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Master Thesis

Homomorphisms and Bounds

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Also I would like to thank all my teachers, my family, and my friends.

Preview

In **Introduction** we will define a few basic things, fix a few notations and introduce the problem we want to study.

In Chapter 2 we will define and discuss **Some Special Graphs**. We will also prove an interesting original result in Section 2.4 of that chapter.

In Chapter 3 we speak about **Planar Ocliques** and oclique numbers. We discuss some existing results and conjectures about them. We also make some conjectures and ask some relevant questions.

In Chapter 4 we discuss about **Oriented Chromatic Number Of Oriented Planar Graphs** and reprove some known results about upper and lower bounds of $\chi(\mathcal{P})$. Also we construct a new example of a graph with oriented chromatic number at least sixteen in Section 4.2. To construct this example, we prove an original result in this section. Then using this example and some known results that we discuss in the same chapter, we construct a new example of an oriented planar graph with oriented chromatic number at least seventeen.

In **Conclusion** we put together what we have done in this master thesis.

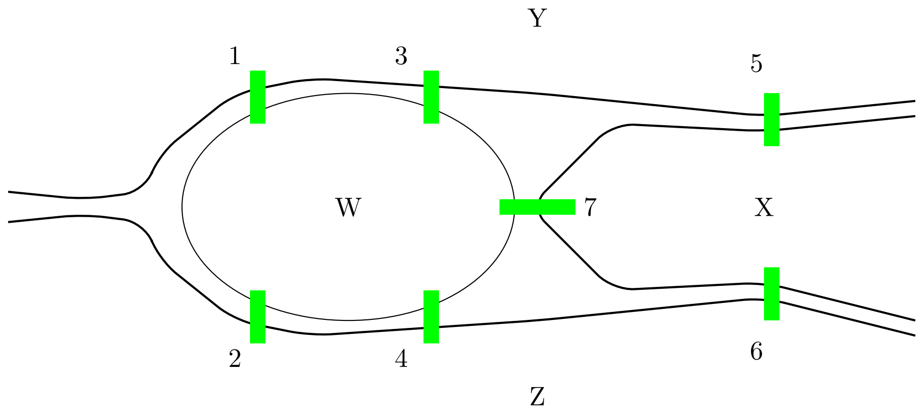
Contents

Chapter 1

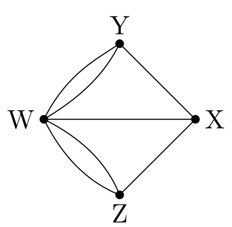
Introduction

We kick-off with a popular problem as motivation.

The Konigsberg Bridge Problem: In Perussia on the river Pregel there was the city of Konigsberg. The city was made up of two islands and some lands on each of the river banks. These four pieces of lands were separated by the river and connected by seven bridges (shown in the diagram below). People wondered if one can start from some point of the city and use every bridge exactly once and come back to the point they started from.



In the diagram we denote the lands by X,Y,Z,W and the bridges (green) by 1,2,3,4,5,6,7.



Solution

Then the problem was reduced and given by the diagram above where the dots (vertices) represent the lands and lines (edges) connecting the dots represent the bridges.

From this diagram we can argue that whenever we enter and leave one piece of land, we use two bridges connected to it. We can also pair the first and the last bridge that we use to leave and enter the piece of land we start from. So to start from any point and travel on each bridge exactly once and come back to the same point, we must have an even number of bridges connected to each of the piece of lands. But we do not have that. So the task cannot be done.

□

This problem, originally modeled (and solved) in the language of graph theory by **Euler** in 1736, is often referred as the beginning of graph theory. Many problems from different areas of studies, as diverse as computer science, coding theory, study of atoms, genetics, chemistry, physics, biology, sociology, architecture etc., can be simplified, studied and solved using graph theory. So we can regard graph theory like a helping tool for other studies.

For me, graph theory is a huge collection of puzzles and riddles and I would like to try solving some of them.

1.1 Basic definitions and notations

Definition 1 A **graph** is an ordered pair $G = (V, E)$ where,

- V is a set of **vertices**
- E is a set of **edges**, where an **edge** is an unordered pair of **vertices**

The vertices in the unordered pair, defining the edge, are called **endpoints** of the edge.

Endpoints are **adjacent** to each other.

The set of **neighbors** of a vertex v in a graph G is the set of vertices adjacent to v in G.

The set of neighbors of a vertex v in a graph is denoted by $N(v)$.

Degree of a vertex v in a graph G is the number of edges having v as their endpoint.

We denote the degree of v by $d(v)$.

Also,

$\Delta(G) = max_{v \in V(G)}\{d(v)\}$ and

$\delta(G) = min_{v \in V(G)}\{d(v)\}$

For a graph G we denote the **vertex set** of the graph by $V(G)$ and the **edge set** of the graph by $E(G)$.

Order of a graph G is given by, $\mid G \mid = \mid V(G) \mid$.

Sometimes we allow $E(G)$ to be a multiset. In this case G is called a multigraph. Notice that the graph in the **Konigsberg Bridge Problem** is a multigraph. But here we will only deal with graphs without multiple edges. Though $V(G)$ can be any set, in this master thesis we will only use finite $V(G)$. Also there are graphs that have edges with the same endpoints. These kind of edges are called loops. But in our definition of a graph we do not allow loops. Basically, the definition we give for a graph is the definition of a **simple graph**. We do this because we will not use graphs with multiple edges or loops in this master thesis.

Definition 2 A **digraph** or **directed graph** is an ordered pair $\overrightarrow{G} = (V, A)$ where,

- V is a set of **vertices**.
- A is a set of **arcs**, where an **arc** is an ordered pair of **vertices**.

For an **arc** (x, y) where $x, y \in V$ we say x is a **successor** of y and y is a **predecessor** of x. x and y are the **endpoints** of the arc.

Endpoints are **adjacent** to each other.

The set of **neighbors** of a vertex v in a graph is the set of vertices adjacent to v in the graph, where a **successor** is an **out (or +)-neighbor** and a **predecessor** is an **in (or -)-neighbor**.

The set of **neighbors** of a vertex v in a digraph is denoted by $N(v)$.

The set of **out-neighbors** of a vertex v in a digraph is denoted by $N^+(v)$.

The set of **in-neighbors** of a vertex v in a digraph is denoted by $N^-(v)$.

Degree of a vertex v in a digraph $\overrightarrow{G}$ is the number of arcs having v as their endpoint.

We denote the degree of v by $d(v)$.Also $d^+(v)$ is the number of vertices which are successors of v and $d^-(v)$ is the number of vertices which are predecessors of v.

If $d^+(v) =\mid V(\overrightarrow{G}) \mid -1$, then v is a **sink**.

If $d^-(v) =\mid V(\overrightarrow{G}) \mid -1$, then v is a **source**.

Also,

$\Delta(\overrightarrow{G}) = max_{v \in V(\overrightarrow{G})}\{d(v)\}$

$\delta(\overrightarrow{G}) = min_{v \in V(\overrightarrow{G})}\{d(v)\}$

For a digraph $\overrightarrow{G}$ we denote the **vertex set** of the graph by $V(\overrightarrow{G})$ and the **arc set** of the graph by $A(\overrightarrow{G})$.

Order of a digraph $\overrightarrow{G}$ is given by, $\mid \overrightarrow{G} \mid = \mid V(\overrightarrow{G}) \mid$

We will sometimes denote an **arc** (x, y) by $x \to y$.

We use this notation in a "free way". For example we may use $\{x \to y \to z \to x\}$ to describe a graph with vertex set $\{x, y, z\}$ and arc set $\{(x, y), (y, z), (z, x)\}$.

For an **arc** (x, y) we have $[x, y] = 1$ and $[y, x] = -1$

If $[x, z][y, z] = 1$, then we say that x, y **agree** on z and if $[x, z][y, z] = -1$, then we say that x, y **disagree** on z.

Like we commented after the definition of a graph, here we can make a similar comment. Sometimes we allow $E(\overrightarrow{G})$ to be a multiset. In this case $\overrightarrow{G}$ is called a multidigraph. Also sometimes we allow loops (i.e. arcs with the same endpoints). So we actually defined **oriented graphs** in the above definition.

We are more interested in simple graphs and oriented graphs. Notice that an oriented graph is not a graph. But sometimes when there is no chance of confusion, we ambiguasly use the term "graph" instead of "oriented graph".

To draw a graph we can draw some points on a plane representing the vertices of the graph and join the points corresponding to the adjacent vertices with a line. These lines represent the edges. We can draw a digraph in a similar way by replacing a the lines by arrows pointing towords the point corresponding to the successor. This drawing is not unique. This is called a **planar embedding** of a graph (digraph). From now on whenever we speak about graphs or digraphs we can think about some planar embedding of it. In this way it will be easy to visualize the graph (digraph). It is good to have the drawing of the graph in mind while reading the theory (at least for this master thesis).

Definition 3 An **underlying graph** of a digraph is the graph that we get by replacing the arcs (ordered pair) of the digraph by edges (unordered pair).

If $\overrightarrow{G}$ is the digraph we denote the **underlying graph** of $\overrightarrow{G}$ by $und(\overrightarrow{G})$.

Sometimes, when there is no chance of confusion, we use a simpler notation. We denote the underlying graph of a digraph $\overrightarrow{G}$ simply by G. That is, we just forget the "arrow" over it.

Remark 4 Underlying graph of an oriented graph is a simple graph.

Definition 5 Let $G = (V, E)$ be a graph. A subgraph induced by the vertex set $V' \subseteq V$ of $G = (V, E)$ is the graph $G' = (V', E')$ where, $E' = \{\{x, y\} \in E \mid x, y \in V'\}$.

We call G' the **induced subgraph of** G **by** V'. We denote it by $G[V']$.

Similarly, let $\overrightarrow{G} = (V, A)$ be a digraph. A subgraph induced by the vertex set $V' \subseteq V$ of $\overrightarrow{G} = (V, A)$ is the digraph $\overrightarrow{G'} = (V', A')$ where, $A' = \{(x, y) \in A \mid x, y \in V'\}$.

We call $\overrightarrow{G'}$ the **induced subdigraph of** $\overrightarrow{G}$ **by** V'. We denote it by $\overrightarrow{G[V']}$.

Definition 6 A subgraph H of a graph G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G[V(H)]$.

A **subdigraph** $\overrightarrow{H}$ of a digraph $\overrightarrow{G}$ is a digraph with $V(\overrightarrow{H}) \subseteq V(\overrightarrow{G})$ and $A(\overrightarrow{H}) \subseteq A(\overrightarrow{G[V(\overrightarrow{H})]})$.

Notice that induced subgraph (induced subdigraph) is also a subgraph (subdigraph). Though here we define subgraph (subdigraph) using the definition of induced subgraph (induced subdigraph), it can be defined independently.

Definition 7 A **planar graph** is a simple graph that can be drawn in a plane in such way that its edges intersect only at their endpoints.

An **oriented planar graph** is an oriented graph whose underlying graph is planar.

Definition 8 An **outerplanar graph** is a planar graph O such that there is a planar graph P with $V(P) = V(O) \cup \{v\}$, v being a vertex adjacent to all the other vertices.

An **oriented outerplanar graph** is an oriented graph whose underlying graph is outerplanar.

1.2 Homomorphisms and bounds

Definition 9 A **homomorphism** of a graph G to a graph H is a map f from $V(G)$ to $V(H)$ such that if $\{a, b\} \in E(G)$, then $\{f(a), f(b)\} \in E(H)$.

We write $f : G \longrightarrow H$ to show that f is a homomorphism of G to H. Also we can write $G \longrightarrow H$ to say that there exist a homomorphism of G to H.

A **homomorphism** $f : G \longrightarrow H$ is an **isomorphism** if f is a bijection from $V(G)$ to $V(H)$.

Let $\mathcal{C}$ be a class of graphs. If for any $G \in \mathcal{C}$, $G \longrightarrow H$, we say H is a **bound** for $\mathcal{C}$ or H **bounds** $\mathcal{C}$ or H is a $\mathcal{C}$**-bound**. If no subgraph of H bounds $\mathcal{C}$, then H is a **minimal** $\mathcal{C}$**-bound**.

A graph is **vertex transitive** if for any pair of vertices $v, u \in V(G)$ there is an automorphism $f : G \longrightarrow G$ such that $f(u) = v$.

