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Dynamics of Cherry flows

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Perchè mi avete ascoltata, sostenuta e amata.
Perchè non conosco altro modo per dirvi

GRAZIE.

Contents

Introduction

In this work we are interested in recurrence of vector fields on surfaces. From the Poincaré-Bendixson theorem the dynamics of vector fields on the sphere is clear: the ω-limit of every point is equal to a point or to a closed orbit. On the torus T^2 this situation is more interesting. If a vector field on T^2 has periodic not contractible orbits then there are again no non-trivial recurrent orbits, because then the situation reduces to a planar one.
We are interested in a particular vectorial field X on the torus with the following properties:

1. X has exactly two singularities, a sink P and a saddle S, both hyperbolic;

2. X has not periodic orbits.

Such a vector field on T^2 is called *Cherry vector field*, Cherry gave in 1938 an example of an analytic vector field on T^2 satisfying these properties.
In particular according to the eigenvalues $\lambda_1 > 0 > \lambda_2$ of the saddle point S, we can distinguish three cases:

- the non-dissipative case if $|\lambda_2| < \lambda_1$;

- the conservative case if $|\lambda_2| = \lambda_1$;

- the dissipative case if $|\lambda_2| > \lambda_1$.

In [11] is treated the non-dissipative case and is proved that if X has positive divergence at the saddle-point then the union U of the bidimensional invariant manifolds of X is dense on the torus and has total Lebesgue measure.
This result is generalized in [12] for a vector field X on the torus without closed orbits and that has a finite number of hyperbolic singularities, sinks and saddles, and positive divergence on the saddle-type singularities. Here is proved that the union U of the stable manifolds of the sinks of X has total Lebesgue measure and that the Hausdorff dimension of $\mathbb{S}^1 \setminus U$ is zero.
The conservative is treated in [16] where is proved that the union U of the

bidimensional invariant manifolds of a Cherry field X is dense on the torus and that the Lebesgue measure and the Hausdorff dimension of $\mathbb{S}^1 \setminus U$ is zero. We are interested instead in the dissipative case.

It is not difficult to obtain for a Cherry vector field a C^∞ curve $\Sigma \cong \mathbb{S}^1$ on T^2 which does not bound a disk and is everywhere transversal to X. We define a continuous map $f : \Sigma \to \Sigma$ putting $f(x)$ as the first return of x to Σ by the flow of X if it exists and is constant on the interval where it is not defined.

In [4] is proved that the non-wandering set K of f (that using the notation above is exactly $K = \mathbb{S}^1 \setminus U$) has zero Lebesgue mesure and if the rotation number is of bounded type (i.e.q_n/q_{n-1} are uniformely bounded) then the Hausdorff dimension of K is strictly less than 1.

Theorem 3.0.3 is the central theorem of this paper. In the case when f has the golden mean rotation number, we prove that the Hausdorff dimension of K is strictly greater than zero.

An application of this result is Theorem 4.4.3 which states that the quasi-minimal set of X has Hausdorff dimension strictly grater than 1 and consequently the vector field X has an infinite number of quasi-periodic orbits.

Follows the main idea of the proof of Theorem 3.0.3 which is shown for a larger class of monotone maps $f : \mathbb{S}^1 \to \mathbb{S}^1$, to which the return map constructed above belongs and whose properties are listed in Section 1.5.

The properties of f are studied by analysing the geometry of the partitions of the circle generated, for every n, by a certain number of preimages of the interval where f is constant, together with all holes (gaps) between two successive preimages.

We have proved that, for every n, the adjacent gaps of the n^{th} dynamic partition are comparable and this allowed us to construct a probability measure ν on the non-wandering set K with the property that, for whatever interval I, $\nu(I) \leq C\,|I|^\alpha$, where $C > 0$ and $0 < \alpha < 1$ are constant.

This establishes the theorem.

Chapter 1

Preliminaries

1.1 Basic Notations

We consider the circle $\mathbb{S}^1$ as the quotient of the real line by the group of translations by integers: $\mathbb{R}/\mathbb{Z}$ and we consider the circular ordering on $\mathbb{S}^1$. Let $\tilde{\pi} : \mathbb{R} \to \mathbb{S}^1$ be the quotient map. In $\mathbb{S}^1$ we consider the metric and the orientation induced from the metric and orientation of the real line via $\tilde{\pi}$.
If $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ we put $|k| = \sup_i |k_i|$ and for $x \in \mathbb{R}$ we put $\|x\| = \inf_{p\in\mathbb{Z}} |x + p|$ that defines a distance on $\mathbb{S}^1$.
We will introduce a simplified notation for backward and forward images of U. Instead of $f^i(U)$ we will simply write $\underline{i}$. For example, $\underline{0} = U$. This convention will also apply to more complex expressions. For example $f^{-3q_n-20}(U)$, will be abbrevieted to $\underline{-3q_n - 20}$.

1.2 Distance between Points-Conventions of Notation

Denoted by $(a, b) = (b, a)$ the open shortest interval between a and b regardless of the order of these two points. The lenght of that interval in the natural metric on the circle will be denoted by $|a - b|$. Let us adopt now the following conventions of notation:

- $|\underline{-i}|$ stands for the length of the interval $\underline{-i}$.
- Consider a point x and an interval $\underline{-i}$ not containing it. Then the distance from x to the closer endpoint of $\underline{-i}$ will be denoted by $|(x, \underline{-i})|$, and the distance to the more distant endpoint by $|(x, \underline{-i}]|$.

- We define the distance between the endpoints of two intervals $\underline{-i}$ and $\underline{-j}$ analogously. For example, $\left|(\underline{-i}, \underline{-j})\right|$ denotes the distance between the closest endpoints of these two intervals while $\left|[\underline{-i}, \underline{-j})\right|$ stands for $|\underline{-i}| + \left|(\underline{-i}, \underline{-j})\right|$.

1.3 Uniform Constants

The letter K with subscripts will be reserved for 'uniform constants'. If we claim a statement which involves such constants we mean precisely that for each occurrence of such a constant a positive value can be inserted which will make the statement true. The choice is uniform in the sense that once f has been fixed, there is a choice of values which makes the statement true in all cases covered. The use of the symbol K will be local, in that the same symbol may signify different uniform constants in different parts of the paper.

1.4 Rotation Number

Proposition 1. *Let $F, G : \mathbb{R} \to \mathbb{R}$ be monotonic continuous functions such that $F(x+1) = F(x) + 1$ and $G(x+1) = G(x) + 1$ for all $x \in \mathbb{R}$. Then*

1. *$\rho(F) = \lim_{n\to\infty} \frac{F^n(0)}{n}$ exists and $\left|\frac{F^n(0)}{n} - \rho(F)\right| < \frac{1}{n}$;*

2. *$\lim_{n\to\infty} \frac{F^n(x)-x}{n}$ exists for all $x \in \mathbb{R}$ and is equal to $\rho(F)$;*

3. *$\rho(F) = \frac{m}{n}$ with $m, n \in \mathbb{Z}$, $n > 0$ if and only if there exists $x \in \mathbb{R}$ such that $F^n(x) = x + m$;*

4. *given $\epsilon > 0$ there exists $\delta > 0$ such that if $\|F - G\|_0 =$ $= \sup_{x\in\mathbb{R}} |F(x) - G(x)| < \delta$ then $|\rho(F) - \rho(G)| < \epsilon$;*

5. *$\rho(F + n) = \rho(F) + n$ for any integer n.*

Proof. Let $M_k = \max_{x\in\mathbb{R}}(F^k(x) - x)$ and $m_k = \min_{x\in\mathbb{R}}(F^k(x) - x)$. We claim that $M_k - m_k < 1$. In fact, as $F(x+1) = F(x) + 1$ we have that $F^k(x+1) = F^k(x) + 1$. Therefore, $\varphi = F^k - id$ is periodic with period 1. Consequently there exist $x_k, X_k \in \mathbb{R}$ with $0 \le x_k - X_k < 1$ such that $\varphi(x_k) = m_k$ and $\varphi(X_k) = M_k$. Since F^k is also monotonic nondecresing we have $F^k(X_k) \le F^k(x_k)$. Hence $M_k + X_k \le m_k + x_k$ and so $M_k - m_k \le x_k - X_k < 1$ which proves our claim.
We are now going to prove that

$$F^k(y) - y - 1 \le F^k(x) - x \le F^k(y) - y + 1, \forall x, y \in \mathbb{R}. \tag{1.4.1}$$

