O-minimal Structures and Problems on Zilber-Pink Conjecture

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July 10, 2015

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1 Introduction

The research about unlikely intersection is an important subject in Diophantine geometry. Roughly speaking, unlikely intersection means varieties do not intersect due to natural dimensional reasons. For example, let Z be an algebraic variety of dimension n, X, Y be two subvarieties of dimension r, s respectively. In general, we expect that $X \cap Y$ has dimension at most $r+s-n$. If the contrary happens, i.e $X \cap Y$ has dimension strictly lager than $r+s-n$, then we say that X, Y have unlikely intersection.

This leads us to the concept of special subvarieties. Consider a variation of the above example, let X be a fixed subvariety of an algebraic variety Z, $\mathcal{Y}$ be a set of algebraic subvarieties of Z with certain conditions. Suppose $\dim X + \dim Y < \dim Z$ for all $Y \in \mathcal{Y}$. Then by the above argument, we expect that $X \cap Y = \emptyset$ for all $Y \in \mathcal{Y}$ except for a small subset. If the opposite happens, we can imagine that there must be some special requirement for X, we call this kind of X special subvariety of Z.

The formulation of special subvarieties depends on the ambient variety we are discussing. For the case of abelian variety and the case of torus $\mathbb{G}_m^n$, special subvarieties are exactly the torsion cosets, i.e the translate of an algebraic subgroup by a torsion point. We also define the weakly special subvarieties be the those coset of the ambient variety.

Theorem 1.1. (Multiplicative Manin-Mumford) *Let $V \subset X = \mathbb{G}_m^n$ be a subvariety. Then the following equivalent assertion holds:*
(a) *If the set of special points of X is Zariski dense in V, then V is a special subvariety.*
(b) *A component of the Zariski closure of a set of special points is special.*
(c) *V contains only finitely many maximal special subvarieties.*

This theorem has an interesting consequence

Corollary 1.2. *For a curve $V \subset \mathbb{C}^{\times 2}$ defined by a polynomial $F(X,Y) \in \mathbb{C}[X,Y]$, if there are infinitely many points $(\zeta, \eta) \in V$ such that both ζ, η are root of unities, then F is of the form $X^n Y^m = \zeta$ or $X^n = \zeta Y^m$ for some root of unity ζ.*

To see this, notice that root of unities are precisely the torsion points of the torus, and both $X^n Y^n = \zeta$ and $X^n = \zeta Y^m$ define a translate of a subtorus by a torsion point.

Now, we replace X above by an abelian variety, then we get the Manin-Mumford conjecture, which has been proved by Raynaud.

Theorem 1.3. (Raynaud) *Let X be an abelian variety, and $V \subset X$ be a subvariety, then (a),(b) and (c) in Theorem 1.1 holds.*

In the context of Shimura variety, we can also define the special and weakly special subvarieties, there is an analogue of Manin-Mumford conjecture in this case.

Conjecture 1.4. (André-Oort) *Let X be an Shimura variety, $V \subset X$ be a subvariety, then (a),(b) and (c) in theorem 1.1 holds.*

To get some feeling about this conjecture, take $X = \mathbb{C}^2$ for example, we have the following statement.

Theorem 1.5. *Let $\Sigma \subset \mathbb{C}^2$ be a set of special points, take Z be a component of the Zariski closure Σ^{zar} of Σ, then Z is one of the following.*
1. a poing (x_1, x_2) such that both coordinates are special points.
2. $\{x_1\} \times \mathbb{C}$ with x_1 special.
3. $\mathbb{C} \times \{x_2\}$ with x_2 special.
4. the image of the Hecke correspondence

$$j : \mathbb{H} \to \mathbb{C}^2, \quad \tau \mapsto (j(\tau), j(n\tau))$$

for some $n \in \mathbb{N}_{\geq 0}$
5. $\mathbb{C}^2$ itself

Where $z \in \mathbb{C}$ is called special if it is of the form $j(\tau)$ for some $\tau \in \mathbb{H}$ quadratic. We will define special subvarieties of Shimura varieties in general later.

Several mathematicians including Ben Moonen, Yves André, Andrei Yafaev, Bas Edixhoven, Laurent Clozel, and Emmanuel Ullmo, proved some partial results for André-Oort conjecture. Most of them were conditional upon the generalized Riemann hypothesis. Recently, Jonathan Pila and his collaborators used the theory of o-minimal structures in model theory, proved some unconditional results of André-Oort conjecture. See [Pil11, Tsi13]

Theorem 1.6. (Pila) *André-Oort conjecture is true for $\mathbb{C}^n$.*

Theorem 1.7. (Pila-Tsimerman) *André-Oort conjecture is true for moduli space of Abelian surfaces.*

The latest progress was made by Shouwu Zhang, his work about Colmez conjecture, combined with a new result of Tsimerman [Tsi]

Theorem 1.8. (Tsimerman) *Colmez conjecture implies the André-Oort conjecture for the moduli of Abelian varieties.*

We obtain that André-Oort conjecture is true for moduli space of Abelian varieties.

In order to get a more common statement, we consider Mix-Shimura varieties, which is a common generalization of Abelian varieties and Shimura varieties. Mixed-Shimura variety is also equipped with a collection of special subvarieties and weakly special subvarieties. While we already got some progress to the André-Oort conjecture, there are still a lot of things worth to explore in Zilber-Pink conjecture. Recently, by using the tools of o-minimal structures, Pila and Habegger obtained some partial results of unlikely intersections in the spirit of Zilber-Pink, see [PH]. In this paper, we are going to introduce their work.

2 Pila-Wilkie counting strategy

2.1 O-minimal structure

One thing to keep in mind is that the idea of definability is central in the Pila and his collaborators' results. In order to understand o-minimal structure, we first review the definition of language and structure. For model theory, see [Mar02]

Definition 2.1. *A language $\mathcal{L}$ is given by specifying the following data*
1. a set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$;
2. a set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$;
3. a set of constant symbols $\mathcal{C}$

The numbers n_f and n_R tell us that f is a function of n_f variables and R is a n_R-ary relation. Next, we describe the structures for a certain language $\mathcal{L}$.

Definition 2.2. *An $\mathcal{L}$ structure $\mathcal{M}$ is given by the following data:*
1. a nonempty set M called the universe, domain or underlying set of $\mathcal{M}$;
2. a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$;
3. a set $R^{\mathcal{M}} \subset M^{n_R}$ for each $R \in \mathcal{R}$;
4. an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$

Once we have an $\mathcal{L}$-structure, we can define sets within that structure by evaluating equalities and the distinguished relations. In other words, we have the notion of definable sets. To be specific:

Definition 2.3. *The set of $\mathcal{L}$-terms is the smallest set $\mathcal{T}$ such that*
1. $c \in \mathcal{T}$ for each constant symbol $c \in \mathcal{C}$;
2. each variable symbol $v_i \in \mathcal{T}$ for $i = 1, ...$ and;
3. for every $f \in \mathcal{F}$, if $t_1, ..., t_{n_f} \in \mathcal{T}$, then $f(t_1, ..., t_{n_f}) \in \mathcal{T}$

With this notion, we can define $\mathcal{L}$-formulas.

Definition 2.4. *We say that ϕ is an atomic $\mathcal{L}$-formula if ϕ is either*
1. $t_1 = t_2$ where t_1, t_2 are $\mathcal{L}$-terms;
2. $\mathcal{R}(t_1, ..., t_R)$ where $R \in \mathcal{R}$ and $t_1, ..., t_R$ are $\mathcal{L}$-terms.
The set of $\mathcal{L}$ formulas is the smallest set $\mathcal{W}$ containing the atomic formulas such that
1. If ϕ is in $\mathcal{W}$, then $\neg\phi \in \mathcal{W}$;
2. If ϕ and ψ are in $\mathcal{W}$, then $\phi \vee \psi$ and $\phi \wedge \psi$ are in $\mathcal{W}$;
3. If ϕ is in $\mathcal{W}$, then $\exists v_i \phi$ and $\forall v_i \phi$ are in $\mathcal{W}$.

We say that a variable v occurs freely in a formula ϕ if it is not inside a $\exists v$ or $\forall v$ quantifier. We call a formula sentence if it has no free variables. In this case, we define what it means for $\phi(v_1, ..., v_n)$ to hold of $(a_1, ..., a_n) \in M^n$

Definition 2.5. *Let ϕ be a formula with free variables from $\bar{v} = (v_{i_1}, ..., v_{i_m})$ and let $\bar{a} = (a_{i_1}, ..., a_{i_m}) \in M^m$, we inductively define $\mathcal{M} \vDash \phi(\bar{a})$ as follows*
1. If ψ is $t_1 = t_2$, then $\mathcal{M} \vDash \phi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$;
2. If ϕ is $R(t_1, ..., t_R)$, then $\mathcal{M} \vDash \phi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), ..., t_{n_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$;
3. If ϕ is $\neg\psi$, then $\mathcal{M} \vDash \phi(\bar{a})$ if $\mathcal{M} \nvDash \psi(\bar{a})$;
4. If ϕ is $\psi \wedge \theta$, then $\mathcal{M} \vDash \phi(\bar{a})$ if $\mathcal{M} \vDash \psi(\bar{a})$ and $\mathcal{M} \vDash \theta(\bar{a})$;
5. If ϕ is $\psi \vee \theta$, then $\mathcal{M} \vDash \phi(\bar{a})$ if $\mathcal{M} \vDash \psi(\bar{a})$ or $\mathcal{M} \vDash \theta(\bar{a})$;
6. If ψ is $\exists v_j, \psi(\bar{v}, v_j)$, then $\mathcal{M} \vDash \phi(\bar{a})$ if there is $b \in M$ such that $\mathcal{M} \vDash \psi(\bar{a}, b)$; 6. If ψ is $\forall v_j, \psi(\bar{v}, v_j)$, then $\mathcal{M} \vDash \phi(\bar{a})$ if $\mathcal{M} \vDash \psi(\bar{a}, b)$ for all $b \in M$

If $\mathcal{M} \vDash \phi(\bar{a})$ we say that $\mathcal{M}$ satisfies $\phi(\bar{a})$ or $\phi(\bar{a})$ is true in $\mathcal{M}$. Now we can say what is a definable set

Definition 2.6. *Let $\mathcal{M} = \{M, ...\}$ be an $\mathcal{L}$-structure. We say that $X \subset M^n$ is definable if and only if there is an $\mathcal{L}$-formula $\phi(v_1, ..., v_n, w_1, ..., w_m)$ and $\bar{b} \in M^n$ such that $X = \{\bar{a} \in M^n : \mathcal{M} \vDash \phi(\bar{a}, \bar{b})\}$. We say that $\phi(\bar{v}, \bar{b})$ defines X. We say that X is A-definable or defined over A if there is a formula $\psi(\bar{v}, w_1, ..., w_l)$ and $\bar{b} \in A^l$ such that $\psi(\bar{v}, \bar{b})$ defines X.*

We can see although these definitions are formal, they really coincide with our intuition. Now we are able to define the o-minimal structure. To know detailed discussion about o-minimal structures, see [vdD98].