Similarly, a **homomorphism** of a digraph $\overrightarrow{G}$ to a digraph $\overrightarrow{H}$ is a map f from $V(\overrightarrow{G})$ to $V(\overrightarrow{H})$ such that if $(a, b) \in A(\overrightarrow{G})$, then $(f(a), f(b)) \in A(\overrightarrow{H})$.

We write $f : \overrightarrow{G} \longrightarrow \overrightarrow{H}$ to show that f is a homomorphism of $\overrightarrow{G}$ to $\overrightarrow{H}$. Also we can write $\overrightarrow{G} \longrightarrow \overrightarrow{H}$ to say that there exist a homomorphism of $\overrightarrow{G}$ to $\overrightarrow{H}$.

A **homomorphism** $f : \overrightarrow{G} \longrightarrow \overrightarrow{H}$ is an **isomorphism** if f is a bijection from $V(\overrightarrow{G})$ to $V(\overrightarrow{H})$.

Let $\overrightarrow{C}$ be a class of digraphs. If for any $\overrightarrow{G} \in \overrightarrow{C}$, $\overrightarrow{G} \longrightarrow \overrightarrow{H}$, we say $\overrightarrow{H}$ is a **bound** for $\overrightarrow{C}$ or $\overrightarrow{H}$ **bounds** $\overrightarrow{C}$ or $\overrightarrow{H}$ is a $\overrightarrow{C}$**-bound**. If no subgraph of $\overrightarrow{H}$ bounds $\overrightarrow{C}$, then $\overrightarrow{H}$ is a **minimal** $\overrightarrow{C}$**-bound**.

An **anti-homomorphism** of a digraph $\overrightarrow{G}$ to a digraph $\overrightarrow{H}$ is a map f from $V(\overrightarrow{G})$ to $V(\overrightarrow{H})$ such that if $(a, b) \in A(\overrightarrow{G})$, then $(f(b), f(a)) \in A(\overrightarrow{H})$.

An **anti-homomorphism** $f : \overrightarrow{G} \longrightarrow \overrightarrow{H}$ is an **anti-isomorphism** if f is a bijection from $(V(\overrightarrow{G})$ to $V(\overrightarrow{H})$.

An **automorphism** of a digraph $\overrightarrow{G}$ is isomorphism to itself.

A digraph is **vertex transitive** if for any pair of vertices $v, u \in V(\overrightarrow{G})$ there is an automorphism $f : \overrightarrow{G} \longrightarrow \overrightarrow{G}$ such that $f(u) = v$.

A digraph is **arc transitive** if for any pair of arcs $(u, v), (x, y) \in A(\overrightarrow{G})$ there is an automorphism $f : \overrightarrow{G} \longrightarrow \overrightarrow{G}$ such that $f(u) = x$ and $f(v) = y$.

If $f : G \longrightarrow H$ ($f : \overrightarrow{G} \longrightarrow \overrightarrow{H}$) and $g : H \longrightarrow S$ ($g : \overrightarrow{H} \longrightarrow \overrightarrow{S}$), then $g \circ f : G \longrightarrow S$ ($g \circ f : \overrightarrow{G} \longrightarrow \overrightarrow{S}$).

That is, homomorphism defines a quasi-order on the class of all graphs (digraphs).

Definition 10 **Core** of a graph (digraph) G is a graph H such that ,

- $G \longrightarrow H$.
- $H \longrightarrow G$.
- no subgraph of H has this property.

A graph (digraph) G is a **core graph (digraph)** if it is a core of itself.

Homomorphism defines a partial order on the class of core graphs (digraphs).

Finding minimal bound for some "nice" class of graphs or digraphs is a very important and interesting problem. Clearly, the minimal bounds are core graphs (digraph). Here we will discuss about the minimal bound of the class of oriented planar graphs. Lets fix a notation $\mathcal{P}$ for the class of oriented planar graphs. So, we will be interested in the **minimal** $\mathcal{P}$**-bound** in this master thesis. This problem is unsolved till now. We will discuss this problem and prove some of the latest results about it.

Definition 11 A graph H is said to be contained in a graph G if H is isomorphic to some subgraph of G.

A digraph $\overrightarrow{H}$ is said to be contained in a digraph $\overrightarrow{G}$ if $\overrightarrow{H}$ is isomorphic to some subdigraph of $\overrightarrow{G}$.

Here we will define some special kinds of graphs and digraphs which we will use frequently.

Definition 12 **Complete graph** on n vertices is a simple graph on n vertices such that there is an edge between any two vertices.

We denote it by K_n.

A **tournament** on n vertices is an oriented graph such that its underlying graph is K_n.

Definition 13 A **path** of **length** n is the graph $P(n)$ such that $V(P(n)) = \{1, 2, 3, ..., n\}$ and $E(P(n)) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, ...\{n-1, n\}\}$

A **directed path** of **length** n is the graph $\overrightarrow{P(n)}$ such that $V(\overrightarrow{P(n)}) = \{1, 2, 3, ..., n\}$ and $A(\overrightarrow{P(n)}) = \{(1, 2), (2, 3), (3, 4), ..., (n-1, n)\}$

A **cycle** of **length** n is the graph $C(n)$ such that $V(C(n)) = \{1, 2, 3, ..., n\}$ and $E(P(n)) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, ...\{n-1, n\}, \{n, 1\}\}$

A **directed cycle** of **length** n is the graph $\overrightarrow{C(n)}$ such that $V(\overrightarrow{C(n)}) = \{1, 2, 3, ..., n\}$ and $A(\overrightarrow{C(n)}) = \{(1, 2), (2, 3), (3, 4), ..., (n-1, n), (n, 1)\}$

We will call a **path/directed path/cycle/directed cycle** of **length** n a n**-path/directed path/cycle/directed cycle**.

We call a directed 3-cycle a **directed triangle**.

A tournament on three vertices with a sink (or source) is called a **transitive triangle**.

These definitions are up to isomorphism.

Definition 14 A **tree** is a graph such that there is exactly one simple path between any two vertices. In other words, any connected cycle free graph is a **tree**.

Disjoint union of **tree(s)** is called **forest**.

A **directed tree (forest)** is a digraph whose underlying graph is a **tree (forest)**.

Now we want to state a famous theorem by Wagner. For that we need to define a few things.

Definition 15 In a graph G we can remove one edge and identify the endpoints of that edge to get another graph. This process is called **edge contraction**.

Definition 16 A graph H obtained (up to isomorphism) by zero or more edge contractions on a subgraph of a graph G is called a **minor** of the graph G.

Definition 17 A **clique** in a graph is a subset of the vertex set of the graph that induces a complete graph.

If the induced complete graph is K_n then we call the **clique** an n**-clique**.

The clique number of a graph G is an integer $w(G)$ such that there is a $w(G)$-clique but not a $(w(G) + 1)$-clique in G.

For a class $\mathcal{C}$ of graphs we have $w(\mathcal{C}) = max\{w(G) \mid G \in \mathcal{C}\}$.

Definition 18 An **independent set in a graph** is a subset of the vertex set that induces a subgraph with no edges.

Definition 19 A **k-partite graph** is a graph whose vertex set can be written as the disjoint union of k independent sets. These k independent sets are called **parts** of the graph. The parts may not be unique.

Definition 20 A **complete k-partite graph** is a simple k-partite graph with all the edges between two different parts.

We call a 2-partite graph a **bipartite graph**.

Moreover, we denote a **complete bipartite graph** with parts of size r and s by $K_{r,s}$.

Here we state a very important theorem due to **Wagner** about planar graphs:

Theorem 21 *A graph (finite) is planar if and only if it does not have K_5 or $K_{2,3}$ as a minor.*

Definition 22 For an oriented graph $\overrightarrow{G}$ we define, $\overrightarrow{G}^2$ by,

•$V(\overrightarrow{G}^2) = V(\overrightarrow{G})$

•$A(\overrightarrow{G}^2) = A(\overrightarrow{G}) \cup \{(x,y) | (x,y)$ and $(y,x) \notin A(\overrightarrow{G})$, but (x,w) and $(w,y) \in A(\overrightarrow{G})$ for some $w \in V(\overrightarrow{G})\}$.

Definition 23 An oclique is a digraph $\overrightarrow{G}$ such that $\overrightarrow{G}^2$ is a tournament.

The oclique number $w_o(\overrightarrow{G})$ of a digraph $\overrightarrow{G}$ is $w(und(\overrightarrow{G}^2))$.

For a class $\mathcal{C}$ of digraphs we have $w_o(\mathcal{C}) = max\{w_o(\overrightarrow{G}) \mid \overrightarrow{G} \in \mathcal{C}\}$.

Definition 24 Let S be a set with $\mid S \mid= k$. Let G be a graph. A function $f : V(G) \longrightarrow S$ is called a **k-coloring** of G.

Elements of the set S are called **colors**.

Definition 25 f is a **proper k-coloring** of G if,

(i) f is a **k-coloring** of G.
$(ii)\{x,y\} \in E(G)$ implies, $f(x) \neq f(y)$.

If G has a **proper k-coloring**, then G is **k-colorable**.

The **chromatic number** $\chi(G)$ of G is the least **k** such that G is **k-colorable**.

For a class $\mathcal{C}$ of graphs we have $\chi(\mathcal{C}) = max\{\chi(G) \mid G \in \mathcal{C}\}$.

Definition 26 Let S be a set with $\mid S \mid= k$. Let $\overrightarrow{G}$ be an oriented graph. An function $f : V(\overrightarrow{G}) \longrightarrow S$ is called an **oriented k-coloring** of $\overrightarrow{G}$.

Elements of the set S are called **colors**.

Definition 27 f is a **proper oriented k-coloring** of $\overrightarrow{G}$ if,

(i) f is an **oriented k-coloring** of $\overrightarrow{G}$.
$(ii)(x, y) \in A(\overrightarrow{G})$ implies, $f(x) \neq f(y)$.
(iii) if $\{(x, y), (u, v)\} \subseteq A(\overrightarrow{G})$ and $f(y) = f(u)$, then $f(x) \neq f(v)$.

If $\overrightarrow{G}$ has a **proper oriented k-coloring**, then $\overrightarrow{G}$ is said to be **oriented k-colorable**.

The **oriented chromatic number** $\chi_o(\overrightarrow{G})$ of $\overrightarrow{G}$ is the least **k** such that $\overrightarrow{G}$ is **oriented k-colorable**.

For a class $\mathcal{C}$ of digraphs we have $\chi_o(\mathcal{C}) = max\{\chi_o(\overrightarrow{G}) \mid \overrightarrow{G} \in \mathcal{C}\}$.

From now on whenever we speak about coloring in the context of a simple graph, we will actually be refering to a proper coloring unless otherwise stated. Also k-coloring of a simple graph will mean proper k-coloring of it unless otherwise stated.

In the context of an oriented graph, coloring/proper coloring/oriented coloring or k-coloring/oriented k-coloring will refer to a proper oriented coloring or proper oriented k-coloring respectively, unless otherwise stated.

The following classic results can be found in the book **Graphs and Homomorphisms** written by **Pavol Hell** and **Jaroslav Nesetril** [1] .

Lemma 28 Let $f : G \longrightarrow H$ be a homomorphism. Then, $w(G) \leq w(H)$.

Proof
There is a $w(G)$-clique in G. The image of this $w(G)$-clique in H is also a $w(G)$-clique. So H has a $w(G)$-clique. Therefore, $w(G) \leq w(H)$. $\square$

Lemma 29 If there is a homomorphism $f : \overrightarrow{G} \longrightarrow \overrightarrow{H}$, then there is a homomorphism $g : und(\overrightarrow{G}^2) \longrightarrow und(\overrightarrow{H}^2)$.

Proof
This is clear from the definitions of homomorphism and $und(\overrightarrow{G}^2)$ of a oriented graph $\overrightarrow{G}$. $\square$

Lemma 30 Let $f : \overrightarrow{G} \longrightarrow \overrightarrow{H}$ be a homomorphism. Then, $w_o(\overrightarrow{G}) \leq w_o(\overrightarrow{H})$.

Proof
By Lemma 29 we have some homomorphism $g : und(\overrightarrow{G}^2) \longrightarrow und(\overrightarrow{H}^2)$. Now by Lemma 28 we have $w_o(\overrightarrow{G}) = w(und(\overrightarrow{G}^2)) \leq w(und(\overrightarrow{H}^2)) = w_o(\overrightarrow{H})$. $\square$

Lemma 31 Let $f : G \longrightarrow H$ be a homomorphism. Then, $\chi(G) \leq \chi(H)$.