In fact, $F^k(y) - y - 1 \leq M_k - 1 \leq m_k \leq F^k(x) - x \leq M_k \leq m_k + 1 \leq F^k(y) - y + 1$. We next put $y = 0$ and $x = F^{k(j-1)}(0)$ in (1.4.1) and obtain

$$F^k(0) - 1 \leq F^{kj}(0) - F^{k(j-1)}(0) \leq F^k(0) + 1.$$

Thus

$$n\left(F^k(0) - 1\right) = \sum_{j=1}^{n}\left(F^k(0) - 1\right) \leq \sum_{j=1}^{n}\left(F^{kj}(0) - F^{k(j-1)}(0)\right) \leq$$

$$\leq n\left(F^k(0) + 1\right).$$

From this we deduce that

$$nF^k(0) - n \leq F^{kn}(0) \leq nF^k(0) + n.$$

We now divide by kn to obtain

$$\frac{F^k(0)}{k} - \frac{1}{k} \leq \frac{F^{kn}(0)}{kn} \leq \frac{F^k(0)}{k} + \frac{1}{k}$$

or

$$\left|\frac{F^{kn}(0)}{kn} - \frac{F^k(0)}{k}\right| \leq \frac{1}{k}. \tag{1.4.2}$$

Similarly

$$\left|\frac{F^{kn}(0)}{kn} - \frac{F^n(0)}{n}\right| \leq \frac{1}{n}.$$

The sequence $\frac{F^k(0)}{k}$ is a Cauchy sequence because $\left|\frac{F^k(0)}{k} - \frac{F^n(0)}{n}\right| \leq \frac{1}{k} + \frac{1}{n}$ and so it converges to some limit $\rho(F)$. By letting n tend to ∞ in (1.4.2) we see that $\left|\rho(f) - \frac{F^k(0)}{k}\right| \leq \frac{1}{k}$ which proves (1).
By putting $y = 0$ in (1.4.1) we obtain $F^k(0) - 1 \leq F^k(0) + 1$.
Thus

$$\frac{F^k(0)}{k} - \frac{1}{k} \leq \frac{F^k(x) - x}{k} \leq \frac{F^k(0)}{k} + \frac{1}{k}.$$

This shows that $\frac{F^k(x)-x}{k}$ converges to $\rho(f)$ and proves (2).
To prove (3) suppose that there exists x, a real number, such that $F^n(x) = x + m$ with $m, n \in \mathbb{Z}$, n grater than zero. It follows easily by induction that $F^{kn}(x) = x + km$. Then $\rho(f) = \lim_{k\to\infty} \frac{F^{kn}(x)-x}{kn} = \frac{m}{n}$.
Now, for the other implication, let $\rho(f) = \frac{m}{n}$ and suppose that the thesis is false; thus $F^n(x) - x > m$ or $F^n(x) - x < m$ for every x in $\mathbb{R}$. For the first inequality, as $F^n - id$ is periodic, there exists $a > 0$ such that

$F^n(x) - x \geq m + a$, thus $F^{kn}(x) - x \geq km + ka$ and so $\rho(f)$ is greater than $\frac{m+a}{n}$ which is a contradiction. The same argument proves the falsity of inequality in the other direction.
To prove (4) we remark that

$$|\rho(F) - \rho(G)| \leq \left|\rho(G) - \frac{G^k(0)}{k}\right| + \left|\frac{G^k(0)}{k} - \frac{F^k(0)}{k}\right| + \left|\frac{F^k(0)}{k} - \rho(F)\right| \leq$$

$$\leq \frac{1}{k} + \frac{1}{k}\left|G^k(0) - F^k(0)\right| + \frac{1}{k}.$$

Fix an integer k such that $\frac{2}{k}$ is less than $\frac{\epsilon}{2}$ and choose $\delta > 0$ such that $\left|G^k(0) - F^k(0)\right| < \frac{k\epsilon}{2}$ if $\|G - F\|_0 < \delta$. In conclusion, $|\rho(F) - \rho(G)|$ is less than ϵ if $\|G - F\|_0$ is less than δ.
It remains to prove the last point. By induction we have that $(F + n)^k(x) = F^k(x) + kn$; thus

$$\rho(F + n) = \lim_{k\to\infty} \frac{(F + n)^k(0)}{k} = \lim_{k\to\infty} \frac{F^k(0) + kn}{k} = \rho(F) + n.$$

□

The previous proposition allows us to introduce the rotation number for degree 1 monotonic endomorphisms of the circle. An endomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ is monotonic and has degree 1 if and only if it has a lift $F : \mathbb{R} \to \mathbb{R}$ which is a continuous monotonic function satisfying $F(x + 1) = F(x) + 1$. We the define the *rotation number* of f as $\rho(f) = \tilde{\pi}\rho(F)$ where $\tilde{\pi}$ is the quotient map. This definition does not depend on the choice of the lift F because of Proposition 1(5).
In the discussion that follows in this work, it will often be convenient to identify f and F and subsets of $\mathbb{S}^1$ with corresponding subsets of $\mathbb{R}$. The dynamics of f is most interesting when the rotation number is irrational.

1.5 Almost Smooth Maps with a Flat Interval

We consider the class of continuous circle endomorphisms f of degree one for which an arc U exists so that the following properties hold:

1. The image of U is one point.

2. The restriction of f to $\mathbb{S}^1 \setminus \overline{U}$ is a C^3-diffeomorphism onto its image.

3. Let (a,b) be a preimage of U under the projection of the real line of $\mathbb{S}^1$. On same right-side neighdorhood of b, f can be represented as

$$h_r\left((x-b)^{l_r}\right)$$

for $l_r \geq 1$, where h_r is a C^3-diffeomorphism on a two-side neighdorhood of b. Analogously, on a left-sided neighdorhood of a, f is

$$h_l\left((a-x)^{l_l}\right).$$

The ordered pair (l_l, l_r) will be called the *critical exponent of the map.* If $l_l = l_r$ the map will be referred to as *symmetric.*
In the future, we will deal exclusively with symmetric maps with critical exponent l strictly greater than 1, and we will assume that $h_r(x) = h_l(x) = x$. Also, we permanently assume that the rotation number is irrational and of bounded type (i.e. if $q_n \setminus q_{n-1}$ are uniformly bounded). As in [2] is easy to see that the non-wandering set of f is $\mathbb{S}^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(U)$.

1.6 The Croos-Ratio Inequality

If $a < b < c < d$, then define their $croos-ratio$ $\mathbf{Cr}$ by

$$\mathbf{Cr}(a,b,c,d) := \frac{|b-a|\,|d-c|}{|c-a|\,|d-b|}.$$

Consider a chain of quadruples

$$\{a_i, b_i, c_i, d_i\}\; i = 0, \ldots, n$$

such that each is mapped onto the next by the map f. If the following conditions hold:

1. Each point of the circle belongs to at most k of the intervals (a_i, d_i).

2. Intervals (b_i, c_i) do not intersect $\underline{0}$.

Then

$$\log \frac{\mathbf{Cr}(a_n, b_n, c_n, d_n)}{\mathbf{Cr}(a_0, b_0, c_0, d_0)} \leq K_{[k]},$$

where the constant $K_{[k]}$ does not depend on the set of quadruples. In order to avoid both ambiguities in notation and long and unreadable formulas while discussing croos-ratio, we adopt the following notation to describe the quadruples used:

$$\{(a,b),(c,d)\} := \{a,b,c,d\}.$$

For example, if $\underline{-i} = (a,b)$ and $\underline{-j} = (c,d)$ we will write $\mathbf{Cr}(\underline{-i}, \underline{-j})$ in place of $\mathbf{Cr}(a,b,c,d)$.

1.7 The Koebe principle

Definition 1.7.1. *Suppose that I is a compact interval and J is an open interval such that $\overline{J} \subset I$. We define $\nu(J,I) := \frac{|J|}{dist(J,\partial I)}$.*

Proposition 2. *(the Koebe principle) Let I be a compact interval and $f : I \to I$ be a C^2 map with all critical points C^2 nonflat. Then there exists a gauge function σ with the following property. If $J \subset T$ are open intervals and $n \in \mathbb{N}$ is such that f^n is a diffeomorphism on T then, for every $x, y \in J$, we have*

$$\frac{(f^n)'(x)}{(f^n)'(y)} \geq \frac{\exp^{-\sigma(\max_{i=0}^{n-1}|f^i(T)|)\sum_{i=0}^{n-1}|f^i(J)|}}{(1+\nu(f^n(J), f^n(T)))^2}.$$

The proof of the Koebe principle can be found in [5].

1.8 Carathéodory's extension theorem

Definition 1.8.1. *For a given set Ω we define a ring R as a subset of the power set of Ω which has the following properties:*

- *$\varnothing \in R$.*
- *For all $A, B \in R$, we have $A \cup B \in R$ (closed under pairwise unions).*
- *For all $A, B \in R$, we have $A \setminus B \in R$ (closed under relative complements).*

Theorem 1.8.2. *(Carathéodory's extension theorem) Let R be a ring on Ω and $\mu : R \to [0, +\infty]$ be a measure; then there exists a measure $\nu : \sigma(R) \to [0, +\infty]$ such that ν is an extension of μ. (That is $\nu_{|R} = \mu$). Here $\sigma(R)$ is the σ-algebra generated by R.*

If μ is σ-finite then the extension ν is unique (and also σ-finite).

