Artin conductors of tori

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under the supervision of Prof. Qing Liu

Abstract

This article is based on the paper "Congruences of Néron models for tori and the Artin conductor" by Ching-Li Chai and Jiu-Kang Yu, published in Annal of Mathematics 154 (2001).

Let $K$ be a complete discrete valuation field with perfect residue field. Let $T$ be a torus over $K$, with Néron model $T^{NR}$ over the ring of integers $O_K$ of $K$. The Néron model does not commutate with the base change in general. Choose a finite Galois extension $L/K$ which splits $T$. One can measure the change of Néron models by comparing $(\text{Lie} T^{NR}) \otimes O_L$ with $\text{Lie}((T \otimes L)^{NR})$. We define an invariant $c(T) \in \mathbb{Q}$ by

$$c(T) = \frac{1}{e_{L/K}} \text{length}_{O_L} \frac{\text{Lie}(T \otimes L)^{NR}}{(\text{Lie} T^{NR}) \otimes O_L}$$

where $e_{L/K}$ is the ramification index of $L/K$ and Lie() denotes the Lie algebra. Let $X_*(T)$ be the ocharacter group of $T$ and let $a(X_*(T) \otimes \mathbb{Q})$ be the Artin conductor of the Galois representation $X_*(T) \otimes \mathbb{Q}$ of $\text{Gal} (\bar{K}/K)$. The main theorem 10.2 states that $c(T)$ is invariant by isogeny and

$$c(T) = \frac{1}{2} a(X_*(T) \otimes \mathbb{Q}),$$

answering a question of B. Gross. Note that in the final step of the proof of theorem 10.2, we restricted ourself to the special case when $K$ has characteristic 0.

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## 1 Notation

- Let \( \mathcal{O} = \mathcal{O}_K \) be a discrete valuation ring with residue field \( \kappa \) and let \( K \) be its field of fractions. Let \( \pi = \pi_K \) be the a prime element of \( \mathcal{O} \). The strict henselization and the completion of \( \mathcal{O} \) are denoted by \( \mathcal{O}^{sh} \) and \( \mathcal{O} \) respectively. Their fields of fraction are denoted by \( K^{sh} \) and \( \hat{K} \) respectively. The residue fields of \( \mathcal{O}^{sh} \) is the separable closure \( \kappa^{sep} \) of \( \kappa \). Denote the algebraic closure of \( K \) by \( \overline{K} \).

- Denote the multiplicative group scheme over a ring \( A \) by \( \mathbb{G}_{m,A} \).

- Let \( T \) be a torus over \( K \). Denote by \( \Lambda \) the cocharacter group

\[
X_\ast(T) = \text{Hom}(\mathbb{G}_{m,\overline{K}}, T \otimes \overline{K})
\]

of \( T \) and by

\[
X^\ast(T) = \text{Hom}(T \otimes \overline{K}, \mathbb{G}_{m,\overline{K}})
\]

the character group of \( T \). We will often denote by \( L/K \) a Galois extension such that \( T \) is split over \( L \) and by \( \Gamma \) the Galois group \( \text{Gal}(L/K) \).
• we will also work with another discrete valuation ring \( \mathcal{O}_0 \). We will analogous constructs by the same notation with a subscript 0. And introduce a series of congruence notation:

- \((\mathcal{O}, \mathcal{O}_L) \equiv_\alpha (\mathcal{O}_0, \mathcal{O}_{L_0}) \) (level N): this means that \( \alpha \) is an isomorphism from \( \mathcal{O}_L / \pi^N \mathcal{O}_L \) to \( \mathcal{O}_{L_0} / \pi^N \mathcal{O}_{L_0} \) and induce an isomorphism \( \mathcal{O} / \pi^N \mathcal{O} \to \mathcal{O}_0 / \pi^N \mathcal{O}_0 \).

- \((\mathcal{O}, \mathcal{O}_L, \Gamma) \equiv_{\alpha, \beta} (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0) \) (level N): this means \((\mathcal{O}, \mathcal{O}_L) \equiv_\alpha (\mathcal{O}_0, \mathcal{O}_{L_0}) \) (level N), \( \beta \) is an isomorphism \( \Gamma \to \Gamma_0 \), and \( \alpha \) is \( \Gamma \)-equivalent relative to \( \beta \): \( \alpha(\gamma \cdot x) = \beta(\gamma) \cdot \alpha(x) \).

- \((\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda) \equiv_{\alpha, \beta, \phi} (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0, \Lambda_0) \) (level N): this means that \((\mathcal{O}, \mathcal{O}_L, \Gamma) \equiv_{\alpha, \beta} (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0) \) (level N), and \( \phi \) is isomorphism \( \Lambda \to \Lambda_0 \) which is \( \Gamma \)-equivalent relative to \( \beta \).

- If it is not necessary to name the isomorphisms \( (\alpha, \beta, \text{etc.}) \), we omit them from the notation.

• In this paper, ”X is determined by \((\mathcal{O} / \pi^N \mathcal{O}, \mathcal{O}_L / \pi^N \mathcal{O}_L, \Gamma, \Lambda) \)” means if \((\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda) \equiv_{\alpha, \beta, \phi} (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0, \Lambda_0)( \text{level N}) \), then there is a canonical isomorphism \( X \to X_0 \) determined by \((\alpha, \beta, \phi) \).

• All rings in this paper are \( \mathcal{O} \)-algebras or \( \mathcal{O}_0 \)-algebra. All maps between two group schemes are the homomorphisms of group schemes.

• If \( X \) is an \( \mathcal{O} \)-scheme, we sometimes denote \( X \times \text{Spec} \mathcal{O} / \pi^N \) by \( X \otimes \mathcal{O} / \pi^N \). Similarly, we have the same meaning for \( X \otimes \mathcal{O}_0 \), etc.

• For a group scheme \( X \) over base scheme \( S \), we denote the module of translation invariant top differential forms on \( X \) by \( \omega(X) \).

2 Basic properties of tori

Definition 2.1. Let \( K \) be a field, a torus \( T \) over \( K \) is an affine group scheme \( T \) over \( K \) such that \( T_{\bar{K}} = T \otimes_K \bar{K} \simeq G^d_{m, \bar{K}} \), where \( d \) is the dimension of \( T \). We say that \( T \) is split over some field extension \( L/K \) if \( T \otimes L \) is isomorphic to \( G^d_L \), and that \( L \) is a splitting field of \( T \).

Assume \( L/K \) is a Galois extension, and \( X, Y \) are \( K \)-schemes, then there exists a right \( \text{Gal}(L/K) \)-action on \( \text{Hom}_L(X_L, Y_L) \). Let \( \sigma \in \text{Gal}(L/K) \), \( \phi \in \text{Hom}_L(X_L, Y_L) \), we have \( \text{id} \otimes \sigma : X \otimes L \to X \otimes L \). Define the action of \( \sigma \) on \( \phi \) to be \( (\text{id}_Y \otimes \sigma) \circ \phi \circ (\text{id}_X \otimes \sigma)^{-1} \), denoted by \( \phi^\sigma \). Then \( \phi^\sigma \) is also an \( L \)-morphism.
If $\phi^\sigma = \phi$ for every $\sigma \in \text{Gal}(L/K)$, there exists $\psi \in \text{Hom}_K(X,Y)$ such that $\hat{\phi} = \psi \otimes id_L$. Hence $\text{Hom}_K(X,Y) = \text{Hom}_L(X_L,Y_L)^{\text{Gal}(L/K)}$, where subscript $\text{Gal}(L/K)$ means the $\text{Gal}(L/K)$-fixed morphisms.

Let $G$ be a group and let $M, N$ be two $\mathbb{Z}[G]$-modules. Then $\text{Hom}_\mathbb{Z}(M, N)$ has a $G$-action defined as follows. Let $f \in \text{Hom}_\mathbb{Z}(M, N), g \in G$. We define $f^g(m) = g(f(g^{-1}(m)))$, for $m \in M$. Then similarly, we have $\text{Hom}_\mathbb{Z}(M, N)^G = \text{Hom}_{\mathbb{Z}[G]}(M, N)$.

**Notation.** In this section the character group $X^*(T)$ of a torus $T$ over $K$ will be denoted by $\hat{T}$.

From the above, we have a $\text{Gal}(\hat{K}/K)$-action on $\hat{T}$. Let $A$ be the affine ring of $T$. Let $\phi \in \hat{T}$, then $\phi$ is determined by the image of $X$ in $A$, where $\mathbb{G}_m,K = \hat{K}[X, X^{-1}]$. Suppose $\phi^\#(X) = \sum_{\text{finite sum}} k_i \otimes a_i$, where $k_i \in \hat{K}, a_i \in A$, then $(\phi^\sigma)^\#(X) = \sum_{\text{finite sum}} \sigma(k_i) \otimes a_i \in \hat{K}' \otimes A$, $K'$ is a finite Galois extension containing all $k_i$, hence the $\text{Gal}(\hat{K}/K)$-action on $\hat{T}$ is continuous.

**Proposition 2.2.** The category of tori over $K$ is anti-equivalent to the category of finitely generated, torsion-free abelian groups with continuous $\Gamma_K = \text{Gal}(\hat{K}/K)$-action.

**Proof.** We have defined a functor $F$ between two categories by $T \longrightarrow \hat{T}$. First, we want to show that $\text{Hom}(T_1, T_2) = \text{Hom}(\hat{T}_1, \hat{T}_2)$.

$$\text{Hom}(T_1, T_2) \simeq \text{Hom}_K(T_1 \times \hat{K}, T_2 \times \hat{K})^{\Gamma_K}$$

$$\simeq \text{Hom}_{\hat{K}}(\mathbb{G}^{d_1}_{m,\hat{K}}, \mathbb{G}^{d_2}_{m,\hat{K}})^{\Gamma_K}$$

$$\simeq \text{Hom}(\mathbb{G}^{d_1}_{m,\hat{K}}, \mathbb{G}^{d_2}_{m,\hat{K}})^{\Gamma_K}$$

$$\simeq \text{Hom}_{\mathbb{Z}[\Gamma_K]}(\hat{T}_2, \hat{T}_1)^{\Gamma_K}$$

$$\simeq \text{Hom}_{\mathbb{Z}[\Gamma_K]}(\hat{T}_2, \hat{T}_1)$$

For any $\mathbb{Z}$-torsion-free and finitely generated $\mathbb{Z}[\Gamma_K]$-module $M$, we want to construct a torus such that $\hat{T} = M$. Let $d = \text{rank}_\mathbb{Z}M$. Consider the group algebra $\hat{K}[M]$, where the group operation on $M$ is written as multiplication. Let $A = \{ x \in \hat{K}[M] : \sigma(x) = x, \forall \sigma \in \Gamma_K \}$. Since $\Gamma_K$-action is continuous, and $M$ is finitely generated, $\Gamma_K$-action factors through $\text{Gal}(L/K)$-action for some finite Galois extension $L/K$. By descend theory, we have $A \otimes \hat{K} = \hat{K}[M]$. Let $T = \text{Spec} A$, then $T$ is a torus over $K$, and $\hat{T} = \text{Hom}(\hat{K}[X, X^{-1}], \hat{K}[M]) = M$. \qed

**Corollary 2.3.** For every torus $T$, there exists a minimal (for the inclusion) splitting field $L/K$. Moreover $L/K$ is a finite Galois extension.
Proof. Since the $\Gamma_K$-action is continuous and $\hat{T}$ is finitely generated, it is enough to take $L$ to be the field fixed by the kernel of the representation $\Gamma_K \to \text{Aut}(\hat{T})$.