Proof
If c is a proper $\chi(H)$-coloring of H, then $c \circ f \mid_{V(G)}$ gives a proper $\chi(H)$-coloring of G. $\square$

Lemma 32 Let $f : \overrightarrow{G} \longrightarrow \overrightarrow{H}$ be a homomorphism. Then, $\chi_o(\overrightarrow{G}) \leq \chi_o(\overrightarrow{H})$.

Proof
If c is a proper oriented $\chi_o(\overrightarrow{H})$-coloring of $\overrightarrow{H}$, then $c \circ f \mid_{V(\overrightarrow{G})}$ gives a proper oriented $\chi_o(\overrightarrow{H})$-coloring of $\overrightarrow{G}$.

□

Lemma 33 For any graph (digraph)G ($\overrightarrow{G}$), $w(G) \leq \chi(G)$ ($w_o(\overrightarrow{G}) \leq \chi_o(\overrightarrow{G})$).

Proof
A graph G has a $w(G)$-clique. Any proper coloring of the subgraph induced by that $w(G)$-clique will use $w(G)$ colors. And we have the inclusion homomorphism from the subgraph to the graph. So by Lemma 28 $w(G) \leq \chi(G)$.

Similarly for the digraph using Lemma 32.

□

Definition 34 The **color graph** $C_c(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ for a proper oriented coloring c of $\overrightarrow{G}$ is given by:
$(i)V(C_c(\overrightarrow{G})) = \{c(x) \mid x \in V(\overrightarrow{G})\}$
$(ii)A(C_c(\overrightarrow{G})) = \{(c(x), c(y)) \mid (x, y) \in A(\overrightarrow{G})\}$

Remark 35 The color graph is also an oriented graph.

In other words, there is a homomorphism from $\overrightarrow{G}$ to $C_c(\overrightarrow{G})$ given by c.

By the remark above we can say that for a class C of oriented graphs, the minimal C-bound will be on at least $\chi_o(C)$ vertices. So to find the minimal C-bound, “what is $\chi_o(C)$?” is a good question to ask.

Also finding $w_o(C)$ helps to answer the question. And for any oclique $\overrightarrow{G}$ is of course a subgraph of the minimal $\mathcal{P}$-bound.

But how good is the question? Sometimes it is very good, especially if we know that the minimal C-bound is actually on $\chi_o(C)$ vertices. But unfortunately this is not always true.

For example, if we take the class of all oriented graphs on n vertices, the class has oriented chromatic number n, but no graph on n vertices will contain all the tournaments on n vertices as subgraph.

We are interested in the class $\mathcal{P}$ of oriented planar graphs. The question is a very good question for this class because of the following result by Sopena:

Theorem 36 *The minimal $\mathcal{P}$-bound has $\chi_o(\mathcal{P})$ vertices.*

Proof
Let $\chi_o(\mathcal{P}) = k$. Suppose the theorem is not true. Then there exists no $\mathcal{P}$-bound on k vertices. That means, for each oriented graph $\overrightarrow{U}$ on k vertices there exist some oriented planar graph $P_{\overrightarrow{U}}$ that does not admit homomorphism to $\overrightarrow{U}$.

Let $\{\overrightarrow{U_1}, ..., \overrightarrow{U_l}\}$ be the set of all oriented graphs on k-vertices.

Let $\overrightarrow{P} = \sqcup_{i=1,2,...,l} P_{\overrightarrow{U_i}}$ be the disjoint union of all $P_{\overrightarrow{U_i}}$s.

$\overrightarrow{P}$ is clearly planar and it has a proper oriented $\chi_o(\mathcal{P})$-coloring c.

Now $C_c(\overrightarrow{P}) = \overrightarrow{U_i}$ for some $i \in \{1, 2, ..., l\}$ is an oriented graph on k vertices to which $\overrightarrow{P}$ admits a homomorphism. Now restrict that homomorphism to $P_{\overrightarrow{U_i}}$ and get a contradiction.

□

For the class P of planar graphs, the minimal bound is K_4 which itself belongs to the class P. So $\chi(P) = w(P) = 4$. This is a graph theoric formulation of the famous **four color theorem**.

Theorem 37 *(Four color) Every planar graph is four colorable.*

In 1852 this statement was proposed as a conjucture. Then there were a few false proofs given over a long period of time until finally in 1989 it was proved by Appel and Haken. The proof was very big and complicated and used computer programming for the very complex calculations. Till now we do not have a hand proof for the theorem.

The equivelant question for the class of oriented-planar graphs ($\mathcal{P}$) also seems to be a very difficult one. We seek for the minimal $\mathcal{P}$-bound. Knowing what $\chi_o(\mathcal{P})$ and $w_o(\mathcal{P})$ is would certainly help in answering the question.

Chapter 2

Some Special Graphs

Here we will define some special oriented graphs and family of oriented graphs. First we will define a highly symetric family of tournaments called the Paley tournaments and state some well known facts about it and reprove few properties of a particular tournament P_7 of that family. Then we will define another highly symmetric family of oriented graphs called Tromp graphs. This family was introduced by Tromp and then generalized by Albiero and Sopena [6] . Then we will define another family of oriented graphs called the Zielonka graphs introduced by Zielonka.

After this we will define property Q_n and clasify all oriented graphs on eight vertices having property Q_2.

These will serve as a tool for what we are going to do in the later part of this master thesis.

2.1 Paley tournaments

Definition 38 Let $q = p^n$ such that $q \equiv 3 (\text{mod } 4)$ for some prime p. We know there exists a field F_q of order q unique upto isomorphism. The Paley torunament P_q is given by,

•$V(P_q) = \{x | x \in F_q\}$

•$A(P_q) = \{(x, y) \mid y - x \text{ is a non-zero square in } F_q\}$

As -1 is not a square in F_q (for $q \equiv 3(\text{mod } 4)$), either $x - y$ or $y - x$ (but not both) is a square for all $x, y \in F_q$.

Hence P_q indeed is a tournament.

From now on, for $q = p$ prime, we will assume $F_p = \mathbf{Z}/p\mathbf{Z}$ without loss of generality in the definition of P_p. The elements of $\mathbf{Z}/p\mathbf{Z}$ will be denoted by $\{0, 1, 2, ..., p - 1\}$ in the natural way.

It is known (for example you can find the proof in Marshall's paper [2]) that P_q is arc transitive .

Definition 39 An oriented graph $\overrightarrow{G}$ has property Q_1 if it has a directed cycle as a subgraph.

An oriented graph $\overrightarrow{G}$ has property Q_n if for every vertex $v \in V(\overrightarrow{G})$, $\overrightarrow{G[N^+(v)]}$ and $\overrightarrow{G[N^-(v)]}$ both has property Q_{n-1}.

This definition was originally given by Marshall [2] . He also proved that the minimal $\mathcal{P}$-bound has property Q_3. So for every vertex v in minimal $\mathcal{P}$-bound the graph induced by $N^+(v)$ and $N^-(v)$ both has property Q_2. So it is important to know oriented graphs with property Q_2. Hence the following lemma. This lemma was stated without proof in Marshall's paper [2] . Here we give a proof.

Lemma 40 If an oriented graph $\overrightarrow{H}$ has property Q_2, then $\overrightarrow{H}$ has at least seven vertices.

Furthermore, P_7 is the only oriented graph on seven vertices that has property Q_2.

Proof
A directed cycle needs at least 3 vertices. So if an oriented graph has property Q_2, then every vertex of the oriented graph should have at least 3 in-neighbors and 3 out -neighbors.

That means, an oriented graph with property Q_2 should have at least 7 vertices and each vertex of the graph should have in-degree and out-degree exactly equal to 3.

Let $\overrightarrow{H}$ be an oriented graph on 7 vertices with property Q_2. We will construct the graph.

Clearly by what we said before, $\overrightarrow{H}$ is a tournament.

Without loss of generality consider a vertex 0 of $\overrightarrow{H}$. Let $\overrightarrow{H[N^+(0)]} = \{1 \to 2 \to 4 \to 1\}$ and $\overrightarrow{H[N^-(0)]} = \{3 \to 5 \to 6 \to 3\}$.

Now notice that each vertex in $N^-(0)$ already has out-degree 2 and each vertex in $N^+(0)$ already has in-degree 2.

So there is a one-one correspondence between $A = \{3, 5, 6\}$ and $B = \{1, 2, 4\}$ such that there is an arc between the corresponding vertices (from the vertex of A to the vertex of B). All the other arcs are from the B to A.

Without loss of generality assume the arc $3 \to 4$. This will imply the arcs $1 \to 3$ and $2 \to 3$.

Now we have $N^-(3) = \{1, 2, 6\}$. This should be a directed cycle. This will force $6 \to 1$.

Now, the one-one correspondence between $A = \{3, 5, 6\}$ and $B = \{1, 2, 4\}$ implies $5 \to 2$. Now we can complete the graph.

This is exactly the graph P_7. $\square$

Now we will prove two more lemmas stated without proof in Marshall's paper [2] .

Lemma 41 For $x, y \in V(P_7)$, x, y agree exactly on 2 vertices and disagree exactly on 3 vertices.

Proof
As P_7 is arc transitive, it is enough to show for $x = 0$ and $y = 1$. Now 0 and 1 agrees on $\{2, 6\}$ and disagrees on $\{5, 3, 4\}$. $\square$

Lemma 42 For any two disjoint directed cycle C, C' in P_7 we have $\{V(C), V(C')\} = \{N^+(v), N^-(v)\}$, for some $v \in V(P_7)$.

Proof
From proof of Lemma 41 we can say that $N^{\sigma}(u) \cap N^{\tau}(v) \neq N^{\sigma}(u)$ and $N^{\sigma}(u) \cap N^{\tau}(v) \neq \phi$ for any two distinct $u, v \in V(P_7)$ and for $\sigma, \tau \in \{+, -\}$

So, $\{P_7[N^+(v)], P_7[N^-(v)]\}_{v \in V(P_7)}$ is a set of 14 distinct directed triangles in P_7.

Now for P_7 we have,

$$\begin{aligned}
\#\text{directed triangles} &= \#\text{triangles} - \#\text{transitive triangles} \\
&= \binom{7}{3} - 7 \times \binom{3}{2} \\
&= 35 - 21 \\
&= 14
\end{aligned}$$

So, the only directed triangles are the directed triangles induced by $N^+(v)$ or $N^-(v)$ for some $v \in V(P_7)$

Now C, C' are disjoint directed cycles in P_7, then one of them, say C, is a directed triangle. Thus $V(C) = N^{\pm}(v)$. Now $C' \subseteq N^{\mp}(v) \cup \{v\}$. So C' cannot contain v. So $C' = N^{\mp}(v)$. □

2.2 Tromp graphs

Here we will define another highly symmetric family of oriented graphs called Tromp graphs. This family was introduced by Tromp and then generalized by Albiero and Sopena [6] .

Definition 43 Let $q = p^n$ such that $q \equiv 3 (\text{mod } 4)$ for some prime p. The Tromp graph T_{2q+2} of order $(2q + 2)$ is given by,

•$V(T_{2q+2}) = \{(x, i) \mid x \in F_q \cup \{\infty\} \text{ and } i \in \{1, -1\}\}$

•$A(T_{2q+2}) = \{(x, i) \to (y, j) \mid y - x$ is a square in F_q and $ij = 1$ or $x - y$ is a square in F_q and $ij = -1$ for $x, y \in F_q\} \cup \{(x, i) \to (y, j) \mid x = \infty, y \in F_q$ and $ij = 1$ or $x \in F_q, y = \infty$ and $ij = -1\}$.

Informally, Tromp graph T_{2q+2} of order $(2q + 2)$ consists of two copies of P_q, along with two vertices $(\infty, 1)$ and $(\infty, -1)$ that disagree with each other completely and $(\infty, 1)$ has one of the P_qs as its in-neighbor and the other as its out-neighbor. To describe the arcs between the two P_qs we first fix an isomorphism $f : N^-((\infty, 1)) \longrightarrow N^+((\infty, 1))$ between the two P_qs. For each vertex v in $N^-((\infty, 1))$ there is no arc between v and $f(v)$, and v and $f(v)$ disagree with each other completely.

Albiero and Sopena [6] had shown that T_{2q+2} is vertex transitive.

Here we state a lemma from Marshall's paper [2] about P_q without proof.

Lemma 44 A vertex set $S \subseteq V(T_{2q+2})$ is such that $T_{2q+2}[S] \cong P_q$ if and only if $S = N^+(v)$ or $N^-(v)$ for some $v \in V(T_{2q+2})$.

