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Perversity of the Nearby Cycles

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Chapter 0

Introduction

In this thesis, I will talk about perverse sheaves as well as nearby and vanishing cycles. In particular, at the end of the thesis, I will prove the perversity of nearby cycles.

The name perverse sheaves is a bit misleading,because neither they are sheaves, nor they are perverse. Perverse sheaves live on spaces with singularities. A perverse sheaf on a topological space X is actually a bounded constructible complex in the derived category that satisfies certain dimension conditions on its cohomology sheaves. The justification of the term sheaf comes from the fact that, they can be glued, form an abelian category, can be used to define cohomology. Actually the category of perverse sheaves is an abelian category of the non-abelian derived category of sheaves (actually of the sheaves of vector spaces with constructible cohomology), equal to the core of a suitable t-structure and preserved by Verdier duality.

Usually, families come with singular fibres and so it is natural to investigate what happens near such a fibre. The notion of nearby and vanishing cycles arise when we consider 1-parameter degenerations. They are complexes defined on the singular fibre. We will show that we have an interpretation of the nearby cycle using the relative De Rham complex. Then we prove that the relative De Rham complex is isomorphism to another complex which comes from the absolute De Rham complex. Finally with the help of the above observation, we can prove the perversity of the nearby cycles. The idea of such a proof of the perversity of the nearby cycles functor comes from an observation given by Illusie in his article [5].

Chapter 1

Nearby and vanishing cycles

1 Constructible sheaves

Let X be a complex analytic space and let k be a field(or more general,Noetherian ring). Denote $Sh(X) := Sh(X, k)$, the category of sheaves of k-vector space over X and $\mathcal{D}(X) = \mathcal{D}(X, k) := \mathcal{D}(Sh(X, k))$, its derived category and $\mathcal{D}^+(X)$ the bounded-below derived category, $\mathcal{D}^-(X)$ the bounded-above derived category,$\mathcal{D}^b(X)$ the bounded derived category.

Definition 1.1. A sheaf $\mathcal{F}$ on X is a local system (or locally constant), if for all $x \in X$, there is a neighborhood U containing x such that $\mathcal{F}|_U$ is a constant sheaf.

Theorem 1.2. *There is a bijection between set of local systems on X up to isomorphism and set of representations of fundamental group $\pi_1(X, x_0)$ up to isomorphism. Even more is true: there is an equivalence of categories.*

Proof. see [4]or[8] □

Definition 1.3. A stratification of a complex analytic space X is a finite set $\mathcal{S}$ (called strata) of X such that

1)X is a disjoint union of all the strata

2)Each stratum $S \in \mathcal{S}$ is a manifold

3) The closure of a stratum $\overline{S}$ is a union of some strata.

We call such X a stratified space.

Definition 1.4. 1)A sheaf $\mathcal{F}$ on a stratified space X is constructible with respect to the stratification $\mathcal{S}$ if for all $S \in \mathcal{S}$, the restriction $\mathcal{F}|_S$ is a locally constant sheaf of A-modules, where A is any ring.

2)A complex of sheaves $\mathcal{C}^\bullet$ is said to be constructible with respect to the stratification $\mathcal{S}$ if all its cohomology sheaves $\mathcal{H}^i(\mathcal{C}^\bullet)$ are constructible with respect to the stratification $\mathcal{S}$, and we write $\mathcal{C}^\bullet \in \mathcal{D}_{\mathcal{S}}(X)$.

3)Define $\mathcal{D}^b_c(X)$ is the full subcategory of $\mathcal{D}^b(X)$ consisting of constructible sheaves, i.e. $\mathcal{C}^\bullet \in \mathcal{D}^b_c(X)$ if $\mathcal{C}^\bullet$ is bounded and there exists stratification $\mathcal{S}$ such that $\mathcal{C}^\bullet \in \mathcal{D}_{\mathcal{S}}(X)$.

Remark. 1) In the case where X is an algebraic variety,denote the analytification of X to be X^{an}. A $\underline{\mathbb{C}}_{X^{an}}$-module $\mathcal{F}$ is called an algebraically constructible sheaf if there exists a stratification $X = \bigsqcup S_\alpha$ of X such that $\mathcal{F}|_{S_\alpha^{an}}$ is a locally constant sheaf on S_α^{an} for any α.

Remark. 2) For an algebraic variety X, we denote by $\mathcal{D}_c^b(X)$ the full subcategory of $\mathcal{D}_c^b(X^{an})$, consisting of bounded complexes of $\underline{\mathbb{C}}_{X^{an}}$-modules whose cohomology groups are algebraically constructible. We apply the same to some notations which we will introduce later, like dualizing complex of X.

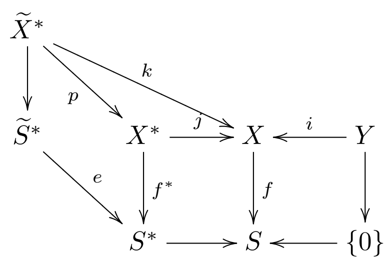
From [1], p 83, the $\mathcal{D}_c^b(X)$ is stable under the six operators. More precisely, we have the following theorem.

Theorem 1.5. *Let $f : X \to Y$ to be a morphism of analytic spaces or of complex algebraic varieties. Then the following holds*
1) If $\mathcal{G}^\bullet \in \mathcal{D}_c^b(Y)$, then $f^{-1}\mathcal{G}^\bullet \in \mathcal{D}_c^b(X)$ and $f^!\mathcal{G}^\bullet \in \mathcal{D}_c^b(X)$.
2) If $\mathcal{F}^\bullet \in \mathcal{D}_c^b(X)$ and f is an algebraic map then $Rf_*(\mathcal{F}^\bullet)$ and $Rf_!(\mathcal{F}^\bullet)$ are constructible. If $\mathcal{F}^\bullet \in \mathcal{D}_c^b(X)$ and f is an analytic map such that the restriction of f to $supp(\mathcal{F}^\bullet)$ is proper, then $Rf_*(\mathcal{F}^\bullet)$ and $Rf_!(\mathcal{F}^\bullet)$ are constructible.
3) If $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathcal{D}_c^b(X)$, then $\mathcal{F}^\bullet \bigotimes^L \mathcal{G}^\bullet \in \mathcal{D}_c^b(X)$ and $R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \in \mathcal{D}_c^b(X)$.

Remark. See chapter 2 for the definition of $f_!$ and $f^!$.

2 Nearby and vanishing cycles

Let S be a unit disk in $\mathbb{C}$, $f : X \to S$ a non-constant analytic function. Then we have the following diagram



Here, $Y = f^{-1}(0)$, $S^* = S - \{0\}$ the punctured disk, $X^* = f^{-1}(S^*)$, $\widetilde{S}^* = \{z \in \mathbb{C} | Im(z) > 0\}$,$e$ is the universal covering

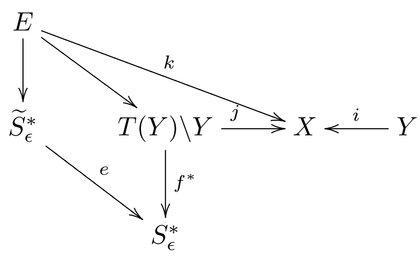
$$e : \widetilde{S}^* \to S^*, e(u) = exp(2\pi i u)$$

and $\widetilde{X}^*$ is the fibre product

$$\widetilde{X}^* = X^* \times_{S^*} \widetilde{S}^*$$

Moreover, we assume that $f^* : X^* \to S^*$ to be smooth, the only possible singularities lies in $Y = f^{-1}(0)$. We wish to investigate how the cohomology of fibers of f with coefficients in $\mathcal{F}^\bullet \in \mathcal{D}_c^b(X)$ changes with respect to the only singular fiber Y.

Remark. As we want to investigate the problems on Y, so we can care about only the situations near Y. So we can actually replace the commutative diagram with the following



Here $T(Y) = f^{-1}(S_\epsilon)$ is the tube about the fiber Y and we replace S^* by S_ϵ^* which is the punctured disk of radius ϵ. (According to [1] p 102)when f is proper on the tube $T(Y)$, ϵ can be chosen such that $f : T(Y)\backslash Y \to S_\epsilon^*$ is a topologically locally trivial fibration. However, even f is not proper, such fibrations exist locally on X, they are precisely the Milnor fibrations of corresponding function germs $f : (X, x) \to (\mathbb{C}, 0)$.$E$ is regarded as the universal fiber of the fibration $f : T(Y)\backslash Y \to S_\epsilon^*$. Now even we have started with algebraic varieties, the new objects $T(Y)$ and E are only analytic spaces. So we assume X to be complex analytic space and moreover we assume that the radius 1 is small enough to be a good ϵ.

Definition 2.1. Let $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ be a complex, We define the nearby cycles of the complex $\mathcal{F}^\bullet$ with respect to the function f and the value $t = 0$ to be the sheaf complex given by

$$\psi_f \mathcal{F}^\bullet = i^{-1} R k_* k^{-1} \mathcal{F}^\bullet.$$

As we have the fundamental group $\pi_1(\widetilde{S}^*) = \mathbb{Z}T$ where $T : z \mapsto z + 1$ in $\widetilde{S}^*$, then there is an associated monodromy deck transformation $h : \widetilde{X}^* \to \widetilde{X}^*$ and h satisfies $p \circ h = p$. This homeomorphism h induces an isomorphism of complexes

$$M : \psi_f(\mathcal{F}^\bullet) \to \psi_f(\mathcal{F}^\bullet)$$

After [11], p.352, one can shows that $\psi_f(\mathcal{F}^\bullet) \in \mathcal{D}_c^b(X)$. In conclusion we get the nearby cycle functor

$$\psi_f : \mathcal{D}_c^b(X) \to \mathcal{D}_c^b(Y)$$

with respect to the function f and the value $t = 0$.

We will give a theorem which can be seen as a interpretation of the name "nearby".

Let $B_\delta(x)$ be an open ball of radius δ in X centered at x, defined by using an embedding of the germ (X, x) in an affine space $\mathbb{C}^N$,let $X_t = f^{-1}(t)$. Then $V_x = B_\delta(x) \bigcap X_t$ for $0 < |t| \ll \epsilon \ll \delta$ is exactly the local Milnor fiber of the function f at the point x. A direct computation using the definition of the complex $\psi_f \mathcal{F}^\bullet$ yields the following:

Theorem 2.2. *For all the points $x \in Y$ there is a natural isomorphism*

$$\mathcal{H}^i(\psi_f \mathcal{F}^\bullet)_x \simeq \mathbb{H}^i(V_x, \mathcal{F}^\bullet)$$

such that the monodromy morphism M_x on the left hand side corresponds to the morphism on the right hand side induced by the monodromy homeomorphism of the local Milnor fibration induced by $f:(X,x)\to(\mathbb{C},0)$.

Proof. Define $V_{r,\eta}$ as in Chapter 3,1 Geometric Set-Up. then the Milnor fiber V_x is the intersection of X_t with $V_{r,\eta}$ for t small enough but non-zero. For t real embeds in $k^{-1}V_{r,\eta}$ through $z\mapsto(z,logt)$ and it can be seen that this is a homotopy equivalence. Hence the inclusion induces

$$\mathbb{H}^i(V_x,\mathcal{F}^\bullet)\simeq Lim_{r,\eta}\mathbb{H}^i(k^{-1}(V_{r,\eta}),k^{-1}(\mathcal{F}^\bullet))\simeq\mathcal{H}^i(i^{-1}Rk_*k^{-1}\mathcal{F}^\bullet)_x=\mathcal{H}^i(\psi_f\mathcal{F}^\bullet)_x$$

To show how this right hand side monodromy operator T is obtained, note that we can work a proper Milnor fibration $f:B_\delta(x)\bigcap f^{-1}(S^*_\epsilon)\to S^*_\epsilon$. It follows that $R^kf_*(\mathcal{F}^\bullet)$ is a local system on S^*_ϵ for ϵ small enough. From theorem 1.2 this local system corresponds to a representation of $\pi_1(S^*_\epsilon)$ which is actually $\rho:\pi_1(S^*_\epsilon)\to Aut(S)$, where $S=R^kf_*(\mathcal{F}^\bullet)_t=\mathbb{H}^k(V_x,\mathcal{F}^\bullet)$. With this notion, the monodromy T is just $\rho([\gamma])$, where $[\gamma]$ is the generator of $\pi_1(S^*_\epsilon)=\mathbb{Z}$. □

Now we consider the adjunction morphism

$$\mathcal{F}^\bullet\to Rk_*k^{-1}(\mathcal{F}^\bullet)$$

and apply the functor i^{-1} to get the natural morphism

$$c:i^{-1}\mathcal{F}^\bullet\to\psi_f\mathcal{F}^\bullet$$

Definition 2.3. Let $\mathcal{F}^\bullet\in\mathcal{D}^b_c(X)$ be a constructible complex. We define the vanishing cycles $\phi_f(\mathcal{F}^\bullet)\in\mathcal{D}^b_c(Y)$ and the canonical morphism $can:\psi_f(\mathcal{F}^\bullet)\to\phi_f(\mathcal{F}^\bullet)$ by inserting the natural map c above to a distinguished triangle

$$i^{-1}\mathcal{F}^\bullet\xrightarrow{c}\psi_f(\mathcal{F}^\bullet)\xrightarrow{can}\phi_f(\mathcal{F}^\bullet)\xrightarrow{[+1]}$$

in the triangulated category $\mathcal{D}^b_c(Y)$. Note here $\phi_f(\mathcal{F}^\bullet)$ is defined up to isomorphism.

The isomorphism of complexes M defined in 2.1 satisfies the equality $M\circ c=c$ implies that there is an induced monodromy isomorphism $M_v:\varphi_f(\mathcal{F}^\bullet)\to\varphi_f(\mathcal{F}^\bullet)$ and an automorphism of the distinguished triangle

$$i^{-1}\mathcal{F}^\bullet\xrightarrow{c}\psi_f(\mathcal{F}^\bullet)\xrightarrow{can}\varphi_f(\mathcal{F}^\bullet)\xrightarrow{[+1]}$$

given by (Id,M,M_v).

Remark. As the triangulated category does only guarantee the existence of M_v, we need to note that the monodromy morphism M_v is not unique.

Chapter 2

Perverse sheaves

1 Verdier duality

Definition 1.1. Let $\mathcal{F}$ be a sheaf on X,let $s \in \mathcal{F}(U)$.The support of s is defined to be

$$\mathrm{supp}(s) = \{x \in U | s_x \neq 0\}$$

This is automatically a closed subset of U.

Definition 1.2. A continuous map $f : X \to Y$ is proper if for every compact set $K \subset Y$, the preimage $f^{-1}(K) \subset X$ is compact.

Definition 1.3. Let $f : X \to Y$ be a continuous map, and let $\mathcal{F}$ be a sheaf on X. The proper push-forward of $\mathcal{F}$,denoted $f_!\mathcal{F}$ is the subsheaf of $f_*\mathcal{F}$ defined by

$$f_!\mathcal{F}(U) = \{s \in f^{-1}(U) | f|_{supp(s)} : supp(s) \to U \text{ is proper}\}$$

Remark. The restriction of a proper map to a closed subset of its domain is always proper. If f is a proper map then the functor $f_!$ and f_* coincide because $f|supp(s)$ is always proper. In particular, if f is an inclusion of a closed subset, proper push-forward is the same as the ordinary push-forward. If f is an inclusion of an open subset, $f_!$ is extension by zero.

Definition 1.4. Given a sheaf $\mathcal{F} \in Sh(X)$,define the group of sections with compact support by

$$\Gamma_c(X, \mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) |\text{ supp } (s) \text{ is compact } \} = a_{X!}\mathcal{F}$$

where $a_{X!}\mathcal{F} : X \to \{point\}$ is the projection to a single point.

The functor $f_! : Sh(X) \to Sh(Y)$ does not have a right adjoint in general, otherwise it would be right exact and therefore exact. But an interesting fact is that the functor $Rf_!$ has a right adjoint.

Theorem 1.5. *(Verdier duality) Let $f : X \to Y$ be a continuous map of locally compact spaces of finite dimension. Then $Rf_! : \mathcal{D}^+(X) \to \mathcal{D}^+(Y)$ admits a right adjoint $f^!$. In fact, we have an isomorphism in $\mathcal{D}^+(k)$:*

$$\mathcal{R}Hom(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \mathcal{R}Hom(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet).$$

where $\mathcal{F}^\bullet \in \mathcal{D}^+(X), \mathcal{G}^\bullet \in \mathcal{D}^+(X)$.