Example 2.4. Let $L/K$ be a finite Galois extension, $G = \text{Gal}(L/K)$. Let $T = \text{Res}_{L/K}(\mathbb{G}_{m,L})$ be the Weil restriction of $\mathbb{G}_{m,L}$ to $K$, then $\hat{T} = \mathbb{Z}[G]$.

Proof. Let $T = \text{Spec} \, A$ be the torus such that $\hat{T} = \mathbb{Z}[G]$, where $A = \mathbb{Z}[\sigma_1, \ldots, \sigma_n]_{\sigma \in G}$. For any $K$-algebra $R$, the $L$-homomorphism $f : A \otimes L = L[\sigma_1, \ldots, \sigma_n]_{\sigma \in G} \to R \otimes L$ is determined by the image of $\sigma$ in $R \otimes L$. If $\sigma \circ f = f \circ \sigma$, this means $\sigma f(x) = f(\sigma x)$. Thus the homomorphism $A \to R$ is naturally corresponding to an invertible element $f(x)_{\sigma}$ in $R \otimes L$, which is also corresponding to a homomorphism from $L[X, X^{-1}] \to R \otimes L$. Hence $T'(X) = \text{Hom}_{L}(X \otimes L, \mathbb{G}_{m,L})$ for any $K$-scheme $X$. Then by definition $T'$ just is $\text{Res}_{L/K}(\mathbb{G}_{m,L})$.

Definition 2.5. Let $T, T'$ be tori over a field $K$. A homomorphism $\alpha : T \to T'$ is an isogeny if $\alpha$ is a surjection with finite kernel. The map $\hat{\alpha} : \hat{T} \to \hat{T}'$ is then injective with finite cokernel. Note that the degree of $\alpha$ is equal to the cardinality of $\text{Coker} \, \hat{\alpha}$.

We write $T \sim T'$ when $T$ is isogenous to $T'$.

For any $n \in \mathbb{Z}$, let us denote by $[n]_G$ the multiplication by $n$ map on a group scheme $G$.

Proposition 2.6. Let $T, T'$ be tori defined over $K$, let $\alpha : T \to T'$ be an isogeny. Then there exists an isogeny $\beta : T' \to T$, such that $\beta \circ \alpha = [\deg \alpha]_T$, and $\alpha \circ \beta = [\deg \alpha]_{T'}$.

Proof. Since $\hat{\alpha} : \hat{T} \to \hat{T}'$ is injective with finite cokernel, then there exists $\hat{\beta} : \hat{T} \to \hat{T}'$, such that $\hat{\beta} \circ \hat{\alpha} = (\deg \alpha) \cdot \text{id}_{\hat{T}'}$, $\hat{\alpha} \circ \hat{\beta} = (\deg \alpha) \cdot \text{id}_{\hat{T}}$. Let $\beta : T' \to T$ be the isogeny corresponding to $\hat{\beta}$. Then $\beta \circ \alpha = [\deg \alpha]_T$, and $\alpha \circ \beta = [\deg \alpha]_{T'}$.

Proposition 2.7. Let $T, T'$ be tori over $K$ and $L$ be a common splitting field of $T$ and $T'$. Let $G = \text{Gal}(L/K)$. Then $T \sim T'$ if and only if $T \otimes \mathbb{Q} \simeq T' \otimes \mathbb{Q}$ as $G$-module.

Proof. If $T \sim T'$, we have an exact sequence

$$0 \longrightarrow \hat{T} \longrightarrow \hat{T}' \longrightarrow M \longrightarrow 0,$$

where $M$ is a finite abelian group. After tensor with $\mathbb{Q}$, we get an exact sequence

$$0 \longrightarrow \hat{T} \otimes \mathbb{Q} \longrightarrow \hat{T}' \otimes \mathbb{Q} \longrightarrow 0.$$
Conversely, if $\hat{T} \otimes \mathbb{Z} \cong \hat{T}' \otimes \mathbb{Z}$, then $n\hat{T} \hookrightarrow \hat{T}'$ (as $\mathbb{Z}[G]$-modules) with finite cokernel for some integer $n$. Let $\hat{\alpha}$ be the composition of $\hat{T} \xrightarrow{n} n\hat{T} \rightarrow \hat{T}'$, then $\hat{\alpha} : \hat{T} \rightarrow \hat{T}'$ is injective with finite cokernel. By Proposition 2.2 it corresponds a homomorphism $\alpha : T' \rightarrow T$ which is a surjection and with finite kernel. Hence $T \sim T'$. \hfill \square

Let $T$ be a torus over $K$, split over $L$. Let $G = \text{Gal}(L/K)$, $g \in G$ and $K_g := L_g = \{x \in L | g(x) = x\}$. Let $\chi_T$ be the character of the representation $\hat{T} \otimes \mathbb{Q}$ over $\mathbb{Q}$ and $T_g = \text{Res}_{K_g/K} (\mathbb{G}_m)$, then $\hat{T}_g$ is $\mathbb{Z} \langle g \rangle$ where $\langle g \rangle$ is the subgroup generated by $g$ in $G$. The character of corresponding representation is denoted by $\chi_{T_g}$.

By a theorem of Artin [Serre2, thm 9.2], there exist positive integers $n_h, n_{h'}$ and subsets $H, H'$ of $G$ such that $H \cap H' = \emptyset$, and

$$n\chi_T + \sum_{h' \in H'} n_{h'}\chi_{T_{h'}} = \sum_{h' \in H'} n_{h}\chi_{T_{h}}.$$

Hence we get:

**Proposition 2.8.** There exist positive integers $n_h, n_{h'}$ such that,

$$T^n \times \prod \text{Res}_{K_{h'}/K}(\mathbb{G}_{m,K_{h'}}) \sim \prod \text{Res}_{K_{h'}/K}(\mathbb{G}_{m,K_{h}}).$$

## 3 Dilatation

Let $K$ be a discrete valuation field with valuation ring $\mathcal{O}$.

**Definition 3.1.** Let $X$ be a $\mathcal{O}$-scheme of finite type, whose generic fibre $X_K$ is smooth over $K$. Let $W$ be a closed subscheme of $X$. The *dilatation of W on X* is a pair $(X', u : X' \rightarrow X)$, where $X'$ is a flat $\mathcal{O}$-scheme of finite type and $u : X' \rightarrow X$ factors through $W$, satisfying the following universal property:

if $Z$ is a flat $\mathcal{O}$-scheme, and if $v : Z \rightarrow X$ is an $\mathcal{O}$-morphism such that its restriction $v_\kappa$ to the special fibre factors through $W$, then $v$ factors uniquely through $u$.

**Construction of dilatation**

Let $\mathcal{J}$ be the sheaf of ideals defining $W$ in $X$. Let $X'$ is an open subset of the blow-up $\text{Bl}(X, W)$ of $X$ with center $W$, where $\text{Bl}(X, W) = \text{Proj} \bigoplus_{t \geq 0} \mathcal{J}^t$ and $X' = \{x \in \text{Bl}(X, W) : (\mathcal{J} \cdot \mathcal{O}_{\text{Bl}(X, W)})_x$ is generated by $\pi\}$. Locally, if $X$ is affine and $A$ is the affine ring of $X$, and the ideal sheaf $\mathcal{J}$ of $W$ is
generated $g_1, \ldots, g_n$, then $X' = \text{Spec } A'$ and let $u : X' \to X$ be the canonical map corresponding to $A \to A'$, where

$$A' = A[\frac{g_1}{\pi}, \ldots, \frac{g_n}{\pi}]/(\pi - \text{torsion})$$

and

$$A[\frac{g_1}{\pi}, \ldots, \frac{g_n}{\pi}] = A[X_1, \ldots, X_n]/(\pi X_1 - g_1, \ldots, \pi X_n - g_n).$$

**Proposition 3.2.** Let $(X', u)$ be constructed as above, then $(X', u)$ is the dilatation of $W$ on $X$.

**Proof.** We just need to show that $(X', u)$ satisfies the universal property of dilatation. Since the problem is local, we can assume $Z = \text{Spec } B$ is affine. Keep the notation as before. The fact that $v_\kappa$ factors through $Y_\kappa$ implies that the ideal $J \cdot B$ is contained in $\pi B$. Hence there exist elements $h_i \in B$ with $v^*(g_i) = h_i$; the elements $h_i$ are unique, for B has no $\pi$-torsion. Thus the $A$-morphism $A[X_1, \ldots, X_n] \to X$ sending $T_i$ to $h_i$ yields a morphism $w^* : A' \to B$ and hence a morphism $w : Z \to X'$ such that $v = u \circ w$. \hfill $\Box$

**Corollary 3.3.** Let $X$ be a closed subscheme of an $\mathcal{O}$-scheme $Z$, and let $Y_\kappa$ be a closed subscheme of $X_\kappa$. Then the dilatation $X'$ of $Y_\kappa$ on $X$ is a closed subscheme of the dilatation $Z'$ of $Y_\kappa$ in $Z$.

**Proof.** This is clear from the construction of dilatation. \hfill $\Box$

**Proposition 3.4.** Let $X$ be a smooth scheme over $\mathcal{O}$, and $W$ be a closed subscheme over $X \otimes \kappa$. Let $X'$ be the dilatation of $W$ on $X$. Then $X' \otimes \mathcal{O}/\pi^N$ depends only on $X \otimes \mathcal{O}/\pi^{N+1} \mathcal{O}$ in a canonical way.

**Remark.** Canonicity. Assume $X_1$ and $X_2$ are $\mathcal{O}$-schemes, and $\phi$ is an isomorphism $X_1 \otimes \mathcal{O}/\pi^{N+1} \mathcal{O} \to X_2 \otimes \mathcal{O}/\pi^{N+1} \mathcal{O}$. Assume also that $W_1 \subseteq X_1 \otimes \kappa; W_2 \subseteq X_2 \otimes \kappa$ are closed smooth subschemes over $\kappa$, and $\phi$ induces an isomorphism from $W_1$ to $W_2$. Form the dilatation $X'_i$ and $Y_i = Bl' (X_i, J_i) = \text{Proj } \bigoplus_{i \geq 0} J_1^i$, $i = 1, 2$. The canonicity statement is that the natural isomorphism $Bl' (\phi) : Y_1 \otimes \mathcal{O}/\pi^N \to Y_2 \otimes \mathcal{O}/\pi^N$ induces an isomorphism from the subschemes $X'_i \otimes \mathcal{O}/\pi^N$ of $Y_1 \otimes \mathcal{O}/\pi^N$ to $X'_2 \otimes \mathcal{O}/\pi^N$.