Definition 2.7. *Let $\mathcal{L}$ be some language including a binary relation symbol $<$ and possibly some other data. We say that an $\mathcal{L}$-structure $\mathcal{M}$ is o-minimal if $<^{\mathcal{M}}$ is a total order of M and every definable subset of M is a finite unions of singletons and intervals.*

The firsthand example of o-minimal structure is

Example 2.8. (Tarski) *The ordered field of real numbers $\bar{\mathbb{R}} := (\mathbb{R}, <, +, \times, -)$ is o-minimal.*

There is also an important case which we will use later

Example 2.9. (van den Dries via Garbrielov) *The structure*

$$\mathbb{R}_{an} := (\mathbb{R}, <, +, \times, \{f\}_{f:[0,1]^n \to \mathbb{R} \quad \text{restricted analytic}})$$

is o-minimal.

Then, it is also worthy to mention that not every structure is o-minimal:

Example 2.10. *The structure*

$$\mathcal{M} = \{\mathbb{R}, +, <, \sin\}$$

is not o-minimal.

To see this, notice that in this case, $\pi i \mathbb{Z} = \sin^{-1}(0)$ is definable set, but it is not a finite unions of singletons and intervals.

For o-minimal structures, the most important properties are finiteness theorem and cell-decomposition theorem. For finiteness result, we consider the definable family.

Theorem 2.11. *Let $\mathcal{M} = \{M, ...\}$ be an o-minimal structure. Let $A \subset M^m \times M^n$ be definable and suppose that for each $x \in M^m$ the fiber*

$$A_x = \{y \in M^n; (x, y) \in A\}$$

is finite. Then there is $N \in \mathbb{N}$ such that $|A_x| \leq N$ for all $x \in M^m$.

The cell decomposition theorem says that we can splits a definable set into finitely many definable subsets of simple forms. This simple form is called a cell.

Definition 2.12. *Fix an o-minimal structure $\mathcal{M}$. Let $(i_1, ..., i_m)$ be a sequence of zeros and ones of length m. An $(i_1, ..., i_m)$-cell is a definable subset of M^n obtained by induction on m as follows:*
1. a (0)-cell is a one-element set $\{r\} \subset M$, a (1)-cell is an interval $(a, b) \subset M$;
2. suppose $(i_1, ..., i_m)$-cells are already defined; then an $(i_1, ..., i_m, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in \mathcal{C}(X)$, where $\mathcal{C}(X)$ is the set of all continues definable functions from X to M and X is an $(i_1, ..., i_m)$-cell; further and $(i_1, ..., i_m, 1)$-cell is a set

$$(f, g)_X := \{(x, r) \in X \times M; f(x) < r < g(x)\}$$

where $f, g \in \mathcal{C}(X)_\infty := \mathcal{C}(X) \cup \{-\infty, +\infty\}$ and $f(x) < g(x)$ for all $x \in X$

For a decomposition, we mean the following thing.

Definition 2.13. *A decomposition of M^n is a special kind of partition of R^n into finitely many cells. The definition is by induction on m:*
1. a decomposition of $M^1 = M$ is a collection

$$\{(-\infty, a_1), (a_1, a_2), ..., (a_k, +\infty), \{a_1\}, ..., \{a_k\}\}$$

where $a_1 < ... < a_k$ are points in R;
2. a decomposition of M^{n+1} is a finite partition of M^{n+1} into cells $\mathcal{A}$ such that the set of projections $\pi(\mathcal{A})$ is a decomposition of M^n.

Further, we say that a decomposition $\mathcal{D}$ of M^n is to partition a set $S \in M^n$ if each cell in $\mathcal{D}$ is either part of S or disjoint from S, in other words, S is a union of cells in $\mathcal{D}$. Now we state the cell decomposition theorem.

Theorem 2.14. (Cell decomposition) *We have*
1. Given any definable sets $A_1, ..., A_k \subset M^n$, there is a decomposition of M^n partitioning each of A_i;
2. For each definable function $f : A \to M, A \subset M^n$, there is a decomposition $\mathcal{D}$ of M^n partitioning A such that the restriction $f|_B : B \to M$ o each cell $B \in \mathcal{D}$ with $B \subset A$ is continuous.

2.2 Counting theorem and its generalizations

In this section, we will work in a fixed o-minimal structre over $\mathbb{R}$. The original Pila-Wilkie counting theorem describes how the number of points of certain bounded height in a definable set grows. First we recall the notion of height of a rational number and the height function.

Definition 2.15. *We define the multiplicative height of a rational number by* $H(0) = 0$ *and* $H(\frac{a}{b}) = \max\{|a|, |b|\}$ *when* a *and* b *are coprime integers. For a tuple* $\bar{x} = (x_1, ..., x_n) \in \mathbb{Q}^n$ *we define* $H(\bar{x}) = \max\{H(x_i); 1 \leq i \leq n\}$.

Definition 2.16. *Let* $X \subset \mathbb{R}^n$ *be any set and* $t \in \mathbb{R}_+$. *We define* $X(\mathbb{Q}, t) := \{\bar{x} \in X \cap \mathbb{Q}^n; H(\bar{x}) \leq t\}$. *Notice that* $X(\mathbb{Q}, t)$ *is always a finite set. We define the counting function* $N(X, t) := \sharp X(\mathbb{Q}, t)$.

We hope that for certain kinds of X, we have for all $\epsilon > 0$, there is a $c > 0$, such that $N(X, t) < ct^{\epsilon}$ for all $t \gg 0$. To do so, we must exclude the case that X contains semi-algebraic subset, the subset with polynomial growth property.

Definition 2.17. *We say that* $Y \subset \mathbb{R}^n$ *is semi-algebraic if it is definable in the structure* $(\mathbb{R}, <, +, \times)$.

Definition 2.18. *Given a set* $X \subset \mathbb{R}^n$ *we define the algebraic part* X^{alg} *to be the union of all infinite, connected semialgebraic subsets* $Y \subset X$. *The transcendental part of* X *is* $X^{tr} := X \backslash X^{alg}$.

With these definition, we may state the counting theorem, see theorem 1.8 in [PW06]

Theorem 2.19. (Pila-Wilkie) *Let* $X \subset \mathbb{R}^n$ *be a definable set in some o-minimal expansion of the real field and let* $\epsilon > 0$. *Then there exists a constant* $c = c(X, \epsilon) > 0$ *such that for* $t \geq 1$ *we have* $N(X^{tr}, t) \leq ct^{\epsilon}$

Later, Pila generalizes this theorem to count algebraic points. Let $k \geq 1$ be an integer. We define the k-height of a real number the corresponding counting function as follows.

Definition 2.20. *Let* $x \in \mathbb{R}^n$, *if* $[\mathbb{Q}(x) : \mathbb{Q}] \leq k$, *then there is a polynomial with coprime integer coefficients* $a_k T^k + ... + a_0$ *which vanishes* x, *let* $H(x) = \max\{|a_k|, ..., |a_0|\}$; *otherwise, let* $H_k(x) = \infty$. *For* $\bar{x} \in \mathbb{R}^n$, *we set* $H_k(\bar{x}) = \max\{H_k(x_1), ..., H_k(x_n)\}$.

Definition 2.21. *Let* $X \subset \mathbb{R}^n$ *be any subset,* $t \geq 1$. *We define*

$$N_k(X, t) := \{\bar{x} \in X; H_k(\bar{x}) \leq t\}$$

Just like Theorem 2.19, we have the following growth condition, see theorem 1.5 in [Pil09]

Theorem 2.22. (Pila) *Let* $X \subset \mathbb{R}^n$ *be a definable set in some o-minimal expansion of the real field and let* $\epsilon > 0$. *Then there exists a constant* $c = c(X, k, \epsilon) > 0$ *such that for* $t \geq 1$ *we have* $N_k(X^{tr}, t) \leq ct^{\epsilon}$

Due to this spirit, in order to get a proposition for us to use, we need the language of blocks. We have the following theorem, see [Pil11]

Theorem 2.23. *Let $F \subset \mathbb{R}^l \times \mathbb{R}^m$ be a definable set, we view F as a definable family parametrised by $\mathbb{R}^l$. Let $\epsilon > 0$, k be a positive integer.There is a finite number $J = J(F, k, \epsilon)$ blocks*

$$W^{(j)} \subset \mathbb{R}^{k_j} \times \mathbb{R}^l \times \mathbb{R}^m$$

each parametrised by $\mathbb{R}^{k_j} \times \mathbb{R}^m$, and a constant $c = c(F, k, \epsilon)$ such that
1. For all $(r, x) \in \mathbb{R}^{k_j} \times \mathbb{R}^l$ and all j, the fiber $W^{(j)}_{(r,x)} \subset F_x$;
2. For all $x \in \mathbb{R}^l$ and $t \geq 1$ the set $N_k(F_x, t)$ is contained in the union of at most ct^ϵ blocks of the form $W^{(j)}_{(r,x)}$ for suitable j and $r \in \mathbb{R}^{k_j}$.