Remark. Here $\mathcal{R}Hom$ is defined as follows. Recall that given chain complexes $A^\bullet, B^\bullet$ of sheaves, one may define a chain complex $Hom^\bullet(A^\bullet, B^\bullet)$, the elements in degree in are given the product $\prod_m Hom(A^m, B^{m+n})$ and the differential sends a collection of maps $\{f_m : A^m \to B^{m+n}\}$ to $df_m + (-1)^{n+1} f_{m+1} d : A^m \to B^{m+n+1}$. Then $\mathcal{R}Hom$ is the derived functor of $Hom^\bullet$.Since the cohomology in degree zero is given by $Hom_{\mathcal{D}^+(X)}(A^\bullet, B^\bullet)$, we see that the last statement of Verdier duality implies the adjointness relation.

We are going to prove the theorem in a few steps following [2].

First, the existence of a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is equivalent to the representability of the functor $C \mapsto Hom_{\mathcal{D}}(FC, D)$ for each $D \in \mathcal{D}$.And about the representability, we have the following lemma:

Lemma 1.6. *An additive functor* $F : Sh(X) \to k - mod^{op}$ *is representable if and only if it sends colimits to limits.*

Proof. If F is representable, then it is clearly it sends colimits to limits. Suppose conversely that F sends colimits to limits, for each open set $U \subset X$, take $k_U = j_!(k)$ where $j : U \to X$ is the inclusion. We can define a sheaf $\mathcal{F} \in Sh(X)$ via $\mathcal{F}(U) = F(k_U)$. Since the k_U have canonical embedding maps(if $U \subset U'$,there is a map $k_U \to k_{U'}$, it is clear that $\mathcal{F}$ is a presheaf, actually, $\mathcal{F}$ is a sheaf. To see this, let $\{U_\alpha\}$ be an open covering of U, then there is an exact sequence of sheaves

$$\textstyle\prod_{\alpha,\beta} k_{U_\alpha \bigcap U_\beta} \to \prod_\alpha k_{U_\alpha} \to k_U \to 0$$

which means that there is an exact sequence

$$0 \to F(k_U) \to \textstyle\prod_\alpha F(k_{U_\alpha}) \to \prod_{\alpha,\beta} F(k_{U_\alpha \bigcap U_\beta})$$

This means that $\mathcal{F}$ is a sheaf. $\mathcal{F}$ is a promising candidate for a representing object, because we know that

$$Hom(k_U, \mathcal{F}) \simeq \mathcal{F}(U) = F(k_U).$$

Now, we need to define a distinguished element of $F(\mathcal{F})$ to show that it is universal. More generally, we can define a natural transformation $Hom(-, \mathcal{F}) \to F(-)$. We can do this because any $\mathcal{G} \in Sh(X)$ is canonically a colimit of sheaves k_U. Namely,form the category whose objects are pairs (U, s) where $U \subset X$ is open and $s \in \mathcal{G}(U)$ and whose morphisms come from inclusions $(V, s') \to (U, s)$ where $V \subset U$ and $s' = s|_V$. For each such pair define a map $k_U \to \mathcal{G}$ by the section s. It is easy to see that this gives a representation of $\mathcal{G}$ functorially as a colimit of sheaves of the form k_U. The natural isomorphism $Hom(k_u, \mathcal{F}) \simeq F(k_U)$ now extends to a natural transformation $Hom(\mathcal{G}, \mathcal{F}) = F(\mathcal{F})$, which is an isomorphism. Indeed, it is an isomorphism when $\mathcal{G} = k_U$, and both functors commute with colimits. □

The functor $\mathcal{F} \mapsto Hom_{Sh(Y)}(f_!\mathcal{F}, \mathcal{G})$ is not in general representable,as $f_!$ is not exact and need not preserve colimits. However a slight variant of the above functor is representable.

Definition 1.7. A soft sheaf $\mathcal{F}$ over X is one such that any section over any closed subset of X can be extended to a global section.

Lemma 1.8. *A sheaf $\mathcal{F} \in Sh(X)$ is soft if and only if $H^1_c(U, \mathcal{F}) = 0$ whenever $U \subset X$ is open.*

Proof. See [12]. □

Lemma 1.9. *If $\mathcal{M}$ is a soft,flat sheaf in $Sh(X)$, then the functor $\mathcal{F} \to f_!(\mathcal{F} \otimes_k \mathcal{M})$ commutes with colimits. In particular, the functor $\mathcal{F} \mapsto Hom_{Sh(Y)}(f_!\mathcal{F} \otimes_k \mathcal{M}, \mathcal{G})$ is representable for any $\mathcal{G} \in Sh(Y)$.*

Proof. We know that $f_!$ commutes with filtered colimits and in particular arbitrary sums. As a result we need only to show that $\mathcal{F} \mapsto f_!(\mathcal{F} \otimes_k \mathcal{M})$ is an exact functor. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence in $Sh(X)$, then so is $0 \to \mathcal{F}' \otimes_k \mathcal{M} \to \mathcal{F} \otimes_k \mathcal{M} \to \mathcal{F}'' \otimes_k \mathcal{M} \to 0$ by flatness. As $\mathcal{M}$ is soft, then it follows from [12],the first term $\mathcal{F}' \otimes_k \mathcal{M}$ is soft. Then after lemma 1.8, the push-forward sequence $0 \to f_!(\mathcal{F}' \otimes_k \mathcal{M}) \to f_!(\mathcal{F} \otimes_k \mathcal{M}) \to f_!(\mathcal{F}'' \otimes_k \mathcal{M}) \to 0$ is exact too. The representability criterion now completes the proof □

Remark. It follows that given $\mathcal{M}$ and $\mathcal{G}$ as above, there is a sheaf $\widehat{f}(\mathcal{M}, \mathcal{G}) \in Sh(X)$ such that

$$Hom_{Sh(X)}(\mathcal{F}, \widehat{f}(\mathcal{M}, \mathcal{G})) \simeq Hom_{Sh(Y)}(f_!(\mathcal{F} \otimes_k \mathcal{M}), \mathcal{G})$$

This is clearly functorial in $\mathcal{G}$ and contravariantly in $\mathcal{M}$. We shall use $\widehat{f}$ to construct $f^!$.

Lemma 1.10. *$\widehat{f}(\mathcal{M}, \mathcal{G})$ is injective whenever $\mathcal{M} \in Sh(X)$ is a soft, flat sheaf and $\mathcal{G} \in Sh(X)$ is injective.*

Proof. $\widehat{f}(\mathcal{M}, \mathcal{G})$ is the object representing the functor

$$\mathcal{F} \mapsto Hom_{Sh(Y)}(f_!(\mathcal{F} \otimes_k \mathcal{M}), \mathcal{G})$$

To say that it is injective is to say that mapping into it is an exact functor,Let $0 \to \mathcal{F}' \to \mathcal{F}$ be an exact sequence. Then the sequence $0 \to f_!(\mathcal{F}' \otimes \mathcal{M}) \to f_!(\mathcal{F} \otimes \mathcal{M})$ is exact too. So injectivity of $\mathcal{G}$ gives that

$$Hom_{Sh(Y)}(f_!(\mathcal{F} \otimes_k \mathcal{M}), \mathcal{G}) \to Hom_{Sh(Y)}(f_!(\mathcal{F}' \otimes_k \mathcal{M}), \mathcal{G}) \to 0$$

is also exact. □

Lemma 1.11. *Let X be a locally compact space, and $n = dimX$, then in any sequence*

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_{n+1} \to 0$$

if $\mathcal{F}_1, \cdots, \mathcal{F}_n$ are all soft, so is $\mathcal{F}_{n+1}$.

Proof. We know that $\mathcal{F}_i, i = 1, \cdots, n$ has no compacted supported cohomology above dimension one. Using the standard dimension shifting method, we get $H^1_c(U, \mathcal{F}_{n+1}) = H^{n+1}_c(U, \mathcal{F}_0) = 0$ for all $U \subset X$ open. By lemma 1.8, we get the softness of $\mathcal{F}_{n+1}$. □

Proof. We now prove the theorem, we choose a soft, flat and bounded resolution $\mathcal{L}^\bullet$ of the constant sheaf k, so a quasi-isomorphism $k \to \mathcal{L}^\bullet$. To see this is possible,we need to check that we can choose $\mathcal{L}^\bullet$ to be bounded. If $\mathcal{L}^\bullet$ is not bounded, we truncate it after the nth stage, where $n = dim X$. e.g. we consider the complex

$$0 \to \mathcal{L}^0 \to \cdots \to \mathcal{L}^n \to Im(\mathcal{L}^n \to \mathcal{L}^{n+1}).$$

This complex will remain soft after lemma 1.11. The final term will also be flat because of the stalkwise split natural of the resolution $\mathcal{L}^\bullet$. Then $\mathcal{F}^\bullet$ and $\mathcal{F} \otimes_k \mathcal{L}^\bullet$ will be isomorphic functors on the level of derived categories, but the latter will be much better behaved. for instance, it will have soft terms.Fix a complex $\mathcal{G}^\bullet \in \mathcal{D}^+(Y)$,we need to show the functor $\mathcal{F}^\bullet \to Hom_{D+(Y)}(Rf_!(\mathcal{F}^\bullet), \mathcal{G}^\bullet)$ is representable. As we have a canonical isomorphism in the derived category:

$$\mathcal{F}^\bullet \simeq \mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet.$$

So alternatively, we may show the functor $\mathcal{F}^\bullet \to Hom_{\mathcal{D}+(Y)}(Rf_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet)$ is representable.

We shall show that there is a complex $\mathcal{K}^\bullet \in \mathcal{D}^+(X)$ such that there is a functorial isomorphism

$$\mathcal{R}Hom(Rf_!(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \simeq \mathcal{R}Hom(Rf_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet) \simeq \mathcal{R}Hom(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

Notice that $\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet$ is already $f_!$-acyclic. In particular $\mathcal{R}f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet) \simeq f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet)$. Moreover we can assume that $\mathcal{G}^\bullet$ is a complex of injectives and we will try to choose $\mathcal{K}^\bullet$ to consists of injectives. In this case, we are just looking for a quasi-isomorphism

$$Hom^\bullet(f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet) \simeq Hom^\bullet(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

However, we know that

$$Hom^n(f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet) = \prod_m \prod_{i+j=m} Hom(f_!(\mathcal{F}^i \otimes_k \mathcal{L}^j), \mathcal{G}^{m+n}) =$$
$$\prod_m \prod_{i+j=m} Hom(\mathcal{F}^i, \widehat{f}(\mathcal{L}^j, \mathcal{G}^{m+n})) = \prod_{i,j} Hom(\mathcal{F}^i, \widehat{f}(\mathcal{L}^j, \mathcal{G}^{m+n}))$$

If we consider the double complex by $C^{rs} = \widehat{f}(\mathcal{L}^{-r}, \mathcal{G}^s)$ with the differential maps being those induced by $\mathcal{L}, \mathcal{G}$. Let $\mathcal{K}^\bullet$ be the associated simple complex. then it follows that there is an isomorphism

$$Hom^n(f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet) \simeq Hom^n(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

In fact, there is an isomorphism of complexes

$$Hom^\bullet(f_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet) \simeq Hom^\bullet(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

This follows from checking through the signs of the differential. This will prove

$$\mathcal{R}Hom(Rf_!(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \simeq \mathcal{R}Hom(Rf_!(\mathcal{F}^\bullet \otimes_k \mathcal{L}^\bullet), \mathcal{G}^\bullet) \simeq \mathcal{R}Hom(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

If we check that $\mathcal{K}^\bullet$ is bounded below complex of injectives. It is bounded below from the definition,as $\mathcal{L}^\bullet$ is bounded in both directions.It is injective because lemma 1.9.

It is now clear we may define the functor $f^! : \mathcal{D}^+(Y) \to \mathcal{D}^+(X)$. Given a bounded-below complex $\mathcal{G}^\bullet \in \mathcal{D}^+(Y)$, we start by replacing it with a complex of injectives, and so just assume that it consists of injectives without loss of generality. We then form the complex $\mathcal{K}^\bullet$ of sheaves on X such that $\mathcal{K}^t =$

$\bigoplus_{r+s=t} \widehat{f}(\mathcal{L}^{-r}, \mathcal{G}^s)$, where $\mathcal{L}^\bullet$ is a fixed soft resolution of the constant sheaf. Then setting $f^! \mathcal{G}^\bullet = \mathcal{K}^\bullet$ finishes the proof. □

Now consider Y to be a one point space and the field k to be the complex field. We have the dualizing complex as following:

Definition 1.12. Let $a : X \to \{point\}$ be the constant map from X to a one-point space. The dualizing complex on X is defined as

$$\omega_X := a^! \underline{\mathbb{C}}_X.$$

Definition 1.13. Let $\mathcal{F}^\bullet \in D^-(X)$, the Verdier dual of $\mathcal{F}^\bullet$ is the complex $\mathbb{D}\mathcal{F}^\bullet = R\mathcal{H}om(\mathcal{F}, \omega_X)$.

It follows from [9],p 112, we have the following theorem:

Theorem 1.14. *1) Let X be an algebraic variety or an analytic space, Then we have $\omega_X \in \mathcal{D}^b_c(X)$. Moreover, the functor $\mathbb{D}_X$ preserves the category $\mathcal{D}^b_c(X)$ and $\mathbb{D}_X \circ \mathbb{D}_X \simeq Id$ on $\mathcal{D}^b_c(X)$.*

2) Let $f : X \to Y$ be a morphism of algebraic varieties or analytic spaces. Then the functor f^{-1} and $f^!$ induce

$$f^{-1}, f^! : \mathcal{D}^b_c(Y) \to \mathcal{D}^b_c(X)$$

and we have

$$f^! = \mathbb{D}_X \circ f^{-1} \circ \mathbb{D}_Y$$

on $\mathcal{D}^b_c(Y)$.

3) Let $f : X \to Y$ be a morphism of algebraic varieties or analytic spaces. We assume that f is proper in the case where f is a morphism of analytic spaces. Then the functors $Rf_*, Rf_!$ induces

$$Rf_*, Rf_! : \mathcal{D}^b_c(X) \to \mathcal{D}^b_c(Y)$$

and we have

$$Rf_! = \mathbb{D}^b_c(Y) \circ Rf_* \circ \mathbb{D}^b_c(X)$$

on $\mathcal{D}^b_c(X)$.

Now we want to recover Poincare duality when X is a manifold from Verdier duality. First we have the theorem:

Theorem 1.15. *(Verdier Duality) $H_c^{-i}(X, \mathcal{F}^\bullet)^\vee \simeq H^i(X, \mathbb{D}\mathcal{F}^\bullet)$.*

Proof. This is just a special case of Verdier duality in which Y is a one point space and f is the constant map $a : X \to \{point\}$. □

Then we have the proposition as follows:

Proposition 1.16. *Let X be a smooth, oriented n-dimensional manifold. Then $\omega_X \simeq \underline{\mathbb{C}}_X[n]$.*

Proof. We need the compare the cohomology, so let compute the cohomology $\mathcal{H}^\bullet(\omega_X)$, The i-th cohomology can be obtained as the sheaf associated to the presheaf

$$U \mapsto Hom(\mathbb{C}_U, \omega_X[i])$$

Here, as $\mathbb{C}_U = j_!(\mathbb{C})$ is the extension by zero of the constant sheaf k from U to X. Indeed, to check this relation, we recall that we assume ω_X a complex of injectives, so the map $\mathbb{C}_U \to \omega_X[i]$ are just homotopy classes of maps $\mathbb{C}_U \to \omega_X[i]$, and the sheaf associated to the presheaf is clearly the homology $\mathcal{H}^i(\omega_X)$.