**Proof of Proposition 3.4.** Let $i = 1, 2$. Let $x'_i$ be a point on $X'_i \otimes \kappa$ which projects to $x_i \in X_i \otimes \kappa$. Since $X_i$ and $W_i$ are smooth, we can choose a system of local coordinates $f_1^{(i)}, \ldots, f_r^{(i)}, g_{r+1}^{(i)}, \ldots, g_n^{(i)}$ at $x_i$ on $X_i$ such that $W_i$ defined by $(\pi, g_{r+1}^{(i)}, \ldots, g_n^{(i)})$ near an affine neighborhood $U_i$ of $x_i$ and $X'_i$ above $U_i$ is Spec($B_i' \otimes \pi^N$-torsion), where $B_i' = \mathcal{O}_{X_i}(U_i)[Y_{r+1}^{(i)}, \ldots, Y_n^{(i)}]/(\pi Y_{r+1}^{(i)} -$
\[ g_{r+1}, \ldots, \pi Y_i^i - g_i^i \). The \( f_1^i, \ldots, f_r^i, Y_{r+1}^i, \ldots, Y_n^i \) form a system of local coordinates at \( x_i^i \) in \( X_i \). We can shrink \( U_i \) such that \( B_i^i \) is free of \( \pi^\infty \)-torsion.

If \( \phi(x_1) = x_2 \), we can assume \( \phi^*(f_j^j \mod \pi^N) \equiv f_j^j \mod \pi^N \) and \( \phi^*(g_k^k \mod \pi^N) \equiv g_k^k \mod \pi^N \), and \( \phi \) induces an isomorphism \( \tilde{\phi}^*: \mathcal{O}_{X_2}(U_2) \otimes \mathcal{O}/\pi^N \to \mathcal{O}_{X_1}(U_1) \otimes \mathcal{O}/\pi^N \). Clearly, there is an isomorphism \( (\phi')^*: B_2' \otimes \mathcal{O}/\pi^N \to B_1' \otimes \mathcal{O}/\pi^N \) which extends \( (\tilde{\phi}^*) \) and sends \( Y_j^j \) to \( Y_j^1 \). It remains to show that \( \phi': X_1' \otimes \mathcal{O}/\pi^N \to X_2' \otimes \mathcal{O}/\pi^N \) is induced by \( B_1'(\phi) \). Above \( U_i \otimes \mathcal{O}/\pi^N \), the affine ring of \( B_1'(X_1, J') \otimes \mathcal{O}/\pi^N \) is \( B''_{i'} = (\prod_{t \geq 0} \text{Sym}_{B_{j''}}^t J_i')_{\pi, 0} \), where \( B_i' = \mathcal{O}_{X_i} \otimes \mathcal{O}/\pi^N \), \( J_i' = (\pi, g_{r+1}^i, \ldots, g_n^i) \otimes \mathcal{O}/\pi^N \), \( \pi \) is regards as a homogeneous element of degree 1. The element \( \pi_i \) is an element of degree 1 in \( \prod_{t \geq 0} \text{Sym}_{B_{j''}}^t J_i' \), and the subscript indicates localization. The ring \( B''_{i'} \) maps to \( B_1' \otimes \mathcal{O}/\pi^N \) by sending \( \pi_{i-1}g_k^i \) to \( Y_k \). Then it is clear that \( (\phi')^* \) is induced by \( B_1'(\phi) \).

\[ \square \]

4 Néron’s measure for the defect of smoothness

Let \( X \) be a scheme of finite type over \( \mathcal{O} \) such that \( X \otimes K \) is smooth over \( K \). Consider \( x \in X(\mathcal{O}^{sh}) \) as a morphism \( \text{Spec} \mathcal{O}^{sh} \to X \).

**Definition 4.1.** Define \( \delta(x) = \text{the length of the torsion part of } x^*\Omega_{X/\mathcal{O}}^1 \) as Néron’s measure for the defect of smooth at \( x \), sometimes we also denote it by \( \delta(x, X) \).

The rank of free part is just the rank of \( \Omega_{X/K}^1 \) at \( x_K \), which is the dimension of \( X_K \) at \( x_K \), since \( X_K \) is smooth.

**Lemma 4.2.** Let \( x \) be an \( \mathcal{O}^{sh} \)-value point of \( X \). Then \( x \) factors through the smooth locus of \( X \) if and only if \( \delta(x) = 0 \).

**Proof.** If \( x \) is contained in the smooth locus \( X_{\text{smooth}} \) of \( X \), then \( x^*\Omega_{X/\mathcal{O}}^1 = x^*\Omega_{X_{\text{smooth}}/\mathcal{O}}^1 \), where \( \Omega_{X_{\text{smooth}}/\mathcal{O}}^1 \) is locally free, so \( \delta(x) = 0 \). Conversely, if \( \delta(x) = 0 \), then \( x^*\Omega_{X/\mathcal{O}}^1 \) can be generated by \( d \) elements where \( d \) is the dimension of \( X_K \) at \( x_K \). In particular, \( x^*\Omega_{X_{\kappa}/\kappa}^1 \) can be generated by \( d \) elements at \( x_{\kappa} \). Since the relative dimension at \( x_{\kappa} \) is at least \( d \). So \( X_{\kappa} \) is smooth over \( \kappa \) at \( x_{\kappa} \) of relative dimension \( d \). Then \( X \) is smooth over \( \mathcal{O} \) at \( x \). \[ \square \]

Let \( U \) be a neighborhood of \( x \) in \( X \) which can be realized as a closed subscheme of an \( \mathcal{O} \)-scheme \( Z \) where \( Z \) is smooth over \( \mathcal{O} \), and has constant relative dimension \( n \). Assume that there exist functions \( z_1, \ldots, z_n \) on \( Z \) such
that \( dz_1, ..., dz_n \) generate \( \Omega^1_{X/\mathcal{O}} \), and let \( g_1, ..., g_m \) be functions which generate the sheaf of ideal of \( \mathcal{O}_Z \) defining \( U \) in \( Z \). Then we have \( d g_u = \sum \frac{\partial g_u}{\partial z_v} dz_v \), and define Jacobian matrix \( J \) of \( g_1, ..., g_m \) to be \(( \frac{\partial g_u}{\partial z_v} )_{m \times n} \). Let \( d \) be the relative dimension of \( X_K \) at \( x_K \), and \( v(a) = \pi \)-order of \( a \) in \( \mathcal{O} \).

**Lemma 4.3.** \( \delta(x) = \min \{ v(\Delta) | \Delta : (n - d) \)-minors of \( J \} \).

**Proof.** By Jacobi criterion, there exist a \((n - d)\)-minors \( \Delta \) with \( x^* \Delta \neq 0 \), and any minor \( \Delta \) of \( J \) with more than \( n - d \) rows will satisfying \( x^* \Delta = 0 \). We know \( x^* \Omega^1_{X/\mathcal{O}} \) is representable as a quotient \( F/M \), where \( F := x^* \Omega^1_{Z/\mathcal{O}} \) is a free \( \mathcal{O}^{sh} \)-module of rank \( n \), and \( M \) is the submodule generated by \( x^* d g_1, ..., x^* d g_m \). Since the rank of \( M \) is \( n - d \) and \( \mathcal{O}^{sh} \) is P.I.D, one can find a base \( e_1, ..., e_n \) of \( x^* \Omega^1_{X} \) such that \( M \) is generated by \( a_{d+1} e_{d+1}, ..., a_n e_n \), where \( a_i \in \mathcal{O} \) and \( a_i \neq 0 \). Thus by the theory of elementary divisors, we have \( \delta(x) = v(a_{d+1}) + ... + v(a_n) \).

Now consider the ideals in \( \mathcal{O}^{sh} \) generated by all elements \( x^* \Delta \), where \( \Delta \) is \((n - d)\)-minor, and this ideal is generated by \( a_{d+1}...a_n \), and there is a minor \( \Delta \) with \( x^*(\Delta) = a_{d+1}...a_n \).

**Proposition 4.4.** Let \( Y \) be the Zariski closure of \( \{ x \mod \pi \in X(\kappa) : x \in X(\mathcal{O}^{sh}) \} \) as a closed subscheme of \( X \times \kappa \). Let \( X' \to X \) be the dilatation of \( Y \) on \( X \). For each \( x \in X(\mathcal{O}^{sh}) \) with \( x_\kappa \in Y \), denote \( x' \in X'(\mathcal{O}^{sh}) \) be the unique lifting of \( x \). Then \( \delta(x') \leq \max \{ 0, \delta(x) - 1 \} \).

**Proof.** The proof takes too many pages, see the details in [BLR, 3.3 Prop 5].

**Lemma 4.5.** 1). Suppose \( X \) is a group scheme over \( \mathcal{O} \), and \( e \in X(\mathcal{O}^{sh}) \) is the identity element. Then \( \delta(e) = \delta(x) \), for any \( x \in X(\mathcal{O}^{sh}) \).

2). Change of base field. Let \( x \in X(\mathcal{O}^{sh}) \), consider \( x \) as a point of \( X \otimes \mathcal{O}_L \), then \( \delta(x) = e(L/K) \cdot \delta(x, X) \), where \( e(L/K) \) is the ramification index of \( L/K \).

3). Closed immersion. Let \( i : X \subseteq X' \) be a closed immersion of \( \mathcal{O} \)-scheme such that \( i \) induce an isomorphism \( X \otimes K \to X' \otimes K \). Then we have a surjection \( \iota^* \Omega^1_{X'/\mathcal{O}} \to \Omega^1_{X/\mathcal{O}} \). Therefore, for any \( x \in X(\mathcal{O}^{sh}) \), we have \( \delta(x, X) \leq \delta(i \circ x; X') \).

**Proof.** Let \( r_x : X \otimes \mathcal{O}^{sh} \to X \otimes \mathcal{O}^{sh} \) be the isomorphism of right multiplication by \( x \). Then \( x = r_x \circ e \), hence \( e^* \Omega^1_{X/\mathcal{O}} = x^* \Omega^1_{X/\mathcal{O}} \), so \( \delta(e) = \delta(x) \). The other two are clear.
5 The construction of the Néron model of a torus

Let $K$ be a discrete valuation field.

**Definition 5.1.** Let $T$ be a torus over $K$, the (finite type) Néron model of $T$ is a smooth group scheme $T^{NR}$ over Spec $\mathcal{O}_K$ with generic fibre isomorphic to $T$, such that the image of $T^{NR}(\mathcal{O}^{sh})$ is in $T(K^{sh})$ is the maximal bounded subgroup of $T(K^{sh})$.

**Remark.** The usual definition of Néron model for a smooth and separated $K$-scheme $X$ of finite type is the following: it is a smooth, separated $\mathcal{O}$-scheme $X$, locally of finite type, satisfying the following universal property:

For each smooth Spec $\mathcal{O}$-scheme $Y$ and each $K$-morphism $u_K : Y_K \to X$, there is a unique Spec $\mathcal{O}$-morphism $u : Y \to X$ extending $u_K$. For more details, see [BLR].

For a torus $T$ over $K$, the (finite type) Néron model $T^{NR}$ is an open subscheme of $T$. Its special fiber consists in the union of the connected components of $T_κ$ which are of finite order in the group of components $\Phi(T)$.

When $T$ is anisotropic (i.e. $T$ does not contain any factor $\mathbb{G}_{m,K}$), then $T^{NR} = T$. In general, both models have the same neutral component.

Follow the construction of the Néron model of $T$ as explained in [BLR].

