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Bertini's Theorem on Generic Smoothness

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Supervisor:
Prof. Qing Liu

Candidate:
Andrea Ricolfi

Introduction

Bertini was an Italian mathematician, who lived and worked in the second half of the nineteenth century. The present dissertation concerns his most celebrated theorem, which appeared for the first time in 1882 in the paper [5], and whose proof can also be found in *Introduzione alla Geometria Proiettiva degli Iperspazi* (E. Bertini, 1907, or 1923 for the latest edition).

The present introduction aims to informally introduce Bertini's Theorem on generic smoothness, with special attention to its recent improvements and its relationships with other kind of results.

Just to set the following discussion in an historical perspective, recall that at Bertini's time the situation was more or less the following:

- there were no schemes,
- almost all varieties were defined over the complex numbers,
- all varieties were embedded in some projective space, that is, they were not intrinsic.

On the contrary, this dissertation will cope with Bertini's theorem by exploiting the powerful tools of modern algebraic geometry, by working with schemes defined over any field (mostly, but not necessarily, algebraically closed). In addition, our varieties will be thought of as abstract varieties (at least when over a field of characteristic zero). This fact does not mean that we are neglecting Bertini's original work, containing already all the relevant ideas: the proof we shall present in this exposition, over the complex numbers, is quite close to the one he gave. By the way, the interested reader can find a detailed description of the original Bertini's theorems in the beautiful and accurate paper [15] by S. L. Kleiman.

Unless otherwise specified we will work with varieties over an arbitrary algebraically closed field k. Bertini's theorem asserts (in its weak form) that the generic hyperplane section of a smooth projective variety preserves the original smoothness; moreover, the set of hyperplanes satisfying this condition forms itself a quasiprojective variety (a behavior recalling us that parameter spaces of algebraic varieties are usually algebraic varieties themselves).

For convenience, we informally say that a (local) property Q of k-varieties is *Bertini* (or weak-Bertini) if it satisfies condition (B) below and it is *strong-Bertini* if it satisfies condition (SB):

> (B): If $X \subset \mathbf{P}^n_k$ is a variety satisfying Q then the general hyperplane section of X satisfies Q.
>
> (SB): Let X be a variety with a finite dimensional linear system S on it and let $X \dashrightarrow S$ be the corresponding rational map. If this map induces separable extensions on residue fields (of closed points), then the generic member of S satisfies Q, away from the base points of S and the points in X that are not Q.

Our aim is to show that Q = smoothness is (B) and to convince ourselves, with some examples rather than by way of precise proofs, that it is also (SB). For most of the time, however, we will be concerned with the weaker (B), so we will become familiar with hyperplane sections in any characteristic; those hyperplanes H giving a smooth intersection with X will be called *good*, otherwise they will be called *bad*. We will also show that in characteristic zero smoothness is (SB) (note that the condition of separability is automatically satisfied). With an explicit example and a reference to a theorem of Kleiman, we hope to clarify that the inseparability of the extensions between residue fields is exactly the obstruction for smoothness to become (SB) in positive characteristic.

Let us precise that smoothness is not the unique "Bertini type" property. Suppose, indeed, you have a property Q on your variety $X \subset \mathbf{P}^n$. Then you may ask yourself several questions:

1. Can I find at least *one* hyperplane H such that $X \cap H$ satisfies Q? (sometimes no such one exists)

2. How many H are there such that $X \cap H$ satisfies Q? (sometimes *few* such exist)

3. Does the answer to the previous questions depend on the characteristic of the base field? (sometimes positive characteristic makes life hard)

Here are examples of Q for which one knows that (B) holds: being reduced, irreducible, normal, Cohen-Macaulay, smooth; and of course it is very interesting to try to enlarge this list as much as possible. It is noteworthy that a property Q has a remarkable hope of being (B) if it is constructible (and the base field is not too small - finite, for example). This will be made clear later but, for the moment, just observe that a constructible set of an irreducible Zariski space either contains an open subset, or is nowhere dense.

In what follows we attempt to give a quick description of some more recent results regarding Bertini's Theorem in its several materializations. Some of them will be discussed later in this work, while others just appear in this introduction as a cue for the reader. The main directions along with the theorem can be improved are:

I. Bertini theorems over $\mathbb{F}_q$;

II. study on new Bertini properties;

III. deduce some new results in Algebraic Geometry;

IV. Bertini over a ring of integers.

Let us analyze each of them a little more carefully.

§ I. *The problem with finite fields.* With finite fields, everything is finite! The number of rational points in the projective space, the set of hyperplanes... When k is any field, and $X \subset \mathbf{P}^n_k$ is a smooth quasiprojective variety of dimension $m \geqslant 0$, then one may consider the set

$$U = \{ u \in \mathbf{P}^{n*}_k \mid X \cap H_u \text{ is smooth of dimension } m-1 \text{ over } k(u) \}$$

where $H_u \subset \mathbf{P}^n_{k(u)}$ is the hyperplane corresponding to the point u, defined over the residue field $k(u)$. It turns out that U is dense in $\mathbf{P}^{n*}_k$. If k is infinite, then one can find a hyperplane in U which is defined over k. Otherwise, if $k = \mathbb{F}_q$, all the finitely many hyperplanes H over k might give a singular intersection $X \cap H$. It is a question by N. M. Katz to see what happens if one considers hypersurfaces of degree greater than one, instead of just looking at hyperplanes. Let S be the homogeneous ring $\mathbb{F}_q[x_0, \dots, x_n]$ of $\mathbf{P}^n$, and for every homogeneous $f \in S_h = \bigcup_{d \geqslant 0} S_d$ let H_f be the hypersurface $\operatorname{Proj} S/f \subset \mathbf{P}^n$. Define the *density*

$$\mu(\mathcal{P}) = \lim_{d \to \infty} \frac{\sharp\, \mathcal{P} \cap S_d}{\sharp\, S_d}$$

for any subset $\mathcal{P} \subset S_h$. Then Poonen in [18] (2004), answering to Katz's question, proves that if $X \subset \mathbf{P}^n$ is smooth and quasiprojective of dimension $m \geqslant 0$ over $\mathbb{F}_q$, then the subset

$$\mathcal{P} = \{\, f \in S_h \mid X \cap H_f \text{ is smooth of dimension } m-1 \,\}$$

has positive density, equal to $\zeta_X(m+1)^{-1}$, where ζ_X is the zeta function attached to the variety X. So, Poonen makes Bertini's statement become true over $\mathbb{F}_q$ by allowing hypersurfaces to play the role of classical hyperplanes.

Another interesting result in characteristic $p > 0$ is due to E. Ballico. In 2003 he proves (see [4]) what follows. He lets k be the algebraic closure of $\mathbb{F}_p$, but in fact he focuses on good hyperplanes $H \subset \mathbf{P}^n_k$ defined over $\mathbb{F}_q$, where q is a power of p. In his theorem, $X \subset \mathbf{P}^n_k$ is an irreducible variety of dimension m and degree d. He proves that if $q \geqslant d(d-1)^m$ then there exists some hyperplane $H \subset \mathbf{P}^n_k$ defined over $\mathbb{F}_q$ and transversal to X.[1] Note that X might not be defined over $\mathbb{F}_q$. However, the result still holds if X is smooth and defined over $\mathbb{F}_q$: there still exists a good hyperplane cutting X transversally at all of its k-points (and not just at $\mathbb{F}_q$-points): this means that for every $x \in X$ one has $T_{X,x} \not\subset H$.

§ II. *New Bertini properties and generalizations.* Let us first focus on the problem of finding new Bertini properties. There exist two properties that we will deal with in this paragraph: weak-normality (WN), and property WN1. Here and in the following a reduced variety X will be said to be WN if every birational universal homeomorphism $Y \to X$ is an isomorphism.[2] Instead, WN1 means WN with the extra condition that the normalization morphism $\tilde{X} \to X$ is unramified in codimension one. One always has that WN1 $\Rightarrow$ WN, and they agree in characteristic zero

[1] This means that $H \notin X^* \subset \mathbf{P}^{n*}_k$. If X is smooth, H is transversal to X if and only if $T_{X,x} \not\subset H$ for all $x \in X$. If it is not smooth and H cuts it transversally, then $T_{X,x} \not\subset H$ for all $x \in X_{\mathrm{sm}}$.

[2] Equivalently, X coincides with its weak normalization, the maximal couple $(\hat{X}, \hat{f})$ among birational universal homeomorphisms $Y \to X$. A universal homeomorphism is a Zariski homeomorphism f such that for every $y \in Y$ the extension $k(f(y)) \subset k(y)$ is purely inseparable (so it is trivial when $\operatorname{char} k = 0$).

or for varieties of dimension one. Cumino, Greco and Manaresi showed (see [6]) that Q = WN is (B) for varieties of every dimension in characteristic zero. By means of an "axiomatic approach" to Bertini properties, the same authors show (see [7]) that in fact WN is (SB), again in characteristic zero. The strategy is as follows. They consider local properties Q of noetherian schemes, satisfying three particular axioms (A1), (A2), (A3): for instance, Q = reduced, normal, regular all happen to satisfy them. Afterwards, they prove (Theorem 1 in [7]) that if Q satisfies these axioms then Q is (SB). One of their results asserts that:

- WN verifies (A1) in characteristic zero;
- WN verifies (A2) and (A3) in any characteristic.

