Neeraja Sahasrabudhe

Grothendieck Ring of Varieties

Thesis advisor: J. Sebag

Université Bordeaux 1

Contents

1. Introduction

First appeared in a letter of Grothendieck in Serre-Grothendieck correspondence (letter of 16/8/64), the Grothendieck ring of varieties is an interesting object lying at the heart of the theory of motivic integration.

A class of variety in this ring contains a lot of geometric information about the variety. For example, the topological euler characteristic, Hodge polynomials, Stably-birational properties, number of points if the variety is defined on a finite field etc. Besides, the question of equality of these classes has given some important new results in birational geometry (for example, Batyrev-Kontsevich's theorem).

The Grothendieck Ring $K_0(Var_k)$ is the quotient of the free abelian group generated by isomorphism classes of k-varieties, by the relation $[X \backslash Y] = [X] - [Y]$, where Y is a closed subscheme of X; the fiber product over k induces a ring structure defined by $[X] \cdot [X'] = [(X \times_k X')_{\text{red}}]$. Many geometric objects verify this kind of relations. It gives many realization maps, called *additive invariants*, containing some geometric information about the varieties.

The main aim of this thesis is to study one of these additive invariants introduced by Larsen and Lunts [LL03] in order to study the behaviour of some generating power series. This map has many applications and allows to obtain some important results on the Grothendieck ring, for example, the fact that the Grothendieck ring is not a domain.

After Larsen and Lunts published this paper, Franziska Bittner came up with another very powerful presentation, in terms of generators and relations, of the Grothendieck ring of varieties. We will recall this new presentation given by Bittner and give a sketch of its proof.

The aim of this thesis is to reprove the result of the existence of the additive invariant of Larsen and Lunts by using Bittner's presentation. This result is known by the specialists of the subject but has never been written in the literature. This theorem leads to some interesting results in birational geometry. Further in this thesis we give some applications of this map and other applications of Bittner's presentation and an introduction to the idea of motives and Grothendieck Ring of Chow Motives.

In section 2, we give the classical as well as Bittner's definiton of Grothendieck Ring of varieties along with some important properties. We also give some other results that demonstrate the importance of $\mathbb{A}^1_k$, which we denote by $\mathbb{L}$, in birational geometry.

In section 3, we prove the the theorem of Larsen-Lunts [LL03] and also some important results associated with the Grothendieck ring of varieties and stable birational geometry.

At the end, there is an appendix which recalls the some important theory on blowing up and some other classical results and a glossary for the terms and definitions used in the text.

2. Grothendieck Ring of Varieties

2.1. **Classical definition.** Let k be a field. We denote by Var_k the category of k-varieties over k. If $S \in Var_k$ we denote by Var_S the category of S-varieties.

2.1.1. ***The relative case.***

Definition 2.1. *Let S be a (not necessarily irreducible) variety over k. Let $K_0(Var_S)$ be the free abelian group on isomorphism classes $[X]_S$ of varieties X over S where we impose the relations $[X]_S = [(X - Y)]_S - [Y]_S$ for $Y \subset X$ a closed subvariety. We call it the Grothendieck group of S-varieties.*

The product of varieties makes $K_0(Var_S)$ a $K_0(Var_k)$-module. We set $\mathbb{L} = [\mathbb{A}^1_k]$. Let $\mathcal{M}_k$ denote $K_0(Var_k)[\mathbb{L}^{-1}]$, let $\mathcal{M}_S$ denote the localisation $K_0(Var_S)[\mathbb{L}^{-1}]$.

There are following two important operations on the Grothendieck group:

(1) Taking products yield a $K_0(Var_k)$-bilinear associative exterior product

$$\boxtimes : K_0(Var_S) \times K_0(Var_T) \longrightarrow K_0(Var_{S\times T})$$

and hence an $\mathcal{M}_k$-bilinear associative map

$$\boxtimes : \mathcal{M}_S \times \mathcal{M}_T \longrightarrow \mathcal{M}_{S\times T}$$

(2) If $f : S \longrightarrow S'$ is a morphism of k-varieties, composition with f yields a $K_0(Var_k)$-linear mapping $f_! : K_0(Var_S) \longrightarrow K_0(Var_{S}')$, hence we get an $\mathcal{M}_k$-linear mapping $f_! : \mathcal{M}_S \longrightarrow \mathcal{M}_{S'}$. Pulling back along f yields a $K_0(Var_k)$-linear mapping $f^* : K_0(Var_{S'}) \longrightarrow K_0(Var_S)$ and hence an $\mathcal{M}_k$-linear mapping $\mathcal{M}_{S'} \longrightarrow \mathcal{M}_S$

For morphisms $f : S \longrightarrow S'$ and $g : T \longrightarrow T'$ of varieties we get the identities $(f \times g)_!(A \boxtimes B) = f_!(A) \boxtimes g_!(B)$ and $(f \times g)^*(C \boxtimes D) = f^*(C) \boxtimes g^*(D)$.

[Bit04]

2.1.2. ***The absolute case.***

Definition 2.2. *The Grothendieck Ring $K_0(Var_k)$ is the quotient of the free abelian group generated by isomorphism classes of k-varieties by the relation $[X\backslash Y] = [X] - [Y]$, where Y is a closed subscheme of X; the fibre product over k induces a ring structure defined by $[X] \cdot [X'] = [(X \times_k X')_{\text{red}}]$.*

Remark 2.1. *It corresponds to the definition 2.1 above when $S = \mathrm{Spec}(k)$.*

Remark 2.2. *In general, the fiber product of reduced varieties is not reduced.*

Lemma 2.1. *Let k be a field. Then in the Grothendieck ring of varieties, we have:*
(1) $[\emptyset] = 0$;
(2) $[\mathrm{Spec}(k)] = 1$;
(3) $[\mathbb{P}_k^n] = 1 + \mathbb{L} + \ldots + \mathbb{L}^n$, *where* $\mathbb{L} := [\mathbb{A}_k^1]$.

Proof. Let $[X] \in K_0(Var_k)$, Then for (1), considering $X \subseteq X$ as a closed subvariety we get:

$$[X\backslash X] = [X] - [X]$$
$$i.e., \quad [\emptyset] = 0$$

For (2), note that $\mathrm{Spec}(k) \times_k Y = Y \times_k \mathrm{Spec}(k) = Y$ for all $Y \in Var_k$. Hence, $\mathrm{Spec}(k) = 1$

For proving (3), we use induction. Note that, $\mathbb{A}_k^1 \subset \mathbb{P}_k^1$ is an open subvariety with ∞ as its compliment , so by definition we get

$$[\mathbb{P}_k^1 \backslash \mathbb{A}_k^1] = [\mathbb{P}_k^1] - [\mathbb{A}_k^1]$$
$$[\mathbb{P}_k^1] = 1 + [\mathbb{A}_k^1] = 1 + \mathbb{L}.$$

We conclude by induction as $\mathbb{P}_k^{n+1} \backslash \mathbb{A}_k^{n+1} \simeq \mathbb{P}_k^n$, for all $n \geq 1$. $\square$

2.2. **Classical properties.** We now give few important properties about the Grothendieck Ring of Varieties:

Lemma 2.2. (1) *If X is a variety and U and V are two locally closed subvarieties in X then,*

$$[U \cup V] + [U \cap V] = [U] + [V];$$

(2) *If a variety X is a disjoint union of locally closed subvarieties X_1, X_2, $\ldots X_n$ for some $n \in \mathbb{N}$, then*

$$[X] = \Sigma_{i=1}^n [X_i];$$

(3) *Let C be a constructible subset of a variety X, then C has a class in $K_0(Var_k)$.*

Proof. (1) Note that $V - U \cap V = U \cup V - U$.
If U and V are both open or closed then from the definition we get that

$$[V] - [U \cap V] = [V - U \cap V] = [U \cup V - U] = [U \cup V] - [U]$$

Thus,

$$[U \cup V] + [U \cap V] = [U] + [V] \qquad \ldots (1)$$

if both U and V are either open or closed.
Now, in general, since U and V are locally closed, we can write them as intersection of an open and a closed set.
Let $U = U_1 \cap F_1$ and $V = U_2 \cap F_2$, where U_1, U_2 are open in X and F_1 and F_2 are closed in X.
Then, by (1),

$$[U_1 \cup U_2] + [U_1 \cap U_2] = [U_1] + [U_2] \qquad \ldots (i)$$

Since $U_1 \cup F_1 \subset U_1$ is a closed subset (similarly, $U_2 \cup F_2 \subset U_2$ is closed), we have,

$$[U_1] = [U_1 \cap F_1] + [U_1 \backslash U_1 \cap F_1]$$

or

$$[U_1] = [U] + [U_1 \backslash U_1 \cap F_1] \qquad \ldots (ii)$$

and

$$[U_2] = [U_2 \cap F_2] + [U_2 \backslash U_2 \cap F_2]$$

or

$$[U_2] = [V] + [U_2 \backslash U_2 \cap F_2] \qquad \ldots (iii)$$

Adding (i), (ii) and (iii) we get

$$[U_1 \cup U_2] + [U_1 \cap U_2] = [U] + [V] + [U_1 \backslash U_1 \cap F_1] + [U_2 \backslash U_2 \cap F_2] \qquad \ldots (iv)$$

Again, note that since $U_1 \cap F_1$ is closed in U_1 (resp. $U_2 \cap F_2$ is closed in U_2), therefore, $U_1 \backslash U_1 \cap F_1 \subset U_1$ is an open subset (resp. $U_2 \backslash U_2 \cap F_2 \subset U_2$ is open subest).