1.9 Golden ratio

Definition 1.9.1. *Two quantities are in the golden ratio if the ratio of the sum of the quantities to the larger quantity is equal to the ratio of the larger quantity to the smaller one.*

The golden ratio is often denoted by the Greek letter φ. Other names frequently used for the golden ratio are the golden section and golden mean.

Proposition 3. *The golden ratio is an irrational mathematical constant, approximately 1.6180339887.*

Proof. By the definition we have that

$$\frac{a+b}{a} = \frac{a}{b} = \varphi.$$

The right equation shows that $a = b\varphi$, which can be substituted in the left part, giving

$$\frac{b\varphi + b}{b\varphi} = \frac{b\varphi}{b}.$$

Dividing out b yields

$$\frac{\varphi + 1}{\varphi} = \varphi.$$

Multiplying both sides by φ and rearranging terms leads to:

$$\varphi^2 - \varphi - 1 = 0.$$

The only positive solution to this quadratic equation is

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887\ldots$$

$\square$

Remark 1.9.2. *The formula $\varphi = 1 + \frac{1}{\varphi}$, deduced from the previous demonstration, can be expanded recursively to obtain a continued fraction for the golden ratio:*

$$\varphi = \frac{1}{1 + \frac{1}{1+\frac{1}{\cdots}}}.$$

Chapter 2

Geometric Bounds

2.1 Continued fractions and dynamics

2.1.1 Continued fraction

The rotation number $\rho(f)$, that we let ρ, can be written as an infinite continued fraction as follow.
Let $G : [0,1] \to [0,1]$ be a function defined by

$$G(x) = \frac{1}{x} - \left[\frac{1}{x}\right] = \frac{1}{x} \mathrm{mod} 1 \text{if } x \neq 0,$$
$$G(0) = 0.$$

Since $\rho \in [0,1]$, the continued fraction expansion back to consider the sequence $(G^n(\rho))$ of iterates n-th of G.
We set:

$$a(x) = \begin{cases} \left[\frac{1}{x}\right] & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

If $\rho \in [0,1] \setminus \mathbb{Q}$ we have that, for all n, $G^n(\rho)$ is different to 0; then we set

$$a_0 = [\rho] = 0$$
$$a_n = a\left(G^{n-1}(\rho)\right) \text{ if } n \geq 1.$$

Finally we have that

$$\rho = [a_0, a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}$$

where the a_i are positive integers greater than 1 and are called the partial quotients of ρ.

2.1.2 The reduced

If $\rho = [a_0, a_1, a_2, \ldots]$, the reduced of ρ are the rationals

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}.$$

For example:

$$\frac{p_0}{q_0} = \frac{a_0}{1} \text{ and } \frac{p_1}{q_1} = \frac{a_0 a_1 + 1}{a_1}.$$

The q_n are called the denominators of reduced ($q_n \geq 1, q_n \in \mathbb{N}$) and they verify, together with the p_n, the following relations:

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2} & \text{for } n \geq 2, p_0 = a_0, p_1 = a_0 a_1 + 1 \\ q_n = a_n q_{n-1} + q_{n-2} & \text{for } n \geq 2, q_0 = 1, q_1 = a_1 \end{cases}$$

So, we have by induction the Lagrange's relation : $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$ that tells us that p_n and q_n are relatively prime.

Lemma 2.1.1. *If $n \geq 1$ then $q_{n+1} > q_n \, (q_1 \geq q_0 = 1)$ and if $n \geq 2$ then $q_n \geq 2^{\frac{n}{2}}$.*

Proof. The first inequality is clear. For the second it suffices to note that $q_n \geq 2q_{n-2}$ and that $q_3 \geq 3 \geq 2^{\frac{3}{2}}$; the lemma follows easily by induction. $\square$

2.1.3 The reduced and estimates by the rationals

If $n \geq 1$ and $t \geq 0$, by induction we have that

$$[a_0, a_1, \ldots, a_n + t] = \frac{p_n + t p_{n-1}}{q_n + t q_{n-1}}.$$

Now, since

$$\rho = [a_0, a_1, \ldots, a_n + G^n(\rho)]$$

then

$$\rho = \frac{p_n + G^n(\rho) p_{n-1}}{q_n + G^n(\rho) q_{n-1}}.$$

By the previous equality and by the Lagrange's relation we have that

$$\rho - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n \left(\frac{1}{G^n(\rho)} q_n + q_{n-1} \right)}.$$

Observing that

$$a_{n+1} \leq \frac{1}{G^n(\rho)} \leq a_{n+1} + 1$$

we have finally the following inequalities:

$$(-1)^n \left(\rho - \frac{p_n}{q_n}\right) > 0,$$

$$\frac{1}{q_n\left(q_n + q_{n+1}\right)} \leq \left|\rho - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n q_{n+1}} < \frac{1}{{q_n}^2}.$$

Then, we can determine a family of solutions $\frac{p}{q} \in \mathbb{Q}\left(q \geq 1, (p,q) = 1\right)$ of the inequality $\left|\rho - \frac{p_n}{q_n}\right| < \frac{1}{{q_n}^2}$; each element of this family is called rational approximation of ρ. Recalling that $q_{n+1} = a_{n+1}q_n + q_{n-1}$ we can write the previous inequality as

$$\frac{1}{\left(a_{n+1} + 2\right){q_n}^2} < \left|\rho - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}{q_n}^2}.$$

Proposition 4. *Let ρ be an irrational number and let $\left(q_n\right)_{n\in\mathbb{N}}$ be the series of the denominators of its reduced; if q is an integer such that $|q| > 0$ and $|q| < q_{n+1}$, then $\|q\rho\| \geq \|q_n\rho\|$. Vice versa the q_n are defined as: $q_0 = 1$, $q_1 = a_1$ and if $n \geq 1$, q_{n+1} is the smallest strictly positive integer such that $\|q_{n+1}\rho\| \leq \|q_n\rho\|$ (attention if $a_1 = 1$, $\|q_0\rho\| = \|q_1\rho\|$).*

For the demonstration, see [8].

Proposition 5. *If $n \geq 1$, $|q_n\rho - p_n| = \|q_n\rho\|$ and if $n \geq 3$ $\|q_{n-2}\rho\| = a_n \|q_{n-1}\rho\| + \|q_n\rho\|$.*

Proof. If $n \geq 2$, from the relations that p_n and q_n occur, it follows that

$$|q_{n-2}\rho - p_{n-2}| = a_n\,|q_{n-1}\rho - p_{n-1}| + |q_n\rho - p_n|\,.$$

Now, since $(-1)^n\left(q_n\rho - p_n\right) > 0$ and since $|q_n\rho - p_n| < \frac{1}{q_{n+1}}$ then $|q_n\rho - p_n| < |q_{n-1}\rho - p_{n-1}|$ and if $n \geq 1$ $|q_n\rho - p_n| = \|q_n\rho\| \leq \frac{1}{2}$. So, if $n \geq 3$ the last two inequalities allow us to write the first relation as,

$$\|q_{n-2}\rho\| = a_n\,\|q_{n-1}\rho\| + \|q_n\rho\|\,.$$

□

We observe that, by the results $|q_{n-2}\rho - p_{n-2}| = a_n\,|q_{n-1}\rho - p_{n-1}| + |q_n\rho - p_n|$ and $|q_n\rho - p_n| < |q_{n-1}\rho - p_{n-1}|$ found in the previous demonstration, if $n \geq 2$ we have that

$$2 < \frac{|q_{n-2}\rho - p_{n-2}|}{|q_n\rho - p_n|} < \left(a_n + 1\right)\left(a_{n+1} + 1\right). \tag{2.1.2}$$

2.1.4 The order of points $n\rho$ on $\mathbb{S}^1$

Proposition 6. *$q_n\rho$ approach the origin from the right for n even, and from the left for n odd.*

Proof. Let n even ; we can write

$$\rho = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \frac{1}{x}}}}}$$

and clearly

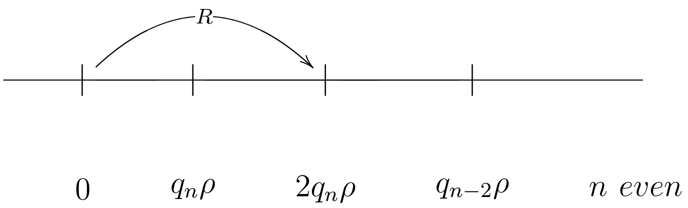
$$a_n + \frac{1}{x} > a_n.$$

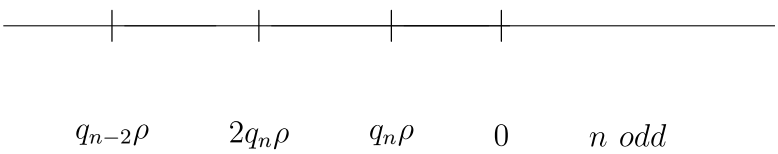
Since, if we take the inverses of both sides an even number of times, the inequality still holds, we have that

$$\rho > \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}} = \frac{p_n}{q_n} \geq \frac{1}{q_n}.$$