Remark 2.24. *One thing to keep in mind is that block is something similar to semi-algebraic sets. To see the relation between theorem2.23 and theorem 2.22, notice that latter means if X satisfies $N_k(X, t) \geq ct^\epsilon$, then the algebraic part of X is nonempty. In theorem2.23, we take $l = 0$,then if $N_k(F, t) > ct^\epsilon$,then this theorem tells us that F must contains a block, so the algebraic part is nonempty.*

We need to express a useful theorem. The proof of which needs a generalization of theorem 2.23, which is stated as following.

Theorem 2.25. *Let F, k, ϵ be in theorem 2.23. There exists a finite number $J = J(F, k, \epsilon)$ of block families*

$$W^{(j)} \subset \mathbb{R}^{k_j} \times \mathbb{R}^l \times \mathbb{R}^m$$

parametrised by $\mathbb{R}^{k_j} \times \mathbb{R}^l$, for each j a continuous, definable function

$$\alpha^{(j)} : W^{(j)} \to \mathbb{R}^n$$

and a constant $c = c(F, k, \epsilon)$ with the following properties.
1. For all $j \in \{1, ..., J\}$ and all $(r, x) \in \mathbb{R}^{k_j} \times \mathbb{R}^l$ we have

$$\Gamma(\alpha^{(j)})_{(r,x)} \subset \{(y, z) \in F_x; z \in F^{iso}_{(x,y)}\}$$

where $F^{iso}_{(x,y)}$ is the set consists all the $z \in F_{(x,y)}$ such that z is isolated in $F_{(x,y)}$;
2. Say $x \in \mathbb{R}^l$ and $Z = F_x$. If $t \geq 1$, the set $Z^{\sim,iso}(k, t)$ is contained in the union of at most ct^ϵ graphs $\Gamma(\alpha^{(j)})_{(r,x)}$ for suitable $j \in \{1, .., j\}$. Where $Z^{\sim,iso}(k, t)$ is set consists of all $(y, z) \in Z$ such that $H_k(y) \leq t$ and z is isolated in Z_y.

Proof. Consider the set

$$F' = \{(x, y, z) \in F; z \in F^{iso}_{(x,y)}\}$$

By theorem 2.14, we write $F = C_1 \cup ... \cup C_N$ be a cell decomposition of F. Then $F_{(x,y)} = (C_1)_{(x,y)} \cup ... \cup (C_N)_{(x,y)}$. Each $(C_i)_{(x,y)}$ is either empty or a cell. Since the local dimension of C_i are constant, so F' is precisely the union of some C_i. Thus F' is definable. Now suppose the theorem is true for F', the corresponding blocks W^j, definable functions $\alpha^{(j)}$ and the constant $c(F', k, \epsilon)$ also satisfied the requirement for F.
By the reason above, from now on, we assume $F = F'$. Notice that for any (x, y), $F_{(x,y)}$ is discrete and definable, hence finite, by theorem 2.11, there is a constant c_1 such that $|F_{(x,y)}| \leq c_1$ for all (x, y).

Write $\pi : \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^l \times \mathbb{R}^m$ be projection map to the first two parts. We make a decompositon of F as follows. Let $E_1 = \pi(F)$, definable. By the property of definability, there is a function $f_1 : E_1 \to \mathbb{R}^n$ such that $\Gamma(f_1) \subset F$. This image is definable, so $F \backslash \Gamma(f_1)$ also. Now replace F by $F \backslash \Gamma(f_1)$, we repeat the same procedure as above. Since every time, the cardinality of $F_{(x,y)}$ decreases strictly, so the procedure terminates in $c_2 \leq c_1$ steps. Thus, we got $E_1, ..., E_{c_1} \subset \mathbb{R}^l \times \mathbb{R}^m$ and definable functions $f_i : E_i \to \mathbb{R}^n$, such that

$$\bigcup_{i=1}^{c_2} \Gamma(f_i) = F$$

Take suitable cell decomposition of $\Gamma(f_i)$ whenever necessary, we may assume that each f_i is definable and continuous. Now we apply theorem 2.23 to each E_i, we got J_i block families $W_i^{(j)} \subset \mathbb{R}^{k_{i,j}} \times \mathbb{R}^l \times \mathbb{R}^m$ parametrized by $\mathbb{R}^{k_{i,j}} \times \mathbb{R}^l$ with properties in that theorem. Now consider the function

$$\alpha_i^{(j)} : W_i^{(j)} \to \mathbb{R}^n \quad (r, x, y) \mapsto f_i(x, y)$$

which is definable and continuous. Now we have $\Gamma(\alpha_i^{(j)})_{(r,x)} \subset F_x$ for all (r, x). For $x \in \mathbb{R}^l, t \geq 1$, let $(y, z) \in Z^{\sim,iso}(k, t) = Z^{\sim}(k, t)$ with $Z = F_x$. Then $(x, y, z) \in \Gamma(f_i)$ for some i. Hence $y \in (E_i)_x(k, t)$, thus by theorem 2.23, there is a constant c_i only depends on E_i, k, ϵ, such that y is inside one of at most $c_i t^\epsilon$ blocks of the form $(W_i^{(j)})_{(r,x)}$. Thus $(y, z) \in \Gamma(\alpha_i^{(j)})_{(r,x)}$. Finally, take c be maximal of all c_i, after renumbering these $\alpha_i^{(j)}$ and $W_i^{(j)}$, the theorem follows. $\square$

Now we are able to state the important theorem which we are going to use later.

Theorem 2.26. *Let $F \subset \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n$ be a definable family parametrised by $\mathbb{R}^l$. Let $\epsilon > 0$, k be a positive integer. Write π_1, π_2 be the projections $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ respectively. Then there exists a constant $c = c(F, k, \epsilon)$ with the following properties. Say $x \in \mathbb{R}^l$ and let Z be the fiber F_x. If $t \geq 1$ and $\Sigma \subset Z^{\sim}(k,t) := \{(y,z) \in Z; H_k(y) \leq t)\}$ with*

$$\#\pi_2(\Sigma) > ct^{\epsilon}$$

there exists a continuous function $\beta : [0,1] \to Z$ such that the following properties hold.
1. The composition $\pi_1 \circ \beta : [0,1] \to \mathbb{R}^m$ is semi-algebraic and its restricition to $(0,1)$ is real analytic;
2. The composition $\pi_2 \circ \beta$ is not constant;
3. We have $\pi_2(\beta(0)) \in \pi_2(\Sigma)$;
4. If the o-minimal structure admits analytic cell decomposition, then $\beta|_{(0,1)}$ is real analytic.

Proof. Take the constant $c = c(F, k, \epsilon)$ as in the theorem 2.25. Let x, Z, Σ, t as in the hypothesis. First, if $\Sigma \not\subseteq Z^{\sim,\mathrm{iso}}(k,t)$, we fix an element $(y,z) \in \Sigma \backslash Z^{\sim,\mathrm{iso}}(k,t)$, then $H_k(y) \leq t$ and the connected component of Z_y containing z has positive dimension. This component is definably connected. Take a definable and continuous path $\alpha : [0,1] \to Z_y$ connecting $\alpha(0) = z$ with any other point $\alpha(1) \neq z$ such that $\Gamma(\alpha)$ is inside this component. Let $\beta : [0,1] \to Z$ defined by $t \mapsto (y, \alpha(t))$. Thus $\pi_1 \circ \beta$ is constant, hence semi-algebraic and its restriction to $(0,1)$ is real analytic. Since $\pi_2 \circ \beta(0) = \alpha(0) = z \neq \alpha(1) = \pi_2 \circ \beta(1)$, so $\pi_2 \circ \beta$ is not constant. Since $(y,z) \in \Sigma$, hence $\pi_2 \circ \beta(0) = z \in \pi_2(\Sigma)$. Since $\exists s \in (0,1)$, such that $\beta|_(0,s)$ is real analytic, hence after rescaling, we are able to make $\beta|_(0,1)$ be real analytic, which verifies part 4. In this case, we are done.

Now we have reduced to the case $\Sigma \subset Z^{\sim,\mathrm{iso}}(k,t)$. By theorem 2.25, the set Σ is contained in the union of at most ct^{ϵ} graphs of continuous and definable functions. Using Pigeonhole principle, there exists $(y,z), (y',z') \in \Sigma$ with $z = \pi_2(y,z) \neq \pi_2(y',z') = z'$ lie in a same graph. To be explicit, there is a block family $W \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^n$ and a continuous, definable function $\alpha : W \to \mathbb{R}^n$, with $(y,z), (y',z') \in \Gamma(\alpha)_{(r,x)}$ for some $r \in \mathbb{R}^k$.

As a block containing y, y', $W_{(r,x)}$ is connected. Hence there exists a continuous and definable map $\gamma : [0,1] \to W_{(r,x)}$ with $\gamma(0) = y, \gamma(1) = y'$. Since block is locally a semi-algebraic set, so for every s, the point $\gamma(s)$ has a semi-algebraic neighborhood in $W_{(r,x)}$. Since $[0,1]$ is compact, hence there is an open subinterval $(a, b) \subset [0, 1]$, such that $\gamma((a, b))$ has an semi-algebraic neighborhood with $\gamma(a) \neq \gamma(b)$. After rescaling, we may assume $(a, b) = (0, 1)$, so γ itself is semi-algebraic. Now we set

$$\beta(s) = (\gamma(s), \alpha(r, x, \gamma(s))$$

so this defines a function $\beta : [0, 1] \to Z$. Since α, γ are continuous and definable, so is β.