Now, since U_1 (resp. U_2) is open in X, therefore, $U_1 \backslash U_1 \cap F_1 \subset U_1$ (resp. $U_2 \backslash U_2 \cap F_2 \subset U_2$) is open in X as well.

Then, from (1) and (iv), we get

$$[U_1 \cup U_2] + [U_1 \cap U_2] = [U] + [V] + [(U_1 \backslash U_1 \cap F_1) \cup (U_2 \backslash U_2 \cap F_2)] + [(U_1 \backslash U_1 \cap F_1) \cap (U_2 \backslash U_2 \cap F_2)]$$

or

$$[U]+[V] = ([U_1 \cup U_2] - [(U_1 \backslash U_1 \cap F_1) \cup (U_2 \backslash U_2 \cap F_2)]) + ([U_1 \cap U_2] - [(U_1 \backslash U_1 \cap F_1) \cap (U_2 \backslash U_2 \cap F_2)])$$

or

$$[U] + [V] = [U_1 \cap F_1 \cup U_2 \cap F_2] + [U_1 \cap F_1 \cap U_2 \cap F_2]$$

i.e.,

$$[U] + [V] = [U \cup V] + [U \cap V]$$

Hence proved.

(2) By induction, it is sufficient to prove that if X is a disjoint union of two locally closed subvarieties X_1 and X_2 i.e., $X = X_1 \bigsqcup X_2$ then $[X] = [X_1] + [X_2]$. Now, using (1), we get

$$[X_1 \cup X_2] + [X_1 \cap X_2] = [X_1] + [X_2]$$

but $X = X_1 \bigsqcup X_2$ implies $[X_1 \cap X_2] = [\emptyset] = 0$ hence, $[X] = [X_1] + [X_2]$.

(3) C is a constructible set in X. We know that a set is constructible if and only if it is a disjoint union of finitely many locally closed sets. So, we can write $C = \bigsqcup_{i=1}^{n} C_i$ and this construction is independent of choices.

□

Proposition 2.1. *Let k be a field of characteristic zero. The Grothendieck ring of varieties is generated by:*

(1) *the smooth varieties;*
(2) *the projective smooth varieties;*
(3) *the projective smooth connected varieties.*

Proof. (1) Let $d = dim(X)$. X can be embedded as an open dense subset of X', where X' is complete using Nagata's theorem. So $[X] = [X'] - [Z]$ with $dim(Z) \leq d - 1$. Then using the Hironaka's theorem we resolve the singularities $\overline{X'} \longrightarrow X'$. We get an expression of the form: $[\overline{X'}] = [X'] - ([C] - [E])$ where the center C is smooth and both $dim(C), dim(E) \leq d-1$. Thus we get that X can be written as finite union of smooth varieties, using induction on the dimension.

(2) This is also proved by using essentially the same argument of induction over the the dimension d of X. The variety X can be embedded into a complete variety which is birational to a projective variety using Nagata's theorem and Chow's lemma respectively. Then resolving the singularities and following the same proof as above, we get the required result.

(3) Using (2), X can be written as a finite union of smooth projective varieties. Let $X = \cup_{i=1}^{n} X_i$ where X_i are smooth and projective. Since, in this case, connected components and irreducible components are the same, we can write each X_i as a finite union of its connected components. Thus X can be written as a finite union of smooth projective connected varieties.

□

Proposition 2.2. *Let X, Y be k-varieties. Let F be a k-variety and let $\pi : Y \longrightarrow X$ be a k morphism of varieties. Assume that for all $x \in X$, $\pi^{-1}(x)$ is $\kappa(x)$-isomorphic to $F \times_k \mathrm{Spec}\kappa(x)$. Then*

$$[Y] = [F] \cdot [X]$$

in $K_0(Var_k)$.

See [Seb04]

Corollary 2.0.1. (1) *If $f : E \longrightarrow X$ is a fibration of fiber F which is locally trivial for the Zariski topology of X, then*

$$[E] = [F] \cdot [X];$$

(2) *Let $f : X \longrightarrow Y$ be a proper morphism of smooth varieties, which is a blow-up with the smooth center $Z \subset Y$ of codimension d. Then $[f^{-1}(Z)] = [Z][\mathbb{P}^{d-1}]$ in the ring $K_0(Var_k)$.*

Proof. It is the direct consequence of proposition 2.2 above. □

2.3. **Bittner's definition.**

Definition 2.3 (Bittner's definition)**.** *The Grothendieck ring of k-varieties has the following alternative presentations:*

(1) ***(sm)****As the abelian group generated by the isomorphism classes of smooth varieties over k subject to the relations $[X] = [X - Y] + [Y]$, where X is smooth and $Y \subset X$ is a smooth closed subvariety.*

(2) ***(bl)*** *As the abelian group generated by the isomorphism classes of smooth complete k-varieties subject to the relation $[\phi] = 0$ and $[Bl_Y X] = [X] - [Y] + [E]$, where X is smooth and complete, $Y \subset X$ is a closed smooth subvariety, $Bl_Y X$ is the blow-up of X along Y and E is the exceptional divisor of this blow-up.*

Remark 2.3. *We get the same group if in case (sm) we restrict to quasi-projective varieties or if in case (bl) we restrict to projective varieties. We can also restrict to connected varieties in both presentations.*

Proof of Bittner's definition. Let us call the group defined in (1) of definition 2.3 above as $K_0^{sm}(Var_k)$. We show that the ring homomorphism

$$K_0^{sm}(Var_k) \longrightarrow K_0(Var_k)$$

defined on the generators by

$$[X]_{sm} \longmapsto [X]$$

is an isomorphism.

We construct an inverse of the above map. Let $[X] \in K_0(Var_k)$. Stratify $X = \cup_{N\in\mathcal{N}} N$ where N smooth and equidimensional and $\overline{N}$ a union of strata for all $N \in \mathcal{N}$. Consider the expression $\Sigma_{N\in\mathcal{N}}[N]_{sm}$ in $K_0^{sm}(Var_k)$. If X is smooth, $\Sigma_{N\in\mathcal{N}}[N]_{sm} = [X]_{sm}$ as can be seen by induction on the number of elements of $\mathcal{N}$:

Let $N \in \mathcal{N}$ be an element of minimal dimension, then $[X]_{sm} = [X - N]_{sm} + [N]_{sm}$, and by the induction hypothesis $[X-N]_{sm} = \Sigma_{N'\in\mathcal{N}-\{N\}}[N']_{sm}$.

For two startifications $\mathcal{N}$ and $\mathcal{N}'$ of X we can always find a common refinement $\mathcal{L}$. The above argument shows that for $N \in \mathcal{N}$ we get $\Sigma_{N\supset L\in\mathcal{L}}[L]_{sm}$. Hence $\Sigma_{L\in\mathcal{L}}[L]_{sm}$ is equal to $\Sigma_{N\in\mathcal{N}}[N]_{sm}$ and analogously it equals $\Sigma_{N\in\mathcal{N}'}[N]_{sm}$, therefore $\Sigma_{N\in\mathcal{N}}[N]_{sm}$ is independent of choice of stratification. Thus we can set $e(X) := \Sigma_{N\in\mathcal{N}}[N]_{sm}$.

If $Y \subset X$ is a closed subvariety we can find a stratification for which Y is a union of strata which yields $e(X) = e(X - Y) + e(Y)$. The induced map on $K_0^{sm}(Var_k)$ is obviously an inverse for $K_0^{sm}(Var_k) \longrightarrow K_0(Var_k)$.