- Step 1, construct a group scheme $T^0$ over $\mathcal{O}$ such that $T^0(\mathcal{O}^{sh}) = T^{NR}(\mathcal{O}^{sh})$ is the maximal bounded subgroup of $T(K^{sh})$.

Let $R = \text{Res}_{L/K}(T \otimes L)$, then there exists a canonical closed embedding $T \to R$, and choose $T^0$ to be the schematic closure of $T$ in $R^{NR} \simeq X_*(T) \otimes (\text{Res}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_{m,\mathcal{O}_L}))$, where $X_*(T)$ is the cocharacter group of $T$.

**Proposition 5.2.** $T^0_K = T$ and $T^0(\mathcal{O}^{sh}) = T^{NR}(\mathcal{O}^{sh})$.

**Proof.** Since all schemes are affine, the first equality is easy from algebraic facts. Let $A, B, C, D$ be the affine rings of $R^{NR}, R, T, T^0$ respectively, and assume $f : A \to B, g : B \to C, h : A \to C$ are the corresponding morphisms and $h = g \circ f$. Then $D = A/\text{Ker}h$ and $h$ induce a mapping $h' : D \to C$. Now we want to show $D \otimes K \to C$ is isomorphic. It is surjective since $A \otimes K = B$ and $g$ is surjective. The injectivity follows from $K$ is flat $\mathcal{O}$-module. Thus $h' \otimes id : D \otimes K \to C \otimes K$ is injective and $C \otimes K = C$. 

\[\text{10}\]
Let \( u \in T^0(O^{sh}) \), then it is in the maximal bounded subgroup of \( R(K^{sh}) \) since it is in \( R^{NR}(O^{sh}) \). So we have \( T^0(O^{sh}) \subseteq T^{NR}(O^{sh}) \). Conversely, let \( t \in T^{NR}(O^{sh}) \), then it lifts \( t' \) in \( R^{NR}(O^{sh}) \), we want to show it factor through \( T^0 \). And this is clear from the universal property of quotient of rings.

- Step 2, apply the smoothening process to \( T^0 \), then we can get the Néron model \( T^{NR} \) of \( T \).

Let \( Z' \) be the Zariski closure of \( \{ x \mod \pi \in T^i(K^{sep}) : x \in T^i(O^{sh}) \} \) as a closed subscheme of \( T^i \otimes \kappa \) with the reduced induced structure. Let \( T^{i+1} \) is the dilatation of \( Z' \) on \( T^i \).

Let \( \delta = \max \{ \delta(x) : x \in T^0(O^{sh}) \} \), where \( \delta(x) \) is the Néron measure for the defect of smoothness. Then \( T^{NR} = T^i \) for \( i \geq \delta \).

Similarly, do the same process to \( R^0 = R^{NR} \). For \( i \geq 0 \), let \( W^i \) be the Zariski closure of

\[
\{ x \mod \pi \in R^i(K^{sep}) : x \in T^0(O^{sh}) \subseteq R^i(O^{sh}) \},
\]

as a subscheme of \( R^i \otimes \kappa \) with the reduced induced structure. Then \( R^{i+1} \) is the dilatation of \( W^i \) on \( R^i \). Clearly, we have \( T^0(O^{sh}) \subseteq R^i(O^{sh}) \subseteq (R^{i+1}(O^{sh})) \).

**Lemma 5.3.** For \( i \geq 0, N \geq 1 \), \( R^{i+1} \otimes \mathcal{O}/\pi N \) depends only on \( R^i \otimes \mathcal{O}/\pi N+1 \mathcal{O} \) in a canonical way.

**Proof.** This is just a corollary of Proposition 3.4. \( \square \)

**Lemma 5.4.** The schematic closure of \( T \) in \( R^i \) is \( T^i \) for all \( i \geq 1 \). In particular, it is \( T^{NR} \) for \( i \gg 0 \).

**Proof.** Prove it by induction on \( i \). \( T^{i-1} \) is a closed subgroup of \( R^{i-1} \), and \( W^{i-1} \) is the image of \( Z^{i-1} \) in \( T^{i-1} \rightarrow R^{i-1} \). Then \( R^i \) is a closed subscheme of subgroup of \( R^i \) by Corollary 3.3. So the schematic closure of \( T^i \)'s generic fibre \( T \) in \( R^i \) is itself. \( \square \)

**Remark.** When \( i \geq \delta(e; T^0) \), \( T^i \) is smooth, hence \( T^{NR} = T^i \). So we want to control \( \delta(e; T^0) \). Let \( T^0_L = T^0 \otimes \mathcal{O}_L \), the schematic closure of \( T \otimes L \) in \( R^{NR} \otimes \mathcal{O}_L \). Let \( R' = R^{NR} \otimes \mathcal{O}_L, R^i = X_s(R \otimes_K L) \otimes \mathbb{Z} \otimes \mathbb{G}_{m/\mathcal{O}_L}, T^i = X_s(T \otimes_K L) \otimes \mathbb{Z} \otimes \mathbb{G}_{m/\mathcal{O}_L} \). There are canonical morphisms \( T^i \rightarrow R^i \), and \( \varphi : R' \rightarrow R^i \). Let \( T^i = T^i \times_{R^i} R' \). Since \( T^i \rightarrow R^i \) is a closed immersion, hence \( T^i \rightarrow R^i \) is also a closed immersion by base change. Since \( T^i \) has generic fibre \( T \otimes L \), \( T^i_L \) is equal to the subscheme closure of \( T \otimes L \) in \( T^i \). By the lemma 4.5, we have \( \delta(e, T^0) \leq \delta(e, T^i) \). So it is enough to control \( \delta(e, T^i) \).
Suppose that \( B \) is a noetherian regular local ring and \( B/J \) is a complete intersection ring. Then \( B/J \) is an ideal of \( B \) if and only if \( J \) is a regular ideal in \( B \).

**Proof.** All following objects are determined only by \((B/J, \pi, \Gamma, \Lambda)\):

\[
R^t \otimes B/J, \quad T^t \otimes B/J, \quad R^t \otimes B/J^t, \quad T^t \otimes B/J^t,
\]

and the matrix \( e^t(M \mod \pi^N) \). And if \( Ne(L/K) > \delta(e, T') \), and by Lemma 4.3, \( \delta(e; T') \) is also determined by \((O/\pi^N, O_{L^t}/\pi^N, \Gamma, \Lambda)\). So the lemma is true.

\( \square \)

## 6 Singularities of commutative group schemes

**Definition 6.1.** Suppose \( A \) is a noetherian local ring. We say that \( A \) is a complete intersection ring if \( \hat{A} \) is isomorphic to a quotient of a complete local regular ring \( B \) by a regular ideal \( J \). We say that a locally noetherian scheme \( X \) is complete intersection at a point \( x \in X \), if \( O_{X,x} \) is a complete intersection ring.

**Definition 6.2.** Suppose \( f : X \rightarrow S \) is a flat, locally of finite presentation morphism. We say that \( X \) is relative complete intersection (r.c.i.) over \( S \) at the point \( x \) if the fibre \( f^{-1}(f(x)) \) is complete intersection at \( x \). We say that \( f \) is an r.c.i. morphism if \( X \) is r.c.i. over \( S \) at all its points.

**Proposition 6.3.** Suppose \( B \) is a noetherian regular local ring, \( J \) is an ideal of \( B \). Then \( A = B/J \) is a complete intersection ring if and only if \( J \) is a regular ideal of \( B \).

**Proof.** If \( J \) is a regular ideal, then \( \hat{J} \hat{B} \) is also a regular ideal in \( \hat{B} \), hence \( A \) is a complete intersection ring.

Conversely, suppose that \( A \) is a complete intersection ring, we need to show \( J \) is a regular ideal. We can assume \( A \) and \( B \) are both complete since \( \hat{A} = \hat{B}/\hat{J} \hat{B} \).

Choose a presentation \( A = B'/J' \), where \( B' \) is a noetherian, complete, regular local ring and \( J' \) is its regular ideal. Denote \( \pi_1 : B \rightarrow A, \pi_2 : B' \rightarrow A \) be the canonical projections. Consider \( B'' = B \times_A B' \), where \( B'' = \{(b, b') \in \)}
$B \times B' | \pi_1(b) = \pi_2(b')$, a subring of $B \times B'$. We claim that $B''$ is complete local noetherian ring. It is easy to seen that $B''$ is a local ring with unique maximal deal $m = \{(b, b') : \pi_1(b) = \pi_2(b') \in m_A\}$. And $(b, b') \in m$ if and only if $b \in m_B$ and $b' \in m_{B'}$, so $B''$ is complete. Let $a$ be an ideal of $B''$, and let $b$ be the kernel of $B'' \to B$. Then we have

$$0 \longrightarrow a \cap b \longrightarrow a \longrightarrow a/a \cap b \longrightarrow 0$$

and $a/a \cap b \simeq (a + b)/b$. Since $(a + b)/b$ is corresponding to an ideal of $B$, and $a \cap b$ is corresponding to an ideal of $B'$; they are both of finite type. Hence $a$ is also finitely generated.

By Cohen’s theorem, there exits a noetherian, complete, regular local ring $C$ such that $B''$ is a quotient of $C$ with regular ideal. Let $I = \text{Ker}(C \to A)$, then $I$ is the preimage of the regular ideal $J'$, hence $I$ is regular. And $J$ is image of $I$ in a regular ring, hence regular.

**Proposition 6.4.** Let $k \subset k'$ be a filed extension. Suppose $X$ is a locally of finite type $k$-scheme and $X' = X \times_k k'$. Suppose $x' \in X'$ and $x$ is its projection on $X$. Then $X$ is complete intersection at $x$ if and only if $X'$ is complete intersection at $x'$.

*Proof.* The problem is local, so we can assume $X = \text{Spec} A$, where $A$ is a quotient of polynomial ring $k[X_1, \ldots, X_n]$ with ideal $I$. ”only if” part is trivial. Assume $\{f_1, \ldots, f_n\}$ be a minimal generators of $I$ at $x$, then they also generate $I' = I \otimes k'$ at $x'$. If they are not regular sequence in $I'_{x'}$, then some $f_i$ is generated by others in $I'_{x'}$. Hence $f_i$ is also generated by others in $I_{x}$ by the faithfully flatness of $k'$ over $k$. This is contradiction with the choice of $f_i$'s. \qed

**Proposition 6.5.** (1). Suppose $f : X \to S$ is an r.c.i morphism. Let $f' = f_{S'} : X \times S' \to S'$ be the base change compatible with $g : S' \to S$. Then $f'$ is also a r.c.i morphism. If $g$ is fpqc (ie. faithfully flat, quasi compact), then vice versa.

(2). If $f : X \to Y, g : Y \to Z$ are both r.c.i morphism. Then so is $g \circ f : X \to Z$.

*Proof.* Clearly from Proposition 6.4. \qed

**Lemma 6.6.** Let $G$ be a commutative group scheme, flat and of finite type over a noetherain base scheme $S$. Then $G \to S$ is an r.c.i morphism.

