrow & \mathrm{H}^1(\mathfrak{g}, k_s[\mathbf{X} \times_k \mathbf{Y}]^*) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Pic}\mathbf{X} \oplus \mathrm{Pic}\mathbf{Y} & \longrightarrow & \mathrm{Pic}(\mathbf{X} \times_k \mathbf{Y}) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Pic}\overline{\mathbf{X}} \oplus \mathrm{Pic}\overline{\mathbf{Y}} & \longrightarrow & \mathrm{Pic}(\overline{\mathbf{X}} \times_{k_s} \overline{\mathbf{Y}}),
\end{array}$$

since the first row and the third row are exact, we conclude the second row is also exact, where we conclude the injectivity. To prove surjectivity, we first note $\mathrm{Pic}\mathbf{X} \simeq \mathrm{Cl}\mathbf{X}$ for $\mathbf{X}$ is a smooth k-variety. Therefore, if we take an nonempty open subset $\mathbf{V}$ of $\mathbf{Y}$, then we have the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Div}_{\mathbf{Y}\setminus\mathbf{V}}\mathbf{Y} & \longrightarrow & \mathrm{Pic}\mathbf{X}\oplus\mathrm{Pic}\mathbf{Y} & \longrightarrow & \mathrm{Pic}\mathbf{X}\oplus\mathrm{Pic}\mathbf{V} & \longrightarrow & 0\,, \\
\downarrow \simeq & & \downarrow \phi_{\mathbf{Y}} & & \downarrow \phi_{\mathbf{V}} & & \\
\mathrm{Div}_{\mathbf{X}\times_k\mathbf{Y}\setminus\mathbf{X}\times_k\mathbf{V}}(\mathbf{X}\times_k\mathbf{Y}) & \longrightarrow & \mathrm{Pic}(\mathbf{X}\times_k\mathbf{Y}) & \longrightarrow & \mathrm{Pic}(\mathbf{X}\times_k\mathbf{V}) & \longrightarrow & 0
\end{array}$$

where $\mathrm{Div}_{\mathbf{Y}\setminus\mathbf{V}}\mathbf{Y} \simeq \mathrm{Div}_{\mathbf{X}\times_k\mathbf{Y}\setminus\mathbf{X}\times_k\mathbf{V}}(\mathbf{X}\times_k\mathbf{Y})$ because X is geometrically integral. Hence, $\phi_{\mathbf{Y}}$ is surjective if and only if $\phi_{\mathbf{V}}$ is surjective. Take an affine open subset $\mathbf{U} = \mathrm{SpecA}$ of $\mathbf{X}$, and an open subset $\mathbf{V}$ of $\mathbf{Y}$ which is isomorphic to an open subset of $\mathbb{A}^n$. Since A is regular, $\mathrm{PicA} \simeq \mathrm{PicA[t]}$. By the above argument, we then have $\mathrm{Pic}\mathbf{U}\oplus\mathrm{Pic}\mathbf{V}$ is surjective to $\mathrm{Pic}\mathbf{U}\times_k\mathbf{V}$, and with a similar argument, we get $\phi_{\mathbf{V}}$ is surjective and finally we conclude ϕ_Y is also surjective. □

Proposition 2.6. *Let* $\mathbf{X}$ *be a smooth* k*-variety and let* K/k *be a Galois extension with* $\mathrm{Gal}(K/k) = \mathrm{G}$*. Suppose* $\mathrm{Pic}\mathbf{X}_K$ *is of finite type. Then there is a nonempty open subset* $\mathbf{V}$ *of* $\mathbf{X}$ *such that* $\mathrm{Pic}\mathbf{V}_K = 0$*. Furthermore,* $\rho(\mathrm{U}_K[\mathbf{V}_K])$ *in* $\mathrm{F_G}$ *does not depend on the representative we choose from the stable* k*-equivalence class of* $\mathbf{X}$*. We note it as* $\rho_{K/k}(\mathbf{X})$*. In particular,* $\rho_{K/k}(\mathbf{X}) = [0]$ *for* $\mathbf{X}$ *is stably* k*-rational.*