Since β semi-algebraic, so is $\pi_1 \circ \beta$. Thus the first assertion in part 1 holds. Since $\pi_2(\beta(0)) = z \neq z' = \pi_2(\beta(1))$, so part 2 follows. Using $\pi_2(\beta(0)) = z \in \pi_2(\Sigma)$, which yields part 3. For the second assertion in part 1, recall that$\mathbb{R}_{\text{alg}}$ admits analytic cell decomposition. There exists $0 = a_0 < a_1 < ... < a_{j+1} = 1$ such that each $\pi_1 \circ \beta|_{(a_i, a_{i+1})} : (a_i, a_{i+1}) \to \mathbb{R}^m$ is real analytic. By continuity and part 2, there is one such interval (a_i, a_{i+1}) such that the restriction of $\pi_1 \circ \beta$ on it is not constant. Let i be the minimal index with this property, so $\pi_2(\beta(a_i)) = \pi_2(\beta(0)) = z$. Now after rescaling, we may assume $a_{i+1} = 1$, then part 3 follows.

For part 4, suppose that the o-minimal structure admits analytic cell decomposition. Using the same method in the proof in part 3, we know that $\pi_2 \circ \beta|_{[0,1]}$ is real analytic. Thus $\beta|_{(0,1)}$ is real analytic. We are done. $\square$

3 Zilber-Pink conjecture

We have already mentioned the Zilber-Pink conjecture in the introduction part. This conjecture has several different expressions. One way to formulate this conjecture is using the language of unlikely intersection.

Conjecture 3.1. (Zilber-Pink, formualtaion 1) *For X a mixed shimura variety over a field K of characteristic zero, $V \subset X$ be a subvariety of dimension r, write*

$$X^{[r+1]} = \cup H(\overline{K})$$

where H in this union runs through all the special subvarieties of X with codimension at least $r+1$. Then

$$V \cap X^{[r+1]}$$

is not zariski dense in V.

Since in X, a subvariety of dimension r and another subvariety of codimension at least $r+1$ "do not likely" have intersection, so this formulation really coincide with the intuition. Another formulation of Zilber-Pink conjecture using the language of optimal subvarieties.

Definition 3.2. *Given $A \subset X$, denote $\langle A \rangle$ be the smallest special subvariety containing A. The defect of A is defined as*

$$\delta(A) = \dim \langle A \rangle - \dim A$$

Notice that $\langle A \rangle$ is well defined since the collection of special subvarieties is closed under taking irreducible components of intersections.

Definition 3.3. *Let X be a mixed Shimura varietiy or a semi-abelian variety defined over $\mathbb{C}$. Let $V \subset X$ be a subvariety. A subvariety $A \subset V$ is called optimal(for V in X) if there is no special subvariety B with $A \subsetneq B \subset V$ such that*

$$\delta(B) \leq \delta(A)$$

We write $\mathrm{Opt}(V)$ for the set of all optimal subvarieties for V. Now we are able to formulate Zilber-Pink conjecture.

Conjecture 3.4. (Zilber-Pink, formualtaion 2)*Let X be a mixed Shimura variety or a semi-abelian variety defined over $\mathbb{C}$. Then $Opt(V)$ is finite.*

Two hypothesis appear to be important to making progress toward Zilber-Pink conjecture. They are: the arithmetic one called "Large Galois Orbit"(LGO), and the transcendental onecalled "Weakly Complex Ax"(WCA). The hypothesis LGO for our cases can be stated as following.

Definition 3.5. *Let K be a field which is a finitely generated $\mathbb{Q}-$algebra. Suppose $X = Y(1)^n$ or an abelian variety defined over K and $V \subset X$ is a subvariety which is also defined over K. Let $s \geq 0$, we say that $LGO_s(V)$ is satisfied if there exists a constant $\kappa > 0$ with the following property. For any $P \in V(\overline{K})$ such that $\{P\}$ is an optimal singleton of V with dimension $\dim \langle P \rangle \leq s$ we have*

$$\Delta(\langle P \rangle) \leq (2[K(P) : K])^{\kappa}$$

If $r \geq 0$, we say that X satisfies LGO_r^s is $LGO_s(V)$ is satisfied for all $V \subset X$ defined over K with $\dim V \leq r$.
Finally, we say that X satisfies LGO if it satisfies LGO_r^s for all $r, s \geq 0$

Note that the number "2" appeared in the right hand side of the inequality above is necessary. Since if this "2" is omitted, then this inequality can not hold for P which is defined over K. We will give the definition of WCA later.

For the later using, it is now worthy to give the notion of geodesic optimal subvarieties. Let X be a mixed Shimura variety or a semi-abelian variety over $\mathbb{C}$. Similar to special subvarieties, the set of weakly special subvarieties are also closed under taking intersections and irreducible components, so for every subvariety$W \subset X$, there is a smallest weakly special subvariety containing W, which we denote as $\langle W \rangle_{\text{geo}}$. We write

$$\delta_{\text{geo}}(W) = \dim \langle W \rangle_{\text{geo}} - \dim W$$

as the geodesic defect of W.

Definition 3.6. *Let $V \subset X$ be a subvariety. A subvariety $W \subset V$ is said to be geodesic optimal(for V in X) if there is no subvariety Y with $W \subsetneq Y \subset V$ such that*

$$\delta_{\text{geo}}(Y) \leq \delta_{\text{geo}}(W)$$

An important proposition is that in our case, optimal subvarieties are always geodesic optimal.

Theorem 3.7. *Let X be an abelian variety or $Y(1)^n$, and let V be a subvariety. An optimal subvariety of V is geodesic optimal in V.*

In the next two chapters, we discuss Pila's progress in the Zilber-Pink conjecture. In the context of Abelian varieties, we will give some partial results in sense of both formulation 1 and formulation 2, while in the context of product of modular curves, we will only gave a conditional result in the sense of formulation 2.

4 Unlikely intersection in abelian varieties

4.1 Special subvarieties and complexity

The special subvarieties of an abelian variety are the torsion cosets, and weakly special subvarieties are the cosets. Now we define the complexity as follows. Let A be an abelian variety defined over $\mathbb{C}$ with dimension $g \geq 1$, and $\mathcal{L}$ is an ample line bundle on A. The degree of A is the intersection number $\deg_{\mathcal{L}} A = (\mathcal{L}^g[A]) \geq 1$.

Definition 4.1. *If W is a special subvariety of A, which is the translate of an abelian subvariety B of A by a torsion point. We define its arithmetic complexity $\Delta_{\text{arith}}(W)$ be the minimum of the order of the torsion points in W. The complexity is defined as*

$$\Delta(W) = \max\{\Delta_{\text{arith}}(W), \deg_{\mathcal{L}} B\}$$

It can be proved that after replacing $\mathcal{L}$ by another ample line bundle, the corresponding complexity only changes up to a controlled factor. Thus we do not have to emphasize the choice of ample line bundle in the definition of complexity.

4.2 Ax-Schanuel type Conjectures

In the case of abelian varieties, the transcendental conjecture of type Ax-Schanuel which we will use are already theorems. Now we state the theorem in Ax's spirit that are sufficient to handle our unlikely intersection problem.

Let A be an abelian variety defined over $\mathbb{C}$. Thus the exponential map $\exp : T_0(A) \to A(\mathbb{C})$ is a compex analytic group homomorphism. The theorem of Ax is the following.

Theorem 4.2. (Ax). *Let $U \subset T_0(A)$ be a complex vector subspace and $z \in T_0(A)$. Let K be an irreducible analytic subset of an open neighborhood of z in $z + U$. Let $Y = \overline{\exp(K)}^{\text{zar}}$ be the zariski closure of $\exp(K)$, then Y is irreducible and*

$$\delta_{\text{geo}}(Y) \leq \dim U - \dim K$$

For the proof, see Corollary 1 of [Ax72]. The following theorem is called Ax-Lindemann-Weierstrass theorem, which we will also use later.

Theorem 4.3. (Ax). *Let $\beta : [0,1] \to T_0(A)$ be real semi-algebraic and continuous with $\beta|_{(0,1)}$ real analytic. Then $\overline{\exp(\beta([0,1]))}^{\text{zar}}$ is a coset.*

Now it is worthy to explain how this theorem connect to the classical Lindemann-Weierstrass theorem, we use the explanation in section 3.1.1 of [Gao14]. Recall that the classical Lindemann-Weierstrass theorem means the following.

Theorem 4.4. (Lindemann-Weierstrass) *Let $x_1, ..., x_n \in \overline{Q}$. If they are $\mathbb{Q}-$linearly independent, then $e^{x_1}, ..., e^{x_n}$ are algebraically independent over $\mathbb{Q}$*

Using geometric language, Lindemann-Weierstrass theorem can be reformulated as following.

Theorem 4.5. (Lindemann-Weierstrass, geometric terms) *Let $unif = (\exp, ..., \exp) : \mathbb{C}^n \to \mathbb{C}^{\times n}$. Let Y be an irreducible algebraic subvariety of $\mathbb{C}^{\times n}$, Z is a maximal irreducible algebraic variety contained in $unif^{-1}(Y)$, then Z is a translate of a $\mathbb{Q}-$linear subspace of $\mathbb{C}^n$.*

Observe that in the above theorem, $unif = (\exp, ..., \exp)$ is precisely the uniformization map in the Shimura sense. In the context of abelian varieties, the uniforization map is precisely the exponential map $T_0(A) \to A(\mathbb{C})$. So we have the analogue statement.

Theorem 4.6. *Let $A, \exp : T_0(A) \to A(\mathbb{C})$ as above. Let Y be an semi-algebraic subset of A, Z is a maximal algebraic subvariety contained in $\exp^{-1}(Y)$, then Z is weakly special.*

Now we show theorem 4.6 implies theorem 4.3. Let $Y = \overline{\exp(\beta([0,1]))}^{\mathrm{zar}}$ as in theorem 4.3, let W be an irreducible algebraic subset of $T_0(A)$ which contains $\beta([0,1])$ and contained in $\exp^{-1}(Y)$, maximal among these properties. Then $Y = \overline{\exp(W)}^{\mathrm{zar}}$, theorem 4.6 implies that W is weakly special. Hence $\exp(W)$ is an irreducible subvariety of $A(\mathbb{C})$. Now we have $Y = \overline{\exp(W)}^{\mathrm{zar}} = \exp(W)$ is weakly special, i.e a coset. This gives us some intuition about the theorem 4.3.