So *in characteristic zero, weak-normality is strong-Bertini*. In positive characteristic, the authors show that WN does not always satisfy (A1). But this does not imply yet that WN is not strong-Bertini. However, this is the case: they show (see [8]) that, in general, WN is not even (B), weak-Bertini, if char $k = p$. The main theorem in [8] states that:

> Let $X \subset \mathbf{P}^n_k$ be projective, WN, equidimensional, of dimension at least 2. If the general hyperplane section $X \cap H$ is WN1 then X is WN1 as well.

If X is as above, but is not WN1, then (the authors show that) intersecting X with a suitable linear subspace $L \subset \mathbf{P}^n_k$ (possibly the whole $\mathbf{P}^n_k$) gives a weakly normal variety $Y = X \cap L$, and the generic hyperplane section of Y is not WN. The conclusion is that weak-normality is really a property whose "Bertini-behavior" depends on the characteristic of the base field.

An important generalization of Bertini's Theorem is a result nowadays known as Kleiman-Bertini Theorem. It appeared in [16], in 1974. Even if we will discuss it in Chapter 2, it is worth mentioning its content now. We work over $k = \overline{k}$ of characteristic zero; the situation (not the most general we will deal with) is as follows: one is given of a homogeneous space (X, G) and of two subvarieties $Y, Z \subset X$. Then one can consider the translates sY, for $s \in G$. As a variety, it is isomorphic to Y, but it embeds in X differently, namely, the point y goes to its translate sy, viewed inside X. The theorem says that if Y and Z are both smooth then a general translate of, say, Y meets Z transversally: $sY \cap Z$ is smooth for general $s \in G$.

§ III. *New "non-Bertini" results*. Bertini's Theorem has several immediate consequences. The easiest ones will be described in Chapter 2. In [1], the book *Arithmetic Geometry*, Milne presents a proof of the (highly nontrivial) fact that any abelian variety over an infinite field is a quotient of a jacobian variety. This proof is based on a repeated use of Bertini's Theorem: one starts with an abelian variety A (of dimension at least 2), and the strategy is to intersect it as many times as $\dim A - 1$ with good hyperplanes. At every step, one finds a smooth variety of dimension one less than at the preceding step, so it is possible to apply Bertini again, by always referring to the same fixed embedding of A into some $\mathbf{P}^n$. Via this procedure, one obtains a curve $C \subset A$ and a morphism from its jacobian J to A; the task reduces finally to showing that such a map is surjective.

§ IV. *Bertini in Arithmetic Geometry*. The bravest generalization of Bertini's Theorem is perhaps that shown in a work of P. Autissier (see [2], 2001). In this case it is not even necessary to work over a field and the base scheme is just $B = \operatorname{Spec} \mathcal{O}_K$, where K is a number field. The focus is now the investigation of arithmetic varieties X over $\mathcal{O}_K$, and here - as for finite fields - the good locus might not contain any B-rational point. In this case there are no hyperplanes cutting X transversally. Autissier proves (more than) the following: starting with X of dimension $d \geqslant 3$, after a suitable extension L/K one finds an arithmetic variety X' over $B' = \operatorname{Spec} \mathcal{O}_L$, closed in $X_{\mathcal{O}_L}$, such that X' has dimension $d - 1$ (it is in fact a hyperplane section of the original X) and for every $\mathfrak{p} \in B$ such that the fiber $X_\mathfrak{p}$ is smooth, the fiber $X'_\mathfrak{q}$ is also smooth for every $\mathfrak{q}$ above $\mathfrak{p}$.

Contents

CHAPTER 1

Linear systems

In this chapter several proofs of Bertini's Theorem will be provided, in the modern language of linear systems. Two different statements will be proved: the first one only holds over the complex numbers, while the second one will turn out to be less general than the previous, but it holds for any characteristic of the (algebraically closed) base field k. Here are the statements.

> THEOREM (Bertini 1) Let X be a smooth complex variety and let $\mathfrak{D}$ be a positive dimensional linear system on X. Then the general element of $\mathfrak{D}$ is smooth away from the base locus $B_{\mathfrak{D}}$. That is, the set
>
> $$\{ H \in \mathfrak{D} \mid D_H \text{ is smooth away from } B_{\mathfrak{D}} \}$$
>
> is a Zariski dense open subset of $\mathfrak{D}$.
>
> THEOREM (Bertini 2) Let $X \subset \mathbf{P}^n_k$ be a smooth projective variety over k. Then the set of hyperplanes $H \subset \mathbf{P}^n_k$ such that $X \cap H$ is a smooth scheme is a Zariski dense open subset of $\mathbf{P}^{n*}_k$.

Before giving any proof, we want to focus on the notion of linear system as it is understood nowadays, and we wish to explain the precise meaning of the word *general*. We begin here to apologize for confusing this word with its cousin *generic*. The subtle difference will be explained later. In this chapter, all that precedes the section named 'Linear Systems' has to be thought of as background material, which we include by a pure clarity need.

$$\mathfrak{Coh}_X \hookrightarrow \mathfrak{QCoh}_X \hookrightarrow \mathscr{O}_X\text{-Mod}$$

We start with a brief introduction to quasi-coherent and coherent sheaves on a noetherian scheme X. These objects correspond, locally, to modules and finitely generated modules respectively, under a certain functor. A quasi-coherent sheaf is constructed by glueing together (sheaves of) modules, exactly as a scheme is made up by glueing together (spectra of) rings.

For the definition of linear system, one is most interested in *line bundles*. The smallest abelian category containing line bundles (resp. locally free sheaves) is $\mathfrak{Coh}_X$ (resp. $\mathfrak{QCoh}_X$): this is the motivation for the first section.

Afterwards, we will relate divisors and line bundles by using sheaf cohomology (that we have available, because our objects are regarded in abelian categories). After the definition of linear system and some remarks, several proofs of Bertini's Theorem will be provided, and a particular attention will be devoted to the case where the base field is of characteristic zero.

1.1 Coherent and Quasi-Coherent Sheaves

In the first part of this section $(X, \mathscr{O}_X)$ is any scheme and all sheaves are sheaves of modules.

For any $\mathscr{O}_X$-module $\mathscr{F}$ and for any $x \in X$, one can consider the bilinear morphism $\beta : \mathscr{F}(X) \times \mathscr{O}_{X,x} \to \mathscr{F}_x$ sending $(s, t_x) \mapsto s_x t_x$. It induces a natural morphism $\mathscr{F}(X) \otimes_{\mathscr{O}_X(X)} \mathscr{O}_{X,x} \to \mathscr{F}_x$.

Definition 1.1.1. An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be generated by its global sections at $x \in X$ if $\mathscr{F}(X) \otimes_{\mathscr{O}_X(X)} \mathscr{O}_{X,x} \to \mathscr{F}_x$ is surjective. If this happens for every $x \in X$ then one says that $\mathscr{F}$ is GENERATED BY ITS GLOBAL SECTIONS. In other words, for every point $x \in X$ the module $\mathscr{F}_x$ is generated over $\mathscr{O}_{X,x}$ by a set of germs of $\mathscr{F}$ coming from $\mathscr{F}(X)$.

We will sometimes refer to the following:

Proposition 1.1.1. An $\mathscr{O}_X$-module $\mathscr{F}$ is generated by its global sections if and only if it is an epimorphic image of a free $\mathscr{O}_X$-module, i.e. there exists a set I together with a surjective morphism of $\mathscr{O}_X$-modules $\mathscr{O}_X^{(I)} \twoheadrightarrow \mathscr{F}$. If $\mathscr{F}$ is generated by a subset $S \subset \mathscr{F}(X)$, that is, for every $x \in X$ the set $\{ s_x \mid s \in S \}$ generates $\mathscr{F}_x$ over $\mathscr{O}_{X,x}$, then one can just take $I = S$.

Proof. See [17], Lemma 1.3, p. 158. □

Definition 1.1.2. An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be QUASI-COHERENT if every $x \in X$ has an open neighborhood U such that there is an exact sequence of $\mathscr{O}_X$-modules

$$\mathscr{O}_X^{(J)}|_U \longrightarrow \mathscr{O}_X^{(I)}|_U \longrightarrow \mathscr{F}|_U \longrightarrow 0.$$

Quasi-coherence is a property of local nature on X. Informally, one could translate the definition as follows: a sheaf is quasi-coherent if, locally, it is the cokernel of an $\mathscr{O}_X$-morphism between free sheaves.

Definition 1.1.3. An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be FINITELY GENERATED if every point $x \in X$ has an open neighborhood U such that, for some $n \geqslant 1$, there is a surjective morphism of $\mathscr{O}_X$-modules $\mathscr{O}_X^n|_U \twoheadrightarrow \mathscr{F}|_U$.

Definition 1.1.4. An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be COHERENT if it is finitely generated and every $\mathscr{O}_X$-morphism $\mathscr{O}_X^n|_U \to \mathscr{F}|_U$ has finitely generated kernel for any open subset U.

The property of being generated by global sections is independent of coherence. For instance, $\widetilde{M}$ is globally generated but it might not be coherent. And any nonzero invertible sheaf of negative degree is coherent but is not generated by its global sections. In particular, a finitely generated sheaf need not be globally generated.

Some important equivalences of categories.