So we need to compute $Hom(\mathbb{C}_U, \omega_X[i]) = Hom(\mathbb{C}_U[-i], \omega_X)$, by taking U small, we may assume that U is a ball in $\mathbb{R}^n$. From the adjoint property, such maps are in natural bijection with maps $R\Gamma_c(\mathbb{C}_U[-i]) \to \mathbb{C}$ in the derived category, so we need to compute $Hom(R\Gamma_c(\mathbb{C}_U[-i]), \mathbb{C})$. So we have the cohomology of dualizing complex $\mathcal{H}^i(\mathcal{D})$ is the sheaf associate to the presheaf $U \mapsto H^i_c(U, \mathbb{C})^\vee$. Then the following lemma will complete the proof of this proposition. □

Lemma 1.17. *Let k be any ring, then we have $H^i_c(\mathbb{R}^n, k) \simeq k$ if $i = n$, and $H^i_c(\mathbb{R}^n, k) = 0$, otherwise.*

Proof. See [6]. □

It follows immediately from the above proposition 1.14 that on a smooth, oriented n-dimensional manifold, we have $\mathbb{D}\underline{\mathbb{C}}_X \simeq \underline{\mathbb{C}}_X[n]$. Then the Verdier duality theorem becomes the following

Theorem 1.18. *(Poincaré Duality). Let X be a smooth, oriented n-dimensional manifold, then $H^{n-i}_c(X, \mathbb{C})^\vee \simeq H^i(X, \mathbb{C})$.*

So the Verdier duality can be seen as a generalization of Poincaré duality. For most spaces and most complexes of sheaves, the complex $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ that appears in Verdier duality are different. What makes Poincaré duality work for manifolds is that the constant sheaf is close to self-dual. e.g. a shifting of it will be self-dual, as we will show in the following section Perverse Sheaves that $\underline{\mathbb{C}}_X[dimX]$ is self-dual. If we could find self-dual complexes of sheaves on some space, then we could achieve a sort of intermediate generalization of Poincaré duality: the duality theorem that is close in the spirit to the original Poincaré duality,now we will have duality not only on manifolds. The search of such self-dual complexes of sheaves is one of the principal motivation for the development of the theory of perverse sheaves. See [8].

2 t-structure and Perverse Sheaves

t-Structure

Definition 2.1. Let $\mathcal{C}$ be a triangulated category. A t-structure on $\mathcal{C}$ is a pair of full subcategories $\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}$ satisfying the axioms below. For any $n \in \mathbb{Z}$, $\mathcal{C}^{\leq n} = \mathcal{C}^{\leq 0}[-n], \mathcal{C}^{\geq n} = \mathcal{C}^{\geq 0}[-n]$

1) $\mathcal{C}^{\leq 0} \subset \mathcal{C}^{\leq 1}$ and $\mathcal{C}^{\geq 1} \subset \mathcal{C}^{\geq 0}$

2) $\bigcap_{n\in\mathbb{Z}} \mathcal{C}^{\leq n} = \bigcap_{n\in\mathbb{Z}} \mathcal{C}^{\geq n} = 0$

3) If $A \in \mathcal{C}^{\leq 0}$ and

4) For any object X in X in $\mathcal{C}$, there is a distinguished triangle $A \to X \to B \to A[1]$ with $A \in \mathcal{C}^{\leq 0}$ and $B \in \mathcal{C}^{\geq 1}$.

Although the last axiom does not say anything about the uniqueness of the distinguished triangle, it turns out to be the unique as a consequence of the other axioms. Specifically, we have

Proposition 2.2. *The distinguished triangle in axiom (4) above is unique up to isomorphism. Indeed, there are functors*

$$^t\tau_{\leq 0} : \mathcal{C} \to \mathcal{C}^{\leq 0}, {}^t\tau_{\geq 1} : \mathcal{C} \to \mathcal{C}^{\geq 1}$$

such that for any object X of $\mathcal{C}$,

$$^t\tau_{\leq 0}X \to X \to {}^t\tau_{\geq 1}X \to ({}^t\tau_{\leq 0}X)[1]$$

is a distinguished triangle.

Proof. let $A \xrightarrow{u} X \xrightarrow{v} B$, $A' \xrightarrow{u} X \xrightarrow{v} B'$ be two distinguished triangles as in axiom (4), Then $A' \in \mathcal{C}^{\leq 0}$, we have $Hom(A', B) = Hom(A', B[-1]) = 0$,Then the exact sequence $Hom(A', B[-1]) \to Hom(A', A) \to Hom(A', X) \to Hom(A', B)$ implies that there is an isomorphism $u_{A'A} : Hom(A', A) \to Hom(A', X)$. Let $A' = A$, then we have an isomorphism $u_A : Hom(A, A) \to Hom(A, X)$. Suppose $u_A(Id_A) = f_A \in Hom(A, X)$. Then we have $u^{-1}_{A'A}(f_{A'}) \circ u^{-1}_{AA'}(f_A) = Id_A$. Thus $A' \simeq A$. This property determines the object $A \in \mathcal{C}^{\leq 0}$ with $u : A \to X$ uniquely. In the same way the object $B \in \mathcal{C}^{\geq 1}$ with $v : X \to B$ is uniquely determined. This implies that the triangle (A, X, B) is uniquely determined. Define ${}^t\tau_{\leq 0}X := A, {}^t\tau_{\geq 1}X := B$, The functoriality follows from the corollary below: for $X \to Y$, compose it with ${}^t\tau_{\leq 0}X \to X$, then we get $Hom({}^t\tau_{\leq 0}X, {}^t\tau_{\leq 0}Y)$ via the isomorphism: $Hom({}^t\tau_{\leq 0}X, {}^t\tau_{\leq 0}Y) \simeq Hom({}^t\tau_{\leq 0}X, Y)$ this complete the proof. □

Corollary 2.3. *Let $\iota_{\leq 0} : \mathcal{C}^{\leq 0} \to \mathcal{C}$,$\iota_{\geq 1} : \mathcal{C}^{\geq 1} \to \mathcal{C}$ be the inclusion functors, Then*

$$(\iota_{\leq 0}, {}^t\tau_{\leq 0}), ({}^t\tau_{\geq 0}, \iota_{\geq 0})$$

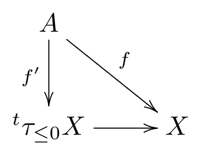
are adjoint pairs.

Proof. For any $A' \in \mathcal{C}^{\leq 0}$, we have

$$Hom(\iota_{\leq 0}A', X) \simeq Hom(A', A) = Hom(A', {}^t\tau_{\leq 0}X)$$

In the same way, one shows the other pair. □

Remark. When doing calculations in a triangulated category with a t-structure, the above proposition typically comes up in the following way: if $A \in \mathcal{C}^{\leq 0}$ and $f : A \to X$ is any morphism in $\mathcal{C}$, then f factors through ${}^t\tau_{\leq 0}X$. That is, there is a unique morphism f' making the following diagram commute:



Similarly, if $B \in \mathcal{C}^{\geq 1}$ and $g : X \to B$ is any morphism, then g factors through ${}^t\tau_{\geq 1}X$:

$$\begin{array}{ccc} X & \longrightarrow & {}^t\tau_{\geq 1}X \\ & \searrow^{g} & \downarrow g' \\ & & B \end{array}$$

Of course, there are corresponding statements with $\leq n$ and $\geq n$ for any $n \in \mathbb{Z}$, obtained by shifting. In particular, there are truncation functor ${}^t\tau_{\leq n}$ and ${}^t\tau_{\geq n}$, and there are distinguished triangles

$${}^t\tau_{\leq n}X \to X \to {}^t\tau_{\geq n+1}X \to ({}^t\tau_{\leq n}X)[1] \text{ for all } n.$$

The relation between truncation and shifting is given by

$${}^t\tau_{\leq n}X = ({}^t\tau_{\leq 0}X[n])[-n]$$

Proposition 2.4. *The following conditions on $X \in \mathcal{C}$ are equivalent*
1) We have $X \in \mathcal{C}^{\leq n}$ (resp. $X \in \mathcal{C}^{\geq n}$).
2) The canonical morphism ${}^t\tau_{\leq n}X \to X$ (resp. $X \to {}^t\tau_{\geq n}X$) is an isomorphism.
3) We have ${}^t\tau_{>n}X = 0$ (resp. ${}^t\tau_{<n}X = 0$)

Proof. Only prove 1) $\Rightarrow$ 2). We may assume $n = 0$, As $X \in \mathcal{C}^{\leq 0}$ so the distinguished triangle

$$X \to X \to 0 \xrightarrow{+1}$$

satisfies the condition that $X \in \mathcal{C}^{\leq 0}$ and $0 \in \mathcal{C}^{\geq 0}$. Then from the uniqueness in proposition 2.2. We get that ${}^t\tau_{\leq 0}X \simeq X$ □

Definition 2.5. Let $\mathcal{C}$ be a triangulated category with a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$. The category $\mathcal{T} = \mathcal{C}^{\leq 0} \bigcap \mathcal{C}^{\geq 0}$ is called the heart (or core) of the t-structure.

Proposition 2.6. *Let*

$$X' \to X \to X'' \xrightarrow{+1}.$$

be a distinguished triangle in $\mathcal{C}$. If $X', X'' \in \mathcal{C}^{\leq 0}$ (resp. $\mathcal{C}^{\geq 0}$), then $X \in \mathcal{C}^{\leq 0}$ (resp. $\mathcal{C}^{\geq 0}$). In particular, if $X', X'' \in \mathcal{T}$, then $X \in \mathcal{T}$.

Proof. Suppose $X', X'' \in \mathcal{C}^{\leq 0}$, by proposition 2.4, we need to show ${}^t\tau_{>0}X = 0$. For the triangle, we have the exact sequence,

$$Hom(X'', {}^t\tau_{>0}X) \to Hom(X, {}^t\tau_{>0}X) \to Hom(X', {}^t\tau_{>0}X)$$

From definition of t-structure, we have $0 = Hom(X'', {}^t\tau_{>0}X) = Hom(X', {}^t\tau_{>0}X)$. Then by corollary 2.3, we have $Hom(X, {}^t\tau_{>0}X) = Hom({}^t\tau_{>0}X, {}^t\tau_{>0}X) = 0$, which implies the desired result. □

Proposition 2.7. *Suppose $n \leq m$, then we have*

$${}^t\tau_{\leq n}{}^t\tau_{\leq m} = {}^t\tau_{\leq m}{}^t\tau_{\leq n} = {}^t\tau_{\leq n}, {}^t\tau_{\geq n}{}^t\tau_{\leq m} = {}^t\tau_{\leq m}{}^t\tau_{\geq n}$$

$${}^t\tau_{\geq n}{}^t\tau_{\geq m} = {}^t\tau_{\geq m}{}^t\tau_{\geq n} = {}^t\tau_{\geq m}, {}^t\tau_{\leq n}{}^t\tau_{\geq m} = {}^t\tau_{\geq m}{}^t\tau_{\leq n} = 0$$

In particular, all truncation functors commute with each other.

Proof. We only prove ${}^t\tau_{\geq n}{}^t\tau_{\leq m} = {}^t\tau_{\leq m}{}^t\tau_{\geq n}$. The other equalities will follow from proposition 2.4 and definition.

Assume the others equalities, Let $X \in \mathcal{C}$, then we have a distinguished triangle

$$ {}^t\tau_{\leq m}{}^t\tau_{\geq n}X \to {}^t\tau_{\geq n}X \to {}^t\tau_{>m}X \xrightarrow{+1} $$

from which we conclude that ${}^t\tau_{\leq m}{}^t\tau_{\geq n}X \in \mathcal{C}^{\geq n}$ by proposition 2.6. Then we get the isomorphisms

$Hom({}^t\tau_{\leq m}X, {}^t\tau_{\geq n}) = Hom({}^t\tau_{\leq m}X, {}^t\tau_{\leq m}{}^t\tau_{\geq n}) = Hom({}^t\tau_{\geq n}{}^t\tau_{\leq m}X, {}^t\tau_{\leq m}{}^t\tau_{\geq n})$.

So we can define a morphism $c \in Hom({}^t\tau_{\geq n}{}^t\tau_{\leq m}X, {}^t\tau_{\leq m}{}^t\tau_{\geq n})$ by the image of the composition ${}^t\tau_{\leq m}X \to X \to {}^t\tau_{\geq n}$ through the above isomorphism. We need to show c is an isomorphism. The distinguished triangle

$$ {}^t\tau_{<n}X \to {}^t\tau_{\leq m}X \to {}^t\tau_{\geq n}{}^t\tau_{\leq m}X \xrightarrow{+1} $$

shows that ${}^t\tau_{\geq n}{}^t\tau_{\leq m}X \in \mathcal{C}^{\leq m}$ by proposition 2.6. On the other hand, applying the octahedral axiom to the three distinguished triangles

$$ {}^t\tau_{<n}X \to {}^t\tau_{\leq m}X \to {}^t\tau_{\geq n}{}^t\tau_{\leq m}X \xrightarrow{+1} $$

$$ {}^t\tau_{<n}X \to X \to {}^t\tau_{\geq n}X \xrightarrow{+1} $$

$$ {}^t\tau_{\leq m}X \to X \to {}^t\tau_{\geq m}X \xrightarrow{+1} $$

We get a distinguished triangle

$$ {}^t\tau_{\geq n}{}^{tt}\tau_{\leq m}X \to {}^t\tau_{\geq n}X \to {}^t\tau_{\geq m}X \xrightarrow{+1} $$

Now from the uniqueness, we get the desired isomorphism. □

Theorem 2.8. *1) The heart $\mathcal{T}$ is an abelian category*
2) An exact sequence

$$ 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 $$

in $\mathcal{T}$ gives rise to a distinguished triangle

$$ X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1} $$

in $\mathcal{C}$

Proof. Apply proposition 2.6 to the distinguished triangle

$$ X \to X \bigoplus Y \to Y \xrightarrow{+1} $$

with $X, Y \in \mathcal{T}$, then by proposition 2.6, we get $X \bigoplus Y \in \mathcal{T}$. For any morphism $F : X \to Y$, embed it into a distinguished triangle

$$ X \to Y \to Z \xrightarrow{+1} $$

Then it follows from proposition 2.6 that $Z \in \mathcal{C}^{\leq 0} \bigcap \mathcal{C}^{\geq -1}$. We will show the kernel and cokernel are given by

$$ Ker \simeq H^{-1}(Z) = {}^t\tau^{\leq 0}(Z[-1]), Coker \simeq H^0(Z) = {}^t\tau^{\geq 0}Z. $$

Consider the exact sequence

$$Hom(X[1], W) \to Hom(Z, W) \to Hom(Y, W) \to Hom(X, W)$$

$$Hom(W, Y[-1]) \to Hom(W, Z[-1]) \to Hom(W, X) \to Hom(W, Y)$$

for $W \in \mathcal{T}$. By the corollary 2.3 and definition, the above sequence turn out to be

$$0 \to Hom({}^t\tau^{\geq 0}Z, W) \to Hom(Y, W) \to Hom(X, W)$$

$$0 \to Hom(W, {}^t\tau^{\leq 0}Z[-1]) \to Hom(W, X) \to Hom(W, Y)$$

This implies that $Ker \simeq H^{-1}(Z) = {}^t\tau^{\leq 0}(Z[-1])$,$Coker \simeq H^0(Z) = {}^t\tau^{\geq 0}Z$. We need to show the morphism $Coimf \to Imf$ is an isomorphism. Embed $Y \to Cokerf$ into a distinguished triangle

$$I \to Y \to Cokerf \xrightarrow{+1}, \text{ then by proposition 2.6, } I \in \mathcal{C}^{\geq 0}.$$

Applying the octahedral axiom to the following distinguished triangles.

$$Y \to Z \to X[1] \xrightarrow{+1}$$

$$Y \to Cokerf \to I[1] \xrightarrow{+1}$$

$$Z \to Cokerf \to Kerf[2] \xrightarrow{+1}$$

we get the distinguished triangle

$$X[1] \to I[1] \to Kerf[2] \xrightarrow{+1}$$

which is equivalent to say we have the distinguished triangle

$$Kerf \to X \to I \xrightarrow{+1}$$

and this implies that $I \in \mathcal{C}^{\leq 0}$. Hence $I \in \mathcal{T}$. Then by the argument used in the proof of the existence of kernel and cokernel we get

$$Imf = Ker(Y \to Cokerf) \simeq I \simeq Coker(Kerf \to X) = Coimf$$

2) Embed $X \xrightarrow{f} Y \to W \xrightarrow{+1}$. Then $Kerf = 0$ and $Cokerf = Z$ we obtain $W \simeq Z$ by the proof for 1). ☐

Definition 2.9. the functor ${}^tH^0 : \mathcal{C} \to \mathcal{T}$ defined by ${}^tH^0 = {}^t\tau_{\geq 0}{}^t\tau_{\leq 0} = {}^t\tau_{\leq 0}{}^t\tau_{\geq 0}$ is called the zeroth t-cohomology. Moreover, for any $i \in \mathbb{Z}$, the functor ${}^tH^i$ defined by ${}^tH^i(X) = {}^tH^0(X[i])$ or equivalently ${}^tH^i = {}^t\tau_{\leq i}{}^t\tau_{\geq i} = {}^t\tau_{\geq i}{}^t\tau_{\leq i}$ is called the i-th t-cohomology.