4.3 A finiteness result

Suppose A is an abelian variety defined over $\mathbb{C}$, we have the following finiteness result for the geodesic optimal subvarieties.

Theorem 4.7. *For any subvariety $V \subset A$, there exists a finite set(only depends on V) of abelian subvarieties of A with the following property. If W is a geodesic optimal subvariety of V, then $\langle W \rangle_{\mathrm{geo}}$ is a translate of the said set.*

In order to prove this theorem, we choose a basis of the period lattice $\Omega_A = H_1(A, \mathbb{Z}) \subset T_0 A$ such that we can identify $T_0 A$ with $\mathbb{R}^n$ as a real vector space, in this basis $\Omega \cong \mathbb{Z}^{2g}$. Now for the subvariety V as in the theorem, we set

$$\mathcal{V} = \exp|_{(-1,1)^{2g}}^{-1}(V(\mathbb{C}))$$

Then $\mathcal{V}$ is a definable subset of $\mathbb{R}^{2g}$. Also, by the isomorphism mentioned above, $\mathcal{V}$ is also a complex analytic subset of $(-1,1)^{2g} \subset T_0 A$. Thus it is also a complex analytic space. The proof of the theorem require the argument of dimension. Now we review the definition of the (local) dimension of definable set in a certain structure.

Definition 4.8. *The dimension of a non-empty definable set $X \subset \mathbb{R}^m$ is defined by maximum of $i_1 + ... + i_m$ such that X contains an $(i_1, ..., i_m)$ cell. The dimension of empty set is $-\infty$.*

For the local dimension, we have

Theorem 4.9. *Let $X \subset \mathbb{R}^m$, $x \in \mathbb{R}^m$. Then there is an integer d such that $\dim(U \cap X) = d$ for all sufficiently small definable neighborhood U of x in $\mathbb{R}^m$.*

Definition 4.10. *Using the above notation, we define $\dim_x X = d$.*

In this section, we add the subscript $\mathbb{C}$ to the dimension symbol to signify the (local) dimension as a complex analytic space. Observing that $\dim \mathbb{C} = 2 = 2 \dim_{\mathbb{C}} \mathbb{C}$, generalize this idea, we have following lemma.

Lemma 4.11. *Let Z be a definable analytic subset of a finite dimension $\mathbb{C}-$vector space. Let $z \in Z$, then we have $\dim_z Z = 2 \dim_{\mathbb{C},z} Z$.*

Now we write $\mathcal{O} = \operatorname{End}(T_0 A)$, the endomorphism of $T_0 A$ as a $\mathbb{C}-$vector space. Suppose $0 \leq r \leq g$ be an integer, for element $(z, M) \in \mathcal{V} \times \mathcal{O}$, we set three conditions:
(a) $\dim_{\mathbb{C}} \ker M = r$;
(b) For all $N \in \mathcal{O}$ with $\ker M \subsetneq \ker N$: $\dim \ker N - \dim_z \mathcal{V} \cap (z + \ker M) < \dim \ker N - \dim_z \mathcal{V} \cap (z + \dim N)$.
(c) For all $M' \in \mathcal{O}$ with $\ker M' \subsetneq \ker M$: $\dim_z \mathcal{V} \cap (z + \ker M') < \dim_z \mathcal{V} \cap (z + \ker M)$.
Now define E_r be the set of (z, M) satisfying all the three conditions (a),(b) and (c). We have the following important lemma.

Lemma 4.12. *Using the above notation.*
1. If $(z, M) \in E_r$, there is an abelian subvariety $B \subset A$ with $T_0 B = \ker M$.

2. The set $\{\ker M; (z, M) \in E_r\}$ *is finite.*
3. Let W *be a geodesic optimal subvariety of* V *and let* $\langle W \rangle_{\mathrm{geo}}$ *be the translate of the ablian subvariety* $B \subset A$. *If* $r = \dim_{\mathbb{C}} B$ *and* $M \in \mathcal{O}$ *with* $T_0 B = \ker M$, *then* $(z, M) \in E_r$ *for some* $z \in (-1,1)^{2g}$.

Before proving this lemma, we give two lemma about basic properties about dimension theory and abelian varieties.

Lemma 4.13. *Let* A, B *be two analytic sets in a complex space* X, *then for all smooth point* $P \in A \cap B$, *we have*

$$\dim_{\mathbb{C},P}(A \cap B) \geq \dim_{\mathbb{C}} A - \dim_{\mathbb{C}} B - \dim_{\mathbb{C}} X$$

Lemma 4.14. *Let* A *be an abelian variety of dimension* g *over* $\mathbb{C}$, $W \subset A$ *is an subvariety contains* 0, *such that* A *is the smallest abelian variety contains* W. *Then the map*

$$\phi : W^g \to A \quad (w_1, ..., w_g) \mapsto w_1 + ... + w_g$$

is surjective.

Now we prove lemma 4.12

Proof of Lemma 4.12. For part 1, Let $(z, M) \in E_r$. We apply the Ax's theorem 4.2 to $U = \ker M$. We take a complex analytically irreducible component $\mathcal{K} \subset \mathcal{V} \cap (z+U)$ with $\dim_{\mathbb{C}} K = \dim_{\mathbb{C},z} \mathcal{V} \cap (z+U)$. By shrinking K to an open neighorhood of z we may assume that K is irreducible and definable in $\mathbb{R}_{\mathrm{an}}$(not necessarily a component). Let $Y \subset A$ be as in Ax's theorem. Then $\langle Y \rangle_{\mathrm{geo}}$ is the translate of an abelian subvariety $B = Y - \exp(z)$ of A. Since $\exp(K) \subset Y + \exp(z)$, we have $K \subset z + T_0(B)$ and so

$$K \subset z + U \cap T_0(B)$$

By the condition (c) of the definition of E_r, we have $U \cap T_0(B) = U$, thus $U \subset T_0(B)$.

Next we prove the equality holds. Suppose the contrary that $U \subsetneq T_0(B)$, select $N \in \mathcal{O}$ such that $\ker N = T_0(B)$. Using the condition (b) of the definition of E_r and the observation that $Y \subset V \cap ((z) + B)$, we have

$$\begin{aligned} \dim U - \dim K &= \dim U - \dim_z \mathcal{V} \cap (z + U) \\ &< \dim T_0(B) - \mathcal{V} \cap (z + T_0(B)) \\ &\leq \dim T_0(B) - \dim_z \exp|^{-1}_{(-1,1)^{2g}}(Y(\mathbb{C})) \end{aligned}$$

Using lemma 4.11, we passe this to complex dimension, we get $\dim_{\mathbb{C}} U - \dim_{\mathbb{C},z} K < \dim_{\mathbb{C}} - \dim_{\mathbb{C},z} = \delta_{\text{geo}}(Y)$. This contradicts with Ax's theorem 4.2. Thus we must have $\ker M = U = T_0(B)$ and part 1 follows.

For part 2, let $\ker M$ as in the question. We fix a $\mathbb{C}-$basis for $T_0(A)$, thus every $\ker M$ correspond to a $g \times g$ matrix. Let $\pi_2 : T_0(A) \times \text{End}(T_0(A)) \to \text{End}(T_0(A))$ be the projection to the second coordinate, then we have a map to the Grassmannian.

$$\phi : \pi_2(E_r) \to \mathbb{G}(r, n) \quad M \mapsto \ker M$$

Since $\pi_2(E_r)$ is definable in $\mathbb{R}_{\text{an}}$, so is $\phi(\pi(E_r))$. By part 1, $\ker M$ is the tangent space o an abelian subvariety of A, but A has at most countably many abeian subvarieties thus $\ker M$ has at most countably many possibilities. Thus $\phi(\pi(E_r))$ is also at most countable. Thus as a definable and at most countable set, $\phi(\pi(E_r))$ is finite, part 2 has been done.

For part 3, let W, B as in the question. Since W is geodesic optimal, it is an irreducible component of $V \cap \langle W \rangle_{\text{geo}}$. Let $z \in \mathcal{V}$ such thatexp(z) is a smooth complex point of W that is not contained in any other irreducible component of $V \cap \langle W \rangle_{\text{geo}}$. We will prove that $(z, M) \in E_r$.

Since $\dim M = \dim B = r$, condition (a) follows.

For condition (b), suppose the contrary that there is a $N \subset \mathcal{O}$ such that $T_0(B) \subset \ker N$ and

$$\dim \ker N - \dim_z \mathcal{V} \cap (z + \ker M) \geq \dim \ker N - \dim_z \mathcal{V} \cap (z + \dim N) \quad (1)$$

We fix K as in the proof of part 1. Let $Y = \overline{\exp(K)}^{\text{zar}}$, then Ax's theorem 4.2 implies that $\delta_{\text{geo}}(Y) \leq \dim_{\mathbb{C}} \ker N - \dim_{\mathbb{C}} K$. Since the exponential map is locally biholomorphic, our choice of z implies that z is a smooth point of the complex analytic set $\mathcal{V} \cap (z + T_0(B))$ which at z has local dimension $\dim_{\mathbb{C}} W$. So by the above inequality, we have

$$\delta_{\text{geo}}(W) = \dim_{\mathbb{C}} T_0(B) - \dim_{\mathbb{C}} W \geq \dim_{\mathbb{C}} \ker N - \dim_{\mathbb{C}} K \geq \delta_{\text{geo}}(Y) \quad (2)$$

By smoothness, $\mathcal{V} \cap (z + \ker N)$ has a unique component K' passing through z. The dimension property in lemma 4.13 implies

$$\dim_{\mathbb{C},z} K \cap (z + T_0(B)) \geq \dim_{\mathbb{C},z} K + \dim_{\mathbb{C},z} T_0(B) - \dim_{\mathbb{C}} \ker N \quad (3)$$

Inequality (2) and the above discussion imply hat the right hand side is at least $\dim_{\mathbb{C}} W$. But $K \cap (z + T_0(B)) \subset \mathcal{V} \cap (z + T_0(B))$, by comparing their local dimensions on z we find that $K \cap (z + T_0(B))$, and hence K, contains a neighborhood of z. This implies that $W \subset Y$. Since W is geodesic optimal, combine this with (2), we get $W = Y$. So $\dim_{\mathbb{C}} K \leq \dim_{\mathbb{C}} W$. But the left hand side of (3) is at most $\dim_{\mathbb{C}} K$, so we also have $\dim_{\mathbb{C}} K \geq \dim_{\mathbb{C}} W$. Thus $\dim_{\mathbb{C}} K = \dim_{\mathbb{C}} W$. Substitute this to (2), we get $\dim_{\mathbb{C}} T_0(B) \geq \dim_{\mathbb{C}} \ker N$, which contradicts our assumption. So condition (b) follows.