2.3 Zielonka graphs

Now we will define another family of oriented graphs called the Zielonka graphs introduced by Zielonka.

Definition 45 Let k be a positive integer. The Zielonka graph $Z(k)$ is given by,

•$V(Z(k)) = \cup_{i=1,2,\dots,k} S_i$, where $S_i = \{x = (x^1, ..., x^k) | x^j \in \{0,1\}$ for $j \neq i$ and $x^{(i)} = *\}$.

•$A(Z(k)) = \{x \rightarrow y \mid$ for $x \in S_i, y \in S_j$ such that either $x^{(j)} = y^{(i)}$ and $i < j$ or $x^{(j)} \neq y^{(i)}$ and $i > j\}$.

Here $x^{(i)}$ refers to the i^{th} coordinate of x.

Note that, the underlying graph of $Z(k)$ is a complete k-partite graph. Clearly $Z(k)$ has $k \times 2^{k-1}$ vertices.

2.4 Oriented graphs on eight vertices having property Q_2

To prove that the minimal $\mathcal{P}$-bound has maximum degree at least 16, Marshall [2] used Lemma 40. To improve the result, it seems that we should know some informations about oriented graphs on 8 vertices having property Q_2.

We would like to classify all graphs on 8 vertices having property Q_2. To this end we first provide examples of such graphs.

Definition 46 A **twin** of a vertex agrees with it on every other vertices of the graph.

An **anti-twin** of a vertex disagrees with it on every other vertices of the graph.

Definition 47 P_7 with a twin $0'$ of 0 gives us new graph on 8 vertices. We will call it P_7^+.

Definition 48 P_7 with an antitwin $0'$ of 0 gives us a new graph on 8 vertices. We will call it P_7^-.

Notice that both P_7^+ and P_7^- has property Q_2 and also T_8 has property Q_2. Now we are going to prove that any graph on 8 vertices having property Q_2 must have one of these graphs as subgraph. To this end we first classify all tournaments on 8 vertices having property Q_2. We show below that each such tournament is obtained from one of these 3 graphs (P_7^+, P_7^- and T_8) plus some more arcs. The new arcs will be called **blue arcs**. This is an original result proved in this master thesis. This result is proved by Sopena, Naserasr and Sen.

Theorem 49 *If G is a tournament on 8 vertics and has property Q_2, then G is isomorphic to one of the following oriented graphs:*

G_1 : P_7^+ *plus* $(0, 0')$,
G_2 : P_7^- *plus* $(0, 0')$,
G_3 : P_7^- *plus* $(0', 0)$,
G_4 : T_8 *plus*

$((1,-1),(1,1))$, $((2,-1),(2,1))$, $((3,-1),(3,1))$ *and* $((\infty,-1),(\infty,1))$,
G_5 : T_8 *plus*
$((1,-1),(1,1))$, $((2,-1),(2,1))$, $((3,-1),(3,1))$ *and* $((\infty,1),(\infty,-1))$.

Before proving the theorem we will prove some lemmas which will help us to prove the theorem.

Fix G to be a tournament on 8 vertices and have property Q_2.

Then each of the vertices of G will have in-degree (out-degree) at least 3.

Remark 50 A **semiregular tournament** is a tournament on $2n$ vertices such that it has n vertices with out-degree n and n vertices with out-degree $(n-1)$. Notice that G is semi-regular.

Lemma 51 For any $v \in V(G)$, $N^+(v)$ and $N^-(v)$ contains a directed triangle.

Proof
If not, then one of $N^+(v)$ and $N^-(v)$ has a directed 4-cycle. But G is a tournament. So there is a directed triangle in the directed 4-cycle. □

Lemma 52 There is a pair $\{u, v\} \subseteq V(G)$ and a directed triangle $T \subseteq G$ such that all the three vertices of T agrees with each other on u (and v).

Proof
For each $x \in V(G)$ we have $N^+(v)$ and $N^-(v)$. Call them $N_1(x)$ and $N_2(x)$ such that $\mid N_1(x) \mid \geq \mid N_2(x) \mid$. So $\mid N_1 \mid = 4$ and have a directed triangle by Lemma 51

But $N_1(x)$ is a tournament. If $N_1(x)$ have exactly one directed triangle, then it has a source or a sink. Then we are done.

If not, then for any $x \in V(G), N^+(x) \cup N^-(x)$ contains three directed triangles. If the lemma is false, then all of these triangles are distinct. So there are $4 \times 3 + 4 \times 3 = 24$ distinct directed triangles.

G is semi-regular.

$$
\begin{aligned}
\#\text{directed triangles} &= \#\text{triangles} - \#\text{transitive triangles} \\
&= \binom{8}{3} - [4 \times \binom{4}{2} + 4 \times \binom{3}{2}], \\
&\text{(we are counting the transitive triangles using sources)} \\
&= 56 - 36 \\
&= 20
\end{aligned}
$$

This is a contradiction. □

Lemma 53 There is a vertex v in G with a twin or an antitwin u.

Proof
Lemma 52 implies that there exists a directed triangle T and vertices u,v such that all the three vertices of T agrees with each other on u (and v).

If u, v agree on T, then asume without loos of generality that u sees v and T in the same way. G has property Q_2. So there is another directed triangle

T' which disagree on u with T. Now there must be a directed triangle that disagree on v with T. This directed triangle cannot use vertex u because other two vertices are form T' and they cannot form a directed triangle with u. So T' is the triangle. Hence u, v are twins.

If u, v disagree on T, then either u (and v) and T agree on v (and u) or u (and v) and T disagree on v (and u).

In the 1^{st} case u and v are antitwin because there must be a directed triangle that disagree on u (or v) with T and the only option for that is the 3 vertex in $V(G) \setminus \{V(T), u, v\}$.

In the 2^{nd} case if u and v are not antitwin, then the following is forced.

Without loss of generality, assume $T \subseteq N^-(u)$ and $v \in N^+(u)$. If u, v are not antitwin, then either v is a vertex of the directed triangle in $N^+(u)$ or u is a vertex of the directed triangle in $N^-(v)$. If v is a vertex of the directed triangle in $N^+(u)$ and let $\{v \to b \to a \to v\} \subseteq N^+(u)$ be the directed triangle. Now $|N^+(v)| = 4$ already. So we get final vertex c (say) to be such that $\{c, u, a\}$ is the directed triangle in $N^-(v)$.

If u is a vertex of the directed triangle in $N^-(v)$, then we force the same structure similarly.

Assume $T = \{1 \to 2 \to 3 \to 1\}$

Now we have two cases.

case1 $\mid N^+(a) \mid= 3$.

Without loss of generality assume $a \to 3$.

This implies $1 \to a$ and $2 \to a$.

Now, $N^+(1) \supseteq \{a, 2, u\}$ where $\{a, u\} \subseteq N^-(2) \Rightarrow\mid N^+(1) \mid= 4$ and $\mid N^-(1) \mid= 3$ but $\{v \to 3\} \subseteq N^-(1)$ and $v \to b$ implies $b \in N^+(1)$ and $c \in N^-(1)$.

Now, to complete the directed triangle in $N^+(1)$ we have $b \to 2$.

Now, $N^-(2) \supseteq \{1 \leftarrow v \to b\}$ implies $c \in N^-(2)$.

To complete the directed triangle in $N^-(2)$ we have $b \to c$.

Now, $N^+(3) \supseteq \{1 \leftarrow c \to u\}$ which cannot make a directed triangle implies $b \in N^+(3)$.

This completes the graph.

Here 1 is antitwin of a.

case2 $\mid N^-(a) \mid= 3$

Without loss of generality assume $3 \to a$.

This implies $a \to 1$ and $a \to 2$.

Now, $\{3 \to u \to b\} \subseteq N^-(a)$. To complete the directed triangle we need $b \to 3$.

Now, $\{2 \leftarrow v \to b\} \subseteq N^-(3)$. We cneed one more to get a directed triangle. This forces $c \to 3$.

$\{v \to 1 \leftarrow a\} \subseteq N^-(2)$ implies $\mid N^+(2) \mid= 3$ but, $N^+(2) \supseteq \{3 \to u\}$ so this forces $b \in N^+(2)$ which implies $c \in N^-(2)$.

To complete the directed triangle in $N^-(2)$ we get $1 \to c$.

$\{c \to u \leftarrow 2\} \subseteq N^+(1)$ implies $b \in N^+(1)$.

Now we have $N^-(b) = \{v, u, 1, 2\}$ implies $b \to c$.

This completes the graph.

Here 2 is antitwin of a.

□

Lemma 54 Let G be a tournament on 8 vertices having property Q_2. If G has a pair u, v of twin vertices, then $G \cong G_1$.

Proof
Any $x \in V(G) \setminus \{u, v\}$ is such that $N^+(x)$ and $N^-(v)$ both contains a directed triangle. (Lemma 51). Now, every $x \in V(G)$ sees u and v in the same way but they cannot both be in the same directed triangle. But, they are twins. So if one of them is in a directed triangle, we can replace it with the other and still have a directed triangle.

Hence $G \setminus \{u\} \cong G \setminus \{v\}$ also have property Q_2.

So by Lemma 40 $G \cong G_1$. □

Proof of Theorem 49

Let u, v be antitwins and $T_1 = \{x \to y \to z \to x\} \subseteq N^+(v)$(and $N^-(u)$) and $T_2 = \{a \to b \to c \to a\} \subseteq N^-(v)$ (and $N^+(u)$) Now as G is a tournament, there are 9 arcs between T_1 and T_2. Let $m = \#\{$ arcs from T_1 to $T_2\}$ and $n = \#\{$ arcs from T_2 to $T_1\}$. without loss of generality we can assume $n \geq m$.

For any $v \in V(T_i)$ there is at least 1 arc from T_i to T_j and at least 1 arc from T_j to T_i having v as one of the endpoints for $i, j = 1, 2$ and $i \neq j$ otherwise, $N^+(v)$ or $N^-(v)$ will be less than 3.

So that means $m, n \geq 3$. But as $n \geq m$, we conclude that either $(m, n) = (3, 6)$ or $(m, n) = (4, 5)$.

case1 Let $(m, n) = (3, 6)$. We know any two arcs from T_1 to T_2 cannot have a common endpoint. Without loss of generality let $x \to a$. Now, there can be two cases.

subcase1: $y \to c$ and $z \to b$. In this case if $u \to v$, then $G \cong G_2$ and if $v \to u$, then $G \cong G_3$ [isomorphisms given by sending (u, v, x, y, z, a, b, c) to $(0, 0', 3, 5, 6, 4, 2, 1)$].

subcase2: $y \to b$ and $z \to c$. In this case if $u \to v$, then $G \cong G_4$ and if $v \to u$, then $G \cong G_5$ [isomorphisms given by sending (a, z, b, x, c, y, u, v) to $((1, -1), (1, 1), (2, -1), (2, 1), (3, -1), (3, 1), (\infty, -1), (\infty, 1))$]

case2 Let $(m, n) = (4, 5)$. We know 3 among the 5 arcs from T_2 to T_1 are such that no two of them have common endpoints. Also no vertex in T_1 can recieve more than 2 arcs from T_2

Without loss of generality assume that x recieves only one arc from T_1 and that from a. Now there can be two subcases.

subcase1:Let $b \to y$ and $c \to z$. Now, if possible, let $c \to y$. Then, $N^-(c) = \{x \to b \leftarrow u\}$ which contradicts property Q_2 of G. So, $y \to c$.

Now let, if possible, $a \to z$. This implies $y \to a$. This implies $V(N^-(a)) = \{c, y, u\}$. But this is not a directed triangle. This is a contradiction. So $z \to a$.

This implies, $b \to z$ and $a \to y$. So we got all the 5 arcs from T_2 to T_1. Hence all the arcs from T_1 to T_2.

Now, if $u \to v$, then $G \cong G_5$ [isomorphism given by sending (x, c, u, v, a, y, z, b) to $((1, -1), (1, 1), (2, -1), (2, 1), (3, -1), (3, 1), (\infty, -1), (\infty, 1))$].