*Proof.* We can assume $S = \text{Spec} k$, where $k$ is algebraically closed. Suppose that $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$ is an exact sequence of commutative
group scheme over \( k \). Assume that \( G' \) and \( G'' \) are r.c.i over \( \text{Spec} \, k \), we claim that \( G \to G'' \) is also an r.c.i morphism, hence \( G \to G'' \to \text{Spec} \, k \) is an r.c.i morphism. By proposition 6.5, it is enough to check after a fpqc base change \( G \to G'' \), that is, look at \( G \times_{\text{Spec} \, k} G'' \to G \). This morphism is canonically isomorphic to \( G \times_{\text{Spec} \, k} G'' \to G \), which is projection to the first factor, and it is an r.c.i morphism since \( G' \to \text{Spec} \, k \) is.

For any \( G \) over \( k \), \( G \) admit a composition series in which the factor are smooth, isomorphic to \( \mu_p \), or \( \alpha_p \). And these factors are clearly r.c.i over \( k \), hence by induction, \( G \to S \) is an r.c.i morphism.

\[ \square \]

**Lemma 6.7.** Suppose that \( X \) is a noetherian scheme and \( X \to \text{Spec} \, \mathcal{O} \) is a flat r.c.i morphism. Then for any \( N \geq 1 \), the collection of points:

\[ \bigcup \{ x \mod \pi^N \in X(C/\pi^NC) : x \in X(C) \}, \]

as \( C \) ranges over local \( \hat{\mathcal{O}} \)-algebra which are flat, and r.c.i over \( \hat{\mathcal{O}} \), is schematic-ally dense in \( X \otimes \mathcal{O}/\pi^N \).

**Proof.** Since \( \mathcal{O} \to \hat{\mathcal{O}} \) is faithfully flat and \( \text{Spec} \, \hat{\mathcal{O}} \to \text{Spec} \, \mathcal{O} \) is surjective, we can assume \( X = \text{Spec} \, A \), and \( A \) is a complete noetherian local ring such that \( \pi \in m_A \).

Choose a presentation \( A = B/I \), where \( B = \langle [X_1,...X_b] \rangle \). Since \( X \) is r.c.i over \( \mathcal{O} \), then \( I \) is generated by a regular sequence \( (t_1,...,t_a) \). Hence, \( (t_1,...,t_a) \otimes \kappa \) is a regular sequence on \( B \otimes \kappa \). Extend \( (t_1,...,t_a) \otimes \kappa \) to a system of regular parameters, and lift the sequence to a sequence \( (t_1,...,t_b) \) in \( B \). Put \( J_n = (t_1^n, ..., t_a^n) \). Then \( \cap_n J_n \subset \cap_n m^n = 0 \). Let \( C_n = B/(I + J_n) \) and \( \text{Spec} \, C_n \to X \) is induced by \( B/I \to B/(I + J_n) \). Then \( \text{Spec} \, C_n \to X; n \geq 1 \) is schematically closed in \( X \). \( I + J_n = (t_1,...,t_a, t_{a+1}^n,...,t_b^n) \) and \( (t_1,...,t_a, t_{a+1}^n,...,t_b^n) \) is also a regular system in \( B \), hence \( C_n \) is r.c.i of relative dimension 0, and then finite over \( \hat{\mathcal{O}} \). Clearly, \( \pi^k \) is not in \( I + J_n \) for any integers \( k \), so \( C_n \) is also flat.

From above, the points \( \text{Spec} \, C_n \otimes \mathcal{O}/\pi^N \to X(\mathcal{O}/\pi^N) : n \geq 1 \) is schematic-

\[ \square \]

**Proposition 6.8.** Let \( G \) be a commutative noetherian group scheme over \( \mathcal{O} \), not necessarily flat. Let \( \hat{G} \) be the schematic closure of \( G \otimes K \) in \( G \). Then \( \hat{G} \otimes \mathcal{O}/\pi^N \) is the schematic closure in \( G \otimes \mathcal{O}/\pi^N \) of the following collection of points

\[ \bigcup \{ x \mod \pi^N \in G(C/\pi^NC) : x \in G(C) \} \]

as \( C \) ranges over local \( \hat{\mathcal{O}} \)-algebras which are flat, finite, and r.c.i over \( \hat{\mathcal{O}} \).
Proof. \( \overline{G}(C) = G(C) \) for any flat \( \mathcal{O} \)-algebra. Then it is clear from the two lemmas before.

Lemma 6.9. The collection of \( \mathcal{O}/\pi^N \)-algebras \( \{ C/\pi^N : C \text{ is a local, flat, finite, r.c.i } \hat{O} \text{-algebra} \} \) is just the collection of all local \( \mathcal{O}/\pi^N \)-algebras which are flat, finite, and r.c.i over \( \mathcal{O}/\pi^N \).

Proof. Since the property of being r.c.i is stable under any base change. So we only need to show that any local flat, finite, r.c.i \( \hat{O} \)-algebra is of the form \( C/\pi^N \) for some C.

Choose a presentation \( A = B/I, B = \mathcal{O}[X_1, ..., X_n], m = (\pi, X_1, ..., X_n), \pi^N \in I \). Since \( B \) is regular and \( A \) is r.c.i, then \( I \) is generated by a regular sequence \( (\pi^N, f_1, ..., f_m) \). Since \( A \) is of dimension 0, we have \( m = n \).

Lift \( f_i \) to \( \tilde{f}_i \in \hat{O}[X_1, ..., X_n]_{\tilde{m}} \), where \( \tilde{m} = (\pi, X_1, ..., X_n) \). Then \( C = \hat{O}[X_1, ..., X_n]_{\tilde{m}}/(\tilde{f}_1, ..., \tilde{f}_n) \) is flat, finite, and r.c.i \( \hat{O} \)-algebra and \( A = C/\pi^N \).

\[ \square \]

7 Elkik’s theory

In this section, let R be a noetherian \( \mathcal{O} \)-algebra, complete with respect to the \( \pi \)-adic topology. Consider \( R[X] = R[X_1, ..., X_n] \), the polynomial ring in \( N \) variables. Let \( I \) be an ideal of \( R[X] \) and put \( B = R[X]/I, Y = \text{Spec } B \).

We assume that \( Y \otimes_{\mathcal{O}} K \to \text{Spec } (R \otimes_{\mathcal{O}} K) \) is smooth of relative dimension \( s \). The Jacobian ideal of \( I \) is defined to be the ideal of \( R[X] \) generated by the \((N - s)\)-minors of \( \left( \frac{\partial f_i}{\partial X_j} \right)_{s \times N} \) for all \( f_1, ..., f_s \) in a generating set of \( I \). By smoothness assumption and Jaccobi Criterion, \( J + I \supset \pi^h R[X] \) for some \( h \geq 0 \). Fix such an \( h \) in the following.

Lemma 7.1 (Elkik). Suppose that \( I \) can be generated by \( N - s \) elements. Then for any \( n > 2h \), the image of \( Y(R) \to Y(R/I^{n-h}) \) is the same as the image of \( Y(R/I^n) \to Y(R/I^{n-h}) \).

Proof. We restate the lemma as following: If \( a = (a_1, ..., a_N) \in R^N \) such that \( I(a) = 0 \mod \pi^n \), where \( I(a) = \{ f(a) : \forall f \in I \} \), then there exists \( a' \in R^N \) such that \( a \equiv a' \mod \pi^{n-h} \) and \( I(a) = 0 \).

Since \( R \) is complete and by approximation, it is enough to find \( y = (y_1, ..., y_N) \in R^N \) such that \( y_i \equiv 0 \mod \pi^{n-h}, \forall i \) and \( I(a - y) \subset (\pi^{2n-2h}) \).

Let \( M \) be the Jacobian matrix of \( I \), and by Taylor’s expansion,

\[
\begin{pmatrix}
  f_1(a - y) \\
  \vdots \\
  f_{N-s}(a - y)
\end{pmatrix} = \begin{pmatrix}
  f_1(a) \\
  \vdots \\
  f_{N-s}(a)
\end{pmatrix} - M(a) \begin{pmatrix}
  y_1 \\
  \vdots \\
  y_N
\end{pmatrix} + \sum y_i y_j Q_{ij}(a - y),
\]

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Where $Q_{ij}$ is an $(N - s)$-column vector whose components are the polynomial in $a$ and $y$. Hence we just need to find a $y = (y_1, ..., y_n)$, such that $y_i \equiv 0 \mod \pi^{n-h}$ and
\[
\begin{pmatrix}
  f_1(a - y) \\
  \vdots \\
  f_{N-s}(a - y)
\end{pmatrix} = M(a) \begin{pmatrix}
  y_1 \\
  \vdots \\
  y_n
\end{pmatrix} \mod \pi^{2n-2h}
\]

Let $\delta$ be a nonzero $(N - s)$-minor of $M$, then exits $N \times (N - s)$ matrix $M_\delta$ such that $MM_\delta = \delta Id$, where Id means the identity matrix. By assumption, we have $\sum \delta P + Q = \pi^h$ in $R[X]$ for some $Q \in I$.

\[
\pi^h \begin{pmatrix}
  f_1(a) \\
  \vdots \\
  f_{N-s}(a)
\end{pmatrix} = (\sum \delta P + Q)(a) \begin{pmatrix}
  f_1(a) \\
  \vdots \\
  f_{N-s}(a)
\end{pmatrix}
\]

\[
= \sum \delta P_\delta(a) \begin{pmatrix}
  f_1(a) \\
  \vdots \\
  f_{N-s}(a)
\end{pmatrix} \mod \pi^{2n}
\]

\[
= \sum P_\delta M(a)M_\delta(a) \begin{pmatrix}
  f_1(a) \\
  \vdots \\
  f_{N-s}(a)
\end{pmatrix} \mod \pi^{2n}
\]

\[
= M(a)[\sum P_\delta M_\delta(a)] \mod \pi^{2n}
\]

Let $y = (\sum P_\delta M_\delta(a) / \pi^h$, then $y$ is what we need. \qed

**Lemma 7.2.** Suppose that $R$ is a local ring, and $Y \to \text{Spec } R$ is a flat r.c.i morphism. Then for any $n \geq 2h$, the image of $Y(R) \to Y(R/\pi^{n-h}R)$ is the same as the image of $Y(R/\pi^nR) \to Y(R/\pi^{n-h}R)$.

**Proof.** Let $y : \text{Spec } R/\pi^n \to Y$ be a closed point of $Y(R/\pi^n)$. Let $m$ be the unique maximal ideal in $R/\pi^n$, $q = y(m)$.

Since $Y \to \text{Spec } R$ is r.c.i, and $\text{Spec } R[X] \to \text{Spec } R$ has regular fibre. Then there exists $f \in R[x]$, such that $q \in Y_f$ and $Y_f$ is cut out by $(N - s)$ equations in $\text{Spec } R[X]_f$, and regard $\text{Spec } R[X]_f$ as a closed subscheme of $\text{Spec } R[X][Z]$ cut out by $Zf - 1$. Then $Y_f$ is cut out by $(N + 1 - s)$ equations in $A^{N+1}$. By Elkik’s lemma, there exists $y' \in Y_f(R) \subset Y(R)$ such that $y \equiv y'$ mod $\pi^{n-h}$. \qed
8 Congruences of Néron models

In this section, assume $K$ is complete for simplicity. Notations are the same as Section 5.