Proposition 2.10. *The functor ${}^tH^0 : \mathcal{C} \to \mathcal{T}$ is a cohomological functor*

Proof. We need to show for a distinguished triangle $X \to Y \to Z \xrightarrow{+1}$ in $\mathcal{C}$,the sequence

$$H^0(X) \to H^0(Y) \to H^0(Z)$$

is exact.

1) suppose $X, Y, Z \in \mathcal{C}^{\geq 0}$, for $W \in \mathcal{T}$, we have the exact sequence

$$Hom(W, Z[-1]) \to Hom(W, X) \to Hom(W, Y) \to Hom(W, Z)$$

from definition, $0 = Hom(W, Z[-1])$, moreover, for $V \in \mathcal{C}^{\geq 0}$, we have ${}^t\tau^{\leq 0}V \simeq {}^t\tau^{\geq 0}{}^t\tau^{\leq 0}V = H^0(V)$. so $Hom(W, V) = Hom(W, {}^t\tau^{\leq 0}V) = Hom(W, H^0(V))$. So we get exact sequence in $\mathcal{T}$:

$$0 \to Hom(W, H^0(X)) \to Hom(W, H^0(Y)) \to Hom(W, H^0(Z))$$

2) suppose only $Z \in \mathcal{C}^{\geq 0}$. Let $W \in \mathcal{C}^{<0}$, Then we have $Hom(W, Z) = Hom(W, Z[-1]) = 0$,hence $Hom(W, X) = Hom(W, Y)$,By proposition, this implies that the canonical morphism ${}^t\tau^{<0}X \to {}^t\tau^{<0}Y$ is an isomorphism.Then apply the octahedral axiom to the following distinguished triangles.

$${}^t\tau^{<0}X \to X \to {}^t\tau^{\geq 0}X \xrightarrow{+1}$$

$${}^t\tau^{<0}X \to Y \to {}^t\tau^{\geq 0}Y \xrightarrow{+1}$$

$$X \to Y \to Z \xrightarrow{+1}$$

We get a new one ${}^t\tau^{\geq 0}X \to {}^t\tau^{\geq 0}Y \to Z \xrightarrow{+1}$ Then we get the situation of 1), so the assertion holds.

3) By the same argument of 2), we can prove the exactness of the sequence

$$H^0(X) \to H^0(Y) \to H^0(Z) \to 0$$

under the assumption that $X \in \mathcal{C}^{\leq 0}$

4) Now let's consider the general case, embed the composition of morphisms ${}^t\tau^{\leq 0}X \to X \to Y$ into a distinguished triangle

$${}^t\tau^{\leq 0}X \to Y \to W \xrightarrow{+1}$$

by 3) we have the exact sequence

$$H^0(X) \to H^0(Y) \to H^0(W)$$

Now applying the octahedral axiom to the distinguished triangles

$${}^t\tau^{\leq 0}X \to X \to {}^t\tau^{>0}X \xrightarrow{+1}$$

$${}^t\tau^{\leq 0}X \to Y \to W \xrightarrow{+1}$$

$$X \to Y \to Z \xrightarrow{+1}$$

we get a distinguished triangle

$$W \to Z \to {}^t\tau^{>0}X[1]$$

hence by 2), we get an exact sequence $0 \to H^(W) \to H^0(Z)$, and this completes the proof. □

The following theorem is the reason we want to introduce the notion of t-structure. For the details, see [8],[9].

Theorem 2.11. *Let $\mathcal{C}$ be a triangulated category with a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$, and let $\mathcal{T}$ be its heart. then $\mathcal{T}$ is an abelian category. Moreover, the functor ${}^tH^0 : \mathcal{C} \to \mathcal{T}$ enjoys the following properties:*

1)For any object $A \in \mathcal{T}$, ${}^tH^0(A) \simeq A$

2) ${}^tH^0$ takes distinguished triangles in $\mathcal{C}$ to long exact sequence in $\mathcal{T}$.

3) A morphism $f : X \to Y$ in $\mathcal{C}$ is an isomorphism if and only if the morphism ${}^tH^i(f)$ are isomorphisms in $\mathcal{T}$ for all $i \in \mathbb{Z}$.

4) We have

$$\mathcal{C}^{\leq 0} = \{X \in \mathcal{C} | {}^tH^i(X) = 0 \text{ for all } i > 0\},$$

$$\mathcal{C}^{\geq 0} = \{X \in \mathcal{C} | {}^tH^i(X) = 0 \text{ for all } i < 0\}$$

Perverse t-structure

Definition 2.12. We define full subcategories ${}^p\mathcal{D}_c^{\leq 0}(X)$ and ${}^p\mathcal{D}_c^{\geq 0}(X)$ of $\mathcal{D}_c^b(X)$ as follows. For $\mathcal{F}^\bullet \in \mathcal{D}_c^b(X)$, define that $\mathcal{F}^\bullet \in {}^p\mathcal{D}_c^{\leq 0}(X)$ if and only if

1) $dim(supp\mathcal{H}^j(\mathcal{F}^\bullet)) \leq -j$, for any $j \in \mathbb{Z}$.

and $\mathcal{F}^\bullet \in {}^p\mathcal{D}_c^{\geq 0}(X)$ if and only if

2) $dim(supp\mathcal{H}^j(\mathbb{D}\mathcal{F}^\bullet)) \leq -j$, for any $j \in \mathbb{Z}$.

we difine the subcategory $Perv(X) = {}^p\mathcal{D}_c^{\leq 0}(X) \bigcap {}^p\mathcal{D}_c^{\geq 0}(X)$

We will show that the pair $({}^p\mathcal{D}_c^{\leq 0}(X), {}^p\mathcal{D}_c^{\geq 0}(X))$ defines a t-structure on $\mathcal{D}_c^b(X)$, and hence $Perv(X)$ turns out to be an abelian category. Since we have $\mathbb{D}\mathbb{D}\mathcal{F}^\bullet \simeq \mathcal{F}^\bullet$ for $\mathcal{F}^\bullet \in \mathcal{D}_c^b(X)$, the Verdier functor $\mathbb{D}$ exchanges ${}^p\mathcal{D}_c^{\leq 0}(X)$ with ${}^p\mathcal{D}_c^{\geq 0}(X)$

Remark. To be more precise, this is called perverse sheaves with respect to middle perversity. We will explain the term "middle perversity" as following: Let $p : 2\mathbb{N} \to \mathbb{N}$ be a decreasing function such that $0 \leq p(n) - p(m) \leq m - n$ for all $n \leq m$. Such a function is called a perversity function. Denote by $p^* : 2\mathbb{N} \to \mathbb{N}$ the dual perversity function given by $p^*(n) = -n - p(n)$ for all $n \in 2\mathbb{N}$. Let X be a complex analytic space and let $\mathcal{S}$ be a stratification of X, For a stratum $S \in \mathcal{S}$ we set $p(S) = p(2dimS)$, where $dimS$ is the dimension of S. Actually we have a generalization of the definition of $({}^p\mathcal{D}_c^{\leq 0}(X), {}^p\mathcal{D}_c^{\geq 0}(X))$ as following. After [1], we have the following result: Let $p : 2\mathbb{N} \to \mathbb{N}$ a perversity function and $\mathcal{F}^\bullet \in \mathcal{D}_c^b(X)$, then the following conditions are equivalent.

1) There exists a stratification $\mathcal{S}$ as above such that $\mathcal{F}^\bullet$ is $\mathcal{S}$ constructible and for any stratum $S \in \mathcal{S}$, one has $\mathcal{H}^j(i_S^{-1}\mathcal{F}^\bullet) = 0$ for all $j > p(S)$ (resp. $\mathcal{H}^j(i_S^!\mathcal{F}^\bullet) = 0$ for all $j < p(S)$), where $i_S : S \to X$ is the inclusion.

2) For any stratification $\mathcal{S}$ as above such that $\mathcal{F}^\bullet$ is $\mathcal{S}$ constructible and for any stratum $S \in \mathcal{S}$, one has $\mathcal{H}^j(i_S^{-1}\mathcal{F}^\bullet) = 0$ for all $j > p(S)$ (resp. $\mathcal{H}^j(i_S^!\mathcal{F}^\bullet) = 0$ for all $j < p(S)$), where $i_S : S \to X$ is the inclusion.

We can define that $\mathcal{F}^\bullet \in {}^p\widetilde{\mathcal{D}}_c^{\leq 0}(X)$ (resp. $\mathcal{F}^\bullet \in {}^p\widetilde{\mathcal{D}}_c^{\geq 0}(X)$) if it satisfies 1) or 2).

Then after [1], we have the following result: The pair $({}^p\widetilde{\mathcal{D}}_c^{\leq 0}(X), {}^p\widetilde{\mathcal{D}}_c^{\geq 0}(X))$ is a t-structure on the triangulated category $\mathcal{D}_c^b(X)$ for any perversity function p.

The above t-structure is called the t-structure of perversity p on $\mathcal{D}_c^b(X)$. Define the category of p-perverse sheaves the heart of this t-structure, namely $Perv(X,p) = {}^p\widetilde{\mathcal{D}}_c^{\leq 0}(X) \bigcap {}^p\widetilde{\mathcal{D}}_c^{\geq 0}(X)$

Now we can see the previous definition of perverse sheaves with respect to the middle perversity as a special case of perverse sheaves $Perv(X, p)$, where $p = p_{1/2}$ and given by $p_{1/2}(2k) = -k$, for all $2k \in 2\mathbb{N}$. We will prove that definition 2.12 coincide with the definition in this remark if we let p to be 1/2.

To see why the perversity is called the middle one, notice that the definition of the perversity function p implies that $p(m) \geq -m + p(0)$, for all integer $m \in 2\mathbb{N}$. If we normalize the perversity functions by setting $p(0) = 0$. then we have $-m \leq p(m) \leq 0$. Hence there is a minimal perversity function p_{min} given by $p_{min}(m) = -m$ and a maximal perversity p_{max} given by $p_{max}(m) = 0$ and half way between these two, the middle perversity $p_{mid} = p_{1/2}$ given by $p_{1/2}(m) = -m/2$ for all $m \in 2\mathbb{N}$. Note also that $p^*_{min} = p_{max}, p_{min} = p^*_{max}$ and $p^*_{1/2} = p_{1/2}$. the last equality shows the self dual of the middle perversity function $p_{1/2}$.

Now I will work only on the middle perversity. By setting $p = p_{mid} = p_{1/2}$, we get $Perv(X) = Perv(X, p_{1/2})$. And we will prove the statements in this remark for the case when $p = p_{1/2}$ in the following context.

Lemma 2.13. *We work on the middle perversity, let $\mathcal{F}^\bullet \in \mathcal{D}^b_c(X)$, then we have*

$$supp(\mathcal{H}^j(\mathbb{D}\mathcal{F}^\bullet)) = \{x \in X | \mathcal{H}^{-j}(i^!_{\{x\}}\mathcal{F}^\bullet) \neq 0\}.$$

for all $j \in \mathbb{Z}$, where $i_{\{x\}} : x \hookrightarrow X$ are inclusion maps.

Proof. From theorem 1.12, chapter 2, for any $x \in X$, the following isomorphism holds

$$i^!_{\{x\}}\mathcal{F}^\bullet \simeq i^!_{\{x\}}\mathbb{D}_X\mathbb{D}_X\mathcal{F}^\bullet \simeq \mathbb{D}_{\{x\}}i^{-1}_{\{x\}}(\mathbb{D}_X\mathcal{F}^\bullet)$$

So we obtain an isomorphism

$$\mathcal{H}^{-j}(i^!_{\{x\}}\mathcal{F}^\bullet) \simeq \mathcal{H}^{-j}(\mathbb{D}_{\{x\}}i^{-1}_{\{x\}}(\mathbb{D}_X\mathcal{F}^\bullet)) = \mathcal{H}^{-j}(Hom(i^{-1}_{\{x\}}\mathbb{D}_X\mathcal{F}^\bullet, \mathbb{C})) = \\ \mathcal{H}^j(i^{-1}_{\{x\}}\mathbb{D}_X\mathcal{F}^\bullet)^\vee = \mathcal{H}^j(\mathbb{D}_X\mathcal{F}^\bullet)^\vee_x$$

□

Follow [9], we have the following propositions and corollaries.

Proposition 2.14. *Let $\mathcal{F}^\bullet \in \mathcal{D}^b_c(X)$ and $X = \bigsqcup X_\alpha$ be a complex stratification of X consisting of connected strata such that $i^{-1}_{X_\alpha}\mathcal{F}^\bullet$ and $i^!_{X_\alpha}\mathcal{F}^\bullet$ have locally constant cohomology sheaves for any α, then*

1) $\mathcal{F}^\bullet \in {}^p\mathcal{D}^{\leq 0}_c(X)$ if and only if $\mathcal{H}^j(i^{-1}_{X_\alpha}\mathcal{F}^\bullet) = 0$ for all α and $j > -d_{X_\alpha}$, where d_{X_α} denotes the dimension of S_α.

2) $\mathcal{F}^\bullet \in {}^p\mathcal{D}^{\geq 0}_c(X)$ if and only if $\mathcal{H}^j(i^!_{X_\alpha}\mathcal{F}^\bullet) = 0$ for all α and $j < -d_{X_\alpha}$.

Proof. 1) $\mathcal{F}^\bullet \in {}^p\mathcal{D}^{\leq 0}_c(X)$ if and only if $dim(supp\mathcal{H}^j(\mathcal{F}^\bullet)) \leq -j$, for any $j \in \mathbb{Z}$. As $dim(supp\mathcal{H}^j(\mathcal{F}^\bullet)) = sup\{dimX_\alpha; |\mathcal{H}^j(\mathcal{F}^\bullet)|_{X_\alpha} \neq 0\}$ and $supp\mathcal{H}^j(\mathcal{F}^\bullet) = \overline{\{x \in X | H^j(i^{-1}_x(\mathcal{F}^\bullet)) = \mathcal{H}^j(\mathcal{F}^\bullet)_x \neq 0\}}$. So 1) holds.

For 2),by the above lemma 2.13 $\mathcal{F}^\bullet \in {}^p\mathcal{D}^{\geq 0}_c(X)$ if and only if

$$dim\{x \in X | H^{-j}(i^!_{\{x\}}\mathcal{F}^\bullet \neq 0)\} \leq -j$$

for any $j \in \mathbb{Z}$. For $x \in X_\alpha$ we can factor the inclusion $i_{\{x\}} : \{x\} \to X$ through $\{x\} \overset{j_{\{x\}}}{\longrightarrow} X_\alpha \overset{i_{X_\alpha}}{\longrightarrow} X$, with the assumption that $i^!_{X_\alpha}\mathcal{F}^\bullet$ have locally constant cohomology sheaf, we get the isomorphism

$$i^!_{\{x\}}\mathcal{F}^\bullet \simeq j^!_{\{x\}} i^!_{X_\alpha}\mathcal{F}^\bullet \simeq j^{-1}_{\{x\}} i^!_{X_\alpha}\mathcal{F}^\bullet[-2d_{X_\alpha}].$$

Hence for any $j \in \mathbb{Z}$, by the connectedness of X_α, $X_\alpha \bigcap suppH^j(\mathbb{D}_X\mathcal{F}^\bullet)$ is X_α or $\varnothing$. Then the following conditions are equivalent

1) $H^j(i^!_{X_\alpha}\mathcal{F}^\bullet) = 0$ for any $j < -d_{X_\alpha}$.
2) $H^{-j}(i^!_{\{x\}}\mathcal{F}^\bullet) = 0$ for any $x \in X_\alpha$ and $j > -d_{X_\alpha}$.
3) $X_\alpha \bigcap suppH^j(\mathbb{D}_X\mathcal{F}^\bullet) = \varnothing$ for any $j > -d_{X_\alpha}$.

Then condition 3) is saying that for any X_α with $X_\alpha \in suppH^j(\mathbb{D}_X\mathcal{F}^\bullet)$, we must have $d_{X_\alpha} \leq -j$. □

Corollary 2.15. *Assume that X is a connected complex manifold and all the cohomology sheaves of $\mathcal{F}^\bullet \in \mathcal{C}^b_c(X)$ are locally constant on X, Then*

1) $\mathcal{F}^\bullet \in {}^p\mathcal{D}^{\leq 0}_c(X)$ if and only if $\mathcal{H}^j(\mathcal{F}^\bullet) = 0$ for all α and $j > -d_X$,
2) $\mathcal{F}^\bullet \in {}^p\mathcal{D}^{\geq 0}_c(X)$ if and only if $\mathcal{H}^j(\mathcal{F}^\bullet) = 0$ for all α and $j < -d_X$.