By the same arguments we calculate the position of $q_n\rho$ for n odd. □

Proposition 7. *If $n \geq 2$, then the intervals $(0, q_n\rho)$ and $R_{q_n\rho}(0, q_n\rho) = (q_n\rho, 2q_n\rho)$ are disjoint and we have that $(0, q_n\rho] \subset (0, q_{n-2}\rho)$ and $[q_n\rho, 2q_n\rho] \subset (0, q_{n-2}\rho)$. The points are in the same order as in the figure*





Proof. By the Proposition 5 and by the inequality 2.1.2 we have that

$$|q_{n-2}\rho - p_{n-2}| = a_n\,|q_{n-1}\rho - p_{n-1}| + |q_{n-2}\rho - p_n| > 2\,|q_{n-2}\rho - p_n|$$

and also

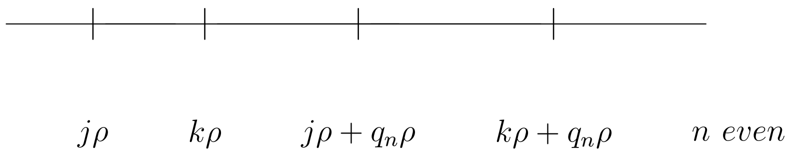
$$|q_{n-2}\rho - p_{n-2}| > 2\,|q_{n-2}\rho - p_n|\,.$$

Follows the proposition. $\square$

Proposition 8. *Let j be an integer between 0 and $q_{n+1}-1$, then the intervals $\{R_{j\rho}\,(0, q_n\rho)\}_{\leq j<q_{n+1}}$ are mutually disjoint.*

Proof. We suppose that $q_{n+1} > 1$, otherwise there is nothing to demonstrate and we suppose, by contradiction that the intervals aren't mutually disjoint; then there exists an integer $k \in [0, q_{n+1}[$ such that:

$$k\rho \in]j\rho, j\rho + q_n\rho[\quad (mod 1).$$



Since R_ρ is an isometry, it preserves the order on $\mathbb{S}^1$, $(k-j)\,\rho \in (0, q_n\rho)$ (mod 1) then $0 < \|(k-j)\,\rho\| < \|q_n\rho\|$; and yet $\|(k-j)\,\rho\| = \||k-j|\,\rho\|$ and $0 < |k-j| < q_{n+1}$ and this is contrary to 4. The proposition follows by contradiction. $\square$

For more details we send back the reader to [14].

Remark 2.1.3. *We suppose $q_{n+1} > 1$ then the intervals modulo 1:*

$$\{R_{j\rho}\,(0, q_n\rho)\}_{0\leq j\leq q_{n+1}-1} \text{ and } \{R_{j\rho}\,(0, q_{n+1}\rho)\}_{0\leq j<q_n-1}$$

cover $\mathbb{S}^1$ and more they are mutually disjoint.

2.2 Scaling near Critical point

We define a sequence of scalings

$$\tau_n := \frac{\left|\underline{0}, \underline{q_n}\right|}{\left|\underline{0}, \underline{q_{n-2}}\right|}.$$

This quantities measure 'the geometry' in the proximity of the critical point. When $\tau_n \to \infty$ we say that the geometry of the mapping is 'degenerate'. When t_n is bounded away from zero we say that the geometry is 'bounded'. 'Universal geometry' is said to occur when the sequence τ_n converges.
All results that follow are valid in the case of the geometry 'bounded'.

2.3 Continued fraction and partitions

By the Poincaré Theorem, because f is order-preserving and has no periodic points, there exists an order-preserving and continuous map $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h \circ f = R_\rho \circ h$, where ρ is the rotation number of f and R_ρ is the rigid rotation over ρ. In particular, the order of points in an orbit of f is the same as the order of points in an orbit of R_ρ; therefore, results about R_ρ, can be translated into results about f, via the semiconjugacy h.
Then, using the results obtained in the previous subsection we can create the following two partitions of $\mathbb{S}^1$ so that to study the properties of f analyzing the geometry of them.
For the first one, we use the orbit of $\underline{0}$ for $0 \leq i \leq q_{n+1} + q_n - 1$ together with open arcs lying between successive points of the orbit. We have so servant a partition of the circle referred to as the n_r^{th} *dynamic partition* $\mathfrak{P}_n$ whose intervalli consist of two types:

- The set of 'long' intervals consists, for n even, of the interval between $\underline{0}$ and $\underline{q_n}$ along with its forward images

  $$\mathfrak{A}_n := \left\{(\underline{i}, \underline{q_n + i}) : 1 \leq i \leq q_{n+1} - 1\right\};$$

  or, for n odd, it consists of the interval between $\underline{q_n}$ and $\underline{0}$ along with its forward images

  $$\mathfrak{A}_n := \left\{(\underline{q_n + i}, \underline{i}) : 1 \leq i \leq q_{n+1} - 1\right\};$$

- The set of 'short' intervals, for n even, consists of the interval between $\underline{q_{n+1}}$ and $\underline{0}$ along with its forward images

  $$\mathfrak{B}_n := \left\{(\underline{q_{n+1} + i}, \underline{i}) : 1 \leq i \leq q_n - 1\right\}$$

  or, for n odd, consists of the interval between $\underline{0}$ and $\underline{q_{n+1}}$ along with its forward images

  $$\mathfrak{B}_n := \left\{(\underline{i}, \underline{q_{n+1} + i}) : 1 \leq i \leq q_n - 1\right\}$$

The second ones is generated by the first $q_n + q_{n+1}$ preimages of U end is denoted $\mathfrak{P}_{-n}$ It consists of

$$\mathfrak{I}_n := \{\underline{-i} : 0 \leq i \leq q_{n+1} + q_n - 1\},$$

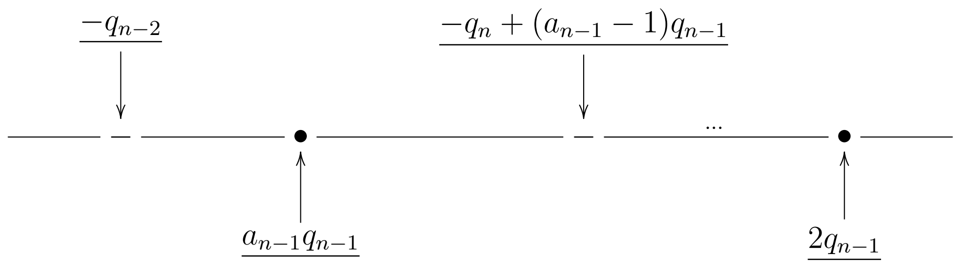
together with the gaps between these sets.
Also in this case, there are two kinds of gaps:

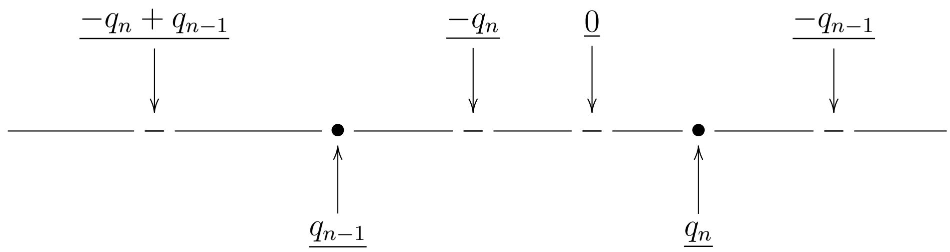
- The 'long' gaps are the interval I_0^n, which is the interval between $\underline{-q_n}$ and $\underline{0}$ for n even or the interval between $\underline{0}$ and $\underline{-q_n}$ for n odd, with its preimages,
$$I_i^n := f^{-i}(I_0^n), i = 0, 1, \ldots q_{n+1} - 1.$$
- The 'short' gaps are the interval I_0^{n+1}, which is the interval between $\underline{0}$ and $\underline{-q_{n+1}}$ for n even or the interval between $\underline{-q_{n+1}}$ and $\underline{0}$for n odd, with its preimages,
$$I_i^{n+1} := f^{-i}(I_0^{n+1}), i = 0, 1, \ldots, q_n - 1.$$

We will briefly explain the structure of the partitions. Take two subsequent dynamical partitions of order n and $n+1$. The latter is clearly a refinement of the former. All 'short' gaps of $\mathfrak{P}_{-\mathfrak{n}}$ become 'long' gaps of $\mathfrak{P}_{-(\mathfrak{n}+1)}$ while all 'long' gaps of $\mathfrak{I}_\mathfrak{n}$ split into a_{n+2} preimages of U and a_{n+2} 'long' gaps and one 'short' gap of the next partition $\mathfrak{I}_{\mathfrak{n}+1}$:

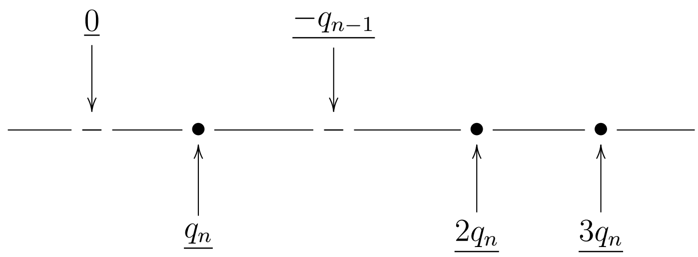
$$I_i^n = \bigcup_{j=1}^{a_{n+2}} f^{-i-q_n-jq_{n+1}}(U) \cup \bigcup_{j=0}^{a_{n+2}-1} I_{i+q_n+jq_{n+1}}^{n+1} \cup I_i^{n+2}.$$

Several of the proofs in the following will depend strongly on the relative positions of the points and intervals of $\mathfrak{P}_n$ and $\mathfrak{P}_{-n}$. In reading the proofs the reader is advised to keep the following pictures in mind, which show some of these objects near the flat interval $\underline{0}$.