We call the group defined in (2) of definition 1.4 as $K_0^{bl}(Var_k)$. That means it is the free abelian group of isomorphism classes $[X]_{bl}$ of smooth complete varieties X over k modulo the relations for blow-ups of smooth complete varieties X along smooth closed subvarieties Y and the relation $[\emptyset]_{bl} = 0$ (then $[X \cup Y]_{bl} = [X]_{bl} + [Y]_{bl}$, which can be seen by blowing up along Y).

Decomposing into connected components and noting that the blow-up along a disjoint union is the successive blow-up along the connected components one sees that this can also be described as the free abelian group on isomorphism classes $[X]_{bl}$ of connected smooth complete varieties with imposed relations $[\emptyset]_{bl} = 0$ and $[Bl_Y X]_{bl} - [E]_{bl} = [X]_{bl} - [Y]_{bl}$, where $Y \subset X$ is a connected closed smooth subvariety.

Also $K_0^{bl}(Var_k)$ carries a commutative ring structure induced by the product of varieties.

We show that the ring homomorphism

$$K_0^{bl}(Var_k) \longrightarrow K_0(Var_k)$$

defined on the generators by

$$[X]_{bl} \longmapsto [X]$$

is an isomorphism.

We again proceed to construct an inverse of the above map.

Let X be a smooth connected variety, let $X \subset \bar{X}$ be a smooth completion(using Chow's lemma and Nagata's theorem, See Appendix) with $D = \bar{X} - X$ a simple normal crossings divisor. As in (1), we need to set $e(X)$ such that $e(X - Y) = e(X) - e(Y)$ and it induces the required inverse map.

We define $e(X) = \Sigma(-1)^l [D^{(l)}]_{bl}$ where $D^{(l)}$ is the normalization of the l-fold intersections of D (where $D^{(0)}$ is understood to be $\bar{X}$). So $D^{(l)}$ is the disjoint union of the l-fold intersections of the irreducible components of D.

The fact that the above expression for $e(X)$ is independent of the completion of X uses weak factorization theorem (See Appendix) and $e(X) = e(X - Y) + e(Y)$ is proved by choosing $\bar{X} \supset X$, smooth and complete, such that $D = \bar{X} - X$ is a simple normal crossing divisor and the closure $\bar{Y}$ of Y in $\bar{X}$ is also smooth and has normal crossings with D. In particular $D \cap Y$ is a simple normal crossings divisor in Y. $\square$

For the entire proof in detail, see [Bit04]

Proposition 2.3. *We have a ring involution $\mathcal{D}_k$ of $\mathcal{M}_k$ that sends $\mathbb{L}$ to $\mathbb{L}^{-1}$ and is characterized by the property that it sends the class of a complete connected smooth variety X to $\mathbb{L}^{-dimX}[X]$.*

Definition 2.4. *We call this involution the duality map.*

Before proving the above proposition we recall that a vector bundle $V \longrightarrow X$ of rank n on a variety X is locally trivial by definition and hence its class $[V]$ in $K_0(Var_k)$ is equal to $\mathbb{L}^n[X]$. Similarly the class of its projectivisation $[\mathbb{P}(V)]$ equals $[\mathbb{P}^{n-1}][X] = (1 + \mathbb{L} + \ldots + \mathbb{L}^{n-1})[X]$.

Lemma 2.3. *Let X be a smooth connected variety and $Y \subset X$ a smooth connected subvariety, let d denote the codimension of Y in X. Let E be the exceptional divisor of the blow-up $Bl_Y X$ of X along Y. Then $[Bl_Y X] - \mathbb{L}[E] = [X] - \mathbb{L}^d[Y]$ in $K_0(Var_k)$.*

Proof. In $K_0(Var_k)$, we have the relation

$$[Bl_Y X] - [E] = [X] - [Y] \tag{1}$$

Furthermore, $[E] = (1 + \mathbb{L} + \ldots + \mathbb{L}^{d-1})[Y]$. Thus,

$$(1 - \mathbb{L})[E] = (1 - \mathbb{L}^d)[Y] \tag{2}$$

Adding (1) and (2) gives the desired expression. $\square$

Proof of Proposition. Using the presentation ***bl*** from Bittner's definition and the above lemma, we define a group homomorphism

$$K_0(Var_k) \longrightarrow \mathcal{M}_k$$

by sending the class of a smooth connected complete variety $[X]$ to $\mathbb{L}^{-dimX}[X]$. This morphism is multiplicative and sends $\mathbb{L}$ to $\mathbb{L}^{-1}$, hence it can be extended uniquely to a ring endomorphism $\mathcal{D}_k$ of $\mathcal{M}_k$. Obviously $\mathcal{D}_k\mathcal{D}_k = id_{\mathcal{M}_k}$. □

2.3.1. ***Why is it interesting to do computation in the Grothendieck ring of varieties?***

Definition 2.5 (Additive Invariants)**.** *Let R and S be rings. An additive invariant λ from the category Var_R of algebraic varieties over R with values in S, assigns to any X in Var_R an element $\lambda(X)$ of S, such that:*

$$\lambda(X) = \lambda(X');$$

for $X \simeq X'$

$$\lambda(X) = \lambda(X') + \lambda(X \backslash X'),$$

for X' closed in X, and

$$\lambda(X \times X') = \lambda(X) \cdot \lambda(X')$$

for every X and X'.

Example 1.

(1) ***Euler characteristic:*** *Let k be a subfield of $\mathbb{C}$, the (topological) Euler characteristic*

$$Eu(X) := \Sigma_i(-1)^i rkH^i_c(X(\mathbb{C}), \mathbb{C})$$

gives rise to an additive invariant $Eu : Var_k \longrightarrow \mathbb{Z}$

(2) ***Hodge polynomial:*** *Let k be a field of characteristic zero. Then it follows from Deligne's Mixed Hodge Theory that there is a unique additive invariant $H : Var_k \longrightarrow \mathbb{Z}[u, v]$, which assignes to a smooth projective variety X over k its usual Hodge polynomial*

$$H_X(u, v) := \Sigma_{p,q}(-1)^{p+q}h^{p,q}(X)u^p v^q,$$

with $h^{p,q}(X) = dimH^q(X, \Omega^p_X)$ the (p, q)-Hodge number of X.

(3) ***Virtual motives:*** *Let k be a field of characteristic zero, there exists a unique additive invariant*

$$\chi_c : Var_k \longrightarrow K_0(CHMot_k)$$

which assigns to a smooth projective variety X over k the class of its Chow motive, where $K_0(CHMot_k)$ denotes the Grothendieck ring of the category of Chow motives over k (with rational coefficients).(See section 4.2)

(4) ***Counting points:*** *Assume $k = \mathbb{F}_q$ and X be a scheme of finite type. For any integer $r \geq 1$, let N_r be the number of point of $\bar{X} = X \times_k \bar{k}$ which are rational over the field $\mathbb{F}_{q^r}$. In other words, N_r is the number of points of $\bar{X}$ whose coordinates lie in $\mathbb{F}_{q^r}$. That is there is a map $N_n : X \longmapsto |X(\mathbb{F}_q^n)|$. This map gives rise to as additive invariant $N_n : Var_k \longrightarrow \mathbb{Z}$.*
[CL05]

Theorem 2.1. *Let X and X' be complex Calabi-Yau varieties of dimension n. Assume X and X' are birationally equivalent. Then they have the same hodge numbers.*

Idea of Proof. Using motivic integration, one can prove that $[X] = [X']$. Apply Hodge polynomial to conclude. □

For a detailed proof explaining the motivic integration part, see [CL05]

Definition 2.6 (Zeta Function)**.** *Let X be a scheme of finite type. Then the zeta funtion of X is defined as*

$$Z(t) = Z(X;t) = exp(\Sigma_{r=1}^{\infty} N_r \frac{t^r}{r}) \in \mathbb{Q}[[t]]$$

For example, let $X = \mathbb{P}^1$. Over any field, $\mathbb{P}^1$ has one more point than the number of elements of the field. Hence $N_r = q^r + 1$. Thus

$$Z(\mathbb{P}^1, t) = exp(\Sigma_{r=1}^{\infty} (q^r + 1) \frac{t^r}{r})$$

It is easy to sum this series, and we find that

$$Z(\mathbb{P}^1, t) = \frac{1}{(1-t)(1-qt)}$$

In particular, it is a rational function of t.
[Har77]

Proposition 2.4. *Let X and Y be two $\mathbb{F}_p$-varieties such that $[X] = [Y]$ in $K_0(Var_{\mathbb{F}_p})$, then they have the same zeta-function.*

Proof. Since $[X] = [Y]$, from (4), we have that $N_r([X]) = N_r([Y])$ for all r. This along with the definition of zeta-function above, proves that X and Y have the same zeta-function. □

Proposition 2.5. *Let k be an algebraically closed field. Let C be an irreducible curve over k, C is rational iff $[C] = \mathbb{L} + \beta$; $\beta \in \mathbb{Z}$*

Proof. Using Nagata's theorem, Chow's lemma and finally resolving the singularities, we get a smooth projective connected curve C' such that C' and C are birational. Then C is a rational curve $\iff C'$ is a rational curve. and $[C] = \mathbb{L} + \beta \iff [C'] = \mathbb{L} + \beta$.