Proof. First, because $\mathbf{X}$ is smooth, we have $\mathrm{Pic}\mathbf{X}_K \simeq \mathrm{Cl}\mathbf{X}_K$. So $\mathrm{Pic}\mathbf{X}_K$ is of finite type implies we can find $\mathbf{X}_1, ..., \mathbf{X}_n$ in $\mathbf{X}_K$, which are integral closed subschemes of codimension one, to generate $\mathrm{Cl}\mathbf{X}_K$. Let $\widetilde{\mathbf{X}_i}$ be the projection of $\mathbf{X}_i$ in $\mathbf{X}$. Note that $\widetilde{\mathbf{X}_i}$'s are still closed since the projection from $\mathbf{X}_K$ to $\mathbf{X}$ is a closed map. Let $\mathbf{V} = \mathbf{X}\setminus(\bigcup\widetilde{\mathbf{X}_i})$. Then $\mathrm{Pic}\mathbf{V}_K = 0$. Next, if there are $\mathbf{V}_1, \mathbf{V}_2$ satisfying $\mathrm{Pic}\mathbf{V}_1 = \mathrm{Pic}\mathbf{V}_2 = 0$, then $\mathrm{Pic}(\mathbf{V}_1\bigcap\mathbf{V}_2) = 0$. Therefore, we only need to prove while $\mathbf{V}\supseteq\mathbf{W}$ and $\mathrm{Pic}\mathbf{V}_K = 0$, $\rho(\mathrm{U}_K(\mathbf{V}_K)) = \rho(\mathrm{U}_K(\mathbf{W}_K))$. In this way, we show $\rho(\mathbf{X})$ is well-defined. Now, let $\mathbf{V}, \mathbf{W}$ as mentioned above, and $\mathbf{Y} = \mathbf{V}\setminus\mathbf{W}$. Then since $\mathrm{Pic}\mathbf{V}_K = 0$, we have the exact sequence: $0 \longrightarrow \mathrm{U}_K(\mathbf{V}_K) \longrightarrow \mathrm{U}_K(\mathbf{W}_K) \longrightarrow \mathrm{Div}_{\mathbf{Y}_K}\mathbf{V}_K \longrightarrow 0$. Because $\mathbf{X}$ is smooth, $\mathrm{Div}_{\mathbf{Y}_K}\mathbf{V}_K$ is a permutation G-module. So by Lemma 1.12(1), $\rho(\mathrm{U}_K(\mathbf{V}_K)) = \rho(\mathrm{U}_K(\mathbf{W}_K))$, which verifies $\rho(\mathbf{X})$ is well-defined. Moreover, if there are two such k-varieties birational to each other, then the existence of isomorphic open subsets proves the invariance of $\rho(\mathbf{X})$. For a k-rational variety $\mathbf{Y}$, let $\mathbf{Z} = \mathbf{X}\times_k\mathbf{Y}$. Then by Lemma 2.3, we have $\mathrm{Pic}\mathbf{Z}_K$ is also of finite type. By Theorem 2.1, we have $\rho(\mathbf{Z}) = \rho(\mathbf{X})\oplus\rho(\mathbf{Y}) = \rho(\mathbf{X})$ in $\mathrm{F_G}$ for $\rho(\mathbf{Y}) = 0$. □

The following is the main Theorem in this article. As we have claimed in the introduction, $\rho(\mathbf{T})$ characterizes stable k-equivalence classes. See also [5] prop. 6, [15] 4.7.

Theorem 2.7. *Let* $\mathbf{T}$ *be an algebraic* k*-torus and define* $\rho(\mathbf{T}) = \rho_{k_s/k}(\mathbf{T})$. *Then* $\rho(\mathbf{T})$ *is an invariant which characterizes the stable* k*-equivalence classes of* $\mathcal{T}_k$*, and is additive and coincides with* $\rho(\hat{\mathbf{T}})$. *If* $\mathbf{T}$ *splits over a Galois extension* K/k *with* $\mathrm{Gal}(K/k) = \mathrm{G}$*, then* $\rho(\mathbf{T})$ *is in* $\mathrm{F_G}$ *and* $\rho(\mathbf{T}) = \rho_{K/k}(\mathbf{T})$. *Moreover, if* $\mathbf{X}$ *is a smooth* k*-compactification of* $\mathbf{T}$*, then* $\rho(\mathbf{T})$ *coincides with* $[\mathrm{Pic}\mathbf{X}_K]$.