In the next picture we have enlarged the right-hand part of this picture to show the location of the points $\underline{q_n}$, $\underline{2q_n}$ and $\underline{3q_n}$ for the case n even and $a_n = 1$.



Fact 2.3.1. *Let three points with y between x and z be arranged so that, of the three, the point x is the closest to the flat interval. If f is a diffeomorphism on (x, z), the following inequality holds:*

$$\frac{|f(z) - f(y)|}{|f(z) - f(x)|} \leq K \frac{|y - z|}{|z - x|},$$

where K is a uniform constant.

Lemma 2.3.2. *The ratio*

$$\frac{\left|(\underline{q_n}, \underline{3q_n})\right|}{\left|(\underline{0}, \underline{q_n})\right|}$$

is uniformly bounded away from zero.

Proof. Let J be the shortest arc belonging to the set $\mathfrak{A}_n$. If J coincides with the interval between $\underline{q_n}$ and $\underline{0}$, then

$$\left|(\underline{0}, \underline{q_n})\right| \leq \left|(\underline{q_n}, \underline{2q_n})\right| + \left|(\underline{2qn}, \underline{3q_n})\right| \leq 2 \left|(\underline{q_n}, \underline{3q_n})\right|$$

end the ratio is larger than $\frac{1}{2}$; so we assume that one of the other elements of $\mathfrak{A}_\mathfrak{n}$ is shorter.

Let J be the i^{th} iterate of $(\underline{q_n+1}, \underline{1})$. We observe that the i^{th} image of each of the intervals $(\underline{q_n+1}, \underline{3q_n+1})$ and $(\underline{q_{n+1}-q_n+1}, \underline{1})$ covers an interval belonging to $\mathfrak{A}_n$ and this images lie on different sides of J. Therefore, by the choice of J, we conclude that the croos-ratio

$$\mathbf{Cr}(\underline{q_{n+1}-q_n+1+i}, \underline{1+i}, \underline{q_n+1+i}, \underline{3q_n+1+i}) \geq \frac{1}{4}.$$

Since all intermediate images of $(\underline{q_{n+1}-q_n+1}, \underline{3q_n+1})$ cover the circle at most three times, we can use the croos-ratio inequality and we have that the croos-ratio $\mathbf{Cr}$ of the initial quadruple is greater than a uniform constant K. Thus, by this argument and by the Fact 2.3.1 we can conclude that

$$K \leq \mathbf{Cr}(\underline{q_{n+1}-q_n+1}, \underline{1}, \underline{q_n+1}, \underline{3q_n+1}) \leq \frac{\left|(\underline{q_n+1}, \underline{3q_n+1})\right|}{\left|(\underline{1}, \underline{3q_n+1})\right|} \leq$$

$$\leq K_1 \frac{\left|(\underline{q_n}, \underline{3q_n})\right|}{\left|(\underline{0}, \underline{3q_n})\right|} \leq \frac{\left|(\underline{q_n}, \underline{3q_n})\right|}{\left|(\underline{0}, \underline{q_n})\right|}.$$

□

Proposition 9. *The sequence* $\left|(\underline{0}, \underline{q_n})\right|$ *tends to zero at least exponentially fast.*

Proof. Lemma 2.3.2 implies that there is a constant $K < 1$ so that

$$\left|(\underline{0}, \underline{q_n})\right| \leq K \left|(\underline{0}, \underline{3q_n})\right|.$$

But $\underline{3q_n}$ lies between $\underline{0}$ and $\underline{q_{n-4}}$. Thus

$$\left|(\underline{0}, \underline{q_n})\right| \leq K \left|(\underline{0}, \underline{q_{n-4}})\right|.$$

□

Proposition 10. *Let $A \in \mathfrak{I}_n$ be a preimage of U and B one of the gaps adjacent to A, then there exists a constant C such that, for every $n \in \mathbb{N}$, $\frac{|A|}{|B|} \geq C$.*

Proof. Let $\underline{-i}$ and $\underline{-j}$ be successive members of I_i^n. In order to apply the croos-ratio inequality we consider the endpoints of these two intervals and their iterate by f until one of the intervals $\underline{-i}$ and $\underline{-j}$ is mapped to $\underline{0}$. By the cross-ratio inequality and by Fact 2.3.1 we have that

$$\mathbf{Cr}(\underline{-i}, \underline{-j}) \geq K_1 \frac{|\underline{-\epsilon}|}{|[\underline{-\epsilon}, \underline{0})|} \geq K_2 \frac{|\underline{-\epsilon+1}|}{|[\underline{-\epsilon+1}, \underline{1})|},$$

where ϵ is equal to either q_n or q_{n+1} and K_1 and K_2 are uniform constants. We now take the quadruple consisting of the endpoints of $\underline{-\epsilon+1}$ along with $\underline{1}$ and $\underline{2\epsilon+1}$ and we iterate this quadruple $\epsilon-1$ times. If we drop one of the factors in the initial croos-ration **Cr** we obtain that the last term of the previous inequality is larger than

$$K_3 \frac{|(\underline{\epsilon}, \underline{3\epsilon})|}{|(\underline{0}, \underline{3\epsilon})|},$$

which is, by Lemma 2.3.2 greater than a uniform constant. Combining all the above inequalities we finally get that the croos-ratio $\mathbf{Cr}(\underline{-i}, \underline{-j})$ is greater than a uniform constant which means in particular that the same holds for each of the two factors

$$\frac{|\underline{-i}|}{\left|[\underline{-i}, \underline{-j})\right|}, \frac{|\underline{-j}|}{\left|(\underline{-i}, \underline{-j}]\right|}.$$

This establishes the proposition. $\square$

Corollary 2.3.3. *Let* $\tau_n = \frac{|(\underline{0}, \underline{q_n})|}{|(\underline{0}, \underline{q_{n-2}})|}$*, then the sequence* $\{\tau\}_{n=1}^{\infty}$ *is bounded away from* 1*.*

Proof. To derive the corollary, it is enough to notice that $\underline{0}$, $\underline{-q_{n-1}}$ and $\underline{-q_{n-1}+q_{n-2}}$ are adjacent elements of $\mathfrak{I}_{n-2}$ and that $\underline{q_n}$ and $\underline{q_{n-2}}$ each lie in one of the gaps between them. $\square$

Corollary 2.3.4. *The lengths of the gaps of the dynamic partition* $\mathfrak{P}_{-n}$ *tend to zero at least exponentially fast with* n*.*

Theorem 2.3.5. *For any* f *with the critical exponent* (l, l)*,* $l > 1$*, the set* $\mathbb{S}^1 \setminus \bigcup_{i=0}^{\infty} f^{-1}(U)$ *has zero Lebesgue measure. Moreover, if the rotation number is of bounded type (i.e.* q_n/q_{n-1} *are uniformly bounded), the Hausdorff dimension of the non-wandering set is strictly than* 1*.*

Proof. By Proposition 10 the complement of all preimages of U does not contain any density point with respect to the Lebesgue measure. Hence, by the Lebesgue Density Lemma the set of non-wandering points is of zero Lebesgue measure.