Thus, we can assume that C is a smooth projective connected curve.

Now, since k is algebraically closed, C is rational $\iff C \simeq \mathbb{P}^1_k$ but we already know that, $[\mathbb{P}^1_k] = 1 + \mathbb{L}$. Hence, if C is rational, then $[C] = \mathbb{L} + 1$.

Conversely, suppose $[C] = \mathbb{L} + \beta$ for some $\beta \in \mathbb{Z}$. Using Hodge decomposition we get,

$$H_C(u, u) = H_{\mathbb{L}}(u, u) + \beta$$

or

$$u^2 + 2g(C)u + 1 = u^2 + \beta \qquad \ldots (i)$$

where $g(C) =$ genus of C.
(i) implies $\beta = 1$ and $g(C) = 0$.
We know that, a curve C is rational $\iff g(C) = 0$. Hence proved. □

Only a few things are known about the Grothendieck ring of varieties. For example, we know that it is not a domain (See section 4.1). Some motivational problems are still open, for example:

(1) Let X, Y be k-varieties such that $[X] = [Y]$. Is it possible to partition X and Y by a finite number of locally closed subvarieties which are pairwise isomorphic? This question was originally asked by Larsen and Lunts. The only results known about it is the work of Liu and Sebag which answers this question positively for curves, surfaces smooth and projective over an algebraically closed field of characteristic zero and variety of higher dimension which does not contain many rational curves.
(2) Is $\mathbb{L}$ a zero divisor in $K_0(Var_k)$?

3. Stable Birational Geometry

The main result of this section is to obtain a unique homomorphism between $K_0(Var_k)$ and $\mathbb{Z}[SB]$, where SB denotes the multiplicative monoid of classes of stable birational equivalence of varierties. This fact was proved by Larsen and Lunts in their paper [LL03]. The proof given by Larsen and Lunts uses induction where the induction step requires checking various constructions and conditions on the image of the map. The proof is quite long and complicated. In this section we give a new proof of the same result using the Bittner's result [Bit04]. This proof is original and not written anywhere upto my knowledge.

From now on we assume that k is an algebraically closed field of characteristic zero.

Definition 3.1. *We say that (irreducible)varieties X and Y are stably birational if $X \times \mathbb{P}^k$ is birational to $Y \times \mathbb{P}^l$ for some $k, l \geq 0$.*

Example 2. *$\mathbb{P}^1_k$ and $\mathbb{P}^2_k$ are stably birational.*

Theorem 3.1 (Larsen-Lunts)**.** *Let $\mathbf{G}$ be an abelian monoid and $\mathbb{Z}[\mathbf{G}]$ be the corresponding monoid ring. Denote by $\mathfrak{M}$ the multiplicative monoid of isomorphism classes of smooth complete irreducible varieties. Let*

$$\Psi : \mathfrak{M} \longrightarrow \mathbf{G}$$

be a homomorphism of monoids such that

(1) $\Psi([X]) = \Psi([Y])$ *if X and Y are birational.*
(2) $\Psi[\mathbb{P}^n] = 1$ *for all $n \geq 0$*

Then there exists a unique homomorphism

$$\Phi : K_0(Var_k) \longrightarrow \mathbb{Z}[\mathbf{G}]$$

such that $\Phi([X]) = \Psi([X])$ for $[X] \in \mathfrak{M}$

Proof. Note that, from Bittner's definition $K_0(Var_k) \cong \mathbb{Z}[\mathfrak{M}]/\sim$, where $\sim$ is given by:

$$[Bl_Y X] - [E] = [X] - [Y]$$

Let ρ denote the quotient map

$$\rho : \mathbb{Z}[\mathfrak{M}] \longrightarrow \mathbb{Z}[\mathfrak{M}]/\sim$$

we write $Z = Bl_Y X$. Let X be a smooth projective variety.

Then let $\pi : Z \longrightarrow X$ be the blow-up map and with Y as its center and $E = \pi^{-1}(Y)$ as the exceptional divisor.

With the above notation the equivalence relation $\sim$ can be written as:

$$[Z] - [\pi^{-1}(Y)] = [X] - [Y]$$

Now, we have a map:

$$\Psi : \mathfrak{M} \longrightarrow \mathbf{G}$$

From this we get a natural extension $\widehat{\Psi} : \mathbb{Z}[\mathfrak{M}] \longrightarrow \mathbb{Z}[\mathbf{G}]$

To get a map $\Phi : K_0(Var_k) \longrightarrow \mathbb{Z}[\mathbf{G}]$ we need to show that $Ker(\rho) \subset Ker(\widehat{\Psi})$ i.e., $\widehat{\Psi}([Z] - [\pi^{-1}(Y)]) = \widehat{\Psi}([X] - [Y])$

It clearly suffices to show that:

(1) $\Psi([Z]) = \Psi([X])$
(2) $\Psi(\pi^{-1}([Y])) = \Psi([Y])$

(1) follows from the fact that blowing up map is a birational map(proved in Appendix). For (2), recall corollary 2.0.1 from section 2. From that we get that $\Psi(\pi^{-1}([Y])) = \Psi([Y])\Psi([\mathbb{P}^r])$ for some r determined by the codimension on the center Y in X. Now, $\Psi([\mathbb{P}^r]) = 1$, thus we get $\Psi(\pi^{-1}([Y])) = \Psi([Y])$.

$\square$

Remark 3.1. *Consider SB, the multiplicative monoid of stable birational equivalence classes of varieties. There is a canonical surjective homomorphism $\Psi_{SB} : \mathfrak{M} \longrightarrow SB$ which satisfies hypotheses (1) and (2) of the above theorem (with $\Psi_{SB} = \Psi, \mathbf{G} = SB$)). By definition, any homomorphism Ψ as in the theorem factors through Ψ_{SB}. Denote by Φ_{SB} the ring homomorphism from $K_0(Var_k)$ to $\mathbb{Z}[SB]$, corresponding to Ψ_{SB} by the theorem.*

Corollary 3.1.1. *Let $X_1, \ldots, X_k, Y_1, \ldots, Y_m$ be smooth complete connected varieties. Let $m_i, n_j \in \mathbb{Z}$ be such that*

$$\Sigma m_i[X_i] = \Sigma n_j[Y_j]$$

in $K_0(Var_{\mathbb{C}})$. Then $k = m$ and after renumbering the varieties X_i and Y_i are stably birational and $m_i = n_i$

Remark 3.2. *There is a mistake in* [LL03]. *In the above corollary, they omit the connected condition on the varieties which is an essential condition for the corollary to hold true.*

Proof. Applying the ring homomorphism Φ_{SB} to the above equality, we obtain the equality in the monoid ring $\mathbb{Z}[SB]$:

$$\Sigma m_i \Psi_{SB}(X_i) = \Sigma n_j \Psi_{SB}(Y_j)$$

and the corollary follows. $\square$

Thus, any variety can be written uniquely (upto stable birational equivalence) as linear combination (in $K_0(Var_{\mathbb{C}})$) of smooth complete varieties.

Proposition 3.1. *The kernel of the (surjective)homomorphism Φ_{SB}: $K_0(Var_{\mathbb{C}}) \longrightarrow \mathbb{Z}[SB]$ is the principal ideal generated by the class $[\mathbb{A}^1]$ of the affine line $\mathbb{A}^1$.*

Proof. Note that Φ_{SB} is obtained uniquely from Ψ_{SB} using the above theorem. Therefore, $\Phi_{SB}([\mathbb{P}^n]) = 1$ for all n.

Hence, we have $\Phi_{SB}([\mathbb{P}^1]) = 1$.