For condition (c). Suppose on the contrary that there is $M' \in \mathcal{O}$ with $\ker M' \subsetneq T_0(B)$ and

$$\dim_z \mathcal{V} \cap (z + \ker M') = \dim_z \mathcal{V} \cap (z + \ker M)$$

The set on the right hand side is a complex analytically space, smooth at z, and contains the one in the left hand side, so these two sets coincide on an open neighborhood of z in $(-1,1)^{2g}$. Therefore, an open neighborhood of 0 in $W(\mathbb{C}) - \exp(z)$ is contained in the group $\exp(\ker M')$. Thus this group contains an non-empty subset of

$$\sum_{i=1}^{\dim A} (W(\mathbb{C}) - \exp(z))$$

By lemma 4.14, this sum equals B. Hence $T_0(B) \subset \ker M'$, contradicts our assumption. Thus condition (c) follows. Now we already proved that $(z, M) \in E_r$. $\square$

Now we are able to prove the finiteness property.

Proof of Theorem 4.7. Suppose W is a geodesic optimal subvariety of V. Take $M \in \mathcal{O}$ such that $\langle W \rangle_{\text{geo}}$ is the translate of an abelian subvariety whose tangent space is $\ker M$. Let $r = \dim_{\mathbb{C}} B$, by part 3 of lemma 4.12, there exists a z, such that $(z, M) \in E_r$. Then, by part 2 of lemma 4.12, these kind of $\ker M$ lie in a finite set. So $\langle W \rangle_{\text{geo}}$ is the translate of an abelian subvariety comes from a finite set. $\square$

A direct application of the finiteness theorem is discussing the anomalous subvarieties, which we will use later.

Definition 4.15. *Let A, V be as above. A subvariety $W \subset V$ is called anomalous if*

$$\dim W \geq \max\{1, \dim \langle W \rangle_{\text{geo}} + \dim V - \dim A + 1\}$$

If in addition W is not contained in any strictly larger anomalous subvariety of V, then we call W be maximal anomalous. The complement in V of the union of all anomalous subvarieties of V is denoted by V^{oa}.

An important property is the openness of V^{oa}.

Theorem 4.16. *Let A, V be as above, then V^{oa} is open in V.*

Proof. We first claim that maximal anomalous varieties are always geodesic-optimal. In fact, let $W \subset$ be an anomalous subvariety, then there exists a geodesic-optimal subvariety Y of V with $\delta_{\mathrm{geo}}(Y) \leq \delta_{\mathrm{geo}}(W)$. So by the definition of anomalous subvariety, we have

$$\dim Y \geq \dim \langle Y \rangle_{\mathrm{geo}} - \dim \langle W \rangle_{\mathrm{geo}} + \dim W \geq \dim \langle Y \rangle_{\mathrm{geo}} + \dim V - \dim A + 1$$

Since $\dim Y \geq \dim V \geq 1$, we see that Y is anomalous. As W is maximal anomalous, we have $W = Y$, so W is geodesic-optimal.

Now by theorem 4.7, $\langle W \rangle_{\mathrm{geo}}$ is the translate of an abelian subvariety coming from a finite set which depends only on V. Let B be such an abelian subvariety. We denote M_B be the set consisting of $Y \subset V$ maximal anomalous such that $\langle Y \rangle_{\mathrm{geo}}$ is a translate of B, U_B be the union of the elements in M_B, and write $q : V \to A/B$ be the restriction of the quotient map $A \to A/B$. Now the points in U_B are precisely those $P \in V(\overline{K})$ such that $\dim(q^{-1}(q(P)) \geq \dim B$. By the property of dimension, we got U_B is closed in V. Thus V^{oa}, as the complement of V by finitely many such U_B, is open in V. □

Finally, we remark that $V^{\mathrm{oa}} = \emptyset$ if and only if there exists a abelian subvariety B such that

$$\dim \phi(V) < \min\{\dim A/B, \dim V\}$$

where $\phi : A \to A/B$ be the quotient map.

4.4 An upper bound of complexity

In this section, we let A be an abelian variety defined over a number field K of dimension g and $\mathcal{L}$ be an ample line bundle on A. After replacing $\mathcal{L}$ by $\mathcal{L} \otimes \mathcal{L}^{\otimes(-1)}$, we may assume $\mathcal{L}$ is symmetric. Let $\hat{h}$ be the Néron-Tate height associated to this line bundle, then it is a quadratic form. Let $d \leq g$, we write $\lambda_A(d)$ be the supremum of all possible $\dim(A/H) \times \mathrm{rank} Hom(A, A/H)$, where H runs through all abelian subvarieties of A defined over $\bar{K}$ with dimension d. With this notation, we have the following theorem

Theorem 4.17. *There is a constant $c = c(A, \mathcal{L}) > 0$ such that for all $P \in A(\bar{K})$*

$$\Delta_{\mathrm{arith}}(\langle P \rangle) \leq c[K(P) : K]^{6g+1}$$

and

$$\deg_{\mathcal{L}} H \leq c[K(P):K]^{60g^4} \max\{1, \hat{h}(P)\}^{\lambda_A(\dim\langle P\rangle)}$$

where $H = \langle P\rangle - P$*. In particular,*

$$\Delta(\langle P\rangle) \leq c[K(P):K]^{60g^4} \max\{1, \hat{h}(P)\}^{\lambda_A(\dim\langle P\rangle)}$$

4.5 Partial results about the optimal subvarieties

In this section, we give some partial result of Zilber-Pink conjecture in the sense of formulation 1. The assumption about Large Galois Orbit implies the Zilber-Pink conjecture for the abelian varieties over number fields. Although the LGO remains open, we are still able to get some weaker unconditional results toward the final conjecture. Now suppose A is an abelian variety defined over a number field K, and $\mathcal{L}$ is a symmetric ample line bundle on A. Let $\hat{h}$ denotes the Néron-Tate height associated to this line bundle.

Definition 4.18. *Let* $V \subset A$ *is a subvariety defined over* K*. Let* $\kappa \geq 0$*. We define* $Opt(V,\kappa)$ *be the set of those* $W \in Opt(V)$ *such that there exists a point* $P \in W(\overline{K})$ *with* $\hat{h}(P) \leq (2[K(W):K])^{\kappa}$*, here* $K(W)$ *is the field of definition of* W*.*

With this notation, we are able to formulate our unconditional result.

Theorem 4.19. *Let* A *be an abelian variety defined over a field* K *which is a finitely generated* $\mathbb{Q}$ *algebra. Let* $V \subset A$ *be a subvariety defined over* K*.*
1. Say $r, s \geq 0$ *and all the quotients of* A *defined over a finite extension of* K *satisfy* LGO_r^s*. Then*

$$\{W \in Opt(V); \mathrm{codim}_V W \leq r, \dim\langle W\rangle - dim\,\langle P\rangle_{\mathrm{geo}} \leq s\}$$

is finite.
2. If K *is a number field, then* $Opt(V,\kappa)$ *is finite for all* $\kappa \geq 0$*.*

Now if LGO is true, then LGO_r^s holds for all $r, s \geq 0$. Let $r = s = g = \dim A$, by theorem 4.19, we have $Opt(V)$ is finite. That means LGO implies Zilber Pink conjecture.

4.6 Intersection with special subvarieties

In this section, we give some partial result of Zilber-Pink conjecture in the sense of formulation 2.

Theorem 4.20. *Let A be an abelian variety defined over a number field K, $\mathcal{L}$ is an ample symmetric line bundle with $\hat{h}$ be the associated Néron-Tate height. Suppose $V \subset A$ is a subvariety defined over K. Then*
1. If $\kappa \geq 0$, then

$$\{P \in V(\overline{K}) \cap A^{[1+\mathrm{codim}_A V+\dim V]}; \hat{h}(P) \leq \kappa\}$$

is not Zariski dense in V.
2. If $\kappa \geq 0$, then

$$\{P \in V(\overline{K}) \cap A^{[1+\dim V]}; \hat{h}(P) \leq \kappa\}$$

is contained in a finite union of proper special subvarieties of A.
3. The set $V^{\mathrm{oa}}(\overline{K}) \cap A^{[1+\dim V]}$ is finite.
4. Suppose $\dim V \geq 1$ and $\dim \phi(V) = \min\{\dim A/B, \dim V\}$ for all abelian subvarieties $B \subset A$, where $\phi : A \to B$ is the quotient map. Then $V(\overline{K}) \cap A^{[1+\dim V]}$ is not Zariski dense in V.
5. Let $\Gamma \subset A(\overline{K})$ be a finitely generated subgroup and let

$$\overline{\Gamma} = \{P \in A(\overline{K}); \exists n \in \mathbb{Z}_{\geq 1}, nP \in \Gamma\}$$

Then

$$\bigcup V^{\mathrm{oa}}(\overline{K}) \cap (H + \overline{\Gamma})$$

is finite, where in the union, H runs through all the special subvarieties of A with codimension at least $1 + \dim V$.
6. Let V as in part 4, and $\overline{\Gamma}$ as in part 5, we have

$$\bigcup V(\overline{K}) \cap (H + \overline{\Gamma})$$

is not Zariski dense in V, where in the union, H also runs through all the special subvarieties of A with codimension at least $1 + \dim V$.