A quasi-coherent sheaf is completely determined by the local *affine* data, that is, a sheaf of modules $\mathscr{F}$ on a scheme X is quasi-coherent if and only if on every open affine subset $U = \operatorname{Spec} A$, there is an isomorphism $\mathscr{F}|_U \cong \mathscr{F}(U)^\sim$, where $\sim$ is the functor sending an A-module M to the sheaf $\widetilde{M}$ associated to M on $\operatorname{Spec} A$. This is a fully faithful exact[1] functor, whose essential image is $\mathfrak{QCoh}_U$. Moreover, $\mathscr{F}$ is coherent whenever the same holds true, and in addition $\mathscr{F}(U)$ is finitely presented over A (or just finitely generated, when A is noetherian). In fact, when restricted to finitely presented A-modules, the equivalence A-Mod $\cong \mathfrak{QCoh}_U$ restricts to an equivalence with the category $\mathfrak{Coh}_U$. It should be clear from these characterizations that any locally free $\mathscr{O}_X$-module is quasi-coherent, and every locally free $\mathscr{O}_X$-module of finite rank is coherent.[2] However, there are examples showing that the converse of these assertions is false: for instance, the

[1] Not only it is exact, but it also reflects exactness of sequences of the shape $M \to L \to N$.

[2] These assertions are not true for *any* scheme, but certainly hold for those X such that the structure sheaf $\mathscr{O}_X$ is coherent; in particular, X locally noetherian is enough.

sheaf $\widetilde{M}$ on $X = \operatorname{Spec} A$, where M is not a free A-module, is quasi-coherent but not locally free.

Remark 1.1. While proving the above characterizations of coherence, one realizes that for a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ the following implications are true:

$$\text{Coherent} \;\Rightarrow\; \text{Finitely generated} \;\Rightarrow\; \begin{array}{l}\mathscr{F}(U) \text{ finitely generated} \\ \text{for every } U \textit{ open affine.}\end{array}$$

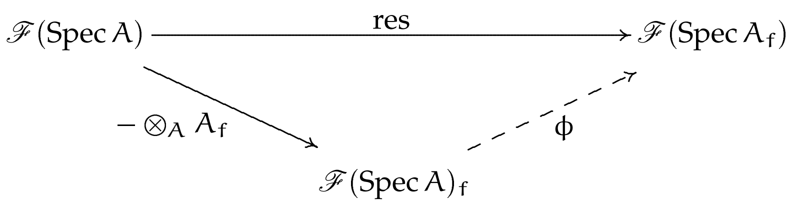
Moreover, these conditions are equivalent whenever X is noetherian.

Remark 1.2. The $\mathscr{O}_X$-module $\widetilde{M}$, the quasi-coherent sheaf on $X = \operatorname{Spec} A$ associated to M, is an example of a sheaf that is generated by its global sections: indeed any set I of generators of M will provide a surjective morphism of $\mathscr{O}_X$-modules $\mathscr{O}_X^{(I)} \twoheadrightarrow \widetilde{M}$. Another way to see this: just apply the definition and observe that for every $\mathfrak{p} \in X$ the canonical map $M \otimes_A A_\mathfrak{p} \to M_\mathfrak{p}$ defined by $m \otimes (a/s) \mapsto (ma)/s$ is surjective, because m/s comes from the tensor $m \otimes (1/s)$.

We now introduce a characterization of quasi-coherence. First, by the local nature of this notion, for knowing a quasi-coherent sheaf on X it is a good start to know its sections over the affine open subsets $U = \operatorname{Spec} A \subset X$. In fact, even better is to know how they restrict from U to any of its *distinguished* affine open subsets, i.e. those of the form $\operatorname{Spec} A_f$ for $f \in A$. We call the open immersion $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$ a *distinguished inclusion*. So, consider an $\mathscr{O}_X$-module $\mathscr{F}$ and define

$$\mathscr{F}(\operatorname{Spec} A)_f = \mathscr{F}(\operatorname{Spec} A) \otimes_A A_f.$$

Now consider the following diagram



Clearly by $- \otimes_A A_f$ we mean the image of $\ell : A \to A_f$ under $\mathscr{F}(\operatorname{Spec} A) \otimes_A -$. Note that $\mathscr{F}(\operatorname{Spec} A)$ is an A-module and $\mathscr{F}(\operatorname{Spec} A_f)$ is an A_f-module, so by the universal property of localization there exists a unique morphism ϕ closing the diagram: for every distinguished inclusion we have a canonical factorization of the restriction map. The following statement holds true.

Proposition 1.1.2. An $\mathscr{O}_X$-module $\mathscr{F}$ is quasi-coherent if and only if for every such diagram the canonical morphism ϕ is an isomorphism.

The following result will be useful.

THEOREM 1.1. Let X be a scheme.

(1) The category of quasi-coherent $\mathscr{O}_X$-modules is abelian.

(2) If $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent $\mathscr{O}_X$-modules, then so too is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$, and we can compute its sections on every open affine subset $U \subset X$ simply by forgetting sheafification, that is $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})(U) = \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$.

Proof. Part (1): the claim is certainly true in the affine case. For any scheme X, one knows that $\mathfrak{QCoh}_X \subset \mathscr{O}_X$-Mod is a subcategory, with $\mathscr{O}_X$-Mod abelian. So one only needs to check that $\mathfrak{QCoh}_X$ contains the 0 object, is closed under finite direct sums, kernels and cokernels. The 0 sheaf is certainly quasi-coherent. If $\mathscr{F}$ and $\mathscr{G}$ are two quasi-coherent $\mathscr{O}_X$-modules, consider an affine open subset Spec A of X and assume that $\mathscr{F} \cong \widetilde{M}$ and $\mathscr{G} \cong \widetilde{N}$ on Spec A. Then $\mathscr{F} \oplus \mathscr{G} \cong (M \oplus N)^\sim$ because, by exactness of localization, one has $(M \oplus N)^\sim(D(f)) = (M \oplus N)_f = M_f \oplus N_f$ for every $f \in A$. So $\mathscr{F} \oplus \mathscr{G}$ is again quasi-coherent. Let us consider a morphism $\alpha : \mathscr{F} \to \mathscr{G}$ of quasi-coherent sheaves. One has to be sure it has both a kernel and a cokernel. Let U be any affine subset of X. Use quasi-coherence: on U, the morphism α is given as an $\mathscr{O}_X(U)$-linear map of modules $\beta : M \to N$. Define $(\ker \alpha)(U) = \ker \beta$ and $(\operatorname{coker} \alpha)(U) = \operatorname{coker} \beta$. Now, if

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\beta} N \longrightarrow P \longrightarrow 0$$

is an exact sequence, then so is the localized sequence

$$0 \longrightarrow L_f \longrightarrow M_f \xrightarrow{\beta_f} N_f \longrightarrow P_f \longrightarrow 0$$

for every $f \in \mathscr{O}_X(U)$. So $(\ker \beta)_f \cong \ker(\beta_f)$ and $(\operatorname{coker} \beta)_f \cong \operatorname{coker}(\beta_f)$. Hence, thanks to Proposition 1.1.2, both $(\ker \beta)_f$ and $(\operatorname{coker} \beta)_f$ define quasi-coherent $\mathscr{O}_X$-modules, and moreover these are exactly the kernel and the cokernel of α, as it is clear by checking on stalks. Part (2): the question is local on X so one may assume $X = U = \operatorname{Spec} A$; suppose that $\mathscr{F}(U) = M$ and $\mathscr{G}(U) = N$, two A-modules (so that $\mathscr{F} = \widetilde{M}$ and $\mathscr{G} = \widetilde{N}$). Then $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})(U) = (\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N})(U) = (M \otimes_A N)^\sim(U) = M \otimes_A N$, as claimed. □

The end of this section is devoted to the definition of the quasi-coherent sheaf associated to the Proj of a graded ring and to twisted sheaves, which will be used from now on. Now S is a graded ring and M a graded S-module. Denote $X = \operatorname{Proj} S$. One can prove that there is a unique sheaf on X, denoted $\widetilde{M}$, such that

$$\widetilde{M}|_{D_+(f)} \cong (M_{(f)})^{\sim} \tag{1.1}$$

for every homogeneous $f \in S_+$. Equivalently, we define it on the principal basis of X by $\widetilde{M}(D_+(f)) = M_{(f)}$, the submodule of M_f consisting of elements of degree 0. This $\widetilde{M}$ is a quasi-coherent $\mathscr{O}_X$-module, by (1.1) and by our characterization in Proposition 1.1.2. If S is noetherian and M is finitely generated then it is coherent.

For every $d \in \mathbb{Z}$ one can consider the graded ring $S(d)$: it is just S (so it is an S-module) with grading $S(d)_n = S_{d+n}$. For $X = \operatorname{Proj} S$ we define $\mathscr{O}_X(d) = S(d)^{\sim}$. In particular, when $d = 1$ one gets the *Twisting Sheaf of Serre*. If $f \in S$ is homogeneous of degree one, then $S(d)_{(f)} = f^d S_{(f)}$, i.e.

$$H^0(D_+(f), \mathscr{O}_X(d)) = f^d H^0(D_+(f), \mathscr{O}_X). \tag{1.2}$$

Clearly the $\mathscr{O}_X$-modules $\mathscr{O}_X(d)$ are quasi-coherent, so one can apply Theorem 1.1 to see that $\mathscr{O}_X(d) \otimes_{\mathscr{O}_X} \mathscr{O}_X(e) \cong \mathscr{O}_X(d+e)$ for every two integers d, e. Indeed, for every $f \in S$ homogeneous of degree one,

$$\begin{aligned}
&(\mathscr{O}_X(d) \otimes_{\mathscr{O}_X} \mathscr{O}_X(e))(D_+(f)) = \\
&\mathscr{O}_X(d)(D_+(f)) \otimes_{\mathscr{O}_X(D_+(f))} \mathscr{O}_X(e)(D_+(f)) = \\
&f^d \mathscr{O}_X(D_+(f)) \otimes_{S_{(f)}} f^e \mathscr{O}_X(D_+(f)) = f^{d+e}(S_{(f)} \otimes_{S_{(f)}} S_{(f)}) = \\
&f^{d+e} S_{(f)} = S(d+e)_{(f)} = \mathscr{O}_X(d+e)(D_+(f)).
\end{aligned}$$

In fact, the following general result is true:

THEOREM 1.2. Let S be a graded ring and $X = \operatorname{Proj} S$. If S is generated by S_1 as an S_0-algebra and if S_0 is noetherian, then:

(a) Every $\mathscr{O}_X(d)$ is an invertible sheaf on X, hence coherent.