And,if $v \to u$, then $G \cong G_5$ [isomorphism given by sending (b, z, v, u, x, c, a, y) to $((1, -1), (1, 1), (2, -1), (2, 1), (3, -1), (3, 1), (\infty, -1), (\infty, 1))$].

subcase2:Let $b \to z$ and $c \to y$. Now $N^-(x) = \{a, z, v\}$ should be a directed triangle. Then $z \to a$. Now, if possible, let $a \to y$. This implies, $y \to b$. This implies, $c \to z$. But then $N^+(b) = \{c \to z \leftarrow v\}$ which contradicts property Q_2. So $y \to a$. So, $b \to y$ and $c \to z$. This is ***subcase1***.

Now we also have Lemma 54. So we are done.

□

Corollary 55 If a graph on 8 vertices has property Q_2, then it has P_7^+, P_7^- or T_8 as a subgraph.

Proof
Note that, adding edges does not kill property Q_2. So any graph on 8 vertices having property Q_2 is a subgraph of G_i for some $i \in \{1, 2, 3, 4, 5\}$ as in Theorem 49. Now notice that if we delete any blue arc from G_i, then we do not kill property Q_2 of the graph (it is easy to check). But after deleting the blue arcs we still have P_7^+, P_7^- or T_8 as a subgraph.

Claim: If we delete any arc other than the blue arcs from G_i, then we kill property Q_2.

If we can prove this claim, then we are done.

In the following proof of the claim, we use arc to mention some arc other than blue arcs.

Proof of the claim: Let u, v be twins or antitwins in G_i. Now let us delete any arc which has u as a an endpoint. Then either $N^+(u)$ or $N^-(u)$ can not have a directed cycle because either it is left with only 2 vertices or with v and 2 other vertices that agree with each other on v. So we kill property Q_2 if we delete any arc which has a vertex having a twin or antitwin as its endpoint.

Now in G_4 and G_5 every arc is endpoint of a vertex having an antitwin. So we are done for G_4 and G_5.

For G_1,G_2 and G_3 we kill property Q_2 if we delete any arc having 0 or $0'$ as endpoints. Also we kill property Q_2 if we delete any arc of the directed triangles induced by $\{1, 2, 4\}$ and $\{3, 5, 6\}$ because otherwise $N^+(0)$ and $N^-(0)$ will not have a directed cycle in them.

Now by symmetry, it will be enough to check if we kill property Q_2 if we delete one of $(3, 4), (1, 3)$ and $(2, 3)$ from G_i for $i = 1, 2, 3$.

Now, if we delete any one of $(3, 4)$, then $N^+(3)$ will not have a directed cycle in it.

If we delete $(1, 3)$, then $N^+(6)$ will not have a directed cycle in it.

If we delete $(2, 3)$, then $N^-(4)$ will not have a directed cycle in it.

This proves the claim.

□

Corollary 56 If a graph on 8 vertices has property Q_2, then it has P_7 or T_8 as a subgraph.

Proof
Immediately from Corollary 55. □

Chapter 3

Planar Ocliques

We know that any planar oclique is a subdigraph of the minimal $\mathcal{P}$-bound. Here we present an oclique on 15 vertices and conjetture about how big can a planar oclique be. Also we discuss about the upper and lower bounds of $w_o(\mathcal{P})$.

3.1 Bounds known for planar ocliques

We call the oriented graph presented by the diagram in the next page, the "butterfly graph". We denote it by $\overrightarrow{B_0}$. It can have some different planar representations.

Remark 57 Any two vertices in $\overrightarrow{B_0}$ have a path of length 1 or 2 between them. So $\overrightarrow{B_0}$ is an oclique on 15 vertices..

From the above remark we immediately conclude $w_o(\mathcal{P}) \geq 15$. But the question is "is there a bigger oclique?". We do not know the answer till now. But we have the following conjecture by Klostermeyer and MacGillivray [7]:

Conjecture 1 *A planar oclique can at most be on 15 vertices.*

We have an upper bound for $w_o(\mathcal{P})$ from the result proved by Raspaud and Sopena [4] , that $\chi_o(\mathcal{P}) \leq 80$. This result and Lemma 33 gives us $w_o(\mathcal{P}) \leq 80$. We will give a proof of this result in the next chapter. This upper bound implies that any oclique has at most 80 vertices.

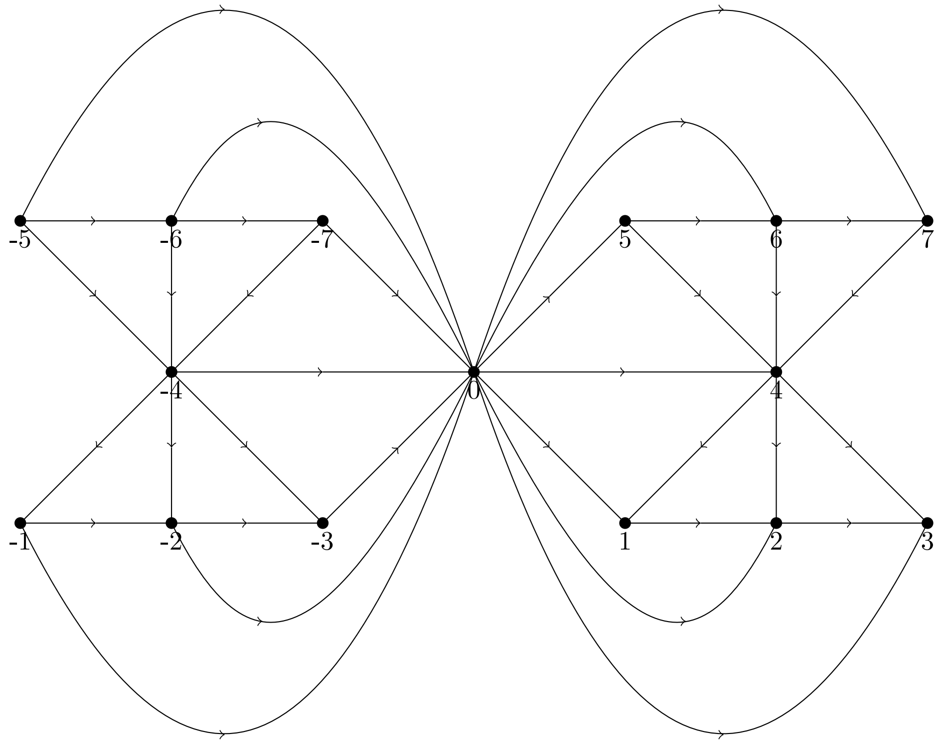
Then there is the following theorem proved by Klostermeyer and MacGillivray [7] :

Theorem 58 *A planar oclique can at most be on 36 vertices.*

We will not prove the theorem but we will give some idea about the proof.

Lemma 59 An indepentent set of size 5 cannot agree on a sixth vertex inside a planar oclique.

Proof
Without loss of generality let $I = \{a, b, c, d, e\} \subseteq N^-(v)$, where $\{a, b, c, d, e, v\} = S$ are all vertex of some planar oclique and I is an independent set in it.



Fix a planar embedding of the planar oclique. Without loss of generality let a, b, c, d, e are arranged in a clockwise order around v. Each pair of vertices of I are connected by a directed 2-path using some intermidiate vertex other than the ones in S.

Without loss of generality assume $a \rightarrow x \rightarrow e$ connects a and e.

case1:if b and d also uses x to connect, then c have to use x to connect with both a, e which is impossible.

case2:if b and d uses $y \neq x$ to connect, then c have to use y to connect with both a, e which is impossible.

So we are done. □

This is a **forbidden configuration** for a planar oclique.

Idea of the proof of Theorem 58: Let $\overrightarrow{D}$ be a planar oclique with more than 36 vertices. Now we know a planar graph has at least one vertex of degree at most 5. Let that vertex be v and its neighbors be $v_1, ..., v_r$. Now as $\overrightarrow{D}$ is an oclique, the graph $D[V(D) \setminus \{v, v_1, .., v_r\}]$ is outerplanar. Hence $D[V(D) \setminus \{v, v_1, .., v_r\}]$ is 3-colorable. Now as this graph has at least 31 vertices, we have a independent set I of at least size 11 in this graph. Now by renumbering the v_is we get a partition of $S_1, ..., S_r$ of I such that,

i)any vertex in S_i is connected to v by a directed 2-path using the vertex v_i

ii)$| S_1 |\geq .... \geq| S_r |$
iii)if $w \in S_i$, then there is no directed 2-path between w and v using v_j for any $j \leq i$

Now we break the situation into some disjoint cases:
(1)$| S_1 |\geq 5$,
(2)$| S_1 |= 4$, $| S_2 |= 4$,
(3)$| S_1 |\geq 3$,$| S_2 |= 3$,$| S_3 |= 3$,
(4)$| S_1 |= 4$,$| S_2 |\geq 2$,$| S_3 |\geq 2$,$| S_4 |\geq 2$, $| S_5 |\geq 1$,
(5)$| S_1 |= 3$,$| S_2 |\geq 2$,$| S_3 |\geq 2$,$| S_4 |\geq 2$,$| S_5 |\geq 1$,
(6)$| S_1 |= 4$,$| S_2 |= 3$,$| S_3 |= 2$,$| S_4 |=| S_5 |= 1$,
(7)$| S_1 |= 4$,$| S_2 |= 3$,$| S_1 |=| S_1 |= 2$.

Then the proof is done by getting contradiction assuming each of the cases.

Case (1) is not possible by Lemma 59. Each of the other cases is non-trivial.

3.2 More on planar ocliques

Clearly $\overrightarrow{B_0[N^+(0)]}$ is an outerplanar oclique on 7 vertices. Moreover, Sopena [5] proved that:

Theorem 60 *Every oriented outerplanar graph admits homomorphism to P_7.*

Hence we have an immediate corollary.

Corollary 61 Let O be the class of oriented outerplanar graphs. Then $w_o(O) = \chi_o(O) = 7$.

Lemma 62 If there is a vertex v that is adjacent to all the other vertices in a planar oclique on k vertices, then $k \leq 15$.

Proof
The planar oclique minus the vertex v is outerplanar. Hence by Theorem 60 we can oriented 7-color it by a coloring c. Now we give a 15-coloring of the oclique by,

$$f(x) = (c(x), \sigma), \text{ where } x \in N^\sigma(v) \text{ for } \sigma \in \{+, -\}$$
$$\text{and } f(v) = 0.$$

It is easy to cheak that this gives a proper oriented coloring. □

Now notice that by deleting vertices of degree 3 from $\overrightarrow{B_0}$ we get a 14 oclique. We keep on deleting degree 3 vertex from the obtained new oclique. This way we can get oclique on k vertices for all $k = 1, 2, ..., 15$.

Now we make two conjectures.

Conjecture 2 *If there exist a planar oclique on k vertices, then there exist a planar oclique on k vertices containing oclique on $(k-1)$ vertices.*

This is true for $k \leq 15$ by the above comment.

Conjecture 3 *There is no oriented planar oclique on 16 vertices.*

And clearly, **Conjecture 1** holds if and only if **Conjecture 2** and **Conjecture 3** both holds.

Lemma 63 If there is vertex v with degree at most 3 in a planar oclique on k vertices, then there exists a planar oclique on k vertices that contains a planar oclique on $(k-1)$ vertices.

Proof
Notice that we can make the neighbors of v pairwise adjacent (if they are already not) and get another planar oclique on k vertices. Then if we remove the vertex v we get a planar oclique on $(k-1)$ vertices. $\square$

We can also ask the question "is there any planar oclique on k vertices, which does not have a proper subgraph which is also a planar oclique on k vertices, with minimum degree greater than 3 ?"

Also some more questions: "is there a planar oclique 15 vertices that does not contain $\overrightarrow{B_0}$ as a subgraph?" and "what is $w_o(\mathcal{P})$?

The best bounds known is $15 \leq w_o(\mathcal{P}) \leq \chi_o(\mathcal{P}) \leq 80$.

Answer to these questions will help us finding the answer to the conjecture.

Chapter 4

Oriented Chromatic Number Of Oriented Planar Graphs

Here we first show the upper bound $\chi_o(\mathcal{P}) \leq 80$ originally done by Sopena and Raspaud [4] .

Then we will show that there is no $\mathcal{P}$-bound on at most 15 vertices. Also we will construct a new example of an oriented planar graph with oriented chromatic number at least 16. This is an original result proved in this master thesis.

Then show the lower bound $\chi_o(\mathcal{P}) \geq 17$. The lower bound follows as a corollary of a theorem by Marshall [2] ,

Theorem 64 *Every $\mathcal{P}$-bound has maximum degree at least 16.*

Marshall [2] actually showed that there is no minimal $\mathcal{P}$-bound with maximum degree at most 15. For this he proved that minimal $\mathcal{P}$-bound will have some certain properties (here we put those properties together and call it the property R_2). Then he showed that the only graph with maximum degree at most 15 is T_{16}. Then he constructed an example of an oriented planar graph that does not admit a homomorphism to T_{16}.