Lemma 2.16. *Let X be a complex manifold and $\mathcal{F}^\bullet \in \mathcal{D}^b_c(X)$. Assume that all the cohomology sheaves of $\mathcal{F}^\bullet$ are locally constant on X and for an integer $d \in \mathbb{Z}$ we have $\mathcal{H}^j(\mathcal{F}^\bullet) = 0$, for $j < d$. then for any locally closed analytic subset Z of X we have*

$$\mathcal{H}^j_Z(\mathcal{F}^\bullet) = 0 \textit{ for any } j < d + 2codim_Y Z$$

Proof. By induction on the cohomological length of $\mathcal{F}^\bullet$, we may assume that $\mathcal{F}^\bullet$ is a local system L on Y. Since the question is local on Y, we may assume that L is the constant sheaf $\underline{\mathbb{C}}_Y$. Hence we are reduced to prove

$$H^j_Z(\underline{\mathbb{C}}_Y) = 0 \text{ for any } j < 2codim_Y Z.$$

This can be proved by induction on the dimension of Z with the aid of the distinguished triangle.

$$R\Gamma_{Z-Z_s}(\underline{\mathbb{C}}_Y) \to R\Gamma_Z(\underline{\mathbb{C}}_Y) \to R\Gamma_{Z_s}(\underline{\mathbb{C}}_Y) \xrightarrow{+1}.$$

Because the dimension of Z_s and $Z - Z_s$ are strictly smaller than Z. Here we denote Z_s the smooth part of Z.

□

Proposition 2.17. *Let $\mathcal{F}^\bullet \in \mathcal{D}^b_c(X)$, then the following are equivalent*

1) $\mathcal{F}^\bullet \in {}^p\mathcal{D}^{\geq 0}_c(X)$
2) For any locally closed analytic subset S of X we have

$$\mathcal{H}^j(i^!_S(\mathcal{F}^\bullet)) = 0, \textit{ for any } j < -d_S.$$

3) For any locally closed analytic subset S of X we have

$$\mathcal{H}^j_S(\mathcal{F}^\bullet) := \mathcal{H}^j R\Gamma_S(\mathcal{F}^\bullet) = 0, \textit{ for any } j < -d_S.$$

4) For any locally closed smooth analytic subset S of X we have

$$\mathcal{H}^j(i^!_S(\mathcal{F}^\bullet)) = 0, \textit{ for any } j < -d_S.$$

Proof. 2) ⇔ 3) This is because we have the equality that $H^i_S(-) = H^i Ri_{S*}i^!_S(-)$.

4) ⇒ 2) Now assume 4) is satisfied, for $\mathcal{F}^\bullet \in \mathcal{D}^b_c(X)$. We will show that

$$H^j(i^!_Z(\mathcal{F}^\bullet)) = 0 \text{ for any } j < -d_Z.$$

for any locally closed analytic subset Z of X by induction on $dimZ$. Denote the smooth part of Z as Z_s and set $Z' = Z/Z_s$. Then $dimZ' < dimZ$, by hypothesis we get

$$H^j_{Z'}(\mathcal{F}^\bullet) = 0 \text{ for any } j < -d_{Z'}.$$

In particular, we have

$$H^j_{Z'}(\mathcal{F}^\bullet) = 0 \text{ for any } j < -d_Z.$$

So the claim follows from 4) and the distinguished triangle.

$$R\Gamma_{Z'}(\mathcal{F}^\bullet) \to R\Gamma_Z(\mathcal{F}^\bullet) \to R\Gamma_{Z_s}(\mathcal{F}^\bullet) \xrightarrow{+1}.$$

4) $\Rightarrow$ 1) Take a complex stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of X consisting of connected strata such that $i^{-1}_{X_\alpha}\mathcal{F}^\bullet$ and $i^!_{X_\alpha}\mathcal{F}^\bullet$ have locally constant cohomology sheaves for all $\alpha \in A$. Then by proposition 2.15, 1) is equivalent to the condition $\mathcal{H}^j(i^!_{S_\alpha}\mathcal{F}^\bullet) = 0$ for all α and $j < -d_{S_\alpha}$. Take $S = X_\alpha$ in 4), we see that 4) implies 1).

1) $\Rightarrow$ 4) Suppose that $\mathcal{H}^j(i^!_{X_\alpha}\mathcal{F}^\bullet) = 0$ for all α and $j < -d_{X_\alpha}$, we need to show that for any locally closed smooth analytic subset S in X, we have

$$\mathcal{H}^j(i^!_S(\mathcal{F}^\bullet)) = 0, \text{ for any } j < -d_S.$$

Let $X_k = \bigsqcup_{dimX_\alpha \leq k} X_\alpha$ in X, then we have

$$\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{d_X} = X.$$

Hence it is enough to show that

$$H^j_{S\bigcap X_k}(\mathcal{F}^\bullet) = 0 \text{ for all } j < -d_S$$

Moreover, by the distinguished triangles,

$$R\Gamma_{S\bigcap X_{k-1}}(\mathcal{F}^\bullet) \to R\Gamma_{S\bigcap X_k}(\mathcal{F}^\bullet) \to R\Gamma_{S\bigcap(X_k - X_{k-1})}(\mathcal{F}^\bullet) \xrightarrow{+1}$$

for $k = 0, \cdots, d_X$, by induction, we are reduced to proof that

$$H^j_{S\bigcap(X_k - X_{k-1})}(\mathcal{F}^\bullet) = 0 \text{ for all } j < -d_S$$

As $X_k - X_{k-1}$ is the union of k-dimensional strata, we obtain a isomorphism:

$$H^j_{S\bigcap(X_k - X_{k-1})}(\mathcal{F}^\bullet) \simeq \bigoplus_{dimX_\alpha = k} H^j_{S\bigcap X_\alpha}(\mathcal{F}^\bullet)$$

So we are reduced to show that $H^j(i^!_{S\bigcap X_\alpha}\mathcal{F}^\bullet) \simeq 0$ for all $\alpha \in A$ and $j < -d_S$. Now factor the inclusion $i_{S\bigcap X_\alpha} : S\bigcap X_\alpha \to X$ through $S\bigcap X_\alpha \xrightarrow{j_{X_\alpha}} X_\alpha \xrightarrow{i_{X_\alpha}} X$, we obtain an isomorphism $i^!_{S\bigcap X_\alpha}\mathcal{F}^\bullet \simeq j^!_{X_\alpha} i^!_{X_\alpha}\mathcal{F}^\bullet$, which means

$$H^j(i^!_{S\bigcap X_\alpha}\mathcal{F}^\bullet) \simeq H^j(j^!_{X_\alpha}(i^!_{X_\alpha}\mathcal{F}^\bullet))$$

Now apply lemma 2.16 to $Y = X_\alpha$ and $G^\bullet = i^!_{X_\alpha}\mathcal{F}^\bullet \in \mathcal{D}^b_c(X)$, we get the desired result.

□

Proposition 2.18. *Let $\mathcal{F}^\bullet \in {}^p\mathcal{D}^{\leq 0}_c(X)$ and $\mathcal{G}^\bullet \in {}^p\mathcal{D}^{\geq 0}_c(X)$*

1) we have $H^i(R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) = 0$, for all $j < 0$.

2) for U open subset of X,the correspondence $U \mapsto Hom_{D^b(U)}(\mathcal{F}^\bullet|U.\mathcal{G}^\bullet|U)$ defines a sheaf on X.

Proof. 1) Let $S = \bigcup_{j<0} supp(H^j(R\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet))) \subset X$. Assume that $S \neq \varnothing$, let $i_S : S \to X$ be the embedding, for $j < 0$, we have

$$supp(H^j(R\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet))) \subset S$$

and hence

$$H^j(R\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) \simeq H^j(R\Gamma_S R\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) \simeq \\ H^j(i_{S*}i_S^! R\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) \simeq i_{S*}H^j(R\mathcal{H}om_{\mathbb{C}_S}(i_S^{-1}\mathcal{F}^\bullet, i_S^{-1}\mathcal{G}^\bullet))$$

The assumption $\mathcal{F}^\bullet \in {}^p\mathcal{D}_c^{\leq 0}(X)$ implies that

$$dimsupp\{\mathcal{H}^k(i_S^{-1}\mathcal{F}^\bullet)\} \leq -k, \text{ for any } k \in \mathbb{Z}$$

and the dimension of

$$Z := \bigcup_{k>-d_S} supp\{\mathcal{H}^k(i_S^{-1}\mathcal{F}^\bullet)\} \subset S$$

is less than d_S. Therefore we obtain $S' = S - Z \neq \varnothing$ and $H^j i_{S'}^{-1}\mathcal{F}^\bullet = 0$ for any $j > -d_S$. On the other hand, we have $H^j i_S^! \mathcal{G}^\bullet = 0$ for any $j < -d_S$. Hence we obtain $H^j R\mathcal{H}om_{\mathbb{C}_S}(i^{-1}\mathcal{F}^\bullet, i_S^!\mathcal{G}^\bullet)|_{S'} = 0$ for any $j < 0$. But this contradicts our definition of S.

2) By 1) we have

$$Hom_{\mathcal{D}^b(U)}(\mathcal{F}^\bullet|_U, \mathcal{G}^\bullet|_U) = H^0(U, R\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) = \\ \Gamma(U, H^0(R\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)))$$

Hence the correspondence $U \mapsto Hom_{\mathcal{D}^b(U)}(\mathcal{F}^\bullet|_U, \mathcal{G}^\bullet|_U)$ gives a sheaf isomorphic to $H^0(R\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet))$. □

Now we prove the main theorem:

Theorem 2.19. *The pair* $({}^p\mathcal{D}_c^{\leq 0}(X)), {}^p\mathcal{D}_c^{\geq 0}(X)$ *defines a* t*-structure on* $\mathcal{D}_c^b(X)$*.*

Proof. To prove this is a t-structure, we need to verify the following conditions.

(T_1) $\mathcal{C}^{\leq 0} \subset \mathcal{C}^{\leq 1}$ and $\mathcal{C}^{\geq 1} \subset \mathcal{C}^{\geq 0}$

(T_2) If $A \in \mathcal{C}^{\leq 0} and B \in \mathcal{C}^{\geq 1}$, then $Hom(A, B) = 0$.

(T_3) For any object X in X in $\mathcal{C}$, there is a distinguished triangle $A \to X \to B \to A[1]$ with $A \in \mathcal{C}^{\leq 0}$ and $B \in \mathcal{C}^{\geq 1}$.

(T_1) is obvious from the definition of $({}^p\mathcal{D}_c^{\leq 0}(X)), {}^p\mathcal{D}_c^{\geq 0}(X)$. (T_2) is proved in proposition 2.18. We only need to show (T_3). For $\mathcal{F}^\bullet \in \mathcal{D}_c^b(X)$, take a stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X such that $i_{X_\alpha}^{-1}\mathcal{F}^\bullet$ and $i_{X_\alpha}^!\mathcal{F}^\bullet$ have locally constant sheaves for any $\alpha \in A$. Let $X_k = \bigsqcup_{dim X_\alpha \leq k} X_\alpha \subset X$ for $k = -1, 0, 1, \cdots$. Now consider the claim which we denote it as C_k,

C_k: **There exists** $\mathcal{F}_0^\bullet \in {}^p\mathcal{D}_c^{\leq 0}(X - X_k)$**,** $\mathcal{F}_0^\bullet \in {}^p\mathcal{D}_c^{\geq 1}(X - X_k)$**, and a distinguished triangle** $\mathcal{F}_0^\bullet \to \mathcal{F}^\bullet|_{X-X_k} \to \mathcal{F}_1^\bullet \xrightarrow{+1}$ **in** $\mathcal{D}_c^b(X - X_k)$ **such that** $\mathcal{F}_0^\bullet|_{X_\alpha}$ **and** $\mathcal{F}_1^\bullet|_{X_\alpha}$ **have locally constant cohomology sheaves for any** $\alpha \in A$ **satisfying** $X_\alpha \subset X - X_k$.

Note that what we need is actually C_{-1}. Now we will prove it by descending induction on $k \in \mathbb{Z}$. It is trivial for $k \gg 0$, now assume that C_k holds, we need to prove C_{k-1}. Take a distinguished triangle $\mathcal{F}_0^\bullet \to \mathcal{F}^\bullet|_{X-X_k} \to \mathcal{F}_1^\bullet \xrightarrow{+1}$ in $\mathcal{D}_c^b(X - X_k)$ as described in C_k, let $j : X - X_k \to X - X_{k-1}$ be the open embedding and $i : X_k - X_{k-1} \to X - X_{k-1}$ be the closed embedding. As $j_!$ is left adjoint to $j^!$, then the morphism $\mathcal{F}_0^\bullet \to \mathcal{F}^\bullet|_{X-X_k} \simeq j^!(\mathcal{F}^\bullet|_{X-X_{k-1}})$ give rise to a morphism $j_!\mathcal{F}_0^\bullet \to \mathcal{F}^\bullet|_{X-X_{k-1}}$. Complete this into a distinguished triangle as following:

$$j_!\mathcal{F}_0^\bullet \to \mathcal{F}^\bullet|_{X-X_{k-1}} \to \mathcal{G}^\bullet \xrightarrow{+1}$$

We also embed the morphism ${}^p\tau^{\leq -k} i_! i^! \mathcal{G}^\bullet \to i_! i^! \mathcal{G}^\bullet \to \mathcal{G}^\bullet$ into a distinguished triangle

$${}^p\tau^{\leq -k} i_! i^! \mathcal{G}^\bullet \to \mathcal{G}^\bullet \to \widetilde{\mathcal{F}}_1^\bullet \xrightarrow{+1}$$

Then embed the morphism $\mathcal{F}^\bullet|_{X-X_{k-1}} \to \mathcal{G}^\bullet \to \widetilde{\mathcal{F}}_1^\bullet$ into a distinguished triangle

$$\widetilde{\mathcal{F}}_0^\bullet \to \mathcal{F}^\bullet|_{X-X_{k-1}} \to \widetilde{\mathcal{F}}_1^\bullet \xrightarrow{+1}$$

By construction, $\widetilde{\mathcal{F}}_0^\bullet|_{X_\alpha}$ and $\widetilde{\mathcal{F}}_1^\bullet|_{X_\alpha}$ have locally constant cohomology sheaves for any $\alpha \in A$ satisfying $X_\alpha \subset X - X_{k-1}$. It remains to show that $\widetilde{\mathcal{F}}_0^\bullet \in {}^p\mathcal{D}_c^{\leq 0}(X - X_{k-1})$ and $\widetilde{\mathcal{F}}_1^\bullet \in {}^p\mathcal{D}_c^{\geq 1}(X - X_{k-1})$. Applying the functor $j^!$ to

$$j_!\mathcal{F}_0^\bullet \to \mathcal{F}^\bullet|_{X-X_{k-1}} \to \mathcal{G}^\bullet \xrightarrow{+1}$$

$${}^p\tau^{\leq -k} i_! i^! \mathcal{G}^\bullet \to \mathcal{G}^\bullet \to \widetilde{\mathcal{F}}_1^\bullet \xrightarrow{+1}$$

we get an isomorphism $j^{-1}\widetilde{\mathcal{F}}^\bullet \simeq j^{-1}\mathcal{G}^\bullet$ and the distinguished triangle

$$\mathcal{F}_0^\bullet \to \mathcal{F}^\bullet|_{X-X_k} \to j^{-1}\widetilde{\mathcal{F}}_1^\bullet \xrightarrow{+1}$$

compare this distinguished triangle withe the following two

$$\widetilde{\mathcal{F}}_0^\bullet \to \mathcal{F}^\bullet|_{X-X_{k-1}} \to \widetilde{\mathcal{F}}_1^\bullet \xrightarrow{+1}$$

$$\mathcal{F}_0^\bullet \to \mathcal{F}^\bullet|_{X-X_k} \to \mathcal{F}_1^\bullet \xrightarrow{+1}$$

by the uniqueness, we get the isomorphisms $j^{-1}\widetilde{\mathcal{F}}_1^\bullet \simeq \mathcal{F}_1^\bullet$ and $j^{-1}\widetilde{\mathcal{F}}_0^\bullet \simeq \mathcal{F}_0^\bullet$. By proposition we need to prove

$$\text{(i): } H^l(i^{-1}\widetilde{\mathcal{F}}_0^\bullet) = 0 \text{ for any } l > -k$$

$$\text{(ii): } H^l(i^!\widetilde{\mathcal{F}}_1^\bullet) = 0 \text{ for any } l < -k+1$$

Now we apply the octahedral axiom to the following three distinguished triangles

$$j_!\mathcal{F}_0^\bullet \to \mathcal{F}^\bullet|_{X-X_{k-1}} \to \mathcal{G}^\bullet \xrightarrow{+1}$$

$$\widetilde{\mathcal{F}}_0^\bullet \to \mathcal{F}^\bullet|_{X-X_{k-1}} \to \widetilde{\mathcal{F}}_1^\bullet \xrightarrow{+1}$$

$${}^p\tau^{\leq -k} i_! i^! \mathcal{G}^\bullet \to \mathcal{G}^\bullet \to \widetilde{\mathcal{F}}_1^\bullet \xrightarrow{+1}$$

We obtain a distinguished triangle

$$j_!\mathcal{F}_0^\bullet \to \widetilde{\mathcal{F}}_0^\bullet \to {}^p\tau^{\leq -k} i_! i^! \mathcal{G}^\bullet \xrightarrow{+1}$$

So we have $i^{-1}\widetilde{\mathcal{F}}_0^\bullet \simeq i^{-1}{}^p\tau^{\leq -k} i_! i^! \mathcal{G}^\bullet \simeq i^{-1} i_! {}^p\tau^{\leq -k} i^! \mathcal{G}^\bullet \simeq {}^p\tau^{\leq -k} i^! \mathcal{G}^\bullet$. So we proved (i). Now apply the functor $i^!$ to

$${}^p\tau^{\leq -k} i_! i^! \mathcal{G}^\bullet \to \mathcal{G}^\bullet \to \widetilde{\mathcal{F}}_1^\bullet \xrightarrow{+1}$$

we get a distinguished triangle

$$i^! {}^p\tau^{\leq -k} i_! i^! \mathcal{G}^\bullet \to i^! \mathcal{G}^\bullet \to i^! \widetilde{\mathcal{F}}_1^\bullet \xrightarrow{+1}$$

Then we get the isomorphism $i^! {}^p\tau^{\leq -k} i_! i^! \mathcal{G}^\bullet \simeq i^! i_! {}^p\tau^{\leq -k} i^! \mathcal{G}^\bullet \simeq {}^p\tau^{\leq -k} i^! \mathcal{G}^\bullet$, and hence $i^! \widetilde{\mathcal{F}}_1^\bullet \simeq {}^p\tau^{\geq -k+1}(i^! \mathcal{G}^\bullet)$. So (ii) is proved.