(b) For d, e any two integers, $\mathscr{O}_X(d) \otimes \mathscr{O}_X(e) \cong \mathscr{O}_X(d+e)$.

Proof. For (a), one has to show that $\mathscr{O}_X(d)$ is locally free of rank one. Take $f \in S_1$, then by the above

$$\mathscr{O}_X(d)|_{D_+(f)} \cong (S(d)_{(f)})^{\sim},$$

the quasi-coherent sheaf on $\operatorname{Spec} S_{(f)}$ associated to the S-module $S(d)_{(f)}$. The claim is that $\mathscr{O}_X(d)|_{D_+(f)}$ is free of rank one. This will be enough because $\{ D_+(f) \mid f \in S_1 \}$ is a covering of X by our assumption. Let us show that $S(d)_{(f)}$ is free of rank one over $S_{(f)}$. In fact, $S_{(f)}$ is the algebra of elements of degree 0 in S_f, and $S(d)_{(f)}$ is the algebra of elements of degree d in S_f (recall that $\deg f = 1$). So there is a group isomorphism $S_{(f)} \to S(d)_{(f)}$ defined by $s \mapsto f^d s$. This is well defined for every $d \in \mathbb{Z}$ since f is invertible in S_f. Hence $S(d)_{(f)}$ is free of rank one over $S_{(f)}$. Part (b) follows from the calculation above. □

Remark 1.3. If $X = \operatorname{Proj} S$ where S is generated by S_1 as an S_0-algebra, then $S_1 \subset \Gamma(X, \mathscr{O}_X(1))$ are global sections *generating* $\mathscr{O}_X(1)$, i.e. they give a surjective $\mathscr{O}_X$-morphism $\mathscr{O}_X^{(S_1)} \twoheadrightarrow \mathscr{O}_X(1)$, so for such an S we have another example of a sheaf that is generated by its global sections.

1.2 Divisors and Line Bundles

This section is devoted to introduce the notion of line bundle by highlighting its (cohomological) relations with divisors. This can be done for any variety X (thought of as a space with a sheaf of functions $\mathscr{O}$), but for simplicity we may assume that X is a noetherian integral and normal scheme (so that we have Weil divisors available). By a *divisor*, one means in fact *Cartier divisor*, i.e. a global section of the quotient sheaf $\mathscr{M}^\times/\mathscr{O}^\times$, where $\mathscr{M}$ is the sheaf of total quotient rings on X (according to complex manifolds, one can call its sections "meromorphic functions"). Recall that one has the following result:

Theorem 1.3. Let X be an integral, separated and noetherian scheme such that $\mathscr{O}_p$ is a UFD for every $p \in X$. Then the group $\operatorname{Div} X$ of Weil divisors is isomorphic to the group $H^0(X, \mathscr{M}^\times/\mathscr{O}^\times)$ of Cartier divisors, and principal Weil divisors correspond to principal Cartier divisors under this isomorphism (so $\operatorname{Cl} X \cong \operatorname{Ca} \operatorname{Cl} X$).

Proof. See [14] II, Proposition 6.11, p. 141. □

Thus, as our interest will be in smooth varieties, we will be almost always allowed to interchange *Weil* with *Cartier*.

A divisor D can be identified with some local data $\{ U_\alpha, f_\alpha \}$, that is, an open covering $\mathfrak{U} = (U_\alpha)$ of X together with a collection of nonzero meromorphic functions $f_\alpha \in \mathscr{M}^\times(U_\alpha)$ satisfying

$$\frac{f_\alpha}{f_\beta} \in \mathscr{O}^\times(U_{\alpha\beta}). \tag{1.3}$$

Here $U_{\alpha\beta}$ denotes the intersection $U_\alpha \cap U_\beta$.

A Cartier divisor is said to be EFFECTIVE in case one can choose the f_α to be *regular* functions.

Recall, as an example, that divisors on $\mathbf{P}^n_k$ are completely determined, up to linear equivalence, by their degree, and the degree map induces an isomorphism $\operatorname{Cl}\mathbf{P}^n_k \cong \mathbb{Z}$. We will see that linear equivalence of divisors corresponds to isomorphism between invertible sheaves (line bundles), and this for any scheme X.

By a (complex) vector bundle of rank r over X one means a surjective morphism of varieties $\pi : E \to X$ such that for all $x \in X$ the fiber $E_x = \pi^{-1}(x)$ is a $\mathbb{C}$-vector space of dimension r, and there exists an open neighborhood U of x together with an isomorphism (called trivialization)

$$\phi : E|_U \xrightarrow{\sim} U \times \mathbb{C}^r$$

where $E|_U = \pi^{-1}(U)$ and $E_y \subset E|_U$ is sent isomorphically to $\{ y \} \times \mathbb{C}^r$ for every point $y \in U$.

THEOREM 1.4. Let X be a variety and r a positive integer. Vector bundles of rank r over X correspond, up to isomorphism, to locally free $\mathscr{O}_X$-modules of rank r.

Proof. See Appendix. □

Giving a vector bundle (E, π) is, more or less,[3] the same as giving a collection of transition functions $\{ g_{\alpha\beta} : U_{\alpha\beta} \to GL(r, \mathbb{C}) \}$, i.e. a one-cochain in the Čech complex $C^\bullet(\mathfrak{U}, \mathscr{O}^\times)$. Explicitly, each $g_{\alpha\beta}$ is a morphism of varieties satisfying $g_{\alpha\beta}(x)g_{\beta\alpha}(x) = 1$ for all $x \in U_{\alpha\beta}$ and the so-called *cocycle condition*

$$g_{\alpha\gamma}(x) = g_{\alpha\beta}(x)g_{\beta\gamma}(x) \quad \forall\, x \in U_{\alpha\beta\gamma}. \tag{1.4}$$

A *line bundle* is just a vector bundle of rank 1.

Let us give ourselves a complex line bundle $L \to X$, so that its transition functions take values in $GL(1, \mathbb{C}) = \mathbb{C}^\times$. Let $\mathfrak{U} = (U_\alpha)$ be an open covering of X and let $f_\alpha \in \mathscr{O}^\times(U_\alpha)$ be nonzero regular functions. If $\phi_\alpha : L|_{U_\alpha} \to U_\alpha \times \mathbb{C}$ are trivializations then the

[3] See (†) to justify this *more or less*.

corresponding transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^\times$ are given by $g_{\alpha\beta}(x) = (\phi_\alpha \circ \phi_\beta^{-1})|_{L_x}$ where

$$\{x\} \times \mathbb{C} \xrightarrow{\phi_\beta|_{L_x}^{-1}} L_x \xrightarrow{\phi_\alpha|_{L_x}} \{x\} \times \mathbb{C}.$$

More precisely, to get $g_{\alpha\beta}(x)$ one has to evaluate this compositions at $(x, 1)$ and look at the second component. Now consider the new trivializations $\psi_\alpha := f_\alpha \phi_\alpha$ (these are again isomorphisms by the choice of f_α). The corresponding transition functions $h_{\alpha\beta}$ are defined by

$$\begin{aligned} h_{\alpha\beta}(x) &= (\psi_\alpha \circ \psi_\beta^{-1})|_{L_x} = \psi_\alpha|_{L_x} \circ \phi_\beta|_{L_x}^{-1} \\ &= f_\alpha(x)\phi_\alpha|_{L_x} \circ (f_\beta^{-1}(x)\phi_\beta|_{L_x}^{-1}) = \frac{f_\alpha(x)}{f_\beta(x)}(\phi_\alpha \circ \phi_\beta^{-1})|_{L_x} \end{aligned}$$

whence the relation

$$h_{\alpha\beta} = \frac{f_\alpha}{f_\beta} g_{\alpha\beta}. \tag{1.5}$$

Of course, we have changed trivializations (and hence transition functions), but L is still the same. On the other hand, there is no other way to "deform" a given collection of trivializations without getting another vector bundle (here *another* means *not isomorphic*), therefore our conclusion is:

> Two one-cocycles $\{ g_{\alpha\beta} \}$ and $\{ h_{\alpha\beta} \}$ determine the same line bundle (up to isomorphism) if and only if there are nonzero holomorphic functions $f_\alpha \in \mathscr{O}^\times(U_\alpha)$ satisfying (1.5), if and only if their difference $\{ g_{\alpha\beta} h_{\alpha\beta}^{-1} \}$ is a Čech one-coboundary. (†)

This leads us to the relation

$$\mathcal{L}_X \cong H^1(X, \mathscr{O}^\times) =: \operatorname{Pic} X \tag{1.6}$$

where $\mathcal{L}_X$ denotes the set of line bundles over X, up to isomorphism. This is really a *group* isomorphism. By Theorem 1.4, we can view Pic X as the group of isomorphism classes of invertible sheaves on X.