The claim concerning the Hausdorff dimension requires a longer argument. Suppose that the rotation number is of bounded type. Take the n^{th} partition $\mathfrak{P}_{-n}$. The elements of $\mathfrak{P}_{-(n+1)}$ subdivide the gaps of $\mathfrak{P}_{-n}$ in the following way:

$$I_i^n \subset \bigcup_{j=0}^{a_{n+2}-1} I_{i+q_n+jq_{n+1}}^{n+1} \cup I_i^{n+2}.$$

We pick α so that $0 < \alpha < 1$ and estimate

$$\sum \left(|I_i^n|^\alpha + \left|I_i^{n+1}\right|^\alpha \right), \tag{2.3.6}$$

where $\sum$ denotes the sum over all gaps of the n^{th} partition $\mathfrak{P}_{-n}$. By Proposition 10 it follows that there is a constant $\beta < 1$ so that

$$\sum_{j=0}^{a_{n+2}-1} \left| I_{i+q_n+jq_{n+1}}^{n+1} \right| \leq \beta \, |I_i^n|$$

holds for all 'long' gaps $I_{i+q_n+jq_{n+1}}^{n+1}$ of the n^{th} partition. In particular it means that the gaps of $\mathfrak{P}_{-n}$ decrease uniformly and exponentially fast to zero while n tends to infinity. We use concavity of the function x^α to obtain that

$$\sum_{j=0}^{a_{n+2}-1} \left| I_{i+q_n+jq_{n+1}}^{n+1} \right|^\alpha \leq |a_{n+2}|^{1-\alpha} \beta^\alpha \, |I_i^n|^\alpha \leq |I_i^n|^\alpha$$

if α is close to 1. Hence 2.3.6 is a decreasing function of n. Consequently, the sum is bounded above. The only remaining point is to prove that for a given ϵ the gaps of $\mathfrak{P}_{-n}$ constitute an ϵ-cover of the non-wandering set if n is large enough. But this is so by Corollary 2.3.3. This completes the proof of the theorem. □

Chapter 3

New Results

Proposition 11. *Let f be a map from our class with the golden mean rotation number. Let $A \in \mathfrak{I}_n$ be a preimage of U and B_1 and B_2 the two gaps adjacent to A. If B_1 and B_2 are contained in a gap of a preceding partition, then there exists a constant C such that, for every n, $\frac{|B_1|}{|A|} \geq C$. Similarly, for B_2.*

Proof. Let $A = \underline{-q_n + 1}$ and let B the interval between A and $\underline{-q_{n-2} + 1}$.
We apply the Koebe principle,
for $T = [\underline{-q_{n-1} + 1}, \underline{-q_{n-2} + 1}]$, $J = (\underline{-q_{n-1} + 1}, \underline{-q_{n-2} + 1})$ and the number the iterates $q_{n-2} - 1$. We observe that $f^{q_{n-2}-1}$ is a diffeomorphism on T. Then there exists a gauge function σ such that

$$\frac{\left(f^{q_{n-2}-1}\right)'(x)}{\left(f^{q_{n-2}-1}\right)'(y)} \geq \frac{e^{-\sigma\left(\max_{i=0}^{q_{n-2}-2}\left|f^i(T)\right|\right)\sum_{i=0}^{q_{n-2}-2}\left|f^i(J)\right|}}{\left(1 + \nu\left(f^{q_{n-2}-1}(J), f^{q_{n-2}-1}(T)\right)\right)^2}$$

for every $x, y \in J$.
The intervals $f^i(T)$ $i = 0, \ldots, q_{n-2} - 1$, are mutually disjoint. Therefore $\max_{i=0}^{q_{n-2}-2} |f^i(T)| < 1$ and $\sum_{i=0}^{q_{n-2}-2} |f^i(J)| < 1$; so, the numerator in the above fraction is greater than a constant. For the denominator we have that

$$\nu\left(f^{q_{n-2}-1}(J), f^{q_{n-2}-1}(T)\right) = \frac{|f^{q_{n-2}-1}(J)|}{dist\left(f^{q_{n-2}-1}(J), \partial f^{q_{n-2}-1}(T)\right)} =$$

$$= \frac{\left|\left(\underline{-q_{n-3}}, \underline{0}\right)\right|}{dist\left(\left(\underline{-q_{n-3}}, \underline{1}\right), \partial\left[\underline{-q_{n-3}}, \underline{0}\right]\right)} \leq \frac{\left|\left(\underline{-q_{n-3}}, \underline{0}\right)\right|}{\left|\underline{-q_{n-3}}\right|}$$

which is smaller than a constant by Proposition 10. Combining all the above results we get that, there exists a constant K such that, for every $x, y \in J$

and $n \in \mathbb{N}$,

$$\frac{\left(f^{q_{n-2}-1}\right)'(x)}{\left(f^{q_{n-2}-1}\right)'(y)} \geq K.$$

Let $A' = \underline{-q_{n-1}}$ and let B' be the interval between A' and $\underline{0}$. By fundamental theorem of calculus and by Koebe lemma

$$\frac{|B'|}{|A'|} = \frac{\int_B \left(f^{q_{n-2}}\right)'(y)\,dy}{\int_A \left(f^{q_{n-2}}\right)'(x)\,dx} \leq \frac{1}{K}\frac{|B|}{|A|}.$$

By the same argument,

$$\frac{|B'|}{|A'|} = \frac{\int_B \left(f^{q_{n-2}}\right)'(y)\,dy}{\int_A \left(f^{q_{n-2}}\right)'(x)\,dx} \geq K\frac{|B|}{|A|}$$

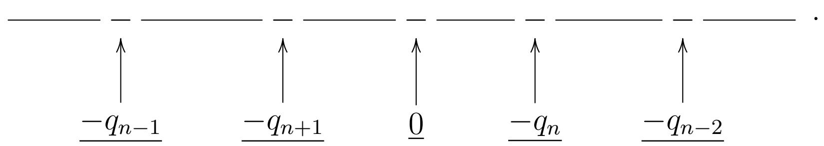
In conclusion, $\frac{|B|}{|A|}$ is greater than a constant and, since f is of the form x^l, we can say the same thing for the interval $\underline{-q_n}$ and for the gap to it adjacent. By the similar arguments we can prove the same result for an arbitrary preimage of U and for a gap adjacent to it, with the same assumptions of proposition. $\square$

Corollary 3.0.1. *Two adjacent gaps of the dynamic partition $\mathfrak{P}_{-\mathfrak{n}}$ which are in the same gap of a previous generation are comparable.*

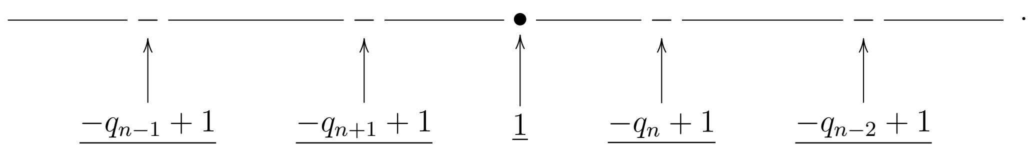
Proof. Derived from the previous proposition and from Proposition 10. $\square$

Lemma 3.0.2. *For every n the two gaps adjacent to interval flat U are comparable*

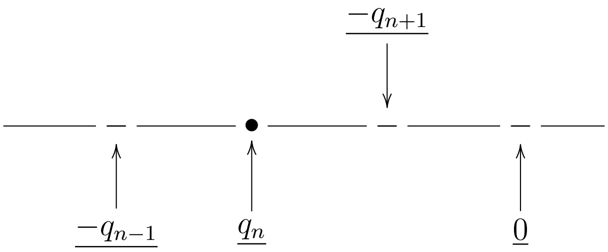
Proof. We prove the theorem with the help of some graph.
The initial situation is



We apply f and we have



After $q_n - 1$ iterate and recalling the position of $\underline{-q_{n+1}}$

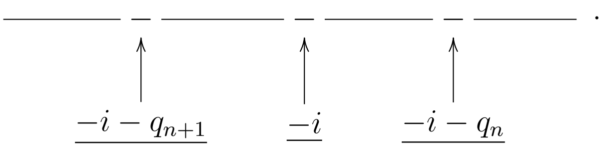


By the Koebe principle, apply for
$T = [\underline{-q_{n-1}}, \underline{-q_{n+1}}]$ and $J = (\underline{-q_{n-1}}, \underline{-q_{n+1}})$, there exists a constant K such that $\left|\left(\underline{-q_{n-1}}, \underline{q_n}\right)\right| \geq K \left|\left(\underline{q_n}, \underline{-q_{n+1}}\right)\right|$. Since f near to the flat interval is of the form x^l, we obtain that the intervals $\left(\underline{-q_{n+1}}, \underline{0}\right)$ and $\left(\underline{0}, \underline{-q_n}\right)$ are comparable.