□

Definition 2.20. The t-structure $({}^p\mathcal{D}_c^{\leq 0}(X)), {}^p\mathcal{D}_c^{\geq 0}(X)$ of the triangulated categroy $\mathcal{D}_c^b(X)$ is called the perverse t-structure. An object of its heart $Perv(X)$ is called a perverse sheaf on X. We denote by

$${}^p\tau^{\leq 0} : \mathcal{D}_c^b(X) \to {}^p\mathcal{D}_c^{\leq 0}(X), {}^p\tau^{\geq 0} : \mathcal{D}_c^b(X) \to {}^p\mathcal{D}_c^{\geq 0}(X)$$

the truncation functor with respect to the perverse t-structure. For $n \in \mathbb{Z}$ we define a functor

$${}^pH^n : \mathcal{D}_c^b(X) \to Perv(X)$$

by ${}^pH^n(\mathcal{F}^\bullet) = {}^p\tau^{\leq 0} {}^p\tau^{\geq 0}(\mathcal{F}^\bullet[n])$, For $\mathcal{F}^\bullet \in \mathcal{D}_c^b(X)$ its image ${}^pH^n(\mathcal{F}^\bullet)$ in $Perv(X)$ is called the n-th perverse cohomology of $\mathcal{F}^\bullet$.

Proposition 2.21. *Assume that X is a smooth algebraic variety or a complex manifold. Then for any local system L on X^{an} we have $L[d_X] \in Perv(X)$*

Proof. Assume that X is a complex manifold, By $\omega_X \simeq \underline{\mathbb{C}}_X[2d_X]$, we have
$\mathbb{D}(L[d_X]) = R\mathcal{H}om(L[d_X], \underline{\mathbb{C}}_X[2d_X]) = L^\vee[d_X]$, here $L^\vee$ denotes the dual local system $\mathcal{H}om(L, \underline{\mathbb{C}}_X)$. Hence the assertion is clear. And the same for the proof of the case when X is an algebraic variety.

□

The following proposition is obvious in the view of the definition of ${}^p\mathcal{D}_c^b(X)$ and ${}^p\mathcal{D}_c^b(X)$.

Proposition 2.22. *The Verdier duality functor $\mathbb{D}_X : \mathcal{D}_c^b(X) \to \mathcal{D}_c^b(X)^{op}$ induces an exact functor*

$$\mathbb{D}_X : Perv(\underline{\mathbb{C}}_X) \to Perv(\underline{\mathbb{C}}_X)^{op}$$

In particular, perverse sheaves are stable under Verdier duality.

3 Riemann-Hilbert Correspondence

Follow [1], we are going to show how perverse sheaves arise from Riemann-Hilbert correspondence. To do this, we need the notation of $\mathcal{D}$-modules.

Let X be a connected n-dimensional complex manifold and $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. We denote by $\mathcal{D}_X$ the sheaf of rings of finite-order holomorphic linear differential operators. This is the non-commutative subalgebra in the algebra $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ generated by $\mathcal{O}_X$ (acting via multiplication) and by the holomorphic vector fields on open sets in X (acting as derivatives). The sheaf of rings $\mathcal{D}_X$ is right and left Noetherian. After [9] the category $mod(\mathcal{D}_X)$ of all the $\mathcal{D}_X$-modules is an abelian category having enough injective objects. Denote by $\mathcal{D}_h^b(\mathcal{D}_X)$ the full triangulated subcategory in $\mathcal{D}^b(mod(\mathcal{D}_X))$ consisting of complexes with holomorphic cohomology sheaves.

Let $x \in X$ be any point and denote by $\hat{\mathcal{O}}_{X,x}$ the completion of the local ring $\mathcal{O}_{X,x}$ at x with respect to the m-adic topology, where m is the unique maximal ideal. Then $\hat{\mathcal{O}}_{X,x}$ is in a natural way a $\mathcal{D}_{X,x}$-module containing $\mathcal{O}_{X,x}$ as a submodule and hence the quotient $\hat{O}_{X,x}/\mathcal{O}_{X,x}$ has a natural structure of $\mathcal{D}_{X,x}$-module.

Definition 3.1. A complex $\mathcal{M}^\bullet \in \mathcal{D}^b_c(\mathcal{D}_X)$ of analytic holomorphic $\mathcal{D}_X$-modules is called regular if for every point $x \in X$ one has

$$R\mathcal{H}om_{\mathcal{D}_{X,x}}(\mathcal{M}^\bullet_x, \hat{\mathcal{O}}_{X,x}/\mathcal{O}_{X,x}) = 0.$$

Let $\mathcal{D}^b_{rh}(\mathcal{D}_X)$ denote the full triangulated subcategory of regular holomorphic complex in $\mathcal{D}^b_h(\mathcal{D}_X)$. This category $\mathcal{D}^b_{rh}(\mathcal{D}_X)$ is endowed with Grothendieck's six operations, exactly as the category $\mathcal{D}^b_c(X)$. The Riemmann-Hilbert correspondence says that these two categories are equivalent.

Definition 3.2. For a complex $\mathcal{M}^\bullet \in \mathcal{D}^b(\mathcal{D}_X)$, we define the de Rham complex of $\mathcal{M}^\bullet$ as following:

$$DR(\mathcal{M}^\bullet) = \Omega^\bullet_X \otimes \mathcal{M}^\bullet[n].$$

Theorem 3.3. *Let X be a connected n-dimensional complex manifold. Consider the triangulated category $\mathcal{D}^b_{rh}(\mathcal{D}_X)$ endowed with the natural t-structure and the triangulated category $\mathcal{D}^b_c(X)$ endowed with the middle perversity t-structure. Then the de Rham functor*

$$\mathcal{D}^b_{rh}(\mathcal{D}_X) \xrightarrow{DR} \mathcal{D}^b_c(X).$$

is t-exact and establishes an equivalence of categories which commutes with direct images, inverse images and duality. In particular

1) DR induces an equivalence of categories between the abelian category $RH(\mathcal{D}_X)$ of regular holonomic $\mathcal{D}_X$-modules on X and the abelian category of middle perversity perverse sheaves $Perv(X)$.

2) For any complex $\mathcal{M}^\bullet \in \mathcal{D}^b_{rh}(\mathcal{D}_X)$, one has an isomorphism

$$DR(\mathcal{H}^m(\mathcal{M}^\bullet)) = \mathcal{H}^m(DR(\mathcal{M}^\bullet)).$$

Proof. See [15],[16]. □

Remark. Recall that we say an abelian subcategory $\mathcal{A}'$ of $\mathcal{A}$ is a thick subcategory if for any exact sequence $X_1 \to X_2 \to X_3 \to X_4 \to X_5$ in $\mathcal{A}$ with $X_i \in \mathcal{A}', i = 1, 2, 4, 5$, then $X_3 \in \mathcal{A}'$. Let $\mathcal{A}$ be an abelian category and $\mathcal{A}'$ a thick abelian subcategory of $\mathcal{A}$, then the full subcategory $\mathcal{D}^b_{\mathcal{A}'}(\mathcal{A})$ of $\mathcal{D}^b(\mathcal{A})$ consisting of objects $\mathcal{F}^\bullet \in \mathcal{D}^b(\mathcal{A})$ satisfying $\mathcal{H}^j(\mathcal{F}^\bullet) \in \mathcal{A}'$ for any j is a triangulated category. The natural t-structure on $\mathcal{D} = \mathcal{D}^b_{\mathcal{A}'}(\mathcal{A})$ is given by

$$\mathcal{D}^{\leq 0} = \{\mathcal{F}^\bullet \in \mathcal{D} | \mathcal{H}^j(\mathcal{F}^\bullet) = 0, \forall j > 0\}; \mathcal{D}^{\geq 0} = \{\mathcal{F}^\bullet \in \mathcal{D} | \mathcal{H}^j(\mathcal{F}^\bullet) = 0, \forall j < 0\}$$

So the theorem shows that the standard t-structure on $\mathcal{D}^b_{rh}(\mathcal{D}_X)$ correspondence to the middle perverse t-structure on $\mathcal{D}^b_c(X)$.

4 Derived category of perverse sheaves

From [14] we see that there exists an isomorphism between the derived category $D^b(Perv(X))$ of the abelian category $Perv(X)$ (the middle perversity) and $\mathcal{D}^b_c(X)$. The result is stated as following:

Theorem 4.1. *For a complex manifold X, we have $\mathcal{D}_c^b(X)$ and $Perv(X)$ (with respect to the middle perversity) as defined earlier, then there exists a canonical t-exact functor $real_X : D^b(Perv(X)) \to \mathcal{D}_c^b(X)$ that induces the identity functor between hearts $Perv(X)$. Moreover $real_X$ is an equivalence of categories*

Proof. For details, see [14]. □

Chapter 3

Log complex and the perversity of the nearby cycles functor

1 Geometric Set-Up

Let X be a complex manifold, $S \subset \mathbb{C}$ be the unit disk and $f : X \to S$ a holomorphic map smooth over the punctured disk $S^* = S - \{0\}$.We say that f is one-parameter degeneration. In general $Y := X_0 = f^{-1}(0)$ can be arbitrarily bad singularities, but after suitable blowing up, Y can be assumed to have only strict normal crossing divisors on X. Let μ be the least common multiple of the multiplicities of the components of the divisor Y and consider the map $m : t \mapsto t^\mu$ sending S to itself. Denote S' the source of the map m and let W be the normalization of the fibre product $X \times_S S'$.Blowing up the singularities we obtain a manifold X' and a morphism $f' : X' \to S'$. We call f' the μ-th root fibration of f.The semistable reduction theorem [13] says the following:

Theorem 1.1. *Let $f : X \to S$ be as above.Then there exists $m \in \mathbb{N}$ such that for the m-th root $f' : X' \to S'$ of f, the special fibre $Y = f^{-1}(0)$has strictly normal crossings and such that all its components are reduced.*

We shall henceforth assume that $f : X \to S$ is smooth over S^* and that $Y = f^{-1}(0)$ is a strictly normal crossing divisor all of whose components are reduced.

With the above assumption, for any given point $x \in Y$, we can choose a system $(z_0, \cdots, z_n)$ of local coordinates on a neighborhood U of x in X centered at x, such that $f(z_0, \cdots, z_n) = z_0 \cdots z_k$. Define

$$V_{r,\eta} = \{z \in U | \|z\| < r, |f(z)| < \eta\}$$

for $0 < \eta \ll r \ll 1$. These form a fundamental system of neighborhood of x in X.

2 Residue Maps

Let $Y = Y_1 \bigcup \cdots \bigcup Y_k$ is a strict normal crossing divisor inside a complex manifold X. We introduce

$$Y_I = Y_{i_1} \bigcap \cdots \bigcap Y_{i_m}, I = \{i_1, \cdots, i_m\};$$

$$Y(I) := \textstyle\sum_{j \notin I} Y_I \cap Y_j;$$

$$a_I : Y_I \hookrightarrow X;$$

$$Y(0) = X;$$

$$Y(m) = \textstyle\coprod_{|I|=m} Y_I, m = 1, \cdots, k;$$

$$a_m = \textstyle\coprod_{|I|=m} a_I : Y(m) \to X.$$

The goal is to define residues along Y_I. So let $p \in Y_I$, then all the m components $Y_i, i \in I = \{i_1, \cdots, i_m\}$ pass through p. Now choose a local coordinate $(U, z_1, \cdots, z_n)$ centered at p in such a way that $Y_{i_j} = \{z_j = 0\}$ for $j = 1, \cdots, m$, and the remaining $k - m$ components of Y are given by $\{z_j = 0\}, j = m+1, \cdots, k$. Any local section ω can be written as

$$\omega = \tfrac{dz_1}{z_1} \wedge \cdots \wedge \tfrac{dz_m}{z_m} \wedge \eta + \eta'$$

where η has at most poles along components $Y_j, j \notin I$, and η' is not divisible by the form $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_m}{z_m}$. The restriction of η is independent of the chosen local coordinates so we get a well-defined map $\omega \mapsto \eta|_{Y_I}$. And the fact

$$d\omega = \tfrac{dz_1}{z_1} \wedge \cdots \wedge \tfrac{dz_m}{z_m} \wedge (-1)^m d\eta + d\eta'$$

implies that this map is compatible with derivatives.

Definition 2.1. The residue map

$$res_I : \Omega_X^\bullet(logY) \to \Omega_{Y_I}^\bullet(logY(I))[-m]$$

is locally defined by sending $\omega = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_m}{z_m} \wedge \eta + \eta'$ to $\eta|_{Y_I}$ as described above.The residue map restricts to filtration W is also denoted as res_I

$$res_I : W_m\Omega_X^\bullet(logY) \to \Omega_{Y_I}^\bullet[-m]$$

here,the filtration W is defined by

$$W_m\Omega_X^p(logY) = \begin{cases} 0 & m < 0; \\ \Omega_X^p(logY) & m \geq p; \\ \Omega_X^m(logY) \wedge \Omega_X^{p-m} & 0 \leq m \leq p \end{cases}$$

Lemma 2.2. *The residue map*

$$res_I : W_m\Omega_X^\bullet(logY) \to \Omega_{Y_I}^\bullet[-m]$$

is surjective and induces an isomorphism of complexes

$$res_m = \textstyle\bigoplus_{|I|=m} res_I : Gr_m^W \Omega_X^\bullet(logY) \xrightarrow{\approx} a_{m*}\Omega_{Y(m)}^\bullet[-m]$$

Proof. We will follow the proof in [10]. One can construct an inverse as follows.Fix $I = \{i_1, \cdots, i_m\}, 1 \leq i_1 < i_2 < \cdots < i_m \leq k$.One defines

$$\begin{array}{l} \rho_I : \Omega_X^p \to Gr_m^W \Omega_X^{p+m}(logY) \\ \rho_I(\beta) = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_m}{z_m} \wedge \beta \end{array}$$

This map is well-defined,since if $w_1, \cdots, w_n$ is another local coordinate with $Y = \{w_1 \cdots w_k = 0\}$,the quotients z_i/w_i are holomorphic (because $D_{i_j} = \{z_j = 0\} = \{w_j = 0\}$) and also the form $dz_i/z_i - dw_i/w_i = (w_i/z_i)d(z_i/w_i)$ are holomorphic, so $\rho_I(\beta)$ in the w-coordinates differs from the the expression in the z-coordinates by a form in $W_{m-1}\Omega_X^{p+m}(logY)$ and so is zero in the quotient.