Definition 1.2.1. To a Cartier divisor $D = \{ U_\alpha, f_\alpha \}$ on a scheme X one can associate the following subsheaf $\mathscr{L}(D)$ of $\mathscr{M}$: for every α, we set $\mathscr{L}(D)(U_\alpha)$ to be the $\mathscr{O}(U_\alpha)$-submodule

$$\mathscr{L}(D)(U_\alpha) = f_\alpha^{-1}\mathscr{O}(U_\alpha) \subset \mathscr{M}(U_\alpha).$$

Since $f_\alpha/f_\beta \in \mathscr{O}^\times(U_{\alpha\beta})$, we have that $f_\alpha\mathscr{O}(U_{\alpha\beta}) = f_\beta\mathscr{O}(U_{\alpha\beta})$, so that $\mathscr{L}(D)$ is a well defined object; it is called the INVERTIBLE SHEAF ASSOCIATED TO D. As a notation, we may also write $\mathscr{O}(D)$ instead of $\mathscr{L}(D)$.

Remark 1.4. Note that each divisor on X can be viewed as a closed subscheme $D \subset X$, and the inclusion is given by the sheaf of ideals $\mathscr{O}_X(-D)$. In fact, there is an exact sequence

$$0 \longrightarrow \mathscr{O}_X(-D) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_D \longrightarrow 0.$$

Aside 1.1. Note that, when X is a regular curve over k with function field K, the global sections of this invertible sheaf are

$$H^0(X, \mathscr{O}_X(D)) = L(D)$$

where $L(D)$ denotes the k-vector space $\{ f \in K^\times \mid (f) + D \geqslant 0 \} \cup \{0\}$. In this case, the meaning of the space $L(D)$ is clear: it contains those rational functions ϕ whose order $\operatorname{ord}_p(\phi) = \nu_p(\phi) \geqslant -n_p$ at every point of X, if D is the divisor $\sum_p n_p p$: the functions in $L(D)$ are those whose poles have order *no worse* that n_p. For an arbitrary scheme X over a field k, even if Weil divisors do not coincide with Cartier divisors, we always have this k-vector space $H^0(X, \mathscr{O}_X(D))$. Now it contains those meromorphic functions $\phi \in \Gamma(X, \mathscr{M}^\times)$ such that the divisor $(\phi) + D$ is an effective Cartier divisor, i.e. such that $(\phi) \geqslant -D$. This means: take (ϕ) to be the image of ϕ under p_* (see below); sum (multiply) by D in $H^0(X, \mathscr{M}^\times/\mathscr{O}_X^\times)$; look at the local data corresponding to this product and check whether they are nonzero regular functions. Note, finally, that when Weil = Cartier, the concept of effective Weil divisor is the same as that of effective Cartier divisor: indeed, a rational function has positive valuation along a prime divisor Z if and only if it is regular (i.e. it has no poles) on Z.

Proposition 1.2.1. Let $(X, \mathscr{O})$ be a scheme and D, E be Cartier divisors on X. Then

$$H^0(X, \mathscr{M}^\times/\mathscr{O}^\times) \hookrightarrow \operatorname{Pic} X$$

(a) The morphism $\mathscr{L}$ induces a bijection

$$H^0(X, \mathscr{M}^\times/\mathscr{O}^\times) \simeq \{ \text{Invertible Subsheaves of } \mathscr{M} \text{ up to } \cong \}.$$

(b) We have $\mathscr{L}(D - E) \cong \mathscr{L}(D) \otimes \mathscr{L}(E)^{-1}$, and

(c) $D \sim E$ if and only if $\mathscr{L}(D) \cong \mathscr{L}(E)$.

Proof. The first point is clear: you cannot have an invertible subsheaf $\mathscr{H} \subset \mathscr{M}$ without a collection $f_\alpha \in \mathscr{M}^\times(U_\alpha)$: simply look at local generators h_α of $\mathscr{H}$, and take their inverses $f_\alpha = h_\alpha^{-1}$ to be the local data of some divisor D. To prove (b) just observe that, $H^0(X, \mathscr{M}^\times/\mathscr{O}^\times)$ being a group, $D - E$ is defined by the quotient of the defining functions, and thus it goes exactly to $\mathscr{L}(D) \otimes \mathscr{L}(E)^{-1}$. For the last part, $\mathscr{L}(D) \in \operatorname{Pic} X$ is trivial if and only the transition functions $\{ h_{\alpha\beta} \}$ of the corresponding line bundle are in $B^1(\mathfrak{U}, \mathscr{O}^\times)$, if and only if $h_{\alpha\beta}$ are quotients of holomorphic functions, i.e. meromorphic functions defined on the open subsets

U_α. And the datum of these meromorphic functions is equivalent to the datum of a single *global* nonzero meromorphic function $f \in \mathscr{M}^\times(X)$, which means exactly that D is the divisor of f. So D is principal if and only if it is in $\ker \mathscr{L}$. Conclude by (b). □

Remark that the short exact sequence

$$1 \longrightarrow \mathscr{O}^\times \longrightarrow \mathscr{M}^\times \xrightarrow{p} \mathscr{M}^\times/\mathscr{O}^\times \longrightarrow 1$$

gives an exact piece in cohomology

$$H^0(X, \mathscr{M}^\times) \xrightarrow{p_*} H^0(X, \mathscr{M}^\times/\mathscr{O}^\times) \xrightarrow{\mathscr{L}} \operatorname{Pic} X$$

and moreover one has an injective group homomorphism

$$\theta : \operatorname{Ca}\operatorname{Cl} X \hookrightarrow \operatorname{Pic} X.$$

which becomes an isomorphism when, for example, X is an integral scheme (See [14] II, Proposition 6.15, p. 145 for a proof). As our varieties will be integral, we will assume that a line bundle will be of the form $\mathscr{L}(D)$.

1.3 Linear Systems

From now on, unless otherwise stated, k denotes an algebraically closed field, and X is an integral smooth projective variety over k. Let us briefly summarize what we know: first of all, for such an X, Weil Divisors coincide with Cartier divisors (consequence of Theorem 1.3); second, linear equivalence of divisors corresponds to isomorphism of invertible sheaves (consequence of Proposition 1.2.1); moreover, as X is integral, there is an isomorphism $\operatorname{Pic} X \cong \operatorname{Ca}\operatorname{Cl} X$, saying that a line bundle $\mathscr{L}$ on X can be written as $\mathscr{O}_X(D)$ for some divisor D on X. Here is another important and nontrivial result that one has to keep in mind.

Theorem 1.5. Let X be a projective scheme over a noetherian ring A and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then $H^0(X, \mathscr{F})$ is a finitely generated A-module.

Proof. [14] II, Theorem 5.19, p. 122. □

The underlying idea of a linear system is the following: giving an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and a set $S \subset H^0(X, \mathscr{L})$ of global sections is the same as giving a certain collection of effective divisors, all linearly equivalent to each other. Let us make this precise: to an invertible sheaf $\mathscr{L}$ on X and a nonzero global section

$s \in H^0(X, \mathscr{L})$, one associates its divisor (called the divisor of zeros), denoted by (s) (or sometimes $(s)_0$, to emphasize it has no negative part). It is defined as follows: for every point $\eta \in X$ such that $Y_\eta = \{\eta\}^-$ is irreducible of codimension one, one has the cyclic module $\mathscr{L}_\eta = e.\mathscr{O}_{X,\eta}$ and the germ s_η writes in the form $e.u$ for a unique $u \in \mathscr{O}_{X,\eta}$. One defines $\nu_\eta(s) = \nu_\eta(u)$, where the ν_η in the right hand side is the discrete valuation associated to the prime divisor Y_η. Finally, $(s) := \sum_\eta \nu_\eta(s) Y_\eta$ is clearly effective.

Proposition 1.3.1. Let X be a smooth projective variety over k and let D be a divisor on X. Consider $\mathscr{O}_X(D) = \mathscr{L}(D)$, the corresponding invertible sheaf. Then

(a) For every nonzero global section $s \in H^0(X, \mathscr{O}_X(D))$, the divisor $(s)_0$ is an effective divisor linearly equivalent to D.

(b) Every effective divisor linearly equivalent to D is the divisor of zeros of some $s \in H^0(X, \mathscr{O}_X(D))$.

(c) Two sections $s, s' \in H^0(X, \mathscr{O}_X(D))$ share the same divisor of zeros if and only if $s' = \lambda s$ for some $\lambda \in k^\times$.