Proof. For part 1, let $P \in V(\overline{K}) \cap A^{[1+\mathrm{codim}_A V+\dim V]}$ with $\hat{h}(P) \leq \kappa$. Then P is contained in an optimal subvariety W of V such that $\delta(A) \leq \delta(\langle P \rangle) = \dim(\langle P \rangle)$. So W contains point P of height at most $\kappa \leq (2[K(W) : K])^{\kappa}$, we have $W \in Opt(V, \kappa)$. By theorem 4.19, the latter set if finite. For every such W, $\delta(W) \leq \dim(\langle P \rangle) < \dim \langle V \rangle - \dim V$. Thus P is contained in a finite union of proper subvarieties of V, hence not Zariski dense.

For part 2, we prove by induction on $\dim V$. If $\dim V = 0$, then V is a point, the set in the assertion is either empty or a point contained in a proper algebraic subgroup of A. Now let $r > 0$, suppose the assertion is true for

V of dimension smaller than r, let $\dim V = r$. If V is already contained in some proper algebraic subgroup of A, then we are done. Otherwise, observe that special subvarieties are those components of proper algebraic subgroups, we have $\langle V \rangle = A$. As in part 1, the set in the assertion is contained in $V_1 \cup ... \cup V_l$ where $V_i \in Opt(V, \kappa)$ which is a proper subvariety of V. Let P be a point in the set in the assertion, then $P \in V_i$ for some i. We have $P \in V_i(\overline{K}) \cap A^{[1+\dim V]} \subset V_i(\overline{K}) \cap A^{[1+\dim V_i]}$, by the induction hypothesis, the latter is contained in finitely many proper algebraic subgroups of A. Thus the set in the assertion is also contained in finitely many proper algebraic subgroups of A. So we proved the case of $\dim V = r$.

For part 3, if $V^{\mathrm{oa}} = \emptyset$, then we are done. Otherwise, $V \neq X$ and is not contained in any proper abelian subvarieties of A. By the main theorem on page 407 in [Hab09], we know the Néron-Tate height is bounded on $V^{\mathrm{oa}}(\overline{K}) \cap A^{[\dim V]}$, thus in particular bounded $V^{\mathrm{oa}}(\overline{K}) \cap A^{[\dim V]}$. Let P inside this intersection, then P is contained in an optimal subvariety W with $\delta(W) \leq \dim \langle P \rangle$. Then, $\dim A - \dim V - 1 \geq \dim \langle P \rangle \geq \dim \langle W \rangle - \dim W$. But $P \in V^{\mathrm{oa}}(\overline{K})$, so W is not anomalous, by definition, we have either $\dim W < 1$ or $\dim W < \dim \langle W \rangle_{\mathrm{geo}} + \dim V - \dim A + 1$, the latter the impossible by the reasoning above, so $\dim W = 0$. Thus $W = \{P\} \in Opt(V, \kappa)$, since $Opt(V, \kappa)$ is finite, we proved this assertion.

For part 4, recall that V^{oa} is Zariski open in V. The condition in the theorem means V^{oa} is not empty. Thus the set in the assertion is the union of a finite set and a proper closed subset in V, which is not Zariski dense in V.

For part 5, we are going to use the method of the proof of theorem 5.3 in [Pin05]. Suppose $P_1, ..., P_t$ are the $\mathbb{Z}-$ independent elements which generate the free part of group Γ. Since every proper subgroup of A has only finitely many components, so after multiply a positive integer, we may assume the Zariski closure in A^t of the subroup generated by $(P_1, ..., P_t)$ is an abelian subvariety B. Write $V' = V \times \{(P_1, ..., P_t)\} \subset A \times B$. For all $x \in \overline{\Gamma} = \{P \in A(\overline{K}); \exists n \in \mathbb{Z}_{\geq 1}, nP \in \Gamma\}$, write $x = h + \gamma$, where h in some H of codimension at least $1 + \dim V$, $\gamma \in \overline{\Gamma}$). Choose integers $n > 0$ and $n_1, ..., n_t$, such that $n\gamma = n_1 P_1 + ... + n_t P_t$. Then we have $n\gamma = \phi(\underline{P})$, where $\phi = (n_1, ..., n_t) : A^t \to A$. Inside $A \times B$, we have

$$(nx, nP) = (nh + n\gamma, n\underline{P}) = (nh + \phi(\underline{P}), n\underline{P}) = (nh, 0) + (\phi, n)(\underline{P})$$

Thus we have

$$(x, \underline{P}) \in G := n^{-1}(H \times \{0\} + (\phi, n)(B))$$

Since $\dim G = \dim H + \dim B$, so $\mathrm{codim}_{A\times B} G = \mathrm{codim}_A H \geq \dim V + 1$. So we have

$$(\bigcup V^{\mathrm{oa}}(\overline{K}) \cap (H + \overline{\Gamma})) \times \{\underline{P}\} \subset V'(\overline{K}) \cap (A \times B)^{[\dim V+1]}$$

So in the above relation, the Néron-Tate height of the left hand side is bounded by a constant κ only depends on V and P_i.

Now for $x \in \bigcup V^{\mathrm{oa}}(\overline{K}) \cap (H+\overline{\Gamma})$, write $P' = (x, \underline{P})$. Then P' is contained in an optimal subvariety $W' \subset V'$ such that $\delta(W') \leq \delta(P')$. So $\dim \langle W' \rangle - \dim W' \leq \delta(P') = \dim \langle P' \rangle \leq \dim(A \times B) - \dim V' - 1$. The image of $\langle W' \rangle$ under the projection $A \times B \to B$ is an irreducible component of an algebraic group which contains $\underline{P}$, so it must equal B. So each fiber of this projection is a coset of dimension $\dim \langle W' \rangle - \dim B$. Observe that $W' = W \times \underline{P}$ is contained in one of such fibers. So we have

$$\dim \langle W \rangle_{\mathrm{geo}} \leq \dim \langle W' \rangle - \dim B \leq \dim W' + \dim A - \dim V' - 1$$

since $\dim A' = \dim A, \dim V' = \dim V$, we have

$$\dim W \geq \dim V + \dim \langle W \rangle_{\mathrm{geo}} - \dim A + 1$$

Since $x \in W(\overline{K}) \cap V^{\mathrm{oa}}(\overline{K})$, so W is not anomalous. Thus $\dim W = 0, W' = \{(x, \underline{P})\}$, we have $W' \in \mathit{Opt}(V', \kappa)$, which is finite by theorem 4.19.

For part 6, similar to part 4, we know that V^{oa} is open and non empty in V, so the set in the assertion is a union of finite set and a proper closed subset of V, hence not Zariski dense. □

The above concern the general subvariety. Now if $V \subset A$ is a curve, we have better results. Before stating the theorem, we first recall the height bound property given by G.Rémond, corollary 1.6, [Ré07]

Theorem 4.21. *Let A be an abelian variety defined over a number field K with an ample symmetric line bundle and its associated Néron-Tate height. Suppose that V is a curve in A which is not contained in any proper algebraic subgroup of A, then the Néron-Tate height is bounded on $V(\overline{K}) \cap A^{[2]}$.*

Now we state the unconditional result for curves.

Theorem 4.22. *Let A be an abelian variety defined over a number field K. Suppose $V \subset A$ is a subvariety defined over K.*
1. The set

$$\{W \in \mathit{Opt}(V); \mathrm{codim}_V W \leq 1\}$$

is finite.
2. If V is a curve then $Opt(V)$ is finite.
3. If V is a curve that is not contained in any proper algebraic subgroup of A,then $V(\overline{K}) \cap A^{[2]}$ is finite.

Proof. For part 1, we first claim that for every $s \geq 0$, LGO_s^1. In fact, let $Y \subset A$ be a subvariety with $\dim Y \leq 1$. Since the case that V is a point is evident, we may assume V is a curve. Let $\{P\} \in V(\overline{K})$ is an optimal singleton, then $\dim \langle P \rangle = \delta(P) < \delta(V) = \dim V - 1$. Write $\langle V \rangle = B + Q, \langle P \rangle = C + Q$, with $C, B \subset A$ be abelian subvarieties and $Q \in A_{\mathrm{tor}}$. Thus $P - Q$ is contained in an abelian subvariety C of B of codimension at least 2, by theorem 4.21, we have $\hat{h}(P - Q)$ is bounded. Thus $\hat{h}(P) = \frac{1}{2}(\hat{h}(P - Q) + \hat{h}(Q) - 2\hat{h}(P - 2Q)) \leq \frac{1}{2}\hat{h}(P - Q)$ is also bounded. Now using this upper bound and theorem 4.17, we get $LGO_s(V)$, thus LGO_s^1 is true. By part 1 of theorem 4.19, we obtain our conclution.

For part 2, if V is a curve, then $\mathrm{codim}_W V \leq 1$ satisfies automatically, so by part 1, we know that $Opt(V)$ is finite.

For part 3, for every $P \in V(\overline{K}) \cap A^{[2]}$, there is a special subvariety $Y \subset A$ of codimension at least 2 such that $P \in V(\overline{K}) \cap Y(\overline{K})$. So $\langle P \rangle \subset Y$, thus $\delta(P) \leq \dim Y \leq \dim A - 2$, since $\langle V \rangle = A$, thus $\delta P \leq \dim \langle P \rangle - 2 < \delta \langle V \rangle$. Therefore, $\{P\}$ is an optimal singleton of V, since $Opt(V)$ is finite by part 2, we only have finitely many such P, we are done. □

5 Unlikely intersection in the products of mudular curves

The structure of this chapter is almost the parallel as we treat abelian varieties. However, there are still lots of differences in details.