Also the elements of the form $\beta = z_{i_j}\beta'$,β' a local section of Ω_X^p,and $dz_{i_j} \wedge \beta''$,β'' a local section of Ω_X^{p-1},map to zero.So the map ρ_I factor through $a_{I*}\Omega_{Y_I}^\bullet[-m]$ and induces a map of complexes denoted as $\widetilde{\rho}_I$,

$$\widetilde{\rho}_I : a_{I*}\Omega_{Y_I}^\bullet[-m] \to Gr_m^W \Omega_X^\bullet(logY)$$

So we get a commutative diagram

$$\begin{array}{ccc} \Omega_X^p & \xrightarrow{\rho_I} & \\ \downarrow{\scriptstyle \theta} & \searrow & \\ a_{I*}\Omega_{Y_I}^\bullet[-m] & \xrightarrow[\widetilde{\rho}_I]{} & Gr_m^W \Omega_X^\bullet(logY) \end{array}$$

Here θ is defined as following,for $\beta = f(z_1, \cdots, z_m, z_{m+1}, \cdots, z_n)dz_{v_1} \wedge \cdots \wedge dz_{v_l}$

$$\theta(\beta) = \begin{cases} 0 & \exists j \in \{1, \cdots, m\}, dz_j | dz_{v_1} \wedge \cdots \wedge dz_{v_l}; \\ f(0, \cdots, 0, z_{m+1}, \cdots, z_n)dz_{v_1} \wedge \cdots \wedge dz_{v_l} & oterwise \end{cases}$$

The sum of morphisms $\widetilde{\rho}_I$ for $|I| = m$ gives a morphism of complexes

$$\rho_m : a_{m*}\Omega_{Y(m)}^\bullet[-m] \to Gr_m^W \Omega_X^\bullet(logY)$$

i.e. for an open set U in X,we have

$$a_{m*}\Omega_{Y(m)}^\bullet[-m](U) = a_m(\coprod_{|I|=m} Y_I \cap U) = \coprod_{|I|=m} a_m(Y_I \cap U) \overset{\oplus_{|I|=m}\widetilde{\rho}_I(U)}{\longrightarrow} Gr_m^W \Omega_X^\bullet(logY)(U)$$

This is the desired inverse for the residue map. As it does not depends on the coordinate, this completes the proof. □

3 The Relative Logarithmic de Rham Complex

Definition 3.1. The relative de Rham complex on X with logarithmic poles along Y is defined as following

$$\Omega_{X/S}^\bullet(logY) := \Omega_X^\bullet(logY)/(f^*\Omega_S^1(log0) \wedge \Omega_X^{\bullet-1}(logY))$$

The cohomology sheaves are given by the following theorem.

Theorem 3.2. *Let $X = \mathbb{C}^{n+1}$ with coordinates $(z_0, \cdots, z_n)$,and let $f : X \to S$ be given by $t = f(z_0, \cdots, z_n) = z_0 \cdots z_k$ for some $k \in \mathbb{N}$ with $0 \le k \le n$. Let Y be the zero set of t.Put $\xi_i = dz_i/z_i$ for $i = 0, \cdots, k$.Then*
1)$\mathcal{H}^0(\Omega^\bullet_{X/S}(logY))_0 = \mathbb{C}\{t\}$;
2)$\mathcal{H}^1(\Omega^\bullet_{X/S}(logY))_0$ is the $\mathbb{C}\{t\}$-module with generators $\xi_0, \cdots, \xi_k$ and the relation $\sum_{i=0}^k \xi_i = 0$;
3)$\mathcal{H}^q(\Omega^\bullet_{X/S}(logY))_0 = \bigwedge^q_{\mathbb{C}\{t\}} \mathcal{H}^1(\Omega^\bullet_{X/S}(logY))_0$ for $q > 1$.

Proof. The complex $\Omega^\bullet_{X/S}(logY)_0$ can be considered as a double complex where the differential d is written as $d_1 + d_2$ where d_1 is the differential with respect to the first $k+1$ variables and d_2 the other variables.The relative Poincaré Lemma implies that the complex $(\Omega^\bullet_{X/S}(logY)_0, d_2)$ is acyclic. So the complex is quasi-isomorphism to $(Ker(d_2), d_1)$. So we may assume $n = k$.

From definition we have $0 = df/f = dt/t$, so $\xi_0 = -\Sigma_{j=1}^k \xi_j$, we see that for $i = 1, \cdots, n$ we have

$$d(f\xi_i) = \Sigma_{j=1}^k D_j(f)\xi_j \wedge \xi_i, D_j = z_j\partial/\partial z_j - z_0\partial/\partial z_0$$

So the complex is isomorphic to the Koszul complex on $R = \mathbb{C}\{z_0, \cdots, z_n\}$ with operators D_j

$$0 \longrightarrow R \xrightarrow{d_0} R \otimes V \longrightarrow \cdots \longrightarrow R \otimes \wedge^n V \xrightarrow{d_n} 0$$

where $V = \mathbb{C}\xi_1 \oplus \cdots \oplus \mathbb{C}\xi_n$ and

$$d_p(f\xi_{i_1} \wedge \cdots \wedge \xi_{i_p}) = \Sigma_{j=1}^k D_j(f)\xi_j \wedge \xi_{i_1} \wedge \cdots \wedge \xi_{i_p}$$

The cohomology of this complex can be computed monomial by monomial because the operators are homogenous.As $Ker(d_0) = \bigcap_{j=1}^k Ker(D_j)$.One only gets a non-zero contribution from those monomial on which the D_j are all zero because $D_j(z_0^{a_0} \cdots z_k^{a_k}) = (a_j - a_0)z_0^{a_0} \cdots z_k^{a_k}$. This is amount to saying that $z_0^{a_0} \cdots z_k^{a_k}$ is a power of t. For H^1, suppose $m = \Sigma_{i=1}^n m_i \otimes \xi_i \in ker(d_1)$, that is to say $D_i(m_i) = D_i(m_j), \forall i, j = 1, \cdots, n$.As $D_j(z_0^{a_0} \cdots z_k^{a_k}) = (a_j - a_0)z_0^{a_0} \cdots z_k^{a_k}$,so without loss of generality, we may assume $m_i = \lambda_i z_0^{a_0} \cdots z_n^{a_n}, m_j = \lambda_j z_0^{a_0} \cdots z_n^{a_n}$,i.e. $\lambda_j(a_i - a_0) = \lambda_i(a_j - a_0), \forall i, j = 1, \cdots, n$,So we get 2) and in a similar way 3). ☐

Corollary 3.3. *1)$\mathcal{H}^0(\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \simeq \underline{\mathbb{C}}_Y$*
2) $\mathcal{H}^1(\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)_0$ is the $\mathbb{C}$-vector space with generators $\xi_0, \cdots, \xi_k$ and the single relation $\sum_{i=0}^k \xi_i = 0$.
3)$\mathcal{H}^q(\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y) = \bigwedge^q_{\underline{\mathbb{C}}_Y} \mathcal{H}^1(\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$ for $q > 1$.

We are going to relate the complex $\Omega^\bullet_{X/S}(logY) \otimes \mathcal{O}_Y$ to $\psi_f(\underline{\mathbb{C}}_X)$.

Definition 3.4. (The Godement Resolution) Let $\mathcal{F}$ be a sheaf on the topological space X. We define a new complex of sheaves $\mathcal{C}^\bullet_{Gdm}$ on X as following. First we have the natural imbedding $\mathcal{F} \to \mathcal{C}^0\mathcal{F}$, defined as below:

$$\text{for } U \text{ open subset of } X,\ \Gamma(U, \mathcal{C}^0\mathcal{F}) = \prod_{x \in U} \mathcal{F}_x$$

here $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at point X, and the restriction map being the natural one. Let $\mathcal{F}^1$ denote the cokernel of $\mathcal{F} \to \mathcal{C}^0\mathcal{F}$ and put $\mathcal{C}^1 = \mathcal{C}^0\mathcal{F}^1$.Iterate this process to get a flasque resolution $\mathcal{F} \to \mathcal{C}^\bullet_{Gdm}\mathcal{F}$, where $\mathcal{C}^\bullet_{Gdm}\mathcal{F}$ is given by

$$0 \to \mathcal{C}^0\mathcal{F} \to \mathcal{C}^1\mathcal{F} \to \mathcal{C}^2\mathcal{F} \to \cdots \to \mathcal{C}^n\mathcal{F} \to \cdots$$

For a complex of sheaves $\mathcal{F}^\bullet$ on X which is bounded,For every $\mathcal{F}^i$,take its Godement resolution $\mathcal{C}^\bullet_{Gdm}\mathcal{F}^i$ so that we have a double complex $\mathcal{C}^\bullet_{Gdm}\mathcal{F}^\bullet$. Since the Godement sheaves are flasque, the associated simple complex $s(\mathcal{C}^\bullet_{Gdm}\mathcal{F}^\bullet)$ gives a flasque resolution of $\mathcal{F}^\bullet$. Its complex of global sections is called the derived complex

$$R\Gamma(X,\mathcal{F}^\bullet) := \Gamma(X, s(\mathcal{C}^\bullet_{Gdm}\mathcal{F}^\bullet)).\text{ This complex computes the hypercohomology.}$$

Then we have the hypercohomology $\mathbb{H}^i(X,\mathcal{F}^\bullet) = R^i\Gamma(\mathcal{F}^\bullet) = H^i(R\Gamma(\mathcal{F}^\bullet)) = H^i(\Gamma(X, s(\mathcal{C}^\bullet_{Gdm}\mathcal{F}^\bullet)))$.

Theorem 3.5. *(The Abstract De Rham Theorem) Let X be a topological space and let $\mathcal{F}$ be a sheaf with a Γ-acyclic resolution $\mathcal{F}^\bullet = \{\mathcal{F}^0 \to \mathcal{F}^1 \to \cdots\}$, i.e. $H^i(X,\mathcal{F}^j) = 0$ for $i \geq 1, j \geq 1$. We have canonical identifications*

$$H^i(X,\mathcal{F}) = \mathbb{H}^i(X,\mathcal{F}^\bullet) = H^i(\Gamma(X,\mathcal{F}^\bullet)).$$

Proof. We have $H^i(X,\mathcal{F}) = \mathbb{H}^i(X,\mathcal{F}^\bullet)$, As any resolution of $\mathcal{F}$ by Γ-acyclic sheaves computes hypercohomology, we get $H^i(X,\mathcal{F}) = \mathbb{H}^i(X,\mathcal{F}^\bullet) = H^i(\Gamma(X,\mathcal{F}^\bullet))$. □

Lemma 3.6. *If X is a Stein manifold,then $H^i(U,\Omega^j_X) = 0, \forall i > 0, j \geq 0$.*

Proof. This is Cartan's theorem B. □

Proposition 3.7. *The inclusion of complexes*

$$\Omega^\bullet_X(logY) \to j_*\Omega^\bullet_{X^*}$$

is a quasi-isomorphism and induces a natural identification

$$H^p(X^*;\mathbb{C}) = \mathbb{H}^p(X,\Omega^\bullet_X(logY))$$

In other words, cohomology of X^* can be calculated using the log-complex.

Proof. The assertion is a local calculation. We take for X is a polydisc Δ^n with coordinates $(z_1,\cdots,z_n)$ and that Y_k is given by $\{z_1\cdots z_k = 0\}$. Then X and X^* are Stein manifolds and hence $H^i(U,\Omega^j_{X^*}) = 0, \forall i > 0, j \geq 0$. From the abstract de Rham theorem it follows that the cohomology can be computed as the de Rham cohomology of the complex $\Omega^\bullet_{X^*}$.

$$H^p(X^*;\mathbb{C}) = H^p_{DR}(\Omega^\bullet_{X^*}) = H^p(\Gamma(X^*,\Omega^\bullet_{X^*})).$$

It suffices therefore to show that

$$H^p(K^\bullet_{n,k}) \cong H^p(X^*;\mathbb{C})$$

$$K^\bullet_{n,k} = \Gamma(\Delta^n, \Omega^\bullet_{\Delta^n}(logY_k)).$$

If we put

$$R^1_{n,k} = \mathbb{C}dz_1/z_1 \oplus \cdots \mathbb{C}dz_k/z_k, R^1_{n,0} = \mathbb{C}, R^0_{n,k} = \mathbb{C}$$

$$R^p_{n,k} = \bigwedge^p R^1_{n,k}$$

then we can form a complex $R^\bullet_{n,k}$, where the differentials are the zero maps. We have the natural inclusion

$$i_{n,k} : R^\bullet_{n,k} \to K^\bullet_{n,k}$$

Consider the following commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^\bullet_{n,k-1} & \longrightarrow & R^\bullet_{n,k} & \longrightarrow & R^\bullet_{n-1,k-1}[-1] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_{n,k-1}} & & \downarrow{\scriptstyle i_{n,k}} & & \downarrow{\scriptstyle i_{n-1,k-1}} & & \\
0 & \longrightarrow & K^\bullet_{n,k-1} & \longrightarrow & K^\bullet_{n,k} & \xrightarrow{res} & K^\bullet_{n-1,k-1}[-1] & \longrightarrow & 0
\end{array}$$

The first row is exact because we have an obvious isomorphism $R^\bullet_{n-1,k-1}[-1] \simeq R^\bullet_{n,k}/R^\bullet_{n,k-1}$, while the second row is exact comes from the definition of residue map.Now if $i_{n,k-1}, i_{n-1,k-1}$ are quasi-isomorphisms, then $i_{n,k}$ is a quasi-isomorphism by the five lemma. By the holomorphic Poincare lemma, $i_{n,0}$ is a quasi-isomorphism for all n, so by induction, $i_{n,k}$ is a quasi-isomorphism for all n, k. As X^* has the homotopy type of a product of k circles, the p-th cohomology of X^* is exactly $R^p_{n,k}$and the p-th cohomology of the complex $R^\bullet_{n,k}$ is also $R^p_{n,k}$. We are done. □

Lemma 3.8. *Let $M^\bullet$ and $L^\bullet$ be bounded complexes of $\mathbb{C}$-vector space with increasing filtrations F resp. G such that $M^\bullet = \bigcup_{n\geq 0} F_n M^\bullet$ and $L^\bullet = \bigcup_{n\geq 0} G_n L^\bullet$. If $\varphi : (M^\bullet, F) \to (L^\bullet, G)$ is a morphism of filtered complexes and $Gr_n(\varphi) : Gr^F_n(M^\bullet) \to Gr^G_n(L^\bullet)$ is a quasi-isomorphism for all n, then φ is a quasi-isomorphism.*

Proof. $Gr_n(\varphi)$ is a quasi-isomorphism for any $n \geq 0$, so we have isomorphisms $H^i(Gr_0(\varphi)) : H^i(F_0M^\bullet) \to H^i(G_0L^\bullet)$ and $H^i(Gr_1(\varphi)) : H^i(F_1M^\bullet/F_0M^\bullet) \to H^i(G_1L^\bullet/G_0L^\bullet)$ which implies the isomorphism $H^i(Gr_1(\varphi)) : H^i(F_1M^\bullet) \to H^i(G_1L^\bullet)$.By induction we have isomorphisms $H^i(Gr_j(\varphi)) : H^i(F_jM^\bullet) \to H^i(G_jL^\bullet)$ for all $i, j \geq 0$. So φ is a quasi-isomorphism. □

The stalk at x of $H^p(\Omega^\bullet_X(logY))$ has already been known from proposition 3.7: the stalk of H^0 at x is $\mathbb{C}$, the stalk of H^1 at x is the vector space spanned by $\xi_0, \cdots, \xi_k$, and the stalk of $H^p = \bigwedge^p H^1$. Now let H^1 be the subspace of $\Omega^1_X(logY)_x$ spanned by $\xi_0, \cdots, \xi_k$ and $H^p = \bigwedge^p H^1 \subseteq \Omega^p_X(logY)_x$.Let $H^p[logt]$ be the subspace of $\Omega^p_X(logY)[logt]_x$ consisting of elements of the form $\sum_{i=0}^s \omega_i(logt)^i$ with $\omega_i \in H^p$. Then $H^\bullet[logt]$ is a subcomplex of $\Omega^\bullet_X(logY)[logt]_x$.Then we have the following results:

Lemma 3.9. *The inclusion $H^\bullet[logt] \hookrightarrow \Omega^\bullet_X(logY)[logt]_x$ is a quasi-isomorphism.*

Proof. Denote F the filtration on $H^\bullet[logt]$ and $\Omega^\bullet_X(logY)[logt]_x$ by the degree of $logt$. Then the map

$$Gr^F_n H^\bullet[logt] \to Gr^F_n \Omega^\bullet_X(logY)[logt]_x$$

is just the inclusion $H^\bullet \to i^{-1}\Omega^\bullet_X(logY)$ which is a quasi-isomorphism as we have already shown. □

Lemma 3.10. *The injection $H^\bullet \to H^\bullet[logt]$ induces surjective maps*

$$H^p \to H^p[logt]$$

with kernels formed by the elements $\eta = dt/t \wedge \omega$ *for* $\omega \in H^{p-1}$.