Proof. For (a), use that $\mathscr{O}_X(D)$ is a subsheaf of the sheaf $\mathscr{M}$ of meromorphic functions on X, that now is the constant sheaf $\underline{K}$. So s can be viewed as a rational function $f \in K^\times$. Assume $\{U_i, f_i\}$ are local data defining D, where $f_i \in K^\times$. Then, by the construction of $\mathscr{O}_X(D)$, we know it is locally generated by f_i^{-1}. Notice that there is a local isomorphism $\phi : \mathscr{O}_X(D) \xrightarrow{\sim} \mathscr{O}_X$, indeed $\mathscr{O}_X(D)(U_i) = f_i^{-1}\mathscr{O}_X(U_i)$ is isomorphic to $\mathscr{O}_X(U_i)$ via multiplication by f_i, i.e. $f_i^{-1}h \mapsto h$. Thus $(s)_0$ is locally defined by $\{U_i, f_i f\}$, i.e. $(s)_0 = D + (f)$, showing that $(s)_0 \sim D$.
For (b), let $E \geqslant 0$ be an effective divisor such that $E = D + (f)$. Then $(f) \geqslant -D$, which is equivalent (by Aside 1.1) to $f \in H^0(X, \mathscr{O}_X(D))$. Finally, the divisor of zeros of f is exactly $(f)_0 = E$.
For (c), suppose $(s)_0 = (s')_0$ where, as above, s and s' correspond to some $f, f' \in K^\times$, and we have $(f/f') = 0$, as f/f' is a meromorphic function. This says that $f/f' \in \ker \Gamma(X, p_*) = H^0(X, \mathscr{O}_X^\times)$, but now X is a projective variety over an algebraically closed field, hence $H^0(X, \mathscr{O}_X) = k$, and then $f/f' = \lambda \in k^\times$. □

If $|D|$ is the set of all effective divisors linearly equivalent to D, the Proposition above says that $|D|$ has the structure of a projective space over k. More precisely, there is a bijective map

$$\begin{aligned} (H^0(X, \mathscr{O}_X(D)) - \{0\})/k^\times &\xrightarrow{\sim} |D| \\ [s] &\longmapsto (s) + D. \end{aligned}$$

Recall: the projectivization of a vector space V, denoted $\mathbf{P}V$, is by definition the projective variety Proj (Sym V); one can think of it as the set of hyperplanes in V; it is in fact the *dual* construction of $(V-\{0\})/k^\times$, the set of lines through the origin in V. It follows by the definition that one always has $(V-\{0\})/k^\times = \mathbf{P}V^*$. Thus one may assume the following identification:

$$(H^0(X, \mathscr{O}_X(D)) - \{0\})/k^\times = \mathbf{P}H^0(X, \mathscr{O}_X(D))^*.$$

Thus, the points of the projective variety $\mathbf{P}H^0(X, \mathscr{O}_X(D))^*$ act as parameters for the set of divisors in $|D|$. A *point* of $|D|$ is a one-dimensional subspace of $H^0(X, \mathscr{O}_X(D))$, namely, such a *line* consists of those global sections which differ one from each other by a nonzero constant, that is, having the same divisor of zeros.

Definition 1.3.1. The set $|D|$, together with its structure of projective variety, is called a COMPLETE LINEAR SYSTEM of divisors on X. A LINEAR SYSTEM on X is a subset $\mathfrak{D} \subset |D|$ which is a linear subspace of $|D|$ when this is viewed as a projective variety. In other words, $\mathfrak{D} = \mathbf{P}V^*$ for some vector subspace $V \subset H^0(X, \mathscr{O}_X(D))$. The dimension of a linear system $\mathbf{P}V^*$ is $\dim_k V - 1$, so it is finite by Theorem 1.5. A linear system of dimension one is called a *pencil*, while in dimension two it is called a *net* and in dimension three a *web*.

REMARK 1.5. Why does one projectivize the *dual*? Because to any inclusion $V \subset H^0(X, \mathscr{O}_X(D))$ there corresponds a surjective morphism $H^0(X, \mathscr{O}_X(D))^* \twoheadrightarrow V^*$ and since Proj is contravariant[4] one recovers an inclusion $\mathbf{P}V^* \hookrightarrow |D|$. In what follows, however, one will identify V to its dual, so $\mathbf{P}V$ will mean the linear system $\mathbf{P}V^*$.

One can figure a linear system as a continuous family of varieties, moving in some bigger ambient variety, say $\mathbf{P}^n_k$; the points that do not move at all when one ranges the set of parameters are the points in the *base locus*, which we now define properly.

Definition 1.3.2. A point $x \in X$ is called a BASE POINT of $\mathfrak{D}$ if it lies in Supp D for every $D \in \mathfrak{D}$, where the support $\operatorname{Supp} D \subset X$ of a divisor D is the union of its prime divisors. We define

$$B_{\mathfrak{D}} = \bigcap_{D \in \mathfrak{D}} \operatorname{Supp} D$$

to be the BASE LOCUS of $\mathfrak{D}$. In other words, a point $x \in X$ is a base point if it lies in every divisor contained in the system. A divisor

[4] Proj is actually not a functor, but it behaves like a functor when one restricts to arrows (of graded rings) that are epimorphisms.

E which is contained in the base locus, i.e. such that $E \leqslant D$ for all $D \in \mathfrak{D}$, is called a FIXED COMPONENT of $\mathfrak{D}$.

EXAMPLE 1.3.1. Let $X = \mathbf{P}^n_k$. For every integer $d > 0$ the invertible sheaf $\mathscr{O}(d)$ has the space of homogeneous polynomials of degree d as global sections: it is a k-vector space of dimension $\binom{n+d}{d}$. One can consider, for an arbitrary hyperplane $H \subset \mathbf{P}^n_k$, the complete linear system $|dH| = \mathbf{P}H^0(\mathbf{P}^n_k, \mathscr{O}(d))$ of dimension $\binom{n+d}{d} - 1$. It is the projective variety of effective divisors linearly equivalent to dH (that is, hypersurfaces of degree d), and it gives an example of a base-point-free linear system. As a notation, one might also denote $|dH|$ by $|\mathscr{O}(d)|$.

EXAMPLE 1.3.2. Let X be a smooth projective curve over k with function field K, and let $D \in \operatorname{Div} X$. Consider the complete linear system $|D|$ associated to the k-vector space

$$L(D) = \{\, \phi \in K^\times \mid (\phi) + D \geqslant 0 \,\} \cup \{\, 0 \,\}\,.$$

That is, $|D| = \mathbf{P}L(D) = \{\, \phi \in K^\times \mid (\phi) + D \geqslant 0 \,\} / k^\times$ is a projective space of dimension $\ell(D) - 1$. A point of this projective space is a class $[\phi] = \{\, a\phi \mid a \in k^\times \,\}$ where all the $a\phi$ share the same $(\phi)_0$. The correspondence is then

$$|D| = \mathbf{P}L(D) \qquad (\phi) + D \longleftrightarrow [\phi]$$

where $(\phi) = \sum_p \operatorname{ord}_p(\phi)p$. Saying that $[\phi]$ is in the base locus amounts to asserting that $(\phi) + D \leqslant D_\lambda$ for all $D_\lambda \in |D|$, i.e. all points (prime divisors) p appearing in $(\phi)+D$ also appear in every D_λ. Note that if D is (the divisor of) a point p then $|p| = \{\, p \,\}$, unless $X \cong \mathbf{P}^1_k$, in which case every point is linearly equivalent to each other, so that $|p| = \mathbf{P}^1$ for every point $p \in \mathbf{P}^1_k$.

Lemma 1.1. Let $\mathfrak{D} = \mathbf{P}V \subset |D|$ be a linear system on X. Then $x \in X$ is a base point of $\mathfrak{D}$ if and only if $s_x \in \mathfrak{m}_x \mathscr{O}_X(D)_x$ for all $s \in V$. In particular, $\mathfrak{D}$ is base-point-free if and only if $\mathscr{O}_X(D)$ is generated by the global sections in V.

Proof. If $X_s = \{\, y \in X \mid s_y \in \mathscr{O}_X(D)^\times_y \,\}$, we can write

$$B_{\mathfrak{D}} = \bigcap_{D \in \mathfrak{D}} \operatorname{Supp} D = \bigcap_{[s] \in \mathfrak{D}} \operatorname{Supp}(s) = \bigcap_{s \in V} (X - X_s),$$

since the support of $(s) = (s)_0$ is exactly the set of points at which s vanishes. It follows that a point $x \in X$ is a base point if $s_x \notin \mathscr{O}_X(D)^\times_x$ for any $s \in V$. That is, $s_x \in \mathfrak{m}_x \mathscr{O}_X(D)_x$ for all $s \in V$.

Suppose the above condition is false for any $x \in X$, i.e. $\mathfrak{D}$ is base-point-free. This is equivalent to saying that for any $x \in X$ there is a global section $s \in V$ such that s_x does not vanish (is invertible) in $\mathscr{O}_X(D)_x$. Such sections are the required generators of $\mathscr{O}_X(D)_x$ over $\mathscr{O}_{X,x}$. □

The sheaf $\mathscr{O}_X(D)$ is not generated by its global sections in general, so a complete linear system might have base points. For instance, consider X a smooth curve of genus > 0 and D the divisor of any point. As remarked above, $|p| = \{ p \}$.

It follows from the lemma (and from Proposition 1.1.1) that one can also define the base locus in this way:

$$B_{\mathfrak{D}} = \{ x \in X \mid V \otimes_k \mathscr{O}_{X,x} \to \mathscr{O}_X(D)_x \text{ is } \textit{not} \text{ surjective} \} .$$

We need two quick observations.

– THE DUAL PROJECTIVE SPACE. We know that $\mathbf{P}^n_k$ is the set of vector lines of k^{n+1}. If one applies the same construction to the dual vector space $(k^{n+1})^*$ then the result is the space of all n-dimensional linear subspaces (hyperplanes) of k^{n+1}. This is called the dual projective space and is denoted $\mathbf{P}^{n*}_k$. Of course, every n-dimensional linear subspace of k^{n+1} is of the form

$$H_a : a_0x_0 + a_1x_1 + \cdots + a_nx_n = 0$$

where not all the $a_i \in k$ are zero, so that $a = (a_0, a_1, \ldots, a_n)$ is a well defined point of $\mathbf{P}^n_k$. Moreover $H_a = H_b$ if and only if $H_b = \lambda H_a$ for some $\lambda \in k^\times$, if and only if $a = b$ in $\mathbf{P}^n_k$. This remark allows us to identify $\mathbf{P}^{n*}_k$ to $\mathbf{P}^n_k$ through $H_a \mapsto a$. Summarizing,

$$\mathbf{P}^{n*}_k = \{ \text{Hyperplanes in } \mathbf{P}^n_k \} = \mathbf{P}H^0(\mathbf{P}^n_k, \mathscr{O}(1)).$$

Finally, note that the vacuous statement 'the points *on* a hyperplane $H \subset \mathbf{P}^n_k$ form a hyperplane' gets dualized to 'the hyperplanes *through* a point $p \in \mathbf{P}^n_k$ form a hyperplane in $\mathbf{P}^{n*}_k$'.