5.1 Special subvarieties and complexity

We have definition of special subvarieties for general Shimura varieties, that is the subvarieties of Hodge type. In the particular case of $X = Y(1)^n$, we are able to describe its special and weakly special subvarieties in an explicit way as follows.

Definition 5.1. *A strongly special curve in $\mathbb{H}^n$ is the image of a map of the form*

$$\mathbb{H} \to \mathbb{H}^n, \quad \mapsto (g_1, ..., g_n)$$

where $g_1 = 1, g_2, ..., g_n \in GL_2^+(\mathbb{Q})$.

Definition 5.2. *Let $R = (R_0, ..., R_k)$ be a partition of $\{1, ..., n\}$ such that only R_0 is allowed to be empty. Denote $\mathbb{H}^{R_j}$ be the corresponding cartesian product. A weakly special subvariety of $\mathbb{H}^n$ is a product*

$$Y = \prod_{j=0}^{k} Y_j$$

where $Y_0 \in \mathbb{H}^{R_0}$ is a special point and, for $j = 1, ..., k$, Y_j is a strongly special curve in $\mathbb{H}^{R_j}$.

Definition 5.3. *A weakly special subvariety is called a special subvariety if each coordinate of Y_0 is a CM point of $\mathbb{H}$*

Recall that $z \in \mathbb{H}$ is a CM point if the corresponding elliptic curve has complex multiplication. In other words, $[\mathbb{Q}(z) : \mathbb{Q}] = 2$, i.e a quadratic point. Now for a CM point z, write $aT^2 + bT + c \in \mathbb{Z}[T]$ be its primitive minimal polynomial, denote $\Delta(z) = b^2 - 4ac$. Also, for all $g \in GL_2^+(\mathbb{Q})$, there exists a modular polynomial Φ_N such that $\Phi_N(j(z), j(gz)) = 0$ for all z. Then we are able to define the complexity of an special subvariety of $\mathbb{H}^n$.

Definition 5.4. *For a special subvariety $Y \in \mathbb{H}^n$ as above we define its complexity*

$$\Delta(Y) = \max\{\Delta(z), N(g)\}$$

where z runs through all the coordinates of Y_0 and g runs through all the element in $GL_2^+(\mathbb{Q})$ appeared in the definition of Y.

The weakly special and special subvarieties of $X = Y(1)^n$ are the image of the ones in $\mathbb{H}^n$ under j invariant, more precisely.

Definition 5.5. *Let $\pi : \mathbb{H}^n \to Y(1)^n$ be the cartesian product of j invariant. A (weakly) special subvariety of $Y(1)^n$ is the image $j(Y)$ for some Y (weakly) special. The complexity of $j(Y)$ is the complexity of Y.*

It can be proved that for a special subvariety $T \subset Y(1)^n$, if $T = \pi(Y) = \pi(Y')$ for $Y, Y' \subset \mathbb{H}^n$ special, then $\Delta(Y) = \Delta(Y')$, thus $\Delta(T)$ is well-defined.

Further, weakly special subvarieties come in families. Given a partition $(R_0, ..., R_k)$ as in the definition of weakly special subvarieties, we may form a new partition S in which the elements in R_0 are made into individual parts, the parts $R_1, ..., R_k$ are retained, and let S_0 be empty. For every $g \in \prod_{i=1}^k GL_2^+(\mathbb{Q})$ correspond to a unique special subvariety T of X. And for this T, every $t \in \mathbb{H}^{R_0}$ corresponds to a unique weakly special subvariety, we call this translate of T by t, which we denoted by T_t.

5.2 Weakly Complex Ax

In the following we formulate the WCA hypothesis in the context of $Y(1)^n$. While the Ax-Schanuel type conjectures we are using in the abelian varieties are already proved, the ones in modular curves are still unknown. We need several definitions.

Definition 5.6. *By a subvariety $U \subset \mathbb{H}^n$ we mean an irreducible complex analytic component of $W \cap U$ for some algebraic subvariety $W \subset \mathbb{C}$.*

Definition 5.7. *By a component we mean a complex-analytically irreducible component of $W \cap \pi^{-1}(V)$ inside $\mathbb{H}^n$ where $W \subset U$ and $V \subset X$ are algebraic varieties.*

By these notations the WCA can be stated as follows.

Conjecture 5.8. (WCA:Formulation 1) *Let U' be a weakly special subvariety of U. Put $X' = \pi(U')$ and let Y be a component of $W \cap \pi^{-1}(V)$, where $W \subset U'$ and $V \subset X'$ are algebraic subvarieties. If Y is not contained in any proper weakly special subvariety of U', then*

$$\dim Y \leq \dim V + \dim W - \dim X'$$

Then we give another formulation of WCA.

Definition 5.9. *Fix a subvariety V of X.*
1. If W is a component, we define its defect by $\delta(W) = \dim \overline{W}^{\mathrm{zar}} - \dim W$.
2. A component W with respect to V is called optimal for V if here is no strictly larger component Y with respect to V such that $\delta(Y) \leq \delta(W)$.
3. a component W with respect to V is called geodesic if it is a component of $Y \cap V$ for some weakly special subvariety Y with $Y = \overline{W}^{\mathrm{zar}}$.

Conjecture 5.10. (WCA:Formulation 2) *Let $V \subset X$ be a subvariety. An optimal component with respect to V is geodesic.*

It can be proved that two formulations are equivalent.

5.3 A finiteness result

Let $X = Y(1)^n$. In order to state the finiteness result. We need the notion of Mobius subvarieties. The definition of Mobius subvarieties is analogue to that of weakly special subvarieties except that the matrices g are allowed to be any element in $GL_2^+(\mathbb{R})$ rather than $GL_2^+(\mathbb{Q})$. Similar to special subvarieties, Mobius also come in families of "translates", which are defined similarly to that of special subvarieties. Any component $Y \subset \mathbb{H}^n$ is contained in a smallest Mobius subvariety L_Y, which has a Mobius defect

$$\delta_M(Y) = \dim L_Y - \dim Y$$

For a subvariety V of X, a component $Y \subset \pi^{-1}(V)$ is called Mobius optimal (for V in X) if there is no component Z with $Y \subsetneq Z \subset \pi^{-1}(V)$ such that $\delta_M(Z) \leq \delta_M(Y)$. Now we have

Theorem 5.11. *Assume WCA. Let $V \subset X$ be a subvariety. Given a partition $(R_0, ..., R_k)$ of $\{1, ..., n\}$ such that only R_0 is allowed to be empty. Then the set of*

$$g \in \prod_{i=1}^{k} GL_2^+(\mathbb{R})^{R_i}$$

such that for some translate of M_g intersects $\pi^{-1}(V)$ in a component which is Mobius optimal for V is finite modulo the action by $\prod_i SL_2(\mathbb{Z})^{R_i}$.

Proof. By WCA, any g in the assertion corresponds to a weakly special subvariety. Thus all the g_i in the definition of g belong to $GL_2^+(\mathbb{Q})$, thus the g in the question comes from a countable set. Notice that for every such g, there is a translate of which under $SL_2(\mathbb{Z})^{R_i}$ such that the optimal component has points of its full dimension in certain fixed fundamental domain, say $\mathbb{F}_0^n$, and thus the optimality can be checked definably by considering dimensions of the intersection of $\pi^{-1}(V) \cap \mathbb{F}_0^n$. Therefore, in the structure of $\mathbb{R}_{\text{an,exp}}$, there is a definable countable and hence finite set of g that represent all the $SL_2(\mathbb{Z})-$orbits. $\square$

Now we are able to show the finiteness result, which can be regarded as an analogue of theorem 4.7.

Theorem 5.12. *Assume WCA. Let $V \subset X$ be a subvariety. Then there is a finite set of "basic special subvarieties" such that every weakly special subvariety which has a geodesic-optimal component in its intersection with V is a translate of one of them.*

Proof. We pick the finite set as in theorem 5.11. Since we assumed WCA, all g in this set correspond to weakly special families. Also, every $g \in \prod GL_2^+(\mathbb{Q})$ havinga translate with a geodesic-optimal intersection will also appear in this set. Thus Take the set of basic special subvarieties be those special subvarieties determined by the g in the above set, we are done. $\square$

5.4 A conditional result for optimal subvarieties

In this section, we prove that by assuming both the hypothesis of Weakly complex Ax and Large Galois Orbit, the Zilber-Pink conjecture for $Y(1)^n$ is true.

Theorem 5.13. *Assuming LGO and WCA for $X = Y(1)^n$. Let $V \subset X$ be a subvariety. Then $Opt(V)$ is finite.*

Proof. We prove the theorem by induction on the dimension of V. If $\dim V = 0$, then V itself is a point, the assertion holds. Now suppose $\dim V = r > 0$ and the assertion is true for all subvarieties of dimension smaller than r. Take K be the field of V which is a finitely generated $\mathbb{Q}$ algebra. Since all optimal subvarieties are automatically geodesically optimal, by the finiteness result, the subvarieties in $Opt(V)$ are the component of the translate of a special subvariety which comes from a set consisting finitely many so called "basic special subvarieties".

Now fix a basic special subvariety $T \subset X$ in this finite set. It suffices to show that only finitely many translates of T intersect V exists an optimal component. Let X_T be the translate spece, i.e the space of parameters of T, by the definition of translation, X_T is a suitable $Y(1)^m$ for some $m \leq n$.

Write $\phi : X \to Y(1)^m$ for the quotient map. By the property of generic smoothness([Har77] III Corollary 10.7), there is a Zariski open and dense subset $V' \subset V$ such that $\phi|_{V'} : V' \to \phi(V')$ is smooth.

$\square$

6 Acknowledgement

I would like to express my deepest thanks to my advisor Professor Emmanuel Ullmo for introducing me to this amazing area of mathematics and supervising me for the master thesis.

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