Since $R^{NR}$ is the Néron model of an induced torus, we can realize $R^{NR}$ as a closed subscheme of $\mathbb{A}^{d(n+1)}_{\hat{O}}$, defined by $n$ explicit equations. Recall that the closed subscheme $T'$ of $R'$ is cut out by $(\dim R^{NR} - \dim T)$ equations, and $R' = R^{NR} \otimes O_L$. Hence, $T'$ can be realized as a closed subscheme of $\mathbb{A}^{d(n+1)}_{\hat{O}_L}$ defined by an ideal $I'$ generated by $(d(n + 1) - \dim T)$ equations.

Let $J'$ be the Jacobian ideal for $I'$. Since the generic fibre of $T'$ is smooth, $I' + J'$ contains $\pi^h$ for some $h > 0$. Let $h = h \left( O, O_L, \Gamma, \Lambda \right)$ be the smallest integer with this property.

**Lemma 8.1.** Suppose $(O, O_L, \Gamma, \Lambda) \equiv (O_0, O_{L_0}, \Gamma_0, \Lambda_0)(\text{level}N)$. Form the Jacobian ideals $J'$ and $J'_0$ and define the integer $h$ and $h_0$ for both data. If $h < N$ or $h_0 < N$, then $h = h_0$.

**Proof.** Suppose $h < N$. Since $T'$ just depends on $(O, O_L, \Gamma, \Lambda)$, hence $I'$ and $J'$ just depends on $(O, O_L, \Gamma, \Lambda)$. Then $J' \otimes O/\pi^N$ just depends on $(O/\pi^N, O_L/\pi^N, \Gamma, \Lambda)$. So $I'_0 + J'_0 \supset \pi^h_0 O_{L_0}[X_1, \ldots, X_{d(n+1)}]$ contains $\pi^h_0$. Then by Nakayama’s Lemma, we have $I'_0 + J'_0 \supset \pi^h_0 O_{L_0}[X_1, \ldots, X_{d(n+1)}]$ Therefore $h_0 \leq h \leq N$. Similarly, $h \leq h_0$, hence $h = h_0$.  

**Definition 8.2.** If $h < n$, define $h \left( O, O_L, \Gamma, \Lambda \right)$ to be $h$; otherwise define $h \left( O, O_L, \Gamma, \Lambda \right) = N$. This is justified by the lemma.

**Proposition 8.3.** The group scheme $T^0_0 \otimes O_L/\pi^{N-h}$ is determined by $(O/\pi^N, O_L/\pi^N, \Gamma, \Lambda)$ if $N > 2h$.

**Proof.** By lemma 6.8, it is enough to show that the collection of points

$$\bigcup_C \text{image } (T'(C) \rightarrow T'(C/\pi^{N-h})), $$

where $C$ ranges over all local finite flat $\hat{O}$-algebra, is determined by $(O/\pi^N, O_L/\pi^N, \Gamma, \Lambda)$. Since $T'$ is complete intersection and by Lemma 7.2, this collection is the same as the union of the image $T'(C/\pi^N) \rightarrow T'(C/\pi^{N-h})$ over all local, flat, r.c.i over $O/\pi^N$ and this is clearly determined by $(O/\pi^N, O_L/\pi^N, \Gamma, \Lambda)$.  

**Corollary 8.4.** The group scheme $T^0 \otimes O/\pi^{N-h}$ is determined by $(O/\pi^N, O_L/\pi^N, \Gamma, \Lambda)$ for $N > 2h$. 

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Proof. We have $T^0_L = T^0 \otimes \mathcal{O}_L$, and by Proposition 8.3, the corollary is clearly derived from the following easy lemma: Suppose $X, X'$ are closed $S$-subschemas of an $S$-scheme $Y$ such that $X \times_S S' = X' \times_S S'$ in $Y \times_S S'$ for some $S' \to S$ faithfully flat. Then $X = X'$.

In the following, we use the notations and procedure in Section 4 and Section 5. $T^0$ is a closed subscheme of $\mathbb{A}^{d(n+1)}$, defined by an ideal $I$ and let $J \subset \mathcal{O}[X_1, ..., X_{d(n+1)}]$ be the Jacobian ideal of $I$. Since $I' \subset I$, we have $J' \subset J$ and $\pi^h \in (J' + I')$.

Proposition 8.5. 1), $T^0 \otimes \mathcal{O}/\pi^N$ is determined by $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ for all $N \geq 1, m \geq \max(N + h, 2h + 1)$.

2), $R^i \otimes \mathcal{O}/\pi^{m-i}$ depends only on $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ for all $m \geq \max(2h + i, 3h + 1)$.

3), $W^i$ depends only on $fourm$, for $m \geq \max(2h + i + 1, 3h + 1)$.

Proof. 1). $T^0 \otimes \mathcal{O}/\pi^N$ is determined by $T^0 \otimes \mathcal{O}/\pi^{\max(n, h+1)}$, and then the proposition follows immediately from Corollary 8.4.

2), By Lemma 5.3 and by induction, $R^i \otimes \mathcal{O}/\pi^{m-i}$ is determined by $R^0 \otimes \mathcal{O}^m$, and $R^0 = \Lambda_\times(T) \otimes \text{Res}_{\mathcal{O}_L/\mathcal{O}_k}(\mathbb{G}_m)$, then the statement is clear.

3), For $i=0$. From definition of $W^0$, $W^0$ is determined by the image of $T^0(\mathcal{O}^h) \to T^0(\mathcal{O}^h/\pi^N)$ for any $N \geq 1$, in particular $N = h + 1$. Moreover, $W^0$ is group scheme, hence is r.c.i. By lemma 8.2, this image is determined by $T^0(\mathcal{O}^h/\pi^{2h+1})$, and the latter is determined by $T^0 \otimes \mathcal{O}^{sh}/\pi^{2h+1}$, which is determined by $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ for $m \geq 3h + 1$, according to Corollary 8.4.

In general, let $B^i$ be the affine ring of $R^i$, and recall the notations in Section 3,

$$B^i = B^{i-1}[Y_1, ..., Y_n]/(\pi Y_1 - g_1, ..., \pi Y_n - g_n) \mod \pi - \text{torsion},$$

where write the image of $Y_i$ as $\overline{y}_i$, suggestively. A point $y$ in $R^i$ is determined by the projection of $y$ on $R^{i-1}$, together with the additional ”coordinates” $(\pi^{-1}g_1(y), ..., \pi^{-1}g_n(y))$.

For $x \in T^0(\mathcal{O}^h)$, by the universal property of dilatations, $x$ is also in $R^i(\mathcal{O}^h)$, denoted by $x_i$. Then $x_i \mod \pi$ is determined by $x_{i-1} \mod \pi^2$. Inductively, the image of $T^0(\mathcal{O}^h) \to T^0((\mathcal{O}^h)/\pi^{i+1})$ determined $W^i$. As in the case $i=0$, this image is determined by $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ whenever $m \geq \max(2h + i + 1, 3h + 1)$.

Let $\delta = \lfloor \delta(e, T) \rfloor$, we have $\delta \leq h$ from Section 4. If $\delta < N$, we define $\delta(\mathcal{O}/\pi^N, \mathcal{O}_L/\pi^N, \Gamma, \Lambda)$ to be $\delta$; otherwise, we define $\delta(\mathcal{O}/\pi^N, \mathcal{O}_L/\pi^N, \Gamma, \Lambda) = N$. The definition is justified by lemma 5.5.
Lemma 8.6. Let $X$ be a smooth group scheme over $\mathcal{O}$. Then the schematic closure of the points $\{x : x \in X(\mathcal{O}^{sh}/\pi^N)\} = \{x \mod \pi^N : x \in X(\mathcal{O}^{sh})\}$ in $X \otimes \mathcal{O}/\pi^N$ is the whole $X \otimes \mathcal{O}/\pi^N$.

Proof. We first show $\{x : x \in X(\mathcal{O}^{sh}/\pi^N)\} = \{x \mod \pi^N : x \in X(\mathcal{O}^{sh})\}$. The notations are the same as Section 7. By lemma 4.2, we have $h = 0$, then the equality is clear by lemma 7.2.

The statement is local, we can assume $X = \text{Spec} A$ is smooth over $\mathcal{O}$. Suppose $f \in A$ satisfies $\pi^h f = 0$, $\forall x$. Then $f \mod \pi$ is zero on every closed points of $X \otimes \kappa^{sep}$, hence $f \in \pi A$. And by induction, we have $f = 0$. □

Theorem 8.7 (Main Theorem). Suppose that $N \geq 1, m \geq \max(N + \delta + 2h, 3h + 1)$, where $h = h(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ as defined at the beginning of this section, $\delta = \delta(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ as defined above. Then, $T^{NR} \otimes \mathcal{O}/\pi^N$ is determined by $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$.

Proof. By lemma 3.3 and remark in Section 5, $T^{NR}$ is the schematic closure of $T$ in $R^\delta$.

Let $Y$ be the image of $T^{NR}(\mathcal{O}^{sh}/\pi^N)$ in $R^\delta(\mathcal{O}^{sh}/\pi^N)$, then the schematic closure of $Y$ in $R^\delta \otimes \mathcal{O}/\pi^N$ is simply $T^{NR} \otimes \mathcal{O}/\pi^N$ by the precious lemma. So we just need to show $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ determine $Y$.

As explained in the proof of Proposition 8.5(3), $Y$ is determined by the image of $T^0(\mathcal{O}^{sh}) \to T^0(\mathcal{O}^{sh}/\pi^{\delta+N})$, which is determined by the image of $T^0(\mathcal{O}^{sh}) \to T^0(\mathcal{O}^{sh}/\pi^{\max(\delta+N,h+1)})$, which is the same as the image of $T^0(\mathcal{O}^{sh}/\pi^{\max(N+\delta,h+1)+h}) \to T^0(\mathcal{O}^{sh}/\pi^{\max(N,h+1)})$ by lemma 7.2. By Corollary 8.4, $T^0(\mathcal{O}^{sh}/\pi^{\max(N+\delta+2h+1)})$ is determined by $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$. Hence, the proof is over. □

9 The invariant $c(T)$ and Artin conductor

Let $K$ be a complete discrete valuation field. We define an invariant of a torus $T$ over $K$ as following: by the universal property of the Néron model, there is a canonical morphism $T \otimes \mathcal{O}_L$ to the (usual) Néron model of $T \otimes L$ extending the identity morphism on the generic fibres. This morphism induces a morphism

$$\Phi_{T,L} : T^{NR} \otimes \mathcal{O}_L \to (T \otimes L)^{NR},$$

Definition 9.1. Let $L$ be a splitting field of $T$, and let $e(L/K)$ be the ramification index of $L/K$. Define

$$c(T) = \frac{1}{e(L/K)} \text{length}_{\mathcal{O}_L} \frac{\omega(T^{NR}) \otimes \mathcal{O}_L}{\Phi_{T,L}^*(\omega((T \otimes L)^{NR}))}.$$
where $\omega(T^{NR})$ (resp. $\omega((T \otimes L)^{NR})$) denotes the module of the translation invariant top differential forms on $T^{NR}$ (resp. $(T \otimes L)^{NR}$). It can easily be seen that this rational number does not depend on the choice of a splitting extension $L/K$.