– THE "GENERIC ELEMENT". Let $\mathfrak{D} = \mathbf{P}^r$ be a linear system of divisors on X. A *general member* of $\mathfrak{D}$ is said to satisfy a property Q if there is a Zariski dense open subset $U \subset \mathbf{P}^r$ such that all divisors corresponding to points of U satisfy Q. The *generic element* of a linear system is the generic point of the projective space $\mathbf{P}^r$ parameterizing the system, and a given property is called generic if it is a property *of* the generic point.
Tricky observation: recall that the generic point of an irreducible scheme is unique, so the contrary of the phrase 'the generic point

ξ satisfies Q' is simply 'ξ does not satisfy Q'. But if one is not necessarily dealing with schemes, and has the assertion

$$\text{'The general member of } \mathfrak{D} \text{ satisfies Q'}, \tag{1.7}$$

then this has a more subtle negation. Call $V_Q \subset \mathfrak{D}$ the set of divisors satisfying Q. The contrary of (1.7) is then:

$$\text{'}V_Q \text{ has empty interior'}.$$

This translates the fact that V_Q does not contain any open set of $\mathbf{P}^r$, that is to say: there is *no* dense open subset of $\mathbf{P}^r$ parameterizing divisors satisfying Q.

Finally, note the relation between generic and general: a first remark is that it is not possible to deduce some generic information from a general one, in the sense that if some property Q is true for general points, we cannot deduce that Q is generic; in the reverse direction, however, one can go

$$\text{generic} \rightsquigarrow \text{general}$$

in this way: let Q be a given property and let $X \to Y$ be a morphism, where Y is an irreducible scheme with generic point ξ. One then checks whether the generic fiber X_ξ has Q; afterwards, if Q is constructible, then there is an open neighborhood U of ξ where the property holds, that is, X_y has Q for all $y \in U$. Let us point out that sometimes one confuses generic with general. Just to make an example: when one says 'a normal variety is generically regular', it means that there is a nonempty open subset where all the points are regular.

Now we are set to begin our discussion on Bertini's Theorem. We give two statements.

Theorem (Bertini 1) Let X be a smooth complex variety and let $\mathfrak{D}$ be a positive dimensional linear system on X. Then the general element of $\mathfrak{D}$ is smooth away from the base locus $B_{\mathfrak{D}}$. That is, the set

$$\{ H \in \mathfrak{D} \mid D_H \text{ is smooth away from } B_{\mathfrak{D}} \} \tag{1.8}$$

is a Zariski dense open subset of $\mathfrak{D}$.

Theorem (Bertini 2) Let $X \subset \mathbf{P}^n_k$ be a smooth projective variety over k. Then the set of hyperplanes $H \subset \mathbf{P}^n_k$ such that $X \cap H$ is a smooth scheme is a Zariski dense open subset of $\mathbf{P}^{n*}_k$.

1.4 Comments before the proofs

As a statement, the first one is more general (the associated line bundle might not be very ample, in particular not globally generated), but only holds in characteristic zero. The second one, instead, describes the complete base-point-free linear system associated to $\mathscr{O}(1)$. Let us see in detail why (Bertini 2) is a particular case of (Bertini 1), of course when $\operatorname{char} k = 0$ is assumed. The first step is showing that, for any linear system $\mathfrak{D} = \mathbf{P}V = \mathbf{P}^r \subset |D|$, of dimension r, one has a morphism

$$X - B_{\mathfrak{D}} \xrightarrow{\quad \psi \quad} \mathfrak{D}.$$

Fix a basis $\{ e_0, \ldots, e_r \} \subset V \subset H^0(X, \mathscr{O}_X(D))$ of V. For any $x \in X - B_{\mathfrak{D}}$ the map $\sigma_x : V \otimes_k \mathscr{O}_{X,x} \to \mathscr{O}_X(D)_x$ is surjective, that is, $\{ s_x \mid s \in V \}$ generates $\mathscr{O}_X(D)_x = e.\mathscr{O}_{X,x}$ over $\mathscr{O}_{X,x}$. One can write the germs $e_i(x) = e.u_i \in \mathscr{O}_X(D)_x$ for some (unique) $u_i \in \mathscr{O}_{X,x}$. At least one u_i is invertible in the ring $\mathscr{O}_{X,x}$, because if every u_i were in the maximal ideal $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ then, for every $s \in V$, one would have $s_x \in \mathfrak{m}_x \mathscr{O}_X(D)_x$, that is, σ_x would not be surjective, contradicting our assumption. So, define

$$\psi(x) = (e_0(x), \ldots, e_r(x)) = (u_0(x), \ldots, u_r(x)).$$

This is well defined because if we choose another local generator e' of $\mathscr{O}_X(D)_x$, say $e' = e.\nu$ for some (unique) $\nu \in \mathscr{O}_{X,x}$, then

$$(e'_0(x), \ldots, e'_r(x)) = (e_0(x)\nu(x), \ldots, e_r(x)\nu(x)) = (e_0(x), \ldots, e_r(x)).$$

This was the first step. Note that the definition of ψ *depends* on the choice of a basis for V. Now start with the closed immersion $i : X \hookrightarrow \mathbf{P}^n_k$ and recall that $\mathbf{P}^{n*}_k = \mathbf{P}H^0(\mathbf{P}^n_k, \mathscr{O}(1))$ is a complete linear system on $\mathbf{P}^n_k$ of dimension n. Write

$$X = \operatorname{Proj} k[x_0, x_1, \ldots, x_n]/\mathfrak{a} = \operatorname{Proj} k[y_0, y_1, \ldots, y_n] \qquad (1.9)$$

where $y_i = x_i \bmod \mathfrak{a}$. If $\mathscr{I}_X$ is the sheaf of ideals corresponding to $i : X \hookrightarrow \mathbf{P}^n_k$, there is an exact sequence

$$0 \longrightarrow \mathscr{I}_X \longrightarrow \mathscr{O}_{\mathbf{P}^n_k} \longrightarrow i_* \mathscr{O}_X \longrightarrow 0.$$

By a twist, that is, applying $- \otimes \mathscr{O}(1)$, one finds another exact[5] sequence

$$0 \longrightarrow \mathscr{I}_X(1) \longrightarrow \mathscr{O}(1) \longrightarrow \mathscr{O}_X(1) \longrightarrow 0$$

[5] Tensoring by an invertible sheaf is an exact functor, because the question is local and, locally, this sheaf is $\cong \mathscr{O}_X$.

where by $\mathscr{O}_X(1)$ we mean $(i_*\mathscr{O}_X)(1)$. Notice that, since i is an immersion, we have $(i_*\mathscr{O}_X)|_X \cong \mathscr{O}_X$, so $(i_*\mathscr{O}_X)(1)|_X \cong \mathscr{O}_X(1)$. By taking global sections on $\mathbf{P}^n_k$ and then restricting to global sections on X one gets

$$H^0(\mathbf{P}^n_k, \mathscr{O}(1)) \longrightarrow H^0(\mathbf{P}^n_k, (i_*\mathscr{O}_X)(1)) \longrightarrow H^0(X, \mathscr{O}_X(1)).$$

Let α denote this composition and let $V = \operatorname{im}\alpha \subset H^0(X, \mathscr{O}_X(1))$. One can show - this is the second step - that the linear system $\mathfrak{D} = \mathbf{P}V \subset |\mathscr{O}_X(1)|$ is base-point-free, i.e. for all $x \in X$ the set $\{ s_x \mid s \in V \}$ generates $\mathscr{O}_X(D)_x$ over $\mathscr{O}_{X,x}$. Once this is done, one can deduce by (Bertini 1) that the general member of $\mathfrak{D}$ is smooth (away from the empty set!), and the statement about (1.8) becomes exactly (Bertini 2).
Let us verify that $\mathbf{P}V$ is base-point-free. As $H^0(\mathbf{P}^n_k, \mathscr{O}(1))$ is generated by the $n+1$ global sections $x_0, \ldots, x_n$, the map α is defined by $x_i \mapsto y_i$ (but notice that, as global sections, the y_i's might be linearly dependent over k). If we let $U_i = D_+(y_i)$ then we see that $U_i = X \cap D_+(x_i)$ form an open (affine) covering of X. Moreover

$$\mathscr{O}(1)|_{D_+(x_i)} = x_i\mathscr{O}_{\mathbf{P}^n_k}|_{D_+(x_i)} \quad \text{implies that} \quad \mathscr{O}_X(1)|_{U_i} = y_i\mathscr{O}_X|_{U_i}.$$

Thus for every $x \in U_i$ one has $\mathscr{O}_X(1)_x = y_i\mathscr{O}_{X,x}$; in fact, to avoid the dependance on i one may write $\mathscr{O}_X(1)_x = \sum_i y_i\mathscr{O}_{X,x}$, but this now holds for every $x \in X$ since (U_i) is a covering. This says that the canonical map $V \otimes \mathscr{O}_{X,x} \to \mathscr{O}_X(1)_x$ sending $v \otimes u \mapsto u.v_x$ is surjective for every $x \in X$. So $\mathfrak{D}$ is base-point-free.