Proof. Let $\omega = \sum_{i=0}^{s} \omega_i (logt)^i$ with $\omega_i \in H^p, i = 0, \cdots, s$.Then $d\omega = 0$ if and only if $dt/t \wedge \omega_i = 0, fori = 1, \cdots, s$.By a lemma of De Rham[2], this implies that $\omega_i = dt/t \wedge \eta_i$ for some $\eta_i \in H^{p-1}$. Put $\eta = \sum_{q=1}^{s}(q+1)^{-1}\eta_q(logt)^{q+1}$, then $\omega - \omega_0 = d\eta$. This shows that the injection induces a surjective map. Moreover, if $\omega_0 \in H^p$ is mapped to be zero if and only if there exists $\xi \in H^{p-1}$ with $\omega_0 = d(\xi logt) = dt/t \wedge \xi$. □

Theorem 3.11. *If* X *is a complex manifold and* $f : X \to S$ *is holomorphic such that* $Y = f^{-1}(0)$ *is a reduced divisor with normal crossings on* X*, then we have*

$$\psi_f(\mathbb{C}_X) \simeq \Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$$

in the derived category D^+*(sheaves of* $\mathbb{C}$*-vector spaces on* Y*).*

Proof. Recall $\psi_f(\mathbb{C}_X) \simeq i^{-1}k_*\Omega^\bullet_{\widetilde{X}^*}$. Then we have two morphisms α, β. Where α is the inclusion.

$$\alpha : i^{-1}\Omega^\bullet_X(logY)[logt] \hookrightarrow i^{-1}k_*\Omega^\bullet_{\widetilde{X}^*}$$

Here local sections of $i^{-1}\Omega^\bullet_X(logY)[logt]$ is of the form $\sum_{i=0}^{s}\omega_s(logt)^i$ with $\omega_i \in i^{-1}\Omega^\bullet_X(logY)$ and $d(logt) = dt/t$.

$$\beta : i^{-1}\Omega^\bullet_X(logY)[logt] \to \Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$$

given by $\sum_{i=0}^{s}\omega_s(logt)^i \to \omega_0$.

We need to show the map α, β are quasi-isomorphisms.This can be checked locally on Y.

From proposition 3.7,lemma 3.9 and lemma 3.10, we know β is a quasi-isomorphism and $i^{-1}\Omega^\bullet_X(logY)[logt]$ has stalk as described in corollary 3.3.

For the morphism α, fix a point $x \in Y$, we have

$$H^q(i^{-1}k_*\Omega^\bullet_{\widetilde{X}^*})_x = lim_{r,\eta}H^q(\Gamma(k^{-1}(V_{r,\eta}), \Omega^\bullet_{\widetilde{X}^*}))$$

Denote $B_r = \{z \in U | \|z\| < r\}$,$S_\eta = \{t \in S | |t| < \eta\}$.Then $V_{r,\eta} = B_r \cap f^{-1}(S_\eta)$. and we have

$$k^{-1}(V_{r,\eta}) = \{(z,u) \in B_r \times \widetilde{S}^*_\eta | \Pi_{i=0}^{k} z_i = exp(2\pi i u)\}$$

Denote $V = \{(z,u) \in \mathbb{C}^{n+1} \times \mathbb{C} | \Pi_{i=0}^{k} z_i = exp(2\pi i u)\}$, the natural inclusion i

$$i : k^{-1}(V_{r,\eta}) \hookrightarrow V$$

can be seen to be a homotopy equivalence. Actually, we can define a map

$$\varphi : B_r \to \mathbb{C}^{n+1}$$

$$(z_0, \cdots, z_n) \mapsto (z_0/(r - |z_0|), \cdots, z_n/(r - |z_n|))$$

This is a one-to-one bijection. So under the map φ,$k^{-1}(V_{r,\eta})$ and V can be seen topologically the same.

Now the restriction map

$$H^q((\mathbb{C}^*)^{k+1} \times \mathbb{C}^{n-k} \times \mathbb{C}; \mathbb{C}) \to H^q(k^{-1}(V_{r,\eta}); \mathbb{C})$$

is surjective. The former is the q-th exterior power of the $\mathbb{C}$-vector space with basis $\xi_0, \cdots, \xi_k$ and the latter is obtained by dividing out the relation $dt/t = 0$. This computes the stalks of the cohomology sheaf of $i^{-1}k_*\Omega^\bullet_{\widetilde{X}^*}$ at x .And it is the same as $i^{-1}\Omega^\bullet_X(logY)[logt]$. This prove the theorem.

□

4 The Filtrations

This section arises from an observation made by Illusie in his article [5].

We have seen the isomorphism $\psi_f(\mathbb{C}_X) \simeq \Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$, so we can have an interpretation of the nearby cycle using the relative De Rham complex. Now we are going to show $\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is isomorphism to another complex which comes from the absolute De Rham complex. With the help of the filtration W defined in chapter 3,2 residue maps, we will show that we can define two filtrations on $\psi_f(\mathbb{C}_X)$ and $\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. In this way we give a new interpretation of the nearby cycle $\psi_f(\mathbb{C}_X)$.

The desired complex comes from the following double complex:

Definition 4.1. Define a bi-filtered double complex of sheaves

$$(A^{\bullet,\bullet}, d_1, d_2, W(M), W)$$

on Y by

$$A^{p,q} = \Omega_X^{p+q+1}(logY)/W_p\Omega_X^{p+q+1}(logY), \text{for } p, q \geq 0$$

with differentials

$$d_1 : A^{p,q} \to A^{p+1,q}, d_2 : A^{p,q} \to A^{p,q+1}$$

defined by

$$d_1(\omega) = (dt/t) \wedge \omega, d_2(\omega) = d\omega$$

and the two filtrations, the weight filtration and the monodromy weight filtration

$$W_m A^{p,q} = \text{image of } W_{m+p+1}\Omega_X^{p+q+1}(logY) \text{ in } A^{p,q}$$

$$W(M)_m A^{p,q} = \text{image of } W_{m+2p+1}\Omega_X^{p+q+1}(logY) \text{ in } A^{p,q}$$

Definition 4.2. We have maps

$$\mu : \Omega^q_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to A^{0,q}$$

$$\omega \mapsto (-1)^q(dt/t) \wedge \omega \ (\text{mod } W_0)$$

defining a morphism of complexes

$$\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to s(A^{\bullet,\bullet})$$

Here $s(A^{\bullet,\bullet})$ is the associated single complex.

Definition 4.3. Define the filtration W on $\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ by

$$W_m\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y = \text{Image of } W_m\Omega^\bullet_X(logY) \text{ in } \Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$$

This is a filtration by subcomplexes. By definition the sheaf $\Omega^\bullet_{X/S}(logY)\otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is the cokernel of the map

$$\theta : \Omega^{\bullet-1}_X(logY) \to \Omega^\bullet_X(logY); \theta(\omega) = dt/t \wedge \omega.$$

Note that dt/t induces a global section of $\Omega^1_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ and hence maps $W_m\Omega^p_X(logY)\otimes_{\mathcal{O}_X}\mathcal{O}_Y$ to $W_{m+1}\Omega^{p+1}_X(logY)\otimes_{\mathcal{O}_X}\mathcal{O}_Y$, $W_{m-1}\Omega^p_X(logY)\otimes_{\mathcal{O}_X} \mathcal{O}_Y$ to $W_m\Omega^{p+1}_X(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. So it induces maps

$$\theta : Gr^W_m \Omega^{p+m}_X(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to Gr^W_{m+1}\Omega^{p+m+1}_X(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$$

So we have the sequence

$$0 \to Gr^W_0 \Omega^p_X(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to Gr^W_1 \Omega^{p+1}_X(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to Gr^W_2 \Omega^{p+2}_X(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \cdots$$

Applying the residue maps to the sequence above we get the sequence

$$0 \to \Omega^p_Y \to a_{1*}\Omega^p_{Y(1)} \to a_{2*}\Omega^p_{Y(2)} \to \cdots$$

In which the maps are nothing but the alternating sum of the restriction maps. Hence the sequences are exact. We find $Gr^W_m(\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$ as the cokernel of the map

$$\theta : Gr^W_{m-1}(\Omega^{\bullet-1}_X(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \to Gr^W_m(\Omega^\bullet_X(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$$

The exactness just proved implies the exactness of the sequence

$$0 \to Gr^W_m(\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \to Gr^W_{m+1}(\Omega^{\bullet+1}_X(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \to Gr^W_{m+2}(\Omega^{\bullet+2}_X(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$$

Applying the residue map:

$$0 \to Gr^W_m(\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \to (a_{m+1})_*\Omega^p_{Y(m+1)} \to (a_{m+2})_*\Omega^p_{Y(m+2)}$$

is exact.

Lemma 4.4. *The sequence of coherent sheaves on Y*

$$0 \to \Omega^q_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \xrightarrow{dt/t} A^{0,q} \xrightarrow{dt/t} A^{1,q} \to \cdots$$

is exact.

Proof. We need to check that dt/t is compatible with the filtration W on $\Omega^\bullet_X(logY)$, which means the exactness of the sequence

$$Gr^W_m(\Omega^{q-1}_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \to Gr^W_{m+1}(\Omega^q_X(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \to Gr^W_{m+2}(\Omega^{q+1}_X(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$$

which we have already shown above. □

Theorem 4.5. *The map $\mu : \Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to s(A^{\bullet,\bullet})$ defined by $\mu(\omega) = (-1)^q(dt/t) \wedge \omega$ for ω section of $\Omega^q_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is a quasi-isomorphism.*

Proof. Follow from the lemma above, we need only to check that μ is a homomorphism of complex: $\mu(d\omega) = (-1)^{q+1}(dt/t) \wedge d\omega = (-1)^q d(dt/t \wedge \omega) = d\mu(\omega)$ for ω a section of $\Omega^q_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. □

We can check that d_1, d_2 are compatible with the filtrations W and F if we equip $s(A^{\bullet,\bullet})$ with the filtration F given by

$$F^m s(A^{\bullet,\bullet}) = \bigoplus_p \bigoplus_{q \geq m} A^{p,q}$$

So $\mu : (\Omega^\bullet_{X/S}(logY) \otimes_{\mathcal{O}_X} \mathcal{O}_Y, W, F) \to (s(A^{\bullet,\bullet}), W, F)$ is a bi-filtered quasi-isomorphism,

So we have:

$$s(A^{\bullet,\bullet}) \simeq \psi_f \underline{\mathbb{C}}_X.$$

Moreover, because $d_1 W(M)_m \subset W(M)_{m-1}$ we find that

$$\begin{gathered} Gr_m^{W(M)} s(A^{\bullet,\bullet}) \simeq \bigoplus_{q \geq 0, -m} Gr^W_{m+2q+1} \Omega^\bullet_X(logY)[1] \simeq \\ \bigoplus_{q \geq 0, -m} (a_{m+2q+1})_* \Omega^\bullet_{Y(m+2q+1)}[-m-2q] = \\ \bigoplus_{q \geq 0, -m} (a_{m+2q+1})_* \underline{\mathbb{C}}_{Y(m+2q+1)}(-m-q)[-m-2q] = \\ \bigoplus_{q \geq 0, -m} R^{m+2q+1} j_* \underline{\mathbb{C}}[-m-2q] \end{gathered}$$

Here $R^l j_* \underline{\mathbb{C}} = \bigoplus \underline{\mathbb{C}}_{Y_{i_1} \bigcap \cdots \bigcap Y_{i_l}}$.

In other words,we have

$$Gr_m^{W(M)} \psi_f(\mathbb{C}) \simeq \bigoplus_{q \geq 0, -m} (a_{m+2q+1})_* \underline{\mathbb{C}}_{Y(m+2q+1)}(-m-q)[-m-2q]$$

So the term $Gr_\bullet^{W(M)} \psi_f(\mathbb{C})$ is the simple complex associated to the following double complex

$$\begin{array}{ccccccccc}
(a_{n+1})_*\underline{\mathbb{C}} & & & & & & & & \\
\uparrow d_2 & & & & & & & & \\
(a_n)_*\underline{\mathbb{C}} & \xrightarrow{d_1} & (a_{n+1})_*\underline{\mathbb{C}}(-1) & & & & & & \\
\uparrow d_2 & & \uparrow d_2 & & & & & & \\
\cdots & & \cdots & & & & & & \\
\uparrow d_2 & & \uparrow d_2 & & & & & & \\
(a_2)_*\underline{\mathbb{C}} & \xrightarrow{d_1} & (a_3)_*\underline{\mathbb{C}}(-1) & \xrightarrow{d_1} & \cdots & \xrightarrow{d_1} & (a_{n+1})_*\underline{\mathbb{C}}(-n+1) & & \\
\uparrow d_2 & & \uparrow d_2 & & & & \uparrow d_2 & & \\
(a_1)_*\underline{\mathbb{C}} & \xrightarrow{d_1} & (a_2)_*\underline{\mathbb{C}}(-1) & \xrightarrow{d_1} & \cdots & \xrightarrow{d_1} & (a_n)_*\underline{\mathbb{C}}(-n+1) & \xrightarrow{d_1} & (a_{n+1})_*\underline{\mathbb{C}}(-n)
\end{array}$$

Here the differentials d_1, d_2 are all nulls.

We can show that under certain shifting, the nearby cycle is perverse.

Theorem 4.6. *The shifted nearby cycle $\psi_f \underline{\mathbb{C}}_X[n]$ is perverse. Here note that we have $dimX = n+1$ as denoted as before.*

Proof. First we show that $Gr_m^{W(M)} \psi_f(\mathbb{C})[n]$ is perverse. With the fact that $Gr_m^{W(M)} \psi_f(\mathbb{C})[n] \simeq \bigoplus_{q \geq 0, -m} (a_{m+2q+1})_* \underline{\mathbb{C}}_{Y(m+2q+1)}[n-m-2q]$, we need only to show that very term $(a_{m+2q+1})_* \underline{\mathbb{C}}_{Y(m+2q+1)}[n-m-2q]$ satisfy the perverse condition, note that $codimY_i = 1$,so $dimY(m+2q+1) = n+1-(m+2q+1) = n-m-2q$. So we need actually to show that $\underline{\mathbb{C}}_Y(m+2q+1)[dimY(m+2q+1)]$ is perverse, which is already shown in proposition 2.21. Now the perversity of $\psi_f \underline{\mathbb{C}}_X[n]$ follows from the following lemma $\square$

Lemma 4.7. *For a filtration of* $A : 0 \subset A_0 \subset A_1 \subset A_2 \cdots \subset A$ *with* $A = \bigcup_i A_i$*, if* $Gr_i A_i$ *are perverse, for any* i *then* A *is perverse.*

Proof. We have the short exact sequence $0 \to A_0 \to A_1 \to A_1/A_0 \to 0$. By the same argument as in proposition 2.7, we show that for a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$, if $A_0, A_1/A_0 \in \mathcal{C}^{\leq 0}$ (resp.$A_0, A_1/A_0 \in \mathcal{C}^{\geq 0}$),then $A_1 \in \mathcal{C}^{\leq 0}$ (resp.$A_1 \in \mathcal{C}^{\geq 0}$). Then it follows theorem 2.19 that the pair $({}^p\mathcal{D}_c^{\leq 0}(X), {}^p\mathcal{D}_c^{\geq 0}(X)$ defines a t-structure on $\mathcal{D}_c^b(X)$ and $Perv(X)$ is defined to be its intersection. So we get A_1 is perverse. Applying the same argument to short exact sequence $0 \to A_1 \to A_2 \to A_2/A_1 \to 0$, we show A_2 is perverse, so for all $i \in \mathbb{N}$, A_i is perverse. As A is the union of all A_i and perversity is a local property, we show that A is perverse. □

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