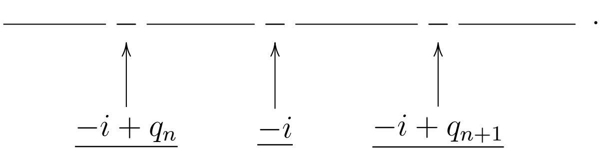
□

Proposition 12. *Two adjacent gaps of the dynamic partition $\mathfrak{P}_{-n}$ are comparable.*

Proof. If the two gaps satisfy the hypotheses of Corollary 3.0.1 or of Lemma 3.0.2 the proposition is proved. Otherwise, if the preimage $\underline{-i}$ is great, the situation is as follows:



Then we apply the principle of Koebe for
$T = \left[\underline{-i - q_{n+1}}, \underline{-i - q_n}\right]$, $J = \left(\underline{-i - q_{n+1}}, \underline{-i - q_n}\right)$ and $n = i$ and we return in the case of Lemma 3.0.2.
If $\underline{-i}$ is small, the situation is as follows:



Then we apply the principle of Koebe for
$T = \left[\underline{-i + q_n}, \underline{-i + q_{n+1}}\right]$, $J = \left(\underline{-i + q_n}, \underline{-i + q_{n+1}}\right)$ and $n = i - q_{n+1}$ and we return in the case of Corollary 3.0.1. □

Theorem 3.0.3. *For a map f from our class with the golden mean rotation number, the Hausdorff dimension of the set $K = \mathbb{S}^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(U)$ is strictly greater than* 0*.*

Proof. We define a probability measure ν on K as follows.
We suppose to have built a probability measure μ on A_n, the algebra generated by the set of all gaps belonging to the n^{th} partition $\mathfrak{P}_{-\mathfrak{n}}$ and from this, we define μ on A_{n+1}.
We recall that the short gaps of the n^{th} partition are the long gaps of $\mathfrak{P}_{-(\mathfrak{n}+1)}$; then for this gaps μ has already been defined.
For the long gap, since $I_i^n = f^{-i-q_n-q_{n+1}}(U) \cup I_{i+q_n}^{n+1} \cup I_i^{n+2}$, we set $\mu(I_{i+q_n}^{n+1}) = \frac{\mu(I_i^n)}{2}$ and $\mu(I_i^{n+2}) = \frac{\mu(I_i^n)}{2}$. In this way we have constructed a probability measure μ on the ring generated by $\bigcup_{n=1}^{\infty} A_n$.
Then by Carathéodory's extension theorem, there exists a measure ν on $\sigma(A_1, A_2, ..)$ that is an extension of μ and which is clearly the measure sought.
Now, by definition of μ it is easy to deduce that $\mu\left(I_i^n\right) \leq \frac{1}{2^{\frac{n}{2}}}$. By the previous Proposition and by Corollary 2.3.4 all gaps satisfy $\lambda_1^n \leq |I_i^n| \leq \lambda_2^n$, where λ_1 and λ_2 are constants. Then, assuming that λ_1 is less then $\frac{1}{\sqrt{2}}$ (this is not restrictive because, if λ_1 is not less then this quantity, then we can replace it by a smaller λ_1) we can pick $\alpha = \log_{\lambda_1} \frac{1}{\sqrt{2}}$ and we have that

$$\nu\left(I_i^n\right) \leq |I_i^n|^{\alpha}$$

Let I be an arbitrary interval and let I_i^{n+1} be the gap with the smallest n that is contained in I; then I is covered by at most two gaps of the n^{th} partition I_j^n and $I_{j'}^n$ and by preimages of U (which are of μ-measure zero). So, we can deduce that

$$\nu\left(I\right) \leq \nu\left(I_j^n\right) + \nu\left(I_{j'}^n\right) \leq C\nu\left(I_j^n\right) \leq C'\nu\left(I_i^{n+1}\right) \leq C'\left|I_i^{n+1}\right|^{\alpha} \leq C'\left|I\right|^{\alpha}$$

where both the second and the third inequalities follows by the fact that all the gaps have comparable measures.
Finally, let $\mathfrak{K}$ an ϵ-cover of K and let $0 < \alpha < 1$ as above, then combining all the than previous results we have that

$$\sum_{I \in \mathfrak{K}} |I|^{\alpha} \geq \frac{1}{C'} \sum_{I \in \mathfrak{K}} \nu\left(I\right) \geq \frac{1}{C'}\nu\left(K\right) > 0.$$

This establishes the theorem. $\square$

Chapter 4

Applications: Cherry Flows

4.1 Limit set

Denote the flow of a vector field X through x by $t \to X_t(x)$. The α and ω *limit set* of x are defined as

$$\alpha(x) = \{y; \exists t_n \to \infty \text{ with } X_{-t_n} \to y\},$$

respectively

$$\omega(x) = \{y; \exists t_n \to \infty \text{ with } X_{t_n} \to y\}.$$

We say that x is *recurrent* if $x \in \alpha(x) \cup \omega(x)$.

4.2 Cherry Flows

Let $\pi : \mathbb{R}^2 \to T^2$ be the covering map; thus π is a C^∞ local diffeomorphism, $\pi(x, y) = \pi(x', y')$ if and only if $x - x' \in \mathbb{Z}$ and $y - y' \in \mathbb{Z}$ and $\pi([0,1] \times [0,1]) = T^2$.
If X is a C^∞ vector field on the torus, we can define a C^∞ field $Y = \pi^* X$ on $\mathbb{R}^2$ by the expression $Y(z) = (d\pi_z)^{-1} X(\pi(z))$. Clearly the field Y defined like this satisfies the condition

$$Y(x + n, y + m) = Y(x, y), \forall (x, y) \in \mathbb{R}^2, \forall (n, m) \in \mathbb{Z}^2.$$

Conversely, if Y is a C^∞ vector field on the plane satisfying the previous condition, then there exists a unique C^∞ field X on the torus such that $Y = \pi^* X$.
We can thus identify the vector fields on the torus with the vector fields on $\mathbb{R}^2$ satisfying the previous condition.
Let $\mathfrak{C}$ be the set of vector fields $Y \in \mathfrak{Y}^\infty(\mathbb{R}^2)$ satisfying the following condition:

1. $Y(x+n, y+m) = Y(x,y), \forall (x,y) \in \mathbb{R}^2, \forall (n,m) \in \mathbb{Z}^2$;
2. Y is transversal to the straight line $0 \times \mathbb{R}$ and has only two singularities p, s in the rectangle $[0,1] \times [0,1]$ where p is a sink and s a saddle, both hyperbolic;
3. there exist $a, b \in \mathbb{R}$ with $a < b < a+1$ such that, if $y \in (b, a+1)$ then the positive orbit of Y through the point $(0,y)$ intersects the line $1 \times \mathbb{R}$ in the point $(1,c)$ while, if $y \in (a,b)$ the positive orbit through $(0,y)$ goes directly to the sink without cutting $1 \times \mathbb{R}$;

Proposition 13. $\mathfrak{C}$ *is nonempty.*

Proof. Consider the vector field $Y(x,y) = \left(2x\left(x+\frac{2}{3}\right), -y\right)$. The nonwandering set of Y consists of two singularities, a saddle $(0,0)$ and a sink $\left(-\frac{2}{3}, 0\right)$. It is easy to check that Y is transversal to the unit circle at all points of the arc $C = \left\{(x,y)\,; x^2+y^2 = 1, x \leq \frac{1}{2}\right\}$. Let

$$Z(x,y) = \left(\varphi(x,y)\left(2x^2 + \frac{4}{3}x\right) + (1-\varphi(x,y))\left(x^2+1\right), -y\right)$$

where φ is a C^∞ function such that $\varphi(\mathbb{R}^2) \subset [0,1]$, $\varphi(x,y) = 1$ if $x > \frac{1}{2}$ or if $(x,y) \in U$, $\varphi(x,y) = 0$ if $x < \frac{1}{4}$ and $(x,y) \in \mathbb{R}^2 \setminus V$. Here U and V are small neighbourhoods of C with $\overline{U} \subset V$. If V is sufficiently small, the nonwandering set of Z is empty. Take $T > 0$ such that $Z_t(C) \subset \{(x,y)\,; x > 1\}$ for all $t \geq T$. Using the flow of Z we can define a diffeomorphism $H : (0,1) \times (0,1) \to W \subset \mathbb{R}^2$ by $H(x,y) = Z_{xT}(h(y))$ where $h : [0,1] \to C$ is a diffeomorphism. If $z \in (0,1) \times (0,1)$ we define $X(z) = dH^{-1}(H(z) \cdot Y(H(z)))$. As $Z = Y$ in a neighbourhood of C and in $\left\{(x,y)\,; x > \frac{1}{2}\right\}$ we have $X(z) = (1,0)$ if z belongs to a small neighbourhood of the boundary of the rectangle $[0,1] \times [0,1]$. We can now extend X to $\mathbb{R}^2$ by defining $X(z) = (1,0)$ if z belongs to the boundary of $[0,1] \times [0,1]$ and $X(x+n, y+m) = X(x,y)$ if $(x,y) \in [0,1] \times [0,1]$ and $(n,m) \in \mathbb{Z}^2$. One checks immediately that X satisfies conditions (1)-(3). □

Let us denote by X the vector field on the torus induced by $Y \in \mathfrak{C}$, that is $Y = \pi^* X$. Then X has the following properties:

1. X has exactly two singularities, a sink P and a saddle S, both hyperbolic;
2. X has not periodic orbits.

We shall call a vector field which satisfies proprieties $(1)-(2)$ a *Cherry vector field*.

4.3 Properties of Cherry flows

Let $\mathfrak{B}$ be the class of Cherry vector fields X on T^2. We will denote the flow through a point x by $t \to X_t(x)$ and by $Sing(X)$ the set of singularities of X.
We denote by $W^s(S)$ the set of points in T^2 that have S as ω-limit (it is called the stable manifold of S) and by $W^u(S)$ the set of points that have S as α-limit (it is called the unstable manifold of S).