Proof. First, we observe that $\mathrm{Pic}\overline{\mathbf{T}} = 0$ and $k_s[\overline{\mathbf{T}}] \simeq k_s[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]$. By Corollary 2.4, $\hat{\mathbf{T}} = \mathrm{U}_{k_s}(\overline{\mathbf{T}}) = (k_s[t_1, t_1^{-1}, ..., t_r, t_r^{-1}])^*/{k_s}^*$. If $\mathbf{T}$ splits over a Galois extension K/k with $\mathrm{Gal}(K/k) = \mathrm{G}$, then $\mathfrak{h} = \mathrm{Gal}(k_s/K)$ acts trivially on $\hat{\mathbf{T}}$, so $\hat{\mathbf{T}}$ is a $\mathfrak{g}/\mathfrak{h}$-module and isomorphic to $\mathrm{U}_K(\mathbf{T}_K)$ as G-module. Hence, $\rho_{K/k}(\mathbf{T}) = \rho(\hat{\mathbf{T}})$ is in $\mathrm{F_G}$ and $\rho_{K/k}(\mathbf{T}) = \rho(\hat{\mathbf{T}}) = \rho(\mathbf{T})$ in $\mathrm{F}_{\mathfrak{g}}$ by Lemma 1.5. From Lemma 2.6, we know $\rho(\mathbf{T})$ is an invariant of the stable k-equivalence classes. Now, suppose there are two algebraic k-tori $\mathbf{T}_1, \mathbf{T}_2$ such that $\rho(\mathbf{T}_1) = \rho(\mathbf{T}_2)$. Then by dualizing Lemma 1.13, we have the following two exact sequences: $1 \to \mathbf{R} \to \mathbf{M} \to \mathbf{T}_1 \to 1$, and $1 \to \mathbf{P} \to \mathbf{M} \to \mathbf{T}_2 \to 1$, where $\mathbf{P}, \mathbf{R}, \mathbf{M}$ are algebraic tori with $\hat{\mathbf{P}}$ and $\hat{\mathbf{R}}$ are permutation $\mathfrak{g}$-modules. Then by Lemma 3.3 in Appendix, the two fibrations $\mathbf{M} \to \mathbf{T}_1$ and $\mathbf{M} \to \mathbf{T}_2$ are locally trivial for Zariski topology. So $\mathbf{R} \times \mathbf{T}_1$ is birational to $\mathbf{P} \times \mathbf{T}_2$. So $\rho(\mathbf{T})$ characterizes the stable k-equivalence classes. Moreover, we can show $\rho(\mathbf{T}) = [\mathrm{Pic}\mathbf{X}_K]$ as what follows. Suppose we are given K, $\mathbf{T}$ as stated above. Then by Brylinski-Künnemann's Theorem [4], we can find a smooth k-compactification $\mathbf{X}$ of $\mathbf{T}$. Note that since $\mathbf{X}_K$ is K-rational, $\mathrm{Pic}\mathbf{X}_K$ is a free $\mathbb{Z}$-module of finite rank (see [6] p. 461). Since $\mathbf{X}$ is smooth and $\mathrm{Pic}\mathbf{T}_K = 0$, we have the exact sequence: $0 \to \hat{\mathbf{T}}(K) = \mathrm{U}_K(\mathbf{T}_K) \to \mathrm{Div}_{\mathbf{Y}_K}\mathbf{X}_K \to \mathrm{Pic}\mathbf{X}_K \to 0$, where $\mathbf{Y} = \mathbf{X} \backslash \mathbf{T}$. Because $\mathbf{X}$ is smooth, $\mathrm{Div}_{\mathbf{Y}_K}\mathbf{X}_K$ is a permutation module, so $\rho(\mathbf{T}) = [\mathrm{Pic}\mathbf{X}_K]$ if $\mathrm{Pic}\mathbf{X}_K$ is a flasque G-module. Since $\mathbf{T}$ splits over some finite extension over k, we can assume G is a finite group.