So $\Phi_{SB}([1] + [\mathbb{A}^1]) = 1$.

and $\Phi_{SB}(\mathbb{A}^1) = 0$

Let $a \in \mathrm{Ker}(\phi_{SB})$. We write a as linear combination

$$a = [X_1] + \ldots + [X_k] - [Y_1] - \ldots - [Y_l]$$

where X$_i$, Y$_j$ are smooth complete and connected varieties. Since

$$\Phi_{SB}(a) = \Sigma\Psi_{SB}(X_i) - \Sigma\Psi_{SB}(Y_j) = 0$$

we get $k = l$ and after renumbering, X_i is stably birational to Y_i. So it is sufficient to show that if X, Y are smooth, complete and stably birational, then $[X] - [Y] \in K_0(Var_k) \cdot [\mathbb{A}^1]$. Note that

$$[X \times \mathbb{P}^k] - [X] = [X] \cdot [\mathbb{A}^1 + \mathbb{A}^2 + \ldots + \mathbb{A}^k]$$

so we may assume that $[X]$ and $[Y]$ are birational. Moreover by weak factorization theorem, we may assume that X is a blow-up of Y with a smooth center $Z \subset Y$ and exceptional divisor $E \subset X$. Then $[E] = [\mathbb{P}^t] \cdot [Z]$ for some t and

$$[X] - [Y] = [E] - [Z] = ([\mathbb{A}^1] + [\mathbb{A}^2] + \ldots + [\mathbb{A}^t])[Z]$$

□

Proposition 3.2. *Let* α*:* $K_0(Var_{\mathbb{C}}) \longrightarrow B$ *be an additive invariant. The following conditions are equivalent:*

(1) $\alpha([\mathbb{A}^1]) = 0$
(2) *If* X*,* Y *are smooth complete vareities that are birational then* $\alpha([X]) = \alpha([Y])$.

If these conditions hold, then $\alpha([Z]) = \alpha([W])$ *for any smooth complete varieties* Z*,* W *which are stably birationally equivalent.*

Proof. Assume (1), then by previous proposition α factors through the homomorphism Φ_{SB} which implies (2). Now, assuming (2), let $\tilde{X}$ be a blow-up of X at a point. Then $[\tilde{X}] = [X] + [\mathbb{A}^1]$. We know that, blow-up map is birational which means $\alpha([\tilde{X}]) = \alpha([X])$. Hence $\alpha([\mathbb{A}^1]) = 0$. Now, suppose Z and W are two stably birationally equivalent smooth complete varieties. This means, $Z \times \mathbb{P}^k$ is birational to W$\times\mathbb{P}^l$ for some $k, l \geq 0$. Now, since the conditions hold, we get that: $\alpha([Z \times \mathbb{P}^k]) = \alpha([W \times \mathbb{P}^l])$ or $\alpha([Z]).\alpha([\mathbb{P}^k]) = \alpha([W]).\alpha([\mathbb{P}^l])$ but condition (1) $\Rightarrow$ $\alpha(\mathbb{P}^n) = 1$ for all n. Thus, $\alpha([Z]) = \alpha([W])$ □

4. Application

4.1. **Grothendieck Ring is not a Domain.** We give an idea of the proof of the fact that Grothendieck Ring is not a domain. This was proved by Poonen [Poo02].

Remark 4.1. *Note that for any extension of fields $k \subseteq k'$, there is a ring homomorphism $K_0(Var_k) \longrightarrow K_0(Var_{k'})$ mapping $[X]$ to $[X_{k'}]$. We have already seen in theorem 3.1 and proposition 3.1 that when $k = \mathbb{C}$, there is a unique ring homomorphism $K_0(Var_k) \longrightarrow \mathbb{Z}[SB_k]$ mapping the class of any smooth projective integral variety to its stable birational class. In fact, this homomorphism is surjective, and its kernel is the ideal generated by $\mathbb{L} := [\mathbb{A}^1]$.*
The set AV_k of isomorphism classes of abelian varieties over k is a monoid. The Albanese functor mapping a smooth, projective, geometrically integral variety to its Albanese variety induces a homomorphism of monoids $SB_k \longrightarrow AV_k$, since the Albanese variety is a birational invariant, since formation of the Albanese variety commutes with products, and since the Albanese variety of $\mathbb{P}^n$ is trivial. Therefore we obtain a ring homomorphism $Z[SB_k] \longrightarrow Z[AV_k]$.

The proof uses the above fact and the following lemmas:

Lemma 4.1. *Let k be an algebraically closed field of characteristic zero. There exists an abelian variety A over k such that $End_k(A) = \mathcal{O}$ where $\mathcal{O}$ is the ring of integers of a number field of class number 2.*

For proof see [Poo02]

Lemma 4.2. *Let k be an algebraically closed field of characteristic zero. There exist abelian varieties A and B over k such that $A \times A \cong B \times B$ but $A_k \ncong B_k$.*

Idea of proof. To prove this one uses the fact that $\mathcal{O}$ as in lemma 4.1, is a dedekind domain which implies that the isomorphism type of a direct sum of fractional ideals $I_1 \oplus \ldots \oplus I_n$ is determined exactly by the nonnegative integer n and the product of the classes of the I_i in the class group $Pic(\mathcal{O})$ and the fact that $Pic(\mathcal{O}) \cong \mathbb{Z}/2$. □

Theorem 4.1. *let k be an algebraically closed field of characteristic zero. The Grothendieck ring of varieties $K_0(Var_k)$ is not a domain.*

Proof. Consider the maps:

$$K_0(Var_k) \longrightarrow \mathbb{Z}[SB_k] \longrightarrow \mathbb{Z}[AV_k]$$

Let A and B as in lemma 4.2. Then since the images of $[A]+[B]$ and $[A]-[B]$ are non-zero under the above composition, both $[A]+[B]$ and $[A]-[B]$ are non-zero but $([A]+[B])([A]-[B]) = 0$. $\square$

4.2. **Grothendieck Ring of motives.** [CL05][GS96]

We explain what Chow motive and the category $K_0(CHMot_k)$ are. Let $\mathbb{V}$ denote the category of smooth projective varieties over $\mathbb{C}$. For an object X in $\mathbb{V}$ and an integer d, $\mathcal{Z}^d(X)$ denotes the free abelian group generated by irreducible subvarities of X of codimension d. We define the (rational) Chow group $\mathcal{A}^d(X)$ as the quotient of $\mathcal{Z}^d(X)\otimes\mathbb{Q}$ modulo rational equivalence. For X and Y in $\mathbb{V}$, we denote by $Corr^r(X,Y)$ the group of correspondences of degree r from X to Y . The category Mot of $\mathbb{C}$-motives may be defines as follows:

• Objects of Mot are triples (X,p,n) where X is in $\mathbb{V}$, p is an idempotent(i.e., $p^2 = p$) in $Corr^0(X,X)$, and n is an integer in $\mathbb{Z}$.

• If (X,p,n) and (Y,q,m) are two motives, then

$$Hom_{Mot}((X,p,n),(Y,q,m)) = qCorr^{m-n}(X,Y)p.$$

Composition of morphisms is given by composition of correspondences. The category Mot is additive, $\mathbb{Q}$-linear, and pseudo-abelian. There is a natural tensor product on Mot, defined on objects by,

$$(X,p,n)\otimes(Y,q,m) = (X\times Y, p\otimes q, n+m)$$

We denote by h the functor $h : \mathbb{V}^\circ \longrightarrow Mot$ which sends an object X to $h(X) = (X, id, 0)$ and a morphism $f : Y \longrightarrow X$ to its graph in $Corr^0(X,Y)$. This functor is compatible with the tensor product and the unit motive $1 = h(\mathrm{Spec}(k))$ is the identity for the product.

Theorem 4.2. *Let k be a field of characteristic zero. There exists a unique morphism of rings*

$$\chi_c : K_0(Var_k) \longrightarrow K_0(CHMot_k)$$

such that $\chi_c([X]) = [h(X)]$ for X projective and smooth.