Here we will not prove the theorem. We will prove that the only oriented graph on 16 vertices having property R_2 is T_{16}. For the proof we will completely follow the ways of the proof of the theorem given by Marshall [2] .

Then using these we will construct example of an oriented planar graph with oriented chromatic number at least 17. This example is an original result of this master thesis. Till now there was no example of an oriented planar graph with chromatic number at least 17.

4.1 Any oriended planar graph admits a homomorphism to $Z(5)$

Definition 65 An acyclic k-coloring of a graph G is a proper k-coloring of G such that any 2-chromatic subgraph of G is cycle-free.

Acyclic chromatic number $a(G)$ of G is the least integer such that G is acyclic $a(G)$-colorable.

Theorem 66 *The class of forest has acylic chromatic number 2.*

It is well known that the class of forest has chromatic number 2. As forests are cycle free we have the theorem. This result is a well known fact.

Now we reprove some results originally done by Raspaud and Sopena [4] . They showed that any oriented forest is oriented 4-colorable. Using this they proved that any oriented graph whose under lying graph is acyclic k-colorable is oriented $k \times 2^{k-1}$-colorable. The following results of this section are due to Raspaud and Sopena [4] (except the one by Borodin).

Lemma 67 Any oriented forest $\overrightarrow{F}$ admits homomorphism $Z(2)$.

Proof
Let c be a proper coloring of the underlying graph F of a directed forest $\overrightarrow{F}$.

WOLG we can assume the image of c to be $\{1, 2\}$.

Now we define the homomorphism f from $\overrightarrow{F}$ to $Z(2)$ by the rules below,

- the $c(v)$th co-ordinate of $f(v)$, $f^{c(v)} = *$
- for an arc (x, y) we have $f(x)^{c(y)} = f(y)^{c(x)}$ if and only if $c(x) < c(y)$.

□

Remark 68 Note that if image of c was $\{a, b\}$, then we would have fixed $a < b$ and pretend to have $a = 1$ and $b = 2$ to do the proof.

Theorem 69 *Let $\overrightarrow{G}$ be an oriented graph such that G has acyclic chromatic number k. Then there is a homomorphism $f : \overrightarrow{G} \longrightarrow Z(k)$.*

Proof
Let c be such an acyclic coloring of $\overrightarrow{G}$ of G such that the image of c is $\{1, 2, ..., k\}$.

Let V_i be the color class of the color i.

Now $\overrightarrow{F_{i,j}} = \overrightarrow{G[V_i \cup V_j]}$ is a forest.

Using Lemma 67 and Remark 68 we have a proper oriented coloring $f_{i,j}$ of $\overrightarrow{F_{i,j}}$ using the proper coloring of the forest to be restriction of c.

Notice that, $\overrightarrow{F_{j,i}} = \overrightarrow{F_{i,j}}$.

Now we will define homomorphism $g : \overrightarrow{G} \longrightarrow Z(k)$.

$$g(x) = (f^1_{i,1}, ..., f^{i-1}_{i,i-1}, *, f^{i+1}_{i,i+1}, ..., f^k_{i,k}) \text{ for } x \in V_i$$

□

We will use the following result due to Borodin:

Theorem 70 *(Borodin) Every planar graph has acyclic 5-coloring.*

Theorem 71 *Any oriented planar graph admits homomorphism to* $Z(5)$.

Proof
This clearly follows from Theorem 69 and Theorem 70. $\square$

Corollary 72 $\chi_o(\mathcal{P}) \leq 80$.

Proof
$Z(5)$ has $5 \times 2^4 = 80$ vertices. Then it clearly follows from Theorem 71. $\square$

4.2 Example of an oriented planar graph with oriented chromatic number at least sixteen

Here we give a new example of a graph with oriented chromatic number at least 16.

Definition 73 Let $T = \{a \to b \to c \to a\}$ and $T' = \{x \leftarrow y \leftarrow z \leftarrow x\}$ be two directed triangles inside a digraph $\overrightarrow{G}$ satisfying the following:
(i) There are exactly three arcs $(a, x), (b, y), (c, z)$ from T to T'.
(ii) $G[V(T) \cup V(T')]$ is a tournament.
We will denote this by $T \longmapsto T'$.

Remark 74 If T and T' are directed triangles in a graph and $T \longmapsto T'$, then for any $u \in V(T)$ there is exactly one $w \in V(T')$ such that $v \to w$. Also for any $x \in V(T')$ there are exactly two vertices $y, z \in V(T)$ such that, $y \to z, x \to y, x \to z$.

Remark 75 If $T \longmapsto T'$ and if we know one arc from T to T', then we know all the arcs between T an T'.

Remark 76 For any $v \in V(P_7)$, $P_7[N^-(v)] \longmapsto P_7[N^+(v)]$.

This is an original result of this master thesis.

Theorem 77 *Let* G *be a graph with property* Q_3. *Then,* $| G |\geq 16$

Proof
Let, G has property Q_3 i.e., for any $v \in V(G), G[N^+(v)]$ and $G[N^-(v)]$ both has property Q_2. Now, $| N^+(v) |\geq 7$ and $| N^-(v) |\geq 7$ by Lemma 40 So, $| G |\geq| N^+(v) | + | N^-(v) | +1 \geq 15$. So, it is enough to show that $| G |\neq 15$.

Assume, $| G |= 15$. Then, by Lemma 40 $G[N^+(v)] \cong P_7 \cong G[N^-(v)], \forall v \in V(G)$. Now, fix $v \in V(G)$. Let v^+ be a vertex in $V(N^+(v))$.

Let $A = G[N^+(v^+) \cap N^+(v)]$ and $B = G[N^-(v^+) \cap N^+(v)]$ We know that A and B are directed triangles. We have, $B \longmapsto A$.

Already, $\{v, V(B)\} \subseteq N^-(v^+)$ and more over v is a source for B. As $G[N^-(v^+)] \cong P_7$ there should be another directed triangle in $G[N^-(v^+) \cap N^-(v)]$. Call it B'. Also note that, $B' \longmapsto B$.

Now, $B' \subseteq G[N^-(v)] \cong P_7$, so there can be 2 cases.

Case1: $B' = G[N^-(v^-) \cap N^-(v)]$, for some $v^- \in V(N^-(v))$. And let $G[N^+(v^-) \cap N^-(v)] = A'$. Note that, A' is also a directed triangle. Also, $B' \longmapsto A'$.

As, $V(B) \cup V(B') \cup \{v\} = N^-(v^+)$ we have, $V(A') \cup \{v^-\} \cup V(A) = N^+(v^+)$. Now, $N^+(v^-) \supseteq V(A')$ and $G[N^+(v^+)] \cong P_7$. So $A \subseteq N^-(v^-)$. Also, $A \longmapsto A'$.

Now, as we already know $N^-(v^-) = V(A) \cup V(B') \cup \{v^+\}$, so $N^+(v^-) = V(A') \cup V(B) \cup \{v\}$.

Now, $N^-(v^-) = V(A) \cup V(B') \cup \{v^+\}$ and $V(A) \subseteq N^+(v^+)$ implies $B' \longmapsto A$.

Also, $N^+(v^-) = V(A') \cup V(B) \cup \}v\}$ and $V(B) \subseteq N^+(v)$ implies $A' \longmapsto B$.

So we have,

$$\begin{tikzcd}
B' \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, mapsto] & B \arrow[d, mapsto] \\
A' \arrow[ur, mapsto] & A \arrow[l, mapsto]
\end{tikzcd}$$

Now, take any $a_1 \in V(A)$. We want to find $N^-(a_1)$. We already have $\{v, v^+\} \subseteq N^-(a_1)$.

Since, $B \longmapsto A$ we have $b_1 \in N^-(a_1)$, for some $b_1 \in V(B)$.

Since, $B' \longmapsto A$ we have $b_1' \in N^-(a_1)$, for some $b_1' \in V(B')$.

Since, $A \longmapsto A'$ we have a_1' and $a_2' \in N^-(a_1)$, for some a_1' and $a_2' \in V(A')$ such that, $a_1' \to a_2'$.

A is a directed triangle. So, we also have $a_2 \in N^-(a_1) \cap V(A)$.

So, we have $N^-(a_1) = \{v, v^+, a_2, b_1, a_1', a_2', b_1'\}$. We know that,$G[N^-(a_1)] \cong P_7$.

Now, $\{a_1', a_2', b_1'\} \subseteq N^-(v)$ and $\{v^+, a_2, b_1\} \subseteq N^+(v)$.

So, $G[a_1', a_2', b_1'] \longmapsto G[v^+, a_2, b_1]$ and $b_1' \to v^+$, where $G[a_1', a_2', b_1'] = \{a_1' \to a_2' \to b_1' \to a_1'\}$ and $G[v^+, a_2, b_1] = \{b_1 \to v^+ \to a_2 \to b_1\}$.

Now we look at, $N^+(a_2)$. We already know $\{v^-, b_1', a_1', a_1, b_1\} \subseteq N^+(a_2)$.

Since, $B \longmapsto A$ we have $b_2 \in N^+(a_2)$, for some $b_2 (\neq b_1) \in V(B)$.

Since, $B' \longmapsto A$ we have $b_2' \in N^+(a_2)$, for some $b_2' (\neq b_1') \in V(B)$.

Now, $\{a_1', b_1, b_1'\} \subseteq N^-(a_1)$ and $\{v^-, b_2, b_2'\} \subseteq N^+(a_1)$.

Also, $G[N^+(a_2)] \cong G[\{a_1', b_1, b_1', a_1, v^-, b_2, b_2'\}] \cong P_7$.

So, $G[a_1', b_1, b_1'] \longmapsto G[v^-, b_2, b_2']$ where, $G[a_1', b_1, b_1'] = \{a_1' \to b_1 \to b_1' \to a_1'\}$ and $G[v^-, b_2, b_2'] = \{v^- \to b_2 \to b_2' \to v^-\}$. But we already know $b_1' \to v^-$.

That means, $b_1 \to b_2$.

Now, let $a_3 \in V(A)$ such that, $a_3 \neq a_1, a_2$. We know, $a_2 \to a_1$. This implies $a_1 \to a_3$.

We know, $B \longmapsto A, b_1 \to a_1$ and $a_2 \to b_2$. This implies, $b_2 \to a_3$.

But this contradicts $B \longmapsto A$.

Case2: $B' = G[N^+(v^-) \cap N^-(v)]$, for some $v^- \in V(N^-(v))$. And let $G[N^+(v^-) \cap N^-(v)] = A'$. Note that, A' is also a directed triangle. Also, $A' \longmapsto B'$.

As, $V(B) \cup V(B') \cup \{v\} = N^-(v^+)$ we have, $V(A') \cup \{v^-\} \cup V(A) = N^+(v^+)$. Now, $G[V(A'), v^-, V(A)] \cong P_7$ and $N^-(v^-) \supseteq V(A')$. So, $V(A) \subseteq N^+(v^-)$. Also, $A' \longmapsto A$.

Now, as we already know $N^+(v^-) = V(A) \cup V(B') \cup \{v\}$, so $N^+(v^-) = V(A') \cup V(B) \cup \{v^+\}$.

Now, $N^-(v^-) = V(A) \cup V(B') \cup \{v\}$ and $V(A) \subseteq N^+(v)$ implies $B' \longmapsto A$.

Also, $N^+(v^-) = V(A') \cup V(B) \cup \{v^+\}$ and $V(B) \subseteq N^-(v^+)$ implies $B \longmapsto A'$.

So we have,

$$\begin{tikzcd} B' \arrow[r, mapsto] \arrow[dr, mapsto] & B \arrow[d, mapsto] \arrow[dl, mapsto] \\ A' \arrow[u, mapsto] \arrow[r, mapsto] & A \end{tikzcd}$$

After this, if we look at $N^+(a'_1)$ for some $a'_1 \in V(A')$ and do similar things as in **case1** (the arcs are in the opposite direction in this case), we will get a contradiction to $A' \longmapsto B'$.

This will end the proof.

□

We know that $C_c(\overrightarrow{G})$ is also an oriented graph for an oriented graph $\overrightarrow{G}$ and a proper oriented coloring c of it.