First, suppose $\mathbf{T}$ is anisotropic. Then we have the exact sequence:

$$0 = \hat{\mathrm{H}}^{-1}(\mathrm{G}, \mathrm{Div}_{\mathbf{Y}_K}\mathbf{X}_K) \to \hat{\mathrm{H}}^{-1}(\mathrm{G}, \mathrm{Pic}\mathbf{X}_K) \to \hat{\mathrm{H}}^0(\mathrm{G}, \hat{\mathbf{T}}) = 0.$$

In the general case, there is an exact sequence of k-tori: $0 \to \mathbf{T}_d \to \mathbf{T} \to \mathbf{T}_a \to 0$, where $\mathbf{T}_d$ is a trivial k-torus; while $\mathbf{T}_a$ is an anisotropic one (see [17] 7.4). Then again, by Lemma 3.3, we have the fibration $\mathbf{T} \to \mathbf{T}_a$ is locally trivial. Let $\mathbf{X}_d, \mathbf{X}_a$ be the smooth k-compactification of $\mathbf{T}_d, \mathbf{T}_a$ respectively. Then $\mathbf{X}$ is birational to $\mathbf{X}_d \times \mathbf{X}_a$, so $[\mathrm{Pic}\mathbf{X}_K] = [\mathrm{Pic}(\mathbf{X}_{d,K} \times \mathbf{X}_{a,K})] = [\mathrm{Pic}\mathbf{X}_{a,K}]$ for $\mathbf{X}_d$ is k-rational. So, $\mathrm{H}^{-1}(\mathrm{G}, \mathrm{Pic}\mathbf{X}_K) = 0$ in general. For a subgroup $\mathrm{H} \subseteq \mathrm{G}$, we replace k by K^{H}. Then we can apply the above argument again to get $\hat{\mathrm{H}}^{-1}(\mathrm{H}, \mathrm{Pic}\mathbf{X}_K) = 0$. So $[\mathrm{Pic}\mathbf{X}_K]$ is a flasque G-module and $\rho(\mathbf{T}) = [\mathrm{Pic}\mathbf{X}_K]$. $\square$

In the following subsection, we will show how to apply those good properties in the first section to our geometrical case, and we will see it indeed simplifies our calculations.

2.3 Some examples

Example 2.8. Let K/k be a finite Galois extension with $\mathrm{Gal}(K/k) = \mathrm{G}$, and $\mathbf{T} = \mathrm{R}^1_{\mathrm{K/k}}(\mathbf{G}_m)$. Then by Theorem 2.7, we know that to find $\rho(\mathbf{T})$ is equivalent to find $\rho(\hat{\mathbf{T}})$. In this special case, we have $\hat{\mathbf{T}} = \mathrm{J_G}$. Therefore, Proposition 1.15(1) provides us a convenient way to find $\rho(\mathbf{T})$, namely, $\rho(\mathbf{T})$ is just $[\mathrm{J_G} \otimes \mathrm{J_G}]$ by Example 1.16. Moreover, by Proposition 1.15(2), we can calculate $\mathrm{H}^1(\mathrm{G}', \mathrm{Pic}\mathbf{T}_K)$ just from $\mathrm{H}^3(\mathrm{G}', \mathbb{Z})$, for any subgroups G' of G. In particular, for $\mathrm{G} \simeq \mathrm{C}_p \times \mathrm{C}_p$, where p is a prime number, we have $\mathrm{H}^1(\mathrm{G}, \mathrm{Pic}\mathbf{T}_K) = \mathrm{H}^1(\mathrm{G}, \rho(\mathbf{T})) = \mathrm{H}^3(\mathrm{G}, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.

Example 2.9. Let K and K' be two non-cyclic Galois extensions of degree four over k. Let $\mathbf{T}_1 = \mathrm{R}^1_{K/k}(\mathbf{G}_m)$, and $\mathbf{T}_2 = \mathrm{R}^1_{K'/k}(\mathbf{G}_m)$. If $K \neq K'$, then $\mathbf{T}_1$ and $\mathbf{T}_2$ are not stably k-equivalent.