Idea of Proof. We already know that by Bittner's theorem, we have a canonical isomorphism

$$K_0(Var_k) \longrightarrow K_0^{bl}(Var_k)$$

The idea is to use this along with the fact that in $K_0(CHMot_k)$, $h([Bl_Y X]-[E]) = h([X]-[Y])$. $\square$

[Ser91], [Jan95]

5. APPENDIX : Tools for Birational geometry

We first give some definitions:

Definition 5.1 (Ideal Sheaf)**.** *Let Y be a closed subscheme of a scheme X and let $i : Y \longrightarrow X$ be the inclusion morphism. We define the ideal sheaf of Y, denoted $\mathcal{I}_Y$, to be the kernel of the morphism $i^\sharp : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$*

Proposition 5.1. *Let X be a scheme. For any closed subscheme Y of X, the correspoding ideal sheaf $\mathcal{I}_Y$ is a quasi-coherent sheaf of ideals on X. If X is noetherian, it is coherent. Conversely, any quasi-coherent sheaf of ideals on X is the ideal sheaf of a uniquely determined closed subscheme of X.*

Definition 5.2 (Prime Ideal Sheaf)**.** *Let $S = \mathrm{Spec}R$ be an affine scheme, given a ring R. Given a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_S$-algebras, we define prime ideal sheaf in $\mathcal{F}$ to be a quasi-coherent sheaf of ideals $\mathcal{I} \subsetneq \mathcal{F}$, such that for each affine open subset $U \subset S$, the ideal $\mathcal{I}(U) \subset \mathcal{F}(U)$ is either prime or the unit ideal.*

Remark 5.1. *For any scheme X, the points of X are simply the prime ideal sheaves of $\mathcal{O}_X$.*

We now define Global Spec and Global Proj:

Definition 5.3 (Global Spec)**.** *The idea is to extend the concept of spectrum of a ring to describe analogous objects in the category of S-schemes for arbitrary S(spectrum of some ring R). That is, given a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_S$-algebras, we define $X = \mathrm{Spec}(\mathcal{F})$.*

There are two alternative ways to construct Glocal Spec:

1. *Cover S by affine open subsets $U_\alpha = \mathrm{Spec}(R_\alpha)$, and define X to be the union of the schemes $\mathcal{F}(U_\alpha)$, with gluing maps induced by the restrictions maps $\mathcal{F}(U_\alpha) \longrightarrow \mathcal{F}(U_\alpha \cup U_\beta)$.*
2. *Define X as the set of prime ideal sheaves in $\mathcal{F}$. Next it is defined as a topological space as follows: for every open $U \subset S$(not necessarily affine) and section $\sigma \in \mathcal{F}(U)$, let $V_{U,\sigma} \subset X$ be the set of prime ideal sheaves $\mathcal{P} \subset \mathcal{F}$ such that $\sigma \in \mathcal{P}(U)$; take these as a basis for the topology.*

Finally, the structure sheaf $\mathcal{O}_X$ on the basic open sets is defined by setting:

$$\mathcal{O}(V_{U,\sigma}) = \mathcal{F}(U)[\sigma^{-1}]$$

For the morphism $f : X \longrightarrow S$: as a set, we assocaite to a prime ideal sheaf $\mathcal{P} \subset \mathcal{F}$ its inverse image in $\mathcal{O}_S \longrightarrow \mathcal{F}$; and the pullback map on functions

$$f^\sharp : \mathcal{O}_S(U) \longrightarrow \mathcal{O}_X(f^{-1}(U)) = \mathcal{F}(U)$$

is just the structure map $\mathcal{O}_S \longrightarrow \mathcal{F}$ on U

Definition 5.4 (Global Proj)**.** *Let $\mathcal{B}$ be any scheme. By the quasicoherent sheaf of graded $\mathcal{O}_B$-algebras we will mean a quasicoherent sheaf $\mathcal{F}$ of algebras on B, and a grading*

$$\bigoplus_{v=0}^{\infty} \mathcal{F}_{\mathcal{V}}$$

such that $\mathcal{F}_\nu \mathcal{F}_\mu \subset \mathcal{F}_{\nu+\mu}$ and $\mathcal{F}_{\prime} = \mathcal{O}_B$. Thus, for every affine open subset $U \subset B$ with coordinate ring $A = \mathcal{O}_B(U)$, the ring $\mathcal{F}(U)$ will be a graded A-algebra with 0-th graded piece $\mathcal{F}(U)_0 = A$. Given such a sheaf $\mathcal{F}$, for each affine open subset $U \subset B$ we will let $X_U \longrightarrow U$ be the scheme $X_U = Proj\mathcal{F}(U)$ with the structure morphism $Proj\mathcal{F}(U) \longrightarrow$ $\mathrm{spec}(A) = U$. For every inclusion $U \subset V$ of open subsets of B, the restriction map $\mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ is a homomorphism of graded rings whose 0-th graded piece is the restriction map $\mathcal{O}_B(V) \longrightarrow \mathcal{O}_B(U)$, and so induces a map $X_U \longrightarrow X_V$ commuting with the structure morphism $X \longrightarrow B$; X is denoted $Proj\mathcal{F}$; and the construction of X is called global proj.

Proposition 5.2. *Let $\mathcal{B}, \mathcal{F}$ as above, let $X =$ **Proj**$\mathcal{F}$, with projection $\pi : X \longrightarrow \mathcal{B}$ and invertible sheaf $\mathcal{O}_X(1)$. Then:*
π is a proper morphism. In particular, it is separated and of finite type.

Blowing Up

Definition 5.5. *Let X be a noetherian scheme, and let $\mathcal{I}$ be a coherent sheaf of ideals on X. Consider the sheaf of graded algebras $\mathfrak{I} = \bigoplus_{d \geq 0} \mathcal{I}^d$, where $\mathcal{I}^d$ is the dth power of ideal $\mathcal{I}$, and we set $\mathcal{I}^0 = \mathcal{O}_X$, then $X, \mathfrak{I}$ clearly satisfy the above, so we can consider $\tilde{X} =$ **Proj**$\mathfrak{I}$. We define $\tilde{X}$ to be the blowing-up of X with respect to the coherent sheaf of odeals $\mathcal{I}$. If Y is the closed subscheme of X corresponding to $\mathcal{I}$, then we call $\tilde{X}$ to the be the blowing-up of X along Y, or with center Y.*

Definition 5.6 (Inverse Image Ideal Sheaf)**.** *Let $f : X \longrightarrow Y$ be a morphism of schemes, and let $\mathcal{I} \subseteq \mathcal{O}_Y$ be a sheaf of ideals on Y. The inverse image ideal sheaf $\mathcal{I}' \subseteq \mathcal{O}_Y$ is defined as follows:*

First consider f as a continuous map of topological spaces $X \longrightarrow Y$ and let $f^{-1}\mathcal{I}$ be the inverse image of the sheaf I. Then $f^{-1}(\mathcal{I})$ is a sheaf of ideals in the sheaf of rings $f^{-1}(\mathcal{O}_Y)$ on the topological space X. Now there is a natural homomorphism of sheaves of rings on X, $f^{-1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X)$, so we define $\mathcal{I}'$ to be the ideal sheaf in $\mathcal{O}_X$ generated by the image of $f^{-1}\mathcal{I}$. We denote $\mathcal{I}'$ by $f^{-1}\mathcal{I} \cdot \mathcal{O}_X$ or simply $\mathcal{I} \cdot \mathcal{O}_X$.

Theorem 5.1 (Universal property of Blowing Up)**.** *Let X be a noetherian scheme, $\mathcal{I}$ a coherent sheaf of ideals, and $\pi : \tilde{X} \longrightarrow X$ the blowing up with respect to $\mathcal{I}$. If $f : Z \longrightarrow X$ is any morphism such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is an invertible sheaf of ideals on Z, then there exists a unique morphism $g : Z \longrightarrow \tilde{X}$ factoring f.*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, dashed, "f"'] & \tilde{X} \\ & X \arrow[u, "\pi"'] \end{tikzcd}$$

Proposition 5.3. *Let X be a noetherian scheme, $\mathcal{I}$ a coherent sheaf of ideals, and let $\pi : \tilde{X} \longrightarrow X$ be the blowing-up of $\mathcal{I}$. Then:*

(1) *The inverse image ideal sheaf $\tilde{\mathcal{I}} = \pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\tilde{X}}$ is and invertible sheaf on $\tilde{X}$.*
(2) *If Y is the closed subscheme corresponding to $\mathcal{I}$, and if $U = X - Y$, then*
$$\pi : \pi^{-1}(U) \longrightarrow U$$
is an isomorphism.

Proof. (1) We know that $\tilde{X}$ is defined as $\mathbf{Proj}\mathfrak{I}$, where $\mathfrak{I} = \bigoplus_{d \geq 0} \mathcal{I}^d$, it comes equipped with a natural invertible sheaf $\mathcal{O}(1)$. For any open affine $U \subseteq X$, this sheaf $\mathcal{O}(1)$ on $\mathbf{Proj}\mathfrak{I}(U)$ is the sheaf associated to the graded $\mathfrak{I}(U)$-module $\mathfrak{I}(U)(1) = \bigoplus_{d \geq 0} \mathcal{I}^{d+1}(U)$. But this is clearly equal to the ideal $\mathcal{I}.\mathfrak{I}(U)$ generated by $\mathcal{I}$ in $\mathfrak{I}(U)$, so we see that the inverse image ideal sheaf $\tilde{I} = \pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\tilde{X}}$ is in fact equal to $\mathcal{O}_{\tilde{X}}(1)$. Hence it is invertible sheaf.