Now we will construct an oriented graph $\overrightarrow{B_2}$ such that if we color it with 15 colors, then $C_c(\overrightarrow{B_2})$ will have property Q_3.

Recall the butterfly graph $\overrightarrow{B_0}$ defined before.

Notice $\overrightarrow{B_0}$ is an oriented graph on 15 vertices and has oriented chromatic number 15. In other words $\overrightarrow{B_0}$ is an oclique on 15 vertices.

Now notice in $\overrightarrow{B_0}$, $d^{\pm}(0) = 7$. Now for each $i \in \{1, ..., 7\}$, we can attach 4 directed 4-path $\overrightarrow{P(4)}$ such that they are in $N^-(0) \cap N^-(-i)$, $N^-(0) \cap N^+(-i)$,$N^+(0) \cap N^-(i)$ and $N^+(0) \cap N^+(i)$. Note that the new oriented graph is an oriented planar graph. Call this new graph $\overrightarrow{B_1}$. Note that $\mid \overrightarrow{B_1} \mid= 127$.

Notice that, to color each of the directed 4-paths we need at least 3 colors. Now if we are using 15 colors to color the graph $\overrightarrow{B_1}$, at most 7 colors can be used to color $\overrightarrow{B_1[N^-(0)]}$.

Fix $i \in \{1, 2, ..., 7\}$. Now notice that, we cannot use the color we used for $-i$ to color any vertex of the directed 4-paths in $N^-(0) \cap N^-(-i)$ and $N^-(0) \cap N^+(-i)$. Also, as the two directed 4-paths disagree on $-i$, we cannot use a one in both the graphs. So we will have to use 6 colors to color both the directed 4-paths. So each of the directed 4-paths will get 3 colors.

Similarly we can conclude the same for all the added directed 4-paths. So, each of these directed 4-paths get 3 colors if we 15 color the graph $\overrightarrow{B_1}$.

So, in $C_c(\overrightarrow{B_1})$, where c is a 15 proper coloring of B_1, each of these $P(4)$ will map to a directed cycle by the homomorphism $c : B_1 \longrightarrow C_c(\overrightarrow{B_1})$.

Now for each $i \in \{-7, -6, ..., -1, 0, 1, ..., 7\}$ add a disjoint copy of $\overrightarrow{B_1}$ and then identify the 0 of this copy with i of $\overrightarrow{B_0}$. Note that the new graph is oriented planar graph. Call the new graph $\overrightarrow{B_2}$. Note that $\mid \overrightarrow{B_2} \mid= 1905$.

Clearly, for a proper 15 coloring c of $\overrightarrow{B_2}$, $C_c(\overrightarrow{B_2})$ will have property Q_3.

Now by Theorem 77 we have:

Theorem 78 $\chi_o(\overrightarrow{B_2}) \geq 16$

Corollary 79 $\chi_o(\mathcal{P}) \geq 16$.

Proof
Clearly follows from Theorem 78.

□

Now we costruct a graph called $\overrightarrow{B_3}$ by adding a disjoint copy of $\overrightarrow{B_0}$ to each vertex v of $\overrightarrow{B_2}$ and then identify vertex v with the vertex 0 of $\overrightarrow{B_0}$. Note that, $|\overrightarrow{B_3}| = 28575$ and $\overrightarrow{B_3}$ is an oriented planar graph.

Now if we have a 16-coloring c of this graph, then the color graph $C_c(\overrightarrow{B_3})$ will have $\delta(C_c(\overrightarrow{B_3})) \geq 14$. This we will use later on.

4.3 The minimal $\mathcal{P}$-bound has order at least seventeen

Definition 80 A class C of oriented graph is k-complete if the graph obtained by identifying the isomorphic l-cliques ($l \leq k$) of two graphs of C is also in C.

The class of oriented planar graphs $\mathcal{P}$ is 2-complete.

Lemma 81 Let C be a k-complete ($k \leq 2$) class of oriented graphs and $\overrightarrow{H}$ be a minimal C-bound. Let $\overrightarrow{D} \in C$, and $K_1 \subseteq \overrightarrow{D}, K_2 \subseteq \overrightarrow{H}$ are k-cliques. Then any isomorphism $\varphi : K_1 \longrightarrow K_2$ can be extended to a homomorphism $\psi : \overrightarrow{D} \longrightarrow \overrightarrow{H}$.

Proof
Let $k = 2$. Let $K_i = \{x_i \to y_i\}$ (for $i = 1, 2$) and $\varphi(x_1) = x_2, \varphi(y_1) = y_2$.

Now there exists a $\overrightarrow{G_1} \in C$ such that for any homomorphism h from $\overrightarrow{G_1}$ to $\overrightarrow{H}$ there is $(x', y') \in A(\overrightarrow{G_1})$ such that, $h(x') = x_2$ and $h(y') = y_2$.

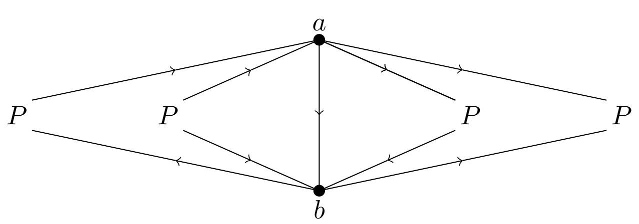
Now for each arc $(x, y) \in A(\overrightarrow{G_1})$ we paste the arc (x_1, y_1) of $\overrightarrow{D}$ and get a new graph $\overrightarrow{G_2}$ (say).

By, 2-completeness $\overrightarrow{G_2} \in C$. Hence there is a homomorphism g $\overrightarrow{G_1}$ to $\overrightarrow{H}$ such that for some $(x^1, y^1) \in A(\overrightarrow{G_1})$, $g(x^1) = x_2$ and $g(y^1) = y_2$

Now restrict g to the $\overrightarrow{D}$ pasted to the arc (x^1, y^1) to get the required homomorphism.

Similarly for $k = 1$. □

Definition 82 An oriented graph $\overrightarrow{G}$ has property R_n if for any k-clique $= \{a_i, a_2, ..., a_k\}$ ($k \leq n$) each of the subgraphs $\overrightarrow{G[\cap_{i=1}^{k} N^{\sigma_i}(a_i)]}$ have a directed cycle in them, where $\sigma_i \in \{+, -\}$ for all $i = 1, 2, ..., k$.



In the above diagram P is a directed path of length k for some integer k.

We can identify the arc (a, b) of the oriented planar graph above with an arc (x, y) of any planar graph and still have a planar graph by 2-completeness of $\mathcal{P}$.

Call this R_2-**fication of length** k of the arc (x, y).

If we do this for all the arcs of some oriented planar graph $\overrightarrow{G}$, then it is called the R_2-**fication of length** k of the oriented planar graph $\overrightarrow{G}$.

Lemma 83 The minimal $\mathcal{P}$-bound has property R_2.

Proof
Let the minimal $\mathcal{P}$-bound be on l vertices. Now by Lemma 81 for each arc (x, y) we have some $\overrightarrow{D(x,y)} \in \mathcal{P}$ such that for any homomorphism from $\overrightarrow{D(x,y)}$ to the minimal $\mathcal{P}$-bound some arc of $\overrightarrow{D(x,y)}$ will map to (x, y). Now by R_2-fication of length l of $\overrightarrow{D(x,y)}$ for all $(x, y) \in A$(minimal $\mathcal{P}$-bound) we have the lemma because by pigeon hole principle, the image of path P will contain a directed cycle.

□

Corollary 84 Endpoints of each arc in an oriented graph $\overrightarrow{G}$ having property R_2, agrees and disagrees on at least 6 vertices.

Proof
A directed cycle has at least 3 vertices. The rest follows from Lemma 83. □

Remark 85 Property R_2 implies property Q_3.

Theorem 86 *The only oriented graphs on 16 vertices having property R_2 is T_{16}.*

Before we prove the theorem we will prove some lemmas.

Fix $\overrightarrow{G}$ to be the minimal $\mathcal{P}$-bound (we fix on this for this section). From the previous section we know that $|\overrightarrow{G}| \geq 15$. If possible let $|\overrightarrow{G}| = 16$.

Now as $\overrightarrow{G}$ has property Q_3 and P_7 is the smallest graph with property Q_2, we have $|N^{\pm}(\overrightarrow{G})| \geq 7$.

Now as $|\overrightarrow{G}| = 16$ we have at least one of $\overrightarrow{G[N^+(v)]}$ and $\overrightarrow{G[N^-(v)]}$ isomorphic to P_7 for any $v \in V(\overrightarrow{G})$.

Let $\overrightarrow{G[N^+(v)]} \cong P_7$. Now for any $x \in N^-(v)$ we have directed cycles in both $\overrightarrow{G[N^+(x) \cap N^+(v)]}$ and $\overrightarrow{G[N^-(x) \cap N^+(v)]}$ (by property R_2). And those directed cycles are $\overrightarrow{G[N^+(y) \cup N^+(v)]}$ and $\overrightarrow{G[N^-(y) \cup N^+(v)]}$ for some $y \in V(\overrightarrow{G[N^+(v)]}$.

Now define a map h_v^+ from $N^-(v)$ to $N^+(v)$ that maps x to y (x, y are as above).

If $\overrightarrow{G[N^-(v)]} \cong P_7$, then we define h_v^- in a similar way.

Note that this map is well defined. Also x and $h_v^{\pm}(x)$ completely agree or completely disagree on the vertices of $V(\overrightarrow{G[N^{\pm}(v) \setminus h_v^{\pm}(x)]}$.

If they agree, then $\theta_v^{\pm}(x) = 1$ and if they disagree, then $\theta_v^{\pm}(x) = -1$.

That is,

$$[h_v^{\pm}(x), w] = \theta_v^{\pm}(x)[x, w], \tag{4.1}$$

for all $w \in V(\overrightarrow{G[N^{\pm}(v) \setminus h_v^{\pm}(x)]}$

Lemma 87 Let $\overrightarrow{G}$ be the minimal $\mathcal{P}$-bound. Let u, v, w be vertices of a triangle in $\overrightarrow{G}$. Then, $\mid \overrightarrow{G} \mid \geq max\{S_1(u,v,w), S_2(u,v,w)\} + 11$,
where,
$S_1(u,v,w) = \{a \in V(\overrightarrow{G}) \mid u, v, w \text{ agree on } a\}$
$S_2(u,v,w) = \{a \in V(\overrightarrow{G}) \mid u, v \text{ disagree with } w \text{ on } a\}$.

Proof

For vertices $x, y \in V(\overrightarrow{G})$ we define,
$A_{x,y} = \{z \in V(\overrightarrow{G}) \mid x, y \text{ agree on } z\}$
$D_{x,y} = \{z \in V(\overrightarrow{G}) \mid x, y \text{ disagree on } z\}$
And let,
$M = N(u) \cap N(v) \cap N(w)$
Now consider six sets $A_{x,y}$ and $D_{x,y}$ such that $\{x, y\} \subseteq \{u, v, w\}$.

Now if $z \in M$, then z is exactly in three of the above sets otherwise in at most one of them.

Hence we get,

$$\sum_{\{x,y\}\subseteq\{u,v,w\}}(\mid A_{x,y} \mid + \mid D_{x,y} \mid) \leq$$
$$3 \mid M \mid +(\mid \overrightarrow{G} \mid - \mid M \mid) = 2 \mid M \mid + \mid \overrightarrow{G} \mid \tag{4.2}$$

Now if $z \in M$, then it is at least in one of the three sets $A_{u,v}, A_{u,w}, A_{w,v}$ and if $z \in S_1(u,v,w)$, then it is in all three of the sets.

Hence we have,

$$2 \mid S_1(u,v,w) \mid + \mid M \mid = 3 \mid S_1(u,v,w) \mid +(\mid M \mid - \mid S_1(u,v,w) \mid) \leq$$
$$\mid A_{u,v} \mid + \mid A_{u,w} \mid + \mid A_{w,v} \mid \tag{4.3}$$

Similarly we get,

$$2 \mid S_2(u,v,w) \mid + \mid M \mid = 3 \mid S_2(u,v,w) \mid +(\mid M \mid - \mid S_2(u,v,w) \mid) \leq$$
$$\mid A_{u,v} \mid + \mid D_{u,w} \mid + \mid D_{w,v} \mid \tag{4.4}$$

Now for all $(x, y) \in A(\overrightarrow{G})$ we have $A_{x,y} \geq 6$ and $D_{x,y} \geq 6$ by Corollary 84
From these equations we get,
$2 \mid S_i(u,v,w) \mid + \mid M \mid \leq 2 \mid M \mid + \mid \overrightarrow{G} \mid +18$ where, $i = 1, 2$
But, $\mid M \mid \leq \mid \overrightarrow{G} \mid -3$
Hence the lemma follows. □

Remark 88 The first three inequalities of the above lemma holds for any oriented graphs.