Proof. As we know from Theorem 2.7, to show $\mathbf{T}_1$ and $\mathbf{T}_2$ are not stably k-equivalent is equivalent to show $\rho(\mathbf{T}_1) \neq \rho(\mathbf{T}_2)$. Take $L = KK'$, i.e. L is the composite field of K and K'. Let $\mathrm{G} = \mathrm{Gal}(L/k)$, $\mathrm{G}_1 = \mathrm{Gal}(L/K)$ and $\mathrm{G}_2 = \mathrm{Gal}(L/K')$. Then it is enough to show $\mathrm{H}^1(\mathrm{G}', \rho(\mathbf{T}_1)) \neq \mathrm{H}^1(\mathrm{G}', \rho(\mathbf{T}_2))$, for some subgroup $\mathrm{G}' \subseteq \mathrm{G}$. First, we consider the case for $K \cap K' = k$, and $\mathrm{G} = \mathrm{G}_1 \times \mathrm{G}_2$ in this case. Then clearly, $\mathrm{H}^1(\mathrm{G}_2, \rho(\mathbf{T}_1)) = \mathrm{H}^3(\mathrm{G}_2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, while $\mathrm{H}^1(\mathrm{G}_2, \rho(\mathbf{T}_2)) = 0$. So $\mathbf{T}_1$ and $\mathbf{T}_2$ are not stably k-equivalent in this case. Next, consider $K \cap K'$ is a degree two extension over k. Then we have $\mathrm{G} \simeq \mathrm{G}_1 \times \mathrm{G}_2 \times \mathrm{G}_3$, where G_3 isomorphic to $\mathrm{Gal}(K \cap K'/k)$. As above, we compute $\mathrm{H}^1(\mathrm{G}_2 \times \mathrm{G}_3, \rho(\mathbf{T}_1)) = \mathrm{H}^3(\mathrm{G}_2 \times \mathrm{G}_3, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$; $\mathrm{H}^1(\mathrm{G}_2 \times \mathrm{G}_3, \rho(\mathbf{T}_2)) = \mathrm{H}^1(\mathrm{G}_2 \times \mathrm{G}_3/\mathrm{G}_2, \rho(\mathbf{T}_2)) = \mathrm{H}^3(\mathrm{G}_3, \mathbb{Z}) = 0$. So, in either case, we have $\mathbf{T}_1$ and $\mathbf{T}_2$ are not stably k-equivalent to each other. □

3 Appendix: some useful facts

Lemma 3.1. *Let* $\mathbf{X}$ *be a* k*-variety. Then we have* $\mathrm{H}^1(\mathrm{G}, L[\mathbf{X}]^*) = \mathrm{Ker}[\mathrm{Pic}(\mathbf{X}) \to \mathrm{Pic}(\mathbf{X}_L)]$*, where* $\mathrm{G} = \mathrm{Gal}(L/k)$*.*