(2) If $U = X - Y$, then $\mathcal{I}\mid_U \cong \mathcal{O}_U$, so $\pi^{-1}U = \mathbf{Proj}\mathcal{O}_U[T] = U$. □

Theorem 5.2. *Let X be a variety over k, let $\mathcal{I} \subseteq \mathcal{O}_X$ be a non-zero coherent sheaf of ideals on X, and let $\pi : \tilde{X} \longrightarrow X$ be the blowing-up with respect to $\mathcal{I}$. Then:*

(a) $\tilde{X}$ is also a variety

(b) π is birational, proper, surjective morphism

(c) if X is quasi-projective(respectively, projective) over k, then $\tilde{X}$ is also. and π is a projective morphism

Proof. First of all since X is integral, the sheaf $\mathfrak{I} = \bigoplus_{d \geq 0} \mathcal{I}^d$ is a sheaf of integral domains on X, so $\tilde{X}$ is also integral. Proposition 1 above implies/shows that π is proper. In particular π is separated and of finite type, so it follows that $\tilde{X}$ is also separated and of finite type, i.e., $\tilde{X}$ is a variety.
Now since $\mathcal{I} \neq 0$, the corresponding closed subscheme Y is not whole of X and so the open set $U = X - Y$ is non-empty. Since π induces an isomorphism from $\pi^{-1}U$ to U (from part 2 of proposition 5.3), we see that π is birational.
Since π is proper, it is a closed map, so the image $\pi(\tilde{X})$ is a closed set containing U, which must be all on X since X is irreducible. Thus π is surjective.
Finally, if X is quasi-projective(respectively, projective), then X admits an ample invertible sheaf. So, by proposition 1, π is a projective morphism. It follows that $\tilde{X}$ is also quasi-projective(respectively, projective) $\square$

Resolution of Singularities

We now state some important results associated with Resolution of Singularities:

Given an ideal sheaf I on a smooth variety X, the first aim is to write down a birational morphism $g : X' \longrightarrow X$ such that X' is smooth and the pulled-back ideal sheaf g^*I is locally principal. This is called the *principalization* of I.

Remark 5.2. *Resolution of singularities implies principalization.*

Theorem 5.3 (Elimination of indeterminacies)**.** *Let X be a smooth variety over a field of characteristic zero and $g : X \dashrightarrow \mathbb{P}$ a rational map to some projective space. Then there is a smooth variety X' and a birational and projective morphism $f : X' \longrightarrow X$ such that the composite $g \circ f : X' \longrightarrow \mathbb{P}$ is a morphism.*

Proposition 5.4. *Let X be a quasi-projective variety. Then there is a smooth variety X' and a birational and projective morphism $g : X' \longrightarrow X$.*

Theorem 5.4 (Hironaka)**.** *Let X be a reduced algebraic variety over a field of characteristic zero, or more generally a reduced scheme that is locally of finite type over an excellent, reduced, locally Noetherian, scheme of characteristic zero (i.e., char$(k(x) = 0)$ for every $x \in X$). Then X admits a desingularization in the strong sense.*

Note that if X is a curve, to resolve the sigularities, it is sufficient to take the normalisation of X.

Below we give a few important theorems and results used in the text:

(1) **Chow's Lemma** For any complete irreducible variety X, there exists a projective variety $\bar{X}$ and a surjective birational morphism $f : \bar{X} \longrightarrow X$.

Proof of Chow's Lemma. Let $X = \cup U_i$ be a finite affine cover. For each affine variety $U_i \subset \mathbb{A}^{n_i}$, denote by Y_i the closure of U_i in projective space $\mathbb{P}^{n_i} \supset \mathbb{A}^{n_i}$. The variety $Y = \prod Y_i$ is obviously projective.
Set $U = \cap U_i$. The inclusions $\psi : U \hookrightarrow X$ and $\psi_I : U \hookrightarrow U_i \hookrightarrow Y_i$ define a morphism

$$\phi : U \longrightarrow X \times Y, \qquad \textit{with} \qquad \phi = \psi \times \prod \psi_i$$

Write X for the closure of $\phi(U)$ in $X \times Y$. The first projection $p_X : X \times Y \longrightarrow X$ defines a morphism $f : \bar{X} \longrightarrow X$. We prove that it is birational. For this it suffices to check that

$$f^{-1}(U) = \phi(U) \qquad (1)$$

Indeed, $p_X \circ \phi = 1$ on U, and in view of (1), f coincides on $f^{-1}(U)$ with the isomorphism ϕ^{-1}. Now (1), is equivalent to

$$(U \times Y) \cap \bar{X} = \phi(U) \qquad (2)$$

that is, to $\phi(U)$ closed in $U \times X$. But this is obvious, since $\phi(U)$ in $U \times Y$ is just graph of the morphism $\prod \psi_i$. The morphism f is surjective, since $f(\bar{X}) \supset U$, and U is dense in X. It remains to prove that $\bar{X}$ is projective. For this we use the second projection $g : X \times Y \longrightarrow Y$, and prove that its restriction $\bar{g} : \bar{X} \longrightarrow Y$ is a closed embedding. Since to be a closed embedding is a local property, it is enough to find open sets $V_i \subset Y$ such that $\cup g^{-1}(V_i) \supset \bar{X}$ and $\bar{g} : \bar{X} \cap g^{-1}(V_i) \longrightarrow V_i$ is a closed embedding. We set

$$V_i = p_i^{-1}(U_i),$$

where $p_i : Y \longrightarrow Y_i$ is the projection. first of all, the $g^{-1}(V_i)$ cover $\bar{X}$. For this it is enough to prove that

$$g^{-1}(V_i) = f^{-1}(U_i) \qquad (3)$$

since $\cup U_i = X$ and $\cup f^{-1}(U_i) = \bar{X}$. In turn, (3) will follow from

$$f = p \circ g \qquad \textit{on } f^{-1}(U) \qquad (4)$$

But it is enough to prove (4) on some open subset $W \subset f^{-1}(U_i)$. We can in particular take $W = f^{-1}(U) = \phi(U)$ (according to (1)), and then (4) is obvious. Thus it remains to prove that

$$\bar{g} : \bar{X} \cup g^{-1}(V_i) \longrightarrow V_i$$

defines a closed embedding. Now recall that

$$V_i = p_i^{-1}(U_i) = U_i \times \widehat{Y}_i \qquad where \widehat{Y} = \prod_{j \neq i} Y_j$$

we get that

$$g^{-1}(V_i) = X \times U_i \times \widehat{Y}_i$$

Write Z_i for the graph of the morphism $U_i \times \widehat{Y}_i \longrightarrow X$, which is the composite of the projection to U_i and the embedding $U_i \hookrightarrow X$. The set Z_i is closed in $X \times U_i \times Y_i = g^{-1}(V_i)$ and its projection to $U_i \times \widehat{Y}_i = V_i$ is an isomorphism. On the other hand, $\phi(U) \subset Z_i$, and since Z_i is closed, $\bar{X} \cup g^{-1}(V_i)$ is closed in Z_i. Hence the restriction of the projection to this set is a closed embedding.

□

Note that analogous result holds true for an arbitrary variety when $\bar{X}$ is quasiprojective.

(2) **Nagata's theorem** Every variety can be embedded as an open dense subset of a complete variety.

(3) **Weak Factorization Theorem** Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between complete smooth varieties over K, let $U \subset X_1$ be and open set where ϕ is an isomorphism. Then ϕ can be factored into a sequence of blow-ups and blow-downs with smooth centers disjoint from U i.e., there exists a sequence of birational maps:

$$X_1 = V_0 \dashrightarrow V_1 \dashrightarrow \ldots \dashrightarrow V_i \dashrightarrow V_{i+1} \dashrightarrow \ldots \dashrightarrow V_{l-1} \dashrightarrow V_l = X_2,$$

where $\phi = \phi_l \circ \phi_{l-1} \circ \ldots \phi_2 \circ \phi_1$, such that each factor ϕ_i is an isomorphism over U, and $\phi_i : V_i \dashrightarrow V_{i+1} or \phi_i^{-1} : V_{i+1} \dashrightarrow V_i$ is a morphism obtained by blowing up a smooth center disjoint from U (here U is identified with an open subset of V_i).
Moreover there is an index i_0 such that for all $i \leq i_0$ the map $V_i \dashrightarrow X_1$ is defined everywhere and projective, and for all $i \geq i_0$ the map $V_i \dashrightarrow X_2$ is defined everywhere and projective.
If $X_1 - U$ (respectively, $X_2 - U$) is a simple normal crossing divisor, then the factorization can be chosen such that the inverse images of this divisor under $V_i \longrightarrow X_1$ (respectively, $V_i \longrightarrow X_2$) are also simple normal crossing divisors, and the centers of blowing up have normal crossings with these divisors.
If ϕ is equivariant under the action of a finite group, then the factorization can be chosen equivariantly.