4.3.1 Poincaré Section

Proposition 14. *Let $X \in \mathfrak{B}$. Then there exists a closed C^∞ curve Σ on $T^2 \setminus Sing(X)$ without self-intersections and with the following properties*
(a) Σ is everywhere transversal to X;
(b) Σ is not retractable to a point.

Proof. It is enough to show that there exists a recurrent orbit γ which is non-trivial (i.e. not equal to a point or a closed curve), see for example page 144 of [13]. Let us call a stable separatrix a component of $W^s(S) \setminus S$ where S is the saddle-point. Let γ be a stable separatrix of S; because of the Denjoy-Schwartz theorem the α-limit set of γ, $\alpha(\gamma)$, must contain S then it must contain also γ. Hence γ is a non-trivial recurrent orbit and as we remarked above this implies the existence of a transversal circle. $\square$

The closed curve Σ built above defines a section on the torus called *Poincaré section.*

4.3.2 Transition map

Let $X \in \mathfrak{B}$ and let Σ be the closed transversal to X on T^2 from the proposition of the previous subsection. Notice that $T^2 \setminus \Sigma$ is an annulus $\Sigma \times (0,1)$ and we can write $T^2 \cong \Sigma \times [0,1] / \sim$, where $(s,0) \sim (s,1)$. Consider X as a flow on $T^2 \cong \Sigma \times [0,1]$ where we identify $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$.
Since X has no sources it follows that
(c) for every $x \in \Sigma \times \{0\}$ which is not contained in the stable manifold of the saddle or of the sink, there exists a $t > 0$ such that $X_t(x) \in \Sigma \times \{1\}$;
(d) there exists at least one $x \in \Sigma \times \{0\}$ such that $X_t(x) \in \Sigma \times \{1\}$ for some $t > 0$.
Now let Σ and Σ' be two closed curves transversal to X. We denote the points $x \in \Sigma$ such that there exists $t > 0$ with $X_t(x) \in \Sigma'$ for some $t > 0$ by Σ_0. For $x \in \Sigma_0$, let $t(x)$ be the minimal $t > 0$ such that $X_t(x) \in \Sigma'$ and we define the map $f : \Sigma_0 \to \Sigma'$ by $f(x) = X_{t(x)}(x)$. This map is called the

transition map between Σ and Σ'.
Now let $\Sigma' = \Sigma$ be the section from Proposition 14 and take the transition f from Σ to Σ. This map is called the *return-map* to Σ.
By Tubular Flow Theorem f is a C^∞ diffeomorphism and since orbits of X cannot intersect, f is order preserving. Let U be $(\Sigma \setminus \Sigma_0) \times \{0\}$. Take $x \in \partial U$. Since the basin of the sink is open, x must be contained in the stable manifold of the saddle-point S; then $W^u(S)$ intersects $\Sigma \times \{1\}$ in some point v and for every point $u \in (\Sigma \setminus I) \times \{0\}$, near ∂U there exists $t > 0$ such that $X_t(u)$ intersects $\Sigma \times \{1\}$ near v. (This can be seen by considering the backward orbits of $X|\,(\Sigma \times [0,1])$ of points in $\Sigma \times \{1\}$ near v intersecting $\Sigma \times \{0\}$. This set consists of a neighbourhood of ∂U in Σ.) In particular $\lim_{x \to \partial U, x \notin U} f(x)$ consists of one single point and we can define f on $\Sigma \setminus \Sigma_0$ to be constant. From the smooth dependence on initial condition, f is then everywhere continuous, and as smooth as the vector field outside boundary points of $\Sigma \setminus \Sigma_0$.

4.3.3 The transition map near singularities

Let $X \in \mathfrak{C}$ which has a hyperbolic singularity at 0 of saddle-type with eigevalues $\lambda_1 > 0 > \lambda_2$ and $|\lambda_2| > 2\lambda_1$ (dissipative case). Let p_1 and p_2 be points in $W^s(0)$ respectively $W^u(0)$. Furthemore let Σ_i be a C^2 curve through p_i which is transversal to X. If we choose Σ_1 sufficiently small, then for every x in one of the components of $\Sigma_1 \setminus \{p_1\}$ there exists $t \geq 0$ such that $X_t(x) \in \Sigma_2$. Call this component Σ_1' and let $t(x) \in \mathbb{R}$ be the smallest number so that $X_{t(x)}(x) \in \Sigma_2$ and define $T : \Sigma_1' \to \Sigma_2$ by

$$T(x) = X_{t(x)}(x).$$

We call this map the *transition map.*
In this section we want to show that $T : \Sigma_1 \to \Sigma_2$ is equal to a map ϕ_α, up to diffeomorphisms C^2. Here $\alpha = \frac{|\lambda_2|}{\lambda_1} > 2$, and ϕ_α is defined by $\phi_\alpha = \pm |z|^\alpha$.

Theorem 4.3.1. *Let $X \in \mathfrak{C}$ which has a hyperbolic singularity at 0 of saddle-type with eigevalues $\lambda_1 > 0 > \lambda_2$ and $|\lambda_2| > 2\lambda_1$. Let $\alpha = \frac{|\lambda_2|}{\lambda_1}$. There exist maps $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ which are C^2, such that the map T from above is of the form*

$$T(x) = \varphi \circ \phi_\alpha \circ \psi(x).$$

Proof. We observe that, according to the Linearisation Theorem of Sternberg (see [1], [15]) there exists a C^2 coordinate system ϕ near 0 such that $\phi_* X$ is linear; then we can suppose that the vector field X is linear near 0.
If Σ_i are the straight lines $\{x = 1\}$, $\{y = 1\}$ and $T_{\{x=1\},\{y=1\}}$ the transition

map from $\{x = 1\}$ and $\{y = 1\}$, then the theorem follows from explicit integration; it is sufficient to solve the differential equations $\dot{x} = \lambda_2 x$ and $\dot{x} = \lambda_1 x$. If Σ_i are different C^2 curves we consider the transition map T_1 from Σ_1 to $\{x = 1\}$, resp. $T_3 : \{y = 1\} \to \Sigma_2$. Then $T_{\Sigma_1,\Sigma_2} = T_2 \circ T_{\{x=1\},\{y=1\}} \circ T_1$. Since X is linear (and in particular the flow map is C^2) it follows from the implicit function theorem that the maps T_i are C^2. This last argument also shows that if this theorem is true for one choice of C^2 curves Σ_i, then it is also true for any other choice of C^2 curves $\tilde{\Sigma_i}$ as above. $\square$

4.4 Applications

Let X be a vector field in $\mathfrak{B}$, with the point of saddle that has eigenvalues $\lambda_1 > 0 > \lambda_2$, $|\lambda_2| > 2\,|\lambda_1|$, let Σ be a Poicaré section and let f be the return-map to Σ.
Then from the results gotten in the previous section is easily deduced that f has the following properties:

1. f is order preserving.

2. It is constant on an interval U.

3. The restriction of f to $\mathbb{S}^1 \setminus \overline{U}$ is a C^∞-diffeomorphism onto its image.

4. Let (a, b) be a preimage of U under the projection of the real line of $\mathbb{S}^1$. On same right-side neighdorhood of b, f can be represented as

$$h\left((x - b)^\alpha\right)$$

for $\alpha = \frac{\lambda_2}{\lambda_1}$, where h is a C^∞-diffeomorphism on a two-side neighdorhood of b. Analogously, on a left-sided neighdorhood of a, f is

$$h\left((a - x)^\alpha\right).$$

5. Since X has no periodic orbits, the rotation number of f, $\rho(f)$ is irrational.

Now, we can suppose that $\rho(f)$ is the golden mean. In fact, if this is false, for every $\epsilon > 0$, we can add ϵ to the orbits of the field X without modifying its proprieties; then for ϵ that tends to infinity, we have an increasing sequence of functions f_ϵ such that $\rho(f_\epsilon) \to \infty$.
In conclusion by the properties of the retour-map f, listed above, it is easy

to see that f is a map of our class with the golden mean rotation number. We also observe that, by Theorem 2 in [4], we are still in the case of the geometry bounded, then all the results obtained in previous sections can be applied to this example.

Definition 4.4.1. *A quasi-minimal set is a set containing finitely many fixed points and such that every positive semiorbit that is not attracted to a fixed point is dense in the set.*

Remark 4.4.2. *By the previous proposition it is easy to see that the non-wandering set of Cherry flow consists of the attractive fixed point and a quasi-minimal set containing the hyperbolic saddle.*

Now we can easily deduce the following theorem.

Theorem 4.4.3. *If X is a vector field on the torus as above, then the quasi-minimal set has the Hausdorff dimension strictly greater than 1.*

Proof. Let Q be the quasi-minimal set of X. In a small neighborhood of the Poincaré section Σ, Q is equivalent, by a C^2 diffeomorphism, to $I \times K$. In conclusion the theorem follows by Theorem 3.0.3 and by the fact that the Hausdorff dimension of the product of two sets is greater than the sum of their Hausdorff dimension.

□

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