Proof. First note $L[\mathbf{X}]^* = \mathrm{Aut}_{\mathbf{X}_L} \mathcal{O}_{\mathbf{X}_L}$. Let $\mathcal{M}$ be an invertible sheaf such that $[\mathcal{M}]$ is in $\mathrm{Ker}[\mathrm{Pic}(\mathbf{X}) \to \mathrm{Pic}(\mathbf{X}_L)]$. Then there is a sheaf isomorphism ϕ: $\mathcal{O}_{\mathbf{X}_L} = \mathcal{O}_{\mathbf{X}} \otimes L \to \mathcal{M} \otimes L$. Let $\sigma \circ \phi = \sigma\phi\sigma^{-1}$. Then define $\alpha_\sigma = \phi^{-1}(\sigma \circ \phi) \in \mathrm{Aut}_{\mathbf{X}} \mathcal{O}_{\mathbf{X}_L}$, for all $\sigma \in \mathrm{G}$. It is clear α defines a 1-cocycle, and if there is another isomorphism φ: $\mathcal{O}_{\mathbf{X}_L} \to \mathcal{M} \otimes L$, we have $\phi^{-1}(\sigma \circ \phi) = f^{-1}\varphi^{-1}(\sigma \circ \varphi)(\sigma \circ f)$, where $f = \varphi^{-1}\phi$. So $[\alpha] \in \mathrm{H}^1(\mathrm{G}, L[\mathbf{X}]^*)$ doesn't depend on the isomorphism we choose. Besides, if $\mathcal{N}$ is another invertible sheaf isomorphic to $\mathcal{M}$ with isomorphism h, and we extend h to $\mathcal{N} \otimes L$ in a trivial way, then $\sigma \circ h = h$, for all $\sigma \in \mathrm{G}$. So the cocycle class defined by $\mathcal{N}$ is the same with the cocycle defined by $\mathcal{M}$. In this way, we define a map Φ: $\mathrm{Ker}[\mathrm{Pic}(\mathbf{X}) \to \mathrm{Pic}(\mathbf{X}_L)] \to \mathrm{H}^1(\mathrm{G}, L[\mathbf{X}]^*)$, which maps an invertible sheaf to the cocycle it defines. On the other side, for $[\alpha] \in \mathrm{H}^1(\mathrm{G}, L[\mathbf{X}]^*)$, let α be a representative cocycle of $[\alpha]$, and define a new G-action on $\mathcal{O}_{\mathbf{X}_L}(\mathbf{U}_L)$ as: $\sigma * a = \alpha_\sigma \sigma a$, for all open subsets $\mathbf{U} \subseteq \mathbf{X}$, $\sigma \in \mathrm{G}, a \in \mathcal{O}_{\mathbf{X}_L}(\mathbf{U}_L)$. Let $\mathcal{M}$ be an $\mathcal{O}_{\mathbf{X}}$-module defined by $\mathcal{O}^\alpha_{\mathbf{X}_L}$, i.e. $\mathcal{M}(\mathrm{U}) = \mathcal{O}_{\mathbf{X}_L}(\mathbf{U}_L)^\alpha$ be the fixed elements of $\mathcal{O}_{\mathbf{X}_L}(\mathbf{U}_L)$ under the G-action twisted by α. Then by Galois descent, we have $\mathcal{O}_{\mathbf{X}_L} \simeq \mathcal{M} \otimes L$, so from α we define an invertible sheaf $\mathcal{M}$ whose class belongs to $\mathrm{Ker}[\mathrm{Pic}(\mathbf{X}) \to \mathrm{Pic}(\mathbf{X}_L)]$. If $\beta = f^{-1}\alpha(\sigma \circ f) \in [\alpha]$, then there is an isomorphism $f^{-1} : \mathcal{O}^\alpha_{\mathbf{X}_L} \to \mathcal{O}^\beta_{\mathbf{X}_L}$, so we get a map Ψ: $\mathrm{H}^1(\mathrm{G}, L[\mathbf{X}]^*) \to \mathrm{Ker}[\mathrm{Pic}(\mathbf{X}) \to \mathrm{Pic}(\mathbf{X}_L)]$. Then one can directly check $\Phi \circ \Psi$ is the identity map on $\mathrm{H}^1(\mathrm{G}, L[\mathbf{X}]^*)$ and $\Psi \circ \Phi$ is the identity map on $\mathrm{Ker}[\mathrm{Pic}(\mathbf{X}) \to \mathrm{Pic}(\mathbf{X}_L)]$ (See also [8] th. 2.3.3). $\square$

Remark 3.2. We can also derive this Lemma 3.1 by using Hochschild-Serre's spectral sequence. See [13] lemma 6.3.

Lemma 3.3. *Let* $\mathbf{P}$ *be a algebraic* k*-tori whose character is a permutation* $\mathfrak{g}$*-module. If we have the exact sequence of algebraic tori:*

$$1 \to \mathbf{P} \to \mathbf{T} \xrightarrow{p} \mathbf{T}' \to 1,$$

then the fibration $\mathbf{T} \xrightarrow{p} \mathbf{T}'$ *is locally trivial for Zariski topology.*

Proof. First, we let $\mathbf{P} = \mathbf{G}_m$. Let $\mathbf{T}' = \mathrm{Spec}\, A$ and all the three tori split over L/k, where L is a finite Galois extension of k with $\mathrm{Gal}(L/k) = \mathrm{G}$. Let η be the generic point of $\mathbf{T}'$. Then we have the following exact sequence:

$$1 \to \mathbf{G}_m(k(\eta)) \to \mathbf{T}(k(\eta)) \to \mathbf{T}'(k(\eta)) \to \mathrm{H}^1(\mathrm{G}, \mathbf{G}_{m,L}(L(\eta))) = 1\ ,$$

where $\mathrm{H}^1(\mathrm{G}, \mathbf{G}_{m,L}(L(\eta))) = \mathrm{H}^1(\mathrm{G}, L(\eta)^*) = 1$ by Hilbert Theorem 90. So there is some $s_0 \in \mathbf{T}(k(\eta))$ mapped to the natural morphism:

$$i : \mathrm{Spec}\, k(\eta) \to \mathbf{T}'.$$

Hence we can find a nonempty affine open subset $\mathbf{U}$ in $\mathbf{T}'$ and a section $s \in \mathrm{Hom}(\mathbf{U}, \mathbf{T})$ which lifts s_0. Define a morphism $f : \mathbf{T} \times_{\mathbf{T}'} \mathbf{U} \to \mathbf{U} \times_k \mathbf{G}_m$ as $f(t, u) = (u, ts(u)^{-1})$. Then $\mathbf{T} \times_{\mathbf{T}'} \mathbf{U} \simeq \mathbf{U} \times_k \mathbf{G}_m$. For k is infinite, $\mathbf{T}(k)$ is dense in $\mathbf{T}$, so $\mathbf{T}' = \bigcup_{x \in \mathbf{T}(k)} p(x)\mathbf{U}$. Define a section $s_x : p(x)\mathbf{U} \to \mathbf{T}$ as

$s_x(p(x)u) = xs(u)$, and an isomorphism $f_x : \mathbf{T} \times_{\mathbf{T}'} p(x)\mathbf{U} \to p(x)\mathbf{U} \times_k \mathbf{G}_m$ as $f_x(t, p(x)u) = (p(x)u, ts_x(p(x)u)^{-1})$. This shows that the fibration $\mathbf{T} \xrightarrow{p} \mathbf{T}'$ is locally trivial for the Zariski topology. For an arbitrary field k, there is a standard argument to permit this result.

In general, $\mathrm{H}^1(\mathrm{G}, \mathbf{P}_L(L(\eta))) = \oplus_j \mathrm{H}^1(\mathrm{G}, \mathbb{Z}[\mathrm{G}/\mathrm{H}_j] \otimes L(\eta)^*)$, where H_j's are subgroups of G. By Shapiro's Lemma and Hilbert Theorem 90,

$$\mathrm{H}^1(\mathrm{G}, \mathbb{Z}[\mathrm{G}/\mathrm{H}_j] \otimes L(\eta)^*) = \mathrm{H}^1(\mathrm{H}_j, L(\eta)^*) = 1,$$

so the above argument also works for $\mathbf{P}$. □

References

[1] K.-S. Brown, *Cohomology of Groups*, GTM 87, Springer-Verlag, 1982.

[2] B. Conrad, *Rosenlicht's unit theorem*, http://math.stanford.edu/ ˜conrad.

[3] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton 1956.

[4] J.-L. Colliot-Thélène, D. Harari, and A. N. Skorobogatov, *Compactification equivariante d'un tore (d'apres Brylinski et Künnemann)*, Expositiones mathematicae 23 (2005), p. 161-170.

[5] J.-L. Colliot-Thélène and J.-J. Sansuc, *La R-équivalence sur les tores*, Ann. Sci. École Norm. Sup. (4) 10 (1977), p. 175-229.

[6] J.-L. Colliot-Thélène and J.-J. Sansuc, *La descente sur les varietes rationnelles II*, Duke Math. J. 54 (1987), p. 375-492.

[7] S. Endo and T. Miyata, *On a Classification of the Function Fields of Algebraic Tori*, Nagoya Math. J., vol. 56, 1974, p. 85-104.

[8] P. Gille and T. Szamuely, *Central Simple Algebra and Galois Cohomology*, Cambridge studies in Adv. Math. 101, Cambridge, 2006.

[9] R. Hartshorne, *Algebraic Geometry*, GTM 52, Springer-Verlag, 1977.

[10] Q. Liu,*Algebraic Geometry and Arithmetic Curves*, Oxford, 2002.

[11] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of Number Fields*, Springer-Verlag, 2000.

[12] J. Rosenberg, *Algebraic K-Theory*, GTM 147, Springer-Verlag, 1994.

[13] J.-J. Sansuc, *Groupe de Brauer et arithmetique des groupes algebriques lineaires sur un corps de nombres*, J. fur die reine und ang. Math. 327 (1981), p. 12-80.

[14] J.-P. Serre, *Local Fields*, GTM 67, Springer-Verlag, 1979.

[15] V. E. Voskresenskiĭ, *Algebraic Groups and Their Birational Invariants*, v. 179, Translations of Math. Monographs, AMS, 1998.

[16] L.-C. Washington, *Introduction to Cyclotomic Fields*, GTM 83, Springer-Verlag, 1982.

[17] W.-C. Waterhouse, *Introduction to Affine Group Schemes*, GTM 66, Springer-Verlag, 1979.