6. Glossary

We first give some definitions:

(a) Abstract Variety: An *abstract variety* is an integral separated scheme of finite type over an algebraically closed field K. If it is proper over K, we say that it is complete. In the text, by a k-variety we mean a reduced separated scheme of finite type over k unless specified otherwise.

(b) Locally closed subset: A subset of X is *locally closed* if it is the intersection of an open subset with a closed subset.

(c) Stratification: Decomposing of a scheme into locally closed regular subschemes, mostly defined through the constancy of some local invariant like dimension.

(d) Vector Bundle

Definition 6.1. *A family of vector spaces over X is a fibration $p : E \longrightarrow X$ such that each fibre $E_x = p^{-1}(x)$ for $x \in X$ is a vector space over $k(x)$, and the structure of algebraic variety of E_x as a vector space coincides with that of $E_x \subset E$ as the inverse image of x under p.*

A morphism of a family of vector space $p : E \longrightarrow X$ into another family $q : F \longrightarrow X$ is a morphism $f : E \longrightarrow F$ for which diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "p"'] & F \arrow[d, "q"] \\ & X \end{tikzcd}$$

commutes (so that in particular f maps E_x to F_x for all $x \in X$), and the map $f_x : E_x \longrightarrow F_x$ is linear over $k(x)$. It's obvious how to define an isomorphism of families.

V is a vector space over k, and p the first projection of $X \times V \longrightarrow X$. A family of this type, or isomorphism to it, is said to be *trivial.*
If $p : E \longrightarrow X$ is a family of vector spaces and $U \subset X$ any open set, the fibration $p^{-1}(U) \longrightarrow U$ is family of vector spaces over U. It is called the *restriction* of E to U and denoted $E|_U$

Definition 6.2. *A family of vector spaces $p : E \longrightarrow X$ is a vector bundle if every point $x \in X$ has a neighbourhood U such that restriction E is trivial.*

The dimension of the fibre E_x of a vector bundle is a locally constant function on X, and, in particular, is constant if X is connected. In this case, the number $dim(E_x)$ is called the *rank* of E, and denoted by $rank(E)$.

(e) Normal Crossing Divisor: Let Y be a regular Noetherian scheme and let D be an effective Cartier divisor on Y. We say that D has *normal crossings* at a point $y \in Y$ if there exists a system of parameters $f_1, f_2, \ldots, f_n$ of Y at y, an integer $0 \leq m \leq n$, and integers $r_1, r_2, \ldots, r_m \geq 1$ such that $\mathcal{O}_Y(-D)_y$ is generated by $f_1^{r_1}, \ldots, f_m^{r_m}$. We say that D has normal crossing if it has normal crossing at every point of $y \in Y$. We say that the prime divisors $D_1, \ldots, D_l$ meet *transversally* at $y \in Y$ is they are pairwise distinct and if the divisor $D_1 + \ldots + D_l$ has normal crossing at y.

(f) Calabi-Yau varieties: Let X be a smooth complex projective variety of dimension n. We say that X is Calabi-Yau of X if X admits a nowhere vanishing degree n algebraic differential form ω, that is, the sheaf Ω^n_X is trivial.

(g) Abelian Variety: An Abelian variety is an algebraic group which is a complete algebraic variety.

(h) Albanese Variety: Let V be a variety, and write $G(V)$ for the set of divisors, $G_l(V)$ for the set of divisors linearly equivalent to 0, and $G_a(V)$ for the group of divisors algebraically equal to 0. Then $G_a(V)/G_l(V)$ is called the Picard variety. The Albanese variety is dual to the Picard variety. It is an Abelian variety which is canonically attached to an algebraic variety (in this case V).

(i) Normal scheme and Normalisation:
First of all recall that an integral domain A is called normal if it is integrally closed in $Frac(A)$.

Definition 6.3 (Normal scheme:). *Let X be a scheme. We say that X is normal at $x \in X$ if $\mathcal{O}_{X,x}$ is normal. We say X is normal if it is irreducible and normal at all its points.*

For example, $\mathbb{A}^n_k$, $\mathbb{P}^n_k$ are normal schemes.

Definition 6.4 (Normalization:). *Let X be an integral scheme. A morphism $\pi X' \longrightarrow X$ is called normalization morphism is X' is normal, and if every dominant morphism $f : Y \longrightarrow X$ with Y normal factors uniquely through π.*

Note that if $\pi X' \longrightarrow X$ is a normalization of X, then for every open set $U \subset X$, the restriction $\pi^{-1}(U) \longrightarrow U$ is a normalization of U.

Proposition 6.1. *Let X be an integral scheme. Then there exists a normalization morphism $\pi X' \longrightarrow X$, and it is unique upto isomorphism (of X-schemes). Moreover, a morphism $f : Y \longrightarrow X$ is the normalization morphism if and only if Y is normal, and f is birational and integral.*

(j) Singularity(or Singular Point): Point of a variety or scheme where local ring in not regular;

(k) Desingularization(Resolution of Singularities): Let X be a reduced locally Noetherian scheme. A proper birational morphism $\pi : Z \longrightarrow X$ with Z regular is called a *desingularization* of X (or a *Resolution of Singularities* of X). If π is an isomorphism above every regular point of X, we say that it is a *desingularization in the strong sense*

References

[And04] Yves André, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses [Panoramas and Syntheses], vol. 17, Société Mathématique de France, Paris, 2004. MR MR2115000 (2005k:14041)

[Bit04] Franziska Bittner, *The universal Euler characteristic for varieties of characteristic zero*, Compos. Math. **140** (2004), no. 4, 1011–1032. MR MR2059227 (2005d:14031)

[CL05] Raf Cluckers and François Loeser, *Ax-Kochen-Eršov theorems for p-adic integrals and motivic integration*, Geometric methods in algebra and number theory, Progr. Math., vol. 235, Birkhäuser Boston, Boston, MA, 2005, pp. 109–137. MR MR2159379 (2006g:12014)

[EH00] David Eisenbud and Joe Harris, *The geometry of schemes*, Graduate Texts in Mathematics, vol. 197, Springer-Verlag, New York, 2000. MR MR1730819 (2001d:14002)

[Gro60] A. Grothendieck, *Éléments de géométrie algébrique. I. Le langage des schémas*, Inst. Hautes Études Sci. Publ. Math. (1960), no. 4, 228. MR MR0217083 (36 #177a)

[GS96] H. Gillet and C. Soulé, *Descent, motives and K-theory*, J. Reine Angew. Math. **478** (1996), 127–176. MR MR1409056 (98d:14012)

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR MR0463157 (57 #3116)

[Hau00] Herwig Hauser, *Resolution of singularities 1860–1999*, Resolution of singularities (Obergurgl, 1997), Progr. Math., vol. 181, Birkhäuser, Basel, 2000, pp. 5–36. MR MR1748616 (2001g:14001)

[Jan95] Uwe Jannsen, *Mixed motives, motivic cohomology, and Ext-groups*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 667–679. MR MR1403967 (97j:14022)

[Liu02] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002, Translated from the French by Reinie Erné, Oxford Science Publications. MR MR1917232 (2003g:14001)

[LL03] Michael Larsen and Valery A. Lunts, *Motivic measures and stable birational geometry*, Mosc. Math. J. **3** (2003), no. 1, 85–95, 259. MR MR1996804 (2005a:14026)

[Poo02] Bjorn Poonen, *The Grothendieck ring of varieties is not a domain*, Math. Res. Lett. **9** (2002), no. 4, 493–497. MR MR1928868 (2003g:14010)

[Seb04] Julien Sebag, *Intégration motivique sur les schémas formels*, Bull. Soc. Math. France **132** (2004), no. 1, 1–54. MR MR2075915 (2005e:14017)

[Ser91] Jean-Pierre Serre, *Motifs*, Astérisque (1991), no. 198-200, 11, 333–349 (1992), Journées Arithmétiques, 1989 (Luminy, 1989). MR MR1144336 (92m:14002)

[SKKT00] Karen E. Smith, Lauri Kahanpää, Pekka Kekäläinen, and William Traves, *An invitation to algebraic geometry*, Universitext, Springer-Verlag, New York, 2000. MR MR1788561 (2001k:14002)