Corollary 89 $\overrightarrow{G}$ cannot be a tournament on 16 vertices.

Proof
If possible let $\overrightarrow{G}$ be a tournament on 16 vertices. Then for any $x \in V(\overrightarrow{G})$ $\{\mid N^+(x) \mid, \mid N^-(x) \mid\} = \{7, 8\}$.

Without loss of generality let $\mid N^+(x) \mid = 8$ and $\mid N^-(x) \mid = 7$. Then the function $h_x^- : N^+(x) \longrightarrow N^-(x)$ is not injective.

Let $h_x^-(y) = h_x^-(z) = w$, for some $y, z \in N^+(x)$ and $w \in N^-(x)$.

This means, either x, y, w all three agree on $N^+(x) \setminus \{w\}$ or two of them disagree with the third on $N^+(x) \setminus \{w\}$.

$\overrightarrow{G}$ is a tournament implies x, y, w are vertices of a triangle.

Now using Lemma 87 we are done. □

This implies $\delta(\overrightarrow{G}) = 14$. Now pick a vertex $(\infty, 1)$ of $\overrightarrow{G}$ such that $((\infty, 1)) = 14$. Call the only non-adjacent vertex to it, $(\infty, -1)$.

Now notice, $\overrightarrow{G[N^{\pm}((\infty, 1))]} \cong P_7$.

Set $h^{\pm}_{(\infty,1)} = h^{\pm}$ and $\theta^{\pm}_{(\infty,1)} = \theta^{\pm}$.

Lemma 90 h^+ and h^- are inverse to each other.

Proof

Let v be a vertex of $N^-((\infty, 1))$.

If v is not adjacent to $h^+(v)$, then $h^-(h^+(v)) = v$ (it has no other option).

Let v and $h^+(v)$ are adjacent and $h^-(h^+(v)) \neq v$. Now by Corollary 41 v, v' agree (disagree) on at most 3 vertices of $N^-((\infty, 1))$. But $h^+(v)$ and v' agree or disagree on all the vertices of $N^-((\infty, 1)) \setminus \{v'\}$. So $v, h^+(v)$ agree (disagree) on at most 3 vertices of $N^-((\infty, 1)) \setminus \{v'\}$. But $v, h^+(v)$ agree or disagree on all the vertices of $N^+((\infty, 1)) \setminus \{h^+(v)\}$. By property R_3, $v, h^+(v)$ should agree (disagree) on at least 6 vertices of $\overrightarrow{G}$. This will force $v, h^+(v)$ to agree (or disagree) on exactly 4 vertices of $N^-((\infty, 1))$ (including v') and $(\infty, \pm 1)$. These six vertices should induce 2 disjoint directed triangle. But there is no arc between $(\infty, \pm 1)$ and the remaining 4 vertices agree with each other on $(\infty, 1)$. So these six vertices cannot induce two disjoint directed cycles. This is a contradiction.

The other case is symmetric. So we are done. □

Lemma 91 For $u \in V(N^-((\infty, 1)))$ and $u, h^+(u)$ adjacent, we have $\theta^+(u) = -\theta^-(h^+(u))$.

Proof

If possible let the theorem not be true. Then $u, h^+(u)$ will agree or disagree on $V(\overrightarrow{G}) \setminus \{u, h^+(u), (\infty, 1), (\infty, -1)\}$ which is a contradiction because $\overrightarrow{G}$ has property R_2. □

Lemma 92 For $u, v \in V(N^{\pm}((\infty, 1)))$ we have $\theta^{\pm}(u) = \theta^{\pm}(v)$.

Proof

It is enough to prove one case. The other case is symmetric.

We have,

$$\begin{aligned}
&[h^+(z), h^+(w)] = -[h^+(w), h^+(z)] = -\theta^+(w)[w, h^+(z)] \\
&= \theta^+(w)[h^+(z), w] = \theta^+(w)\theta^-(h^+(z))[h^-(h^+(z)), w] \\
&= \theta^+(w)\theta^-(h^+(z))[z, w]
\end{aligned} \tag{4.5}$$

Now let, $X^{\pm} = \{z \in V(\overrightarrow{G[N^-((\infty, 1))]} \mid \theta^+(z) = \pm 1\}$

If one of the sets are empty, we are done. If not, then choose v from the set which contains even number of vertices.

This will give us, $\mid N^-((\infty,1)) \cap N^+(v) \cap X^+ \mid \neq \mid N^-((\infty,1)) \cap N^-(v) \cap X^+ \mid$.

If $\theta^-(h^+(v)) = 1$ by 4.5 we get,

$\mid N^+((\infty,1)) \cap N^+(h^+(v)) \mid =$

$\mid N^-((\infty,1)) \cap N^+(v) \cap X^+ \mid + \mid N^-((\infty,1)) \cap N^-(v) \cap X^- \mid$

$= \mid N^-((\infty,1)) \cap N^+(v) \cap X^+ \mid + (3 - \mid N^-((\infty,1)) \cap N^-(v) \cap X^+ \mid) \neq 3$,

which is a contradiction.

If $\theta^-(h^+(v)) = -1$, then we have a similar way of getting contradiction for $\mid N^+((\infty,1)) \cap N^-(h^+(v)) \mid$. □

Lemma 93 For each vertex $v \in N^-((\infty,1))$, $\theta^+(v) = \theta^-(h^+(v))$.

Proof

If possible let for all $v \in N^-((\infty,1)), \theta^+(v) = 1$ and for all $w \in N^+((\infty,1)), \theta^-(w) = -1$.

Now let all vertices in $N^{\pm}((\infty,1))$ have the same name for both T_{16} and $\overrightarrow{G}$ (notice that we know for both they induce P_7).

Now by 4.5 we know that h^+ is an anti-autopmorphism. We assume $h^+(n,-1) = (-n,1)$.

Hence we have,

$N^-((0,-1)) \supseteq \{(1,1),(2,1),(4,1),(1,-1),(2,-1),(4,-1)\}$

$= \{(1,1),(2,1),(4,1)\} \cup \{(1,-1),(2,-1),(4,-1)\}$

$= \{(1,-1),(1,1),(2,1)\} \cup \{(2,-1),(4,-1),(4,1)\}$

And,

$N^+((0,-1)) \supseteq \{(3,1),(5,1),(6,1),(3,-1),(5,-1),(6,-1)\}$

$= \{(3,1),(5,1),(6,1)\} \cup \{(3,-1),(5,-1),(6,-1)\}$

$= \{(3,-1),(5,-1),(3,1)\} \cup \{(6,-1),(5,1),(6,1)\}$

that is, both in and out neighbors of $(0,-1)$ contains a set of vertices that can be written as union of two directed cycles in two different ways. So none of them can be P_7. Which is a contradiction.

Now, let If possible let for all $v \in N^-((\infty,1)), \theta^+(v) = -1$ and for all $w \in N^+((\infty,1)), \theta^-(w) = 1$.

But this case is symmetric. □

Corollary 94 G is 14-regular.

Proof

From Lemma 91 and Lemma 92 we have $v \in N^{\mp}((\infty,1))$ and $h^{\pm}(v)$ are not adjacent. So no vertex have degree 15. □

Now we prove Theorem 86

Proof

From the previous lemmas we have h^+ is an isomorphism. We can assume $h^+((n,-1)) = (n,1)$. Now if $\theta^+((n,-1) = \theta^-((m,1)) = 1$, then $\overrightarrow{G[N^+((n,1))]}$ is not a tournament hence not P_7. This is a contradiction.

So $\theta^+((n,-1) = \theta^-((m,1)) = -1$. Which implies that, $\overrightarrow{G}[V(\overrightarrow{G}) \setminus \{(\infty,-1)\} = T_{16}[V(T_{16}) \setminus \{(\infty,-1)\}]$.

By Lemma 44 we have $\{N^+((\infty,1)), N^-((\infty,1))\} = \{N^+((\infty,-1)), N^-((\infty,-1))\}$

So $(\infty, 1)$ and $(\infty, -1)$ either completely agree or completely disagree on all the other vertices of $\overrightarrow{G}$.

If $(\infty, 1)$ and $(\infty, -1)$ completely agree then we can identify them and get an oriented graph on 15 vertices having property Q_2. Which is not possible by Theorem 77. So $(\infty, 1)$ and $(\infty, -1)$ disagrees completely. Then we are done.

□

Therefore, on 16 vertices T_{16} is the only candidate for being the minimal $\mathcal{P}$-bound.

But Marshall in his paper [2] has constructed an oriented planar graph G_9 which does not admit a homomorphism to T_{16}. This is shown by some explicit calculations which we skip here.

Assuming that example we have the following,

Theorem 95 *The minimal $\mathcal{P}$-bound is at least on 17 vertices.*

4.4 Example of an oriented planar graph with oriented chromatic number at least seventeen

Take an oriented planar graph $\overrightarrow{G}$ and do R_2-fication of length l on it. Call the new oriented planar graph $\overrightarrow{G_{1,l,R_2}}$. If we repeat this process k times, then the new oriented planar graph is called $\overrightarrow{G_{k,l,R_2}}$.

Let there be a m-coloring (for some $m \leq l$) c_k of the graph $\overrightarrow{G_{k,l,R_2}}$ and c_{k-1} be the restriction of c_k to $\overrightarrow{G_{(k-1),l,R_2}}$. Note that, if $C_{c_k}(\overrightarrow{G_{k,l,R_2}}) = C_{c_{k-1}}(\overrightarrow{G_{(k-1),l,R_2}})$, then the oriented graph $C_{c_k}(\overrightarrow{G_{k,l,R_2}})$ has property R_2.

Now let $\overrightarrow{H} = \overrightarrow{B_3} \sqcup G_9$ (G_9 is the oriented planar graph that Marshall [2] constructed and proved that it does not admit a homomorphism to T_{16}). Of course $\overrightarrow{H}$ is a oriented planar graph with oriented chromatic number at least 16. Now let $\overrightarrow{H_{k,16,R_2}} = \overrightarrow{H_k}$ Now we claim that,

Lemma 96 $\chi_o(\overrightarrow{H_9}) \geq 17$

Proof
We have $\overrightarrow{B_2} \subseteq \overrightarrow{B_3} \subseteq \overrightarrow{H_0} \subseteq ...... \subseteq \overrightarrow{H_9}$

Now $\chi_o(\overrightarrow{H_9}) \geq \chi_o(\overrightarrow{B_3}) \geq 16$.

Let, if possible, c_9 be an proper oriented 16-coloring of H_9. Then the restriction c_k of it to $\overrightarrow{H_k}$ is also an oriented 16-coloring. Now $C_{c_0}(\overrightarrow{H_0})$ has at least 14×8 arcs. Now $C_{c_9}(\overrightarrow{H_9})$ can have at most 8 more arcs (then it will become a tournament).

So at least for some $k \in \{1, 2, ..., 9\}$ we have $C_{c_k}(\overrightarrow{H_k}) = C_{c_{k-1}}(\overrightarrow{H_{k-1}})$ Then $C_{c_k}(\overrightarrow{H_k})$ is an oriented graph on 16 vertices with property R_2.

That means $C_{c_k}(\overrightarrow{H_k}) = T_{16}$ which will give us a homomorphism from $\overrightarrow{H_k}$ to T_{16}. But then the restriction of that homomorphism to G_9 will give us a homomorphism from G_9 to T_{16}. This is a contradiction. Hence proved. □

Chapter 5

Conclusion

We saw that the oriented chromatic number of the class of oriented planar graphs is between 17 and 80. It is more likely that $\chi_o(\mathcal{P})$ is closer to 17 than 80.

It maybe possible to use the results proved in Section 2.4 to improve the lower bound.

Another potential way to improve the lower bound can be by finding $\chi_o(H_9)$. It maybe possible to run a computer programming to find out this number. I think this number is more than 17.

We can also look for more properties that the minimal $\mathcal{P}$-bound must have. It will be nice to know if the minimal $\mathcal{P}$-bound is vertex transitive/arc transitive.

We end with a question Marshall [2] asked:

"Is the Paley tournament P_{19} a $\mathcal{P}$-bound?"

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