Note that $\omega(G)$ is the dual of $\bigwedge^{\text{top}} \text{Lie}(G)$ for any smooth group scheme $G$ over $\mathcal{O}_L$.

**Artin conductors of representations**

Let $L/K$ be a finite Galois extension with Galois group $G$. Let $v_L$ be the normalized valuation of $L$ and $\pi_L$ be a prime element of $\mathcal{O}_L$. Let $f$ be the residue degree of $L/K$. Let $\sigma \in G$ and set

$$a_G(\sigma) = -f \cdot v_L(\sigma(\pi_L) - \pi_L) \quad \text{if } \sigma \neq 1$$

$$a_G(1) = f \sum_{\sigma \neq 1} v_L(\sigma(\pi_L) - \pi_L)$$

Then the function $a_G$ is the character of a linear representation $\rho : G \to GL(V)$ by [Serre1, VI.2 Thm 1].

**Definition 9.2.** The **Artin conductor** $a(V)$ of the presentation $\rho : G \to GL(V)$ is defined to be the number

$$\frac{1}{\text{Card}(G)} \sum_{\sigma \in G} a_G(\sigma) \chi(\sigma^{-1}),$$

where $\chi$ is the character of the presentation.

Let $G_i$ be the $i$-th ramification group of $L/K$, of cardinality $g_i$. Then

$$a(V) = \sum_{i \geq 0} \frac{g_i}{g_0} \dim(V/V^{G_i}).$$

**Example 9.3.** Let $T = \text{Res}_{L/K}(\mathbb{G}_m)$, then

$$c(T) = \frac{1}{2} a(X_*(T) \otimes \mathbb{Q}) = \frac{1}{2} v_K(\Delta)$$

where $a(-)$ is the Artin conductor of a module over $\mathbb{Q}[[\text{Gal}(K^{\text{sep}}/K)]]$, $\Delta$ is the discriminant of $L/K$, and $v_K$ is the normalized valuation of $K$.

**Proof.** In Section 2, we saw that $X_*(T) = \mathbb{Z}[G]$, where $G = \text{Gal}(L/K)$. Hence $a(\mathbb{Q}[G]) = f v_L(\mathfrak{D}) = v_K(\Delta)$, where $\mathfrak{D}$ is the different of $L/K$. The first equality is attained by [Serre1, IV. Prop 4] and the second one follows from $N_{L/K}(\mathfrak{D}) = \Delta$, where $N_{L/K}$ is the norm of $L/K$. 

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Let \( n = [L : K] \). Assume \( G = \{\sigma_1, \ldots, \sigma_n\} \) and \( \{\alpha_i, i = 1, \ldots, n\} \) is a base of \( \mathcal{O}_L/\mathcal{O}_K \), then the norm \( N \) of \( \sum (x_i; \alpha_i) \) is a polynomial on the \( x_i \)'s. Let \( A = \mathcal{O}_K[X_1, \ldots, X_n, 1/N] \) and let \( R \) be any \( \mathcal{O}_K \)-algebra. If \( f \in \text{Hom}(A, R) \), then \( \sum f(X_i) \otimes \alpha_i \) is a unit in \( R \otimes \mathcal{O}_L \), and vice versa. Hence \( \text{Hom}(A, R) \simeq (R \otimes \mathcal{O}_L)^\times \) for any \( \mathcal{O}_K \)-algebra \( R \), and \( \text{Res}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_m) = \text{Spec} \ A \). Similarly, \( \text{Res}_{L/K}(\mathbb{G}_m_L) = \text{Spec} \ K[X_1, \ldots, X_n, 1/N] \) with the same polynomial \( N \). And the identity map \( A \to A \) induce a unit \( \sum_j (X_j \alpha_j) \) in \( A \otimes \mathcal{O}_L \). Fix the isomorphism \( \Psi : T \otimes L \to \mathbb{G}_m^m \) which is associated to the ring homomorphism \( \Psi^\#: L[X_{\sigma_1}, X_{\sigma_1}^{-1}] \to L[X_1, \ldots, X_n, 1/N] \) given by \( X_{\sigma_i} \to \sum_j \sigma_i(\alpha_j)X_j \).

The map \( \Psi \) induces an isomorphism \( (T \otimes L)^{NR} \to \mathbb{G}_m^m \) and we define the composition \( \Theta \) of \( T^{NR} \otimes \mathcal{O}_L \to (T \otimes L)^{NR} \to \mathbb{G}_m^m \) as following. Let \( \Theta^\# \) be the ring homomorphism associated to \( \Theta \). The map \( \Theta^\# \) is defined as following:

\[
\Theta^\#: \mathcal{O}_L[X_{\sigma_1}, X_{\sigma_1}^{-1}] \to A \otimes \mathcal{O}_L, \quad X_{\sigma_i} \to \sum_j \sigma_i(\alpha_j)X_j.
\]

Now, it is clear that \( c(T) = v_K(\det(\sigma_i(\alpha_j))) = \frac{1}{2}v_K(\Delta) \).

**Proposition 9.4.** The following two statements are equivalent:

1. \( c(T_1) = c(T_2) \) for any tori \( T_1, T_2 \) over \( K \) such that \( T_1 \) is isogenous to \( T_2 \) over \( K \).

2. \( c(T) = \frac{1}{2}a(X_*(T) \otimes \mathbb{Q}) \) for any torus \( T \) over \( K \), where \( a(-) \) is the Artin conductor of a module over \( \mathbb{Q}[\text{Gal}(K^{sep}/K)] \).

**Proof.** Clearly (2) implies (1) by the Proposition 2.7.

Assume (1). We have seen (2) is true when \( T \) is an induced torus. Since \( c(-) \) and \( a(-) \) are both additive with respect to fibre product. And by Proposition 2.8, we have (2). \( \square \)

Let \( \alpha : T_1 \to T_2 \) be an isogeny over \( K \). Let \( L \) be a common splitting field of \( T_1 \) and \( T_2 \), then \( T_1 \otimes L \simeq X_*(T_1) \otimes \mathbb{G}_m^m \) and \( \Omega_{T_1/K}^1 = X^*(T_1) \otimes \Omega_{\mathbb{G}_m^m/K}^1 \).

We have the commutative diagram

\[
\omega((T_2 \otimes L)^{NR})^{(\alpha \otimes L)^*} \xrightarrow{\Phi_{T_2}} \omega((T_1 \otimes L)^{NR})^{\Phi_{T_1}^{-1}}
\]

with injective vertical maps. When \( \text{char}(K) = 0 \), then the horizontal maps are also injective.
For any homomorphism \( g : M \rightarrow N \) of \( \mathbb{Z} \)-modules with finite cokernel, we define
\[
c(g) = \text{length}(N/g(M)).
\]
Clearly
\[
c(g \circ h) = c(g) + c(h).
\]
Hence \( c(\Phi T_2^*) = c(\Phi T_1^*) \) if and only if \( c((\alpha \otimes L)^*) = c(\alpha^* \otimes \mathcal{O}_L) \). We have \( c(\Phi T_1) = e(L/K)c(T_1) \), and \( c((\alpha \otimes L)^*) = v_L(\deg \alpha) \), where \( v_L \) is the normalized valuation of \( L \). Hence,

**Proposition 9.5.** \( c(T_1) = c(T_2) \) if and only if \( c(\alpha^*) = v_K(\deg \alpha) \), where \( v_K \) is the discrete valuation of \( K \) with \( v_K(\pi) = 1 \), and \( \alpha^* : \omega(T_1) \rightarrow \omega(T_2) \).

**Corollary 9.6.** If the residue field \( \kappa \) of \( \mathcal{O} \) has characteristic 0, then \( c(T_1) = c(T_2) \) for any two isogenous tori \( T_1 \) and \( T_2 \).

**Proof.** Let \( \alpha : T_1 \rightarrow T_2 \) be an isogeny. By Proposition 2.6, there exists an isogeny \( \beta : T_2 \rightarrow T_1 \), such that \( \beta \circ \alpha = [\deg \alpha]_{T_1} \), and \( \alpha \circ \beta = [\deg \alpha]_{T_2} \). Since \( \text{char}(\kappa) = 0 \), \( \deg \alpha \) is invertible in \( \mathcal{O}_K \), hence \( (\alpha \otimes L)^* \) and \( \alpha^* \otimes \mathcal{O}_L \) are both isomorphisms. Then \( c(\alpha^*) = c((\alpha \otimes L)^*) = c(\alpha^* \otimes \mathcal{O}_L) = 0 \), thus \( c(T_1) = c(T_2) \). \( \square \)

## 10 Isogeny invariance in characteristic 0

In this section, we will prove that \( c(T) \) is invariant by isogeny when \( K \) has characteristic 0. As we have already proved this when the residue field \( \kappa \) of \( \mathcal{O}_K \) has characteristic 0, we can assume that \( \text{char} \kappa = p > 0 \).

**Lemma 10.1.** Let \( K \) be a field equipped with a discrete valuation and let \( T \) be a torus over \( K \). Let \( T_s \) be the maximal split subtorus of \( T \), and let \( T_a \) be the quotient torus \( T/T_s \). Then the canonical sequence
\[
1 \rightarrow T_s^{NR} \rightarrow T^{NR} \rightarrow T_a^{NR} \rightarrow 1
\]
is exact.

**Proof.** By [SGA 7 VIII. Cor. 6.6 ], we can extend the sequence
\[
1 \rightarrow T_s \rightarrow T \rightarrow T_a \rightarrow 1
\]
to an exact sequence of smooth group schemes
\[
1 \rightarrow T_s^{NR} \rightarrow T^{NR} \rightarrow T_a^{NR} \rightarrow 1.
\]
Hence we have the commutative diagram
\[
\begin{array}{cccc}
1 & \rightarrow & T^s(\mathcal{O}^sh) & \rightarrow T^s(\mathcal{O}^sh) \\
& & \simeq & \\
1 & \rightarrow & T_a(\mathcal{O}^sh) & \rightarrow T_a(\mathcal{O}^sh)
\end{array}
\]
\[
\begin{array}{cccc}
1 & \rightarrow & T^s(K^sh) & \rightarrow T(K^sh) \\
& & \simeq & \\
1 & \rightarrow & T_a(K^sh) & \rightarrow T_a(K^sh)
\end{array}
\]
Since $T^* \rightarrow T_a$ is smooth, and by [BLR. 2.2 Prop 14], the first low is exact. Thus $T^*(\mathcal{O}^sh) = T(K^sh)$, and by [BLR. 7.1 Thm 1], we have $T^* = T^{NR}$. □

**Theorem 10.2.** Let $K$ be a complete discrete valuation field with mixed characteristic $(0, p)$ and perfect residue field. Let $T_1, T_2$ be two tori over $K$, and let $\alpha : T_1 \rightarrow T_2$ be a $K$-isogeny. Then two tori have the same invariant:
\[c(T_1) = c(T_2) = \frac{1}{2} a(X_*(T_1) \otimes \mathbb{Q}).\]

**Remark.** I will restrict myself to the case when $K$ is a finite extension of $\mathbb{Q}_p$. For the general case, see the original paper of Ching-Li Chai and Jiu-Kang Yu.

**Proposition 10.3.** Consider the pull-back map $\alpha^*: \omega(T_2^{NR}) \rightarrow \omega(T_1^{NR})$. There exists an element $a \in \mathcal{O}_K$, unique up to $\mathcal{O}_K^\times$, such that $\alpha^*(\omega(T_2^{NR})) = a \cdot \omega(T_1^{NR})$. Denote the rational number $p^{\text{ord}_p(a)}$ by $\deg_{\text{diff}}(\alpha)$. Then
\[\deg_{\text{diff}}(\alpha) \leq p^{\text{ord}_p(\deg_{\alpha})}.
\]
In the above, $\text{ord}_p$ denotes the valuation on $K$ with $\text{ord}_p(p) = 1$.

**Proof.** Suppose $K$ is a finite extension of $\mathbb{Q}_p$.

By lemma 10.1, we may assume that $T_1$ and $T_2$ are anisotropic over the maximal unramified extension of $K$ (replacing $K$ by a finite unramified extension $L/K$ if necessary). Then $T_i^{NR}(\mathcal{O}_L) = T_i(L)$ for any unramified extension $L/K$, $i = 1, 2$.

Let $T_i^{NR}$ be the neutral component of the Néron model $T_i^{NR}$, $i = 1, 2$. Let $\omega_i$ be an $\mathcal{O}_K$-generator of $\omega(T_i)^{NR}$, $i = 1, 2$. Let $\text{ord}_K$ be the valuation of $K$ with $\text{ord}_K(\pi) = 1$. Let $M = \text{Ker}(\alpha)$, the kernel of isogeny $\alpha$. Consider finite unramified extension $L/K$, and let $q_L$ be the cardinality of the residue field $\kappa_L$ of $\mathcal{O}_L$. Let $|\omega_i|$ be the Haar measure on $T_i^{NR}$ attached to $\omega_i$, $i = 1, 2$.

Hence we have
\[|\alpha^* \omega_2|(T_1^{NR}(\mathcal{O}_L)) = \text{Card}(M(L) \cap T_1^{NR}(\mathcal{O}_L)) \cdot |\omega_2|(\alpha(T_1^{NR}))\]
By definition, for $i = 1, 2$, $|\omega_i|(T_i^{NR}(\mathcal{O}_L))$ is equal to the number of $\kappa_L$-rational points of the closed fibre of $T_i^{NR}$, divided by $q_L^{\dim T_i}$. Since $T_i$
is anisotropic, its closed fibre is a unipotent group over \( \kappa_L \), and has the same number of \( \kappa_L \)-rational points as \( \mathbb{A}^{\dim(T_i)} \). Hence \(|\omega_1|(T_1^{NR}(\mathcal{O}_L)) = |\omega_2|(T_2^{NR}(\mathcal{O}_L))\), and

\[ [T_2^{NR}(\mathcal{O}_L) : \alpha(T_1^{NR})] = \text{Card}(M(L) \cap T_1^{NR}(\mathcal{O}_L)) \cdot q_L^{\text{ord}_K(\alpha)} \]

Let \( C_{T_i} \) be the group of geometric connected components of the closed fibre of \( T_i^{NR}, i = 1, 2 \). For sufficiently large finite unramified extension \( L \) of \( K \), we have

\[ [T_2^{NR}(\mathcal{O}_L) : \alpha(T_1^{NR}(\mathcal{O}_L))] = \frac{\text{Card}(C_{T_i})}{\text{Card}(C_{T_2})}[T_2^{NR}(\mathcal{O}_L) : \alpha(T_1^{NR}(\mathcal{O}_L))]. \]

On the other hand, by Tate’s formula for the Euler-Poincaré characteristic for the Galois cohomologies of local fields, we have

\[ \text{Card}(H^1(L, M)) = q_L^{\text{ord}_K(\deg \alpha)} \cdot \text{Card}(M(L)) \cdot \text{Card}(H^2(L, M)). \]

By the local duality for Galois cohomology of local fields ([Milne, I, Cor. 2.3]), \( H^2(L, M) \) is the dual of \( M_D(L) \), where \( M_D \) is the Cartier dual of the finite group scheme \( M \) over \( K \).

From the long exact sequence of Galois cohomologies attached to the isogeny \( \alpha \), we get an injection from \( T_2(L)/\alpha(T_1(L)) \) to \( H^1(L, M) \). Thus we have

\[ \frac{\text{Card}(C_{T_2})}{\text{Card}(C_{T_1})} \text{Card}(M(L) \cap T_1^{NR}(\mathcal{O}_L)) \cdot q_L^{\text{ord}_K(\alpha)} \leq q_L^{\text{ord}_K(\deg(\alpha))} \cdot \text{Card}(M(L)) \cdot \text{Card}(H^2(L, M)). \]

As \( L \) tends to \( K^{sh} \), we have \( q_L \to +\infty \). Hence, we get \( \text{ord}_K(\alpha) \leq \text{ord}_K(\deg \alpha) \).

Since \( \text{ord}_K = \text{ord}_K(\rho) \cdot \text{ord}_p \), we have

\[ \text{deg}_{diff}(\alpha) \leq p^{\text{ord}_p(\deg(\alpha))}. \]

\[ \square \]

Proof of Theorem 10.2. Choose an isogeny \( \beta : T_2 \to T_1 \) such that \( \beta \circ \alpha = [n]_{T_1} \). Let \( d = \dim T_1 = \dim T_2 \). Write \( n = p^m u \), where \( m = \text{ord}_p(n) \). We have

\[ p^{md} = \text{deg}_{diff}(\beta \circ \alpha) = \text{deg}_{diff}(\beta) \text{deg}_{diff}(\alpha) \leq p^{\text{ord}_p(\deg(\alpha))}p^{\text{ord}_p(\deg(\beta))} = p^{md}. \]

So the equality holds throughout the above inequality. Hence by Proposition 9.5, we have \( c(T_1) = c(T_2) \).

\[ \square \]
11 Isogeny invariance in characteristic $p$
——Application of Deligne’s theory

Deligne’s theory
Let $K$ be a complete local field with a perfect residue field $\kappa$. Let $\mathcal{O}$ be the ring of integers of $K$, and let $e \geq 1$. A Galois extension $L/K$ is at most $e$-ramified if $\text{Gal}(L/K)^e = 1$, where $e$ refers to the upper numbering filtration of the ramifications groups. In other words, $\text{Gal}(L/K)$ is a quotient of $\text{Gal}(K_{\text{sep}}/K)/\text{Gal}(K_{\text{sep}}/K)^e$.

Deligne [Deligne] shows that $\text{Gal}(K_{\text{sep}}/K)/\text{Gal}(K_{\text{sep}}/K)^e$ is canonically determined by $\text{Tr}_eK = (\mathcal{O}/\mathfrak{p}^e, \mathfrak{p}/\mathfrak{p}^{e+1}, \epsilon)$, where $\mathfrak{p}$ is the prime ideal of $\mathcal{O}$, and $\epsilon$ is the canonical map from $\mathfrak{p}/\mathfrak{p}^{e+1}$ to $\mathcal{O}/\mathfrak{p}^e$. Denote $\text{Gal}(K_{\text{sep}}/K)/\text{Gal}(K_{\text{sep}}/K)^e$ by $\Gamma(\text{Tr}_eK)$.

Suppose $\text{Tr}_eK$ is isomorphic to $\text{Tr}_eK_0$ and $L/K$ is at most $e$-ramified. Then there exits a corresponding $L_0/K_0$ and $(\mathcal{O}, \mathcal{O}_L) \equiv (\mathcal{O}_0, \mathcal{O}_{L_0})$ (level $e$). We can construct $L_0$ as following:

Suppose $\phi: \mathcal{O}/\pi^e \to \mathcal{O}/\pi_0^e$ and $\eta: \mathfrak{p}/\mathfrak{p}^{e+1} \to \mathfrak{p}_0/\mathfrak{p}_0^{e+1}$ define the isomorphism $\text{Tr}_eK \to \text{Tr}_eK_0$. Let $\pi_L$ be a prime element of $\mathcal{O}_L$ satisfying the Eisenstein equation

$$X^n + \sum_{i=0}^{n-1} a^{(i)} X^i = 0, \quad a^{(i)} \in \mathfrak{p}.$$ 

Let $a^{(i)}_0 \in \mathcal{O}_0$ be the lifting of $\eta(a^{(i)} \mod \mathfrak{p}^{e+1})$. Then the equation $X^n + \sum_{i=0}^{n-1} a^{(i)}_0 X^i = 0$ defines the extension $L_0/K_0$.

**Proposition 11.1.** Let $T$ be a torus over $K$, then the invariant $c(T)$ is determined by $\text{Tr}_eK$ for $e \gg 0$.

**Proof.** Let $e \gg N \gg 0$ and $\Lambda = X_*(T)$. Since $(\text{Tr}_e(K), \Gamma = \Gamma(\text{Tr}_eK), \Lambda)$ determines $(\mathcal{O}/\pi^e, \mathcal{O}_L/\pi^e, \Gamma, \Lambda)$, hence determines the following morphisms by Section 8: $T^n_L \otimes \mathcal{O}_L/\pi^N \to (T \otimes L)^{NR} \otimes \mathcal{O}_L$, $R^{i+1} \otimes \mathcal{O}/\pi^N \to R^i$; $T^{NR} \otimes \mathcal{O}/\pi^N \to R^i \otimes \mathcal{O}/\pi^N$. The last morphism factors through the closed immersion $T^{NR} \otimes \mathcal{O}/\pi^N \to T^0 \otimes \mathcal{O}/\pi^N$, hence the morphism $T^{NR} \otimes \mathcal{O}/\pi^N \to T^0 \otimes \mathcal{O}/\pi^N$ is determined by $(\text{Tr}_e, \Lambda)$. Finally, we conclude that the morphism $T^{NR} \otimes \mathcal{O}_L/\pi^N \to (T \otimes L)^{NR} \otimes \mathcal{O}_L/\pi^N$ is determined by $(\text{Tr}_e(K), \Lambda)$ for $e \gg N$. Hence $c(T)$ is determined by $(\text{Tr}_e(K), \Lambda)$ for $e \gg N \gg 0$. \hfill $\Box$

**Theorem 11.2.** Assume that $K$ is of equal-characteristic $p$ and the residue field of $\mathcal{O}_K$ is perfect. Let $T$ be a torus over $K$. Then $c(T) = \frac{1}{2} a(X_*(T) \otimes \mathbb{Q})$. In particular, it is invariant under isogeny.
Proof. Since $T^{NR} \otimes \hat{O} \simeq (T \otimes \hat{K})^{NR}$, we can assume $K$ is complete.

By Deligne’s theory, choose a local field $K_0$ of characteristic 0 such that $Tr_e K_0 \simeq Tr_e K$, then $c(T) = c(T_0) = \frac{1}{2} a(X_*(T_0) \otimes \mathbb{Q})$. Since $X_*(T_0)$ is isomorphic to $X_*(T)$ as $\Gamma(Tr_e K) \simeq \Gamma(Tr_e K_0)$-module, we have $a(X_*(T_0) \otimes \mathbb{Q}) = a(X_*(T) \otimes \mathbb{Q})$. Hence $c(T) = \frac{1}{2} a(X_*(T) \otimes \mathbb{Q})$. \qed

References