In addition, there is a morphism $\psi : X \to \mathfrak{D}$ and one verifies that by composing

$$X \xrightarrow{\ \psi\ } \mathfrak{D} \hookrightarrow \mathbf{P}^n_k$$

one recovers the closed immersion $i : X \hookrightarrow \mathbf{P}^n_k$ one started with. Again, it is enough to check this fact locally on the covering (U_i). In fact, to the sequence of ring homomorphisms

$$\mathscr{O}(U_i) \xleftarrow{\ \sim\ } k[y_0, \ldots, y_n]_{(y_i)} \twoheadleftarrow k[x_0, \ldots, x_n]_{(x_i)}$$

there corresponds the sequence of morphisms of affine schemes

$$U_i \xrightarrow{\ \sim\ } \operatorname{Spec} k[y_0, \ldots, y_n]_{(y_i)} \hookrightarrow \operatorname{Spec} k[x_0, \ldots, x_n]_{(x_i)}$$

which is exactly the local translation of what we want. In particular, in the above situation the morphism ψ is a closed immersion, and $\dim X \leqslant \dim \mathfrak{D}$.

The existence of the morphism $\psi : X - B_{\mathfrak{D}} \to \mathfrak{D}$ allows to formulate another definition of the base locus, that we put in the shape of a proposition, as follows.

Proposition 1.4.1. The base locus of a linear system $\mathfrak{D}$ on X is the subset of X at which the rational map $X \dashrightarrow \mathfrak{D}$ is not defined.

Proof. Assume $\mathfrak{D} = \mathbf{P}V \subset |D|$. The above map is not defined at $x \in X$ if and only if all the sections (in a basis) of V vanish on x. This happens if and only if they all lie in $\mathfrak{m}_x \mathscr{O}_X(D)_x$, which means that the sections in V do not generate $\mathscr{O}_X(D)$. Conclude by Lemma 1.1. □

Example 1.4.1. If X is a smooth curve, then any linear system on X is base-point-free, because - more generally - for a curve C, any rational map $C \dashrightarrow V$, for a subvariety $V \subset \mathbf{P}^n_k$, is defined at all the smooth points.

Remark 1.6. Denote again by ψ the morphism $X \to \mathbf{P}^r$ associated to some base-point-free $\mathfrak{D}$. Then, if one wants to see the effective divisors in $\mathfrak{D}$ as subvarieties of X, it is enough to consider the fibers of ψ. It means that

$$\mathfrak{D} = \{ \psi^{-1}(H) \mid H \in \mathbf{P}^{r*} \}.$$

This will become clear after Lemma 1.2. But if one accepts the fact that $\Sigma = \{ (x, H) \in X \times \mathbf{P}^{r*} \mid \psi(x) \in H \}$, together with the projection $\pi : \Sigma \to \mathbf{P}^{r*}$, is a family (parameterizing the divisors in $\mathfrak{D}$) with members the fibers $\pi^{-1}(H)$, then we can already understand this remark because $X \times_{\mathbf{P}^r} H = \psi^{-1}(H) \cong \pi^{-1}(H)$. Finally, what Bertini says is just that there is a dense open subset $U \subset \mathbf{P}^{r*}$ such that $\psi^{-1}(H) - B_{\mathfrak{D}}$ is smooth for every $H \in U$.

Example 1.4.2. Let k be an algebraically closed field. Consider the 3-uple Veronese embedding of $\mathbf{P}^1_k$, that is

$$\begin{array}{ccc} \mathbf{P}^1_k & \hookrightarrow & \mathbf{P}^3_k \\ (t, u) & \longmapsto & (t^3, t^2u, tu^2, u^3) \end{array}$$

Its image $C \subset \mathbf{P}^3_k$ is called the *twisted cubic curve* in $\mathbf{P}^3_k$. We claim that a smooth projective curve $X \subset \mathbf{P}^3_k$ of degree 3 such that X is not contained in a hyperplane and $X \cong \mathbf{P}^1_k$ as abstract algebraic varieties is of the form $X = \sigma C$ for some automorphism $\sigma \in \mathrm{PGL}(3, k)$ of $\mathbf{P}^3_k$. By viewing such an X as a closed subscheme of $\mathbf{P}^3_k$, we know that we can recover this closed immersion by

$$\begin{array}{ccc} X & \stackrel{i}{\hookrightarrow} & \mathbf{P}^3_k \\ & \searrow \quad \nearrow & \\ & \mathfrak{D} & \end{array}$$

where $\mathfrak{D} = \mathbf{P}V$ and $V = \operatorname{im} H^0(\mathbf{P}^3_k, \mathscr{O}(1)) \subset H^0(X, \mathscr{O}_X(1))$. By our assumption that X is not contained in any hyperplane, we have that α is injective, thus $V \cong H^0(\mathbf{P}^3_k, \mathscr{O}(1))$, a k-vector space of dimension 4. So $\dim \mathfrak{D} = 3$. In addition, $\mathfrak{D}$ has degree[6] $\deg \mathfrak{D} = 3$, and $X \cong \mathbf{P}^1_k$ says that V must correspond to a 4-dimensional vector space $W \subset H^0(\mathbf{P}^1_k, \mathscr{O}(3))$, which is already of dimension 4, so $V \cong H^0(\mathbf{P}^1_k, \mathscr{O}(3))$ and $\mathfrak{D}$ is complete. Since the closed immersion $i : X \hookrightarrow \mathbf{P}^3_k$ is determined by V and the choice of a basis for V (Theorem 1.6 below), we have that i is the 3-uple embedding up to the choice of that basis, i.e. up to a linear automorphism of $\mathbf{P}^3_k$.

Aside 1.2. We want to say more about the rational map $\psi : X \dashrightarrow \mathfrak{D}$. When $X = \mathbf{P}^n_k$, we have the base-point-free linear system $|\mathscr{O}(d)|$ on X, of dimension $m = \binom{n+d}{d} - 1$. So we have a morphism

$$\mathbf{P}^n_k \hookrightarrow \mathbf{P}^m_k$$

which is constructed in this way: take any basis of $H^0(\mathbf{P}^n_k, \mathscr{O}(d))$, for example the set of monomials of total degree d in the coordinates on $\mathbf{P}^n_k$. Then $x \in \mathbf{P}^n_k$ is sent to the point of $\mathbf{P}^m$ whose coordinates are evaluations of the monomials at x, for instance in lexicographic order. This is exactly the d-uple Veronese embedding. The Example 1.4.2 above works with $n = 1$ and $d = 3$; when $n = d = 2$, we have the following situation: the 2-uple embedding

$$\mathbf{P}^2_k \hookrightarrow \mathbf{P}^5_k$$

is exactly the morphism corresponding to the base-point-free linear system $|\mathscr{O}(2)|$ on $\mathbf{P}^2_k$, i.e. the complete linear system of conics. Its image is called the Veronese surface.

Before passing to the proofs of Bertini's Theorem, let us describe the relation between linear systems on a projective variety X and morphisms to projective space. We have the following result.

Theorem 1.6. Let X be a scheme over a ring A. Then

(i) If $\phi : X \to \mathbf{P}^n_A$ is an A-morphism then $\phi^*(\mathscr{O}(1))$ is an invertible sheaf on X generated by the $n+1$ global section $s_i = \phi^*(x_i)$.

(ii) If $\mathscr{L}$ is an invertible sheaf on X generated by $n+1$ global sections $s_0, \dots, s_n \in H^0(X, \mathscr{L})$ then there exists a unique A-morphism $\phi : X \to \mathbf{P}^n_A$ such that $\mathscr{L} \cong \phi^*(\mathscr{O}(1))$ and

[6] The degree of a linear system is the degree of any of its divisors: it is well defined because the degree of a divisor on a smooth projective curve depends only on its linear equivalence class.

$s_i = \phi^*(x_i)$ for each $i = 0, \ldots, n$ under this isomorphism of sheaves.

Proof. See [14], II, Theorem 7.1, p. 150. □

By the above, giving $X \to \mathbf{P}^n_k$ is the same as giving a set

$$S = \{ s_0, \ldots, s_n \in H^0(X, \mathscr{L}) \text{ gererating } \mathscr{L}, \text{ with } \mathscr{L} \text{ invertible} \}.$$

Now, by Proposition 1.3.1, giving a morphism $X \to \mathbf{P}^n_k$ is the same as giving a base-point-free linear system $\mathbf{P}V$ on X and a basis $\{ s_0, \ldots, s_n \}$ of V. If another basis is chosen, the new morphism will differ from the first one by an automorphism of $\mathbf{P}^n_k$.

If one requires, in addition, that a morphism $X \to \mathbf{P}^n_k$ be a closed immersion, then one has to add two conditions: the k-vector space generated by $s_0, \ldots, s_n$ must separate points and tangent vectors. However, if one has recourse to ample line bundles, a characterization of immersions inside the projective space comes almost for free. A line bundle $\mathscr{L}$ on a Y-scheme X is said to be *very ample* if there exists an immersion $i : X \hookrightarrow \mathbf{P}^n_Y$ such that $\mathscr{L} \cong i^*\mathscr{O}(1)$. When $Y = \operatorname{Spec} k$, a line bundle $\mathscr{L}$ is very ample on a k-scheme X if and only if there are finitely many global sections $s_0, \ldots, s_n \in H^0(X, \mathscr{L})$, generating $\mathscr{L}$, such that the induced morphism to $\mathbf{P}^n_k$ is an immersion. Summary: we have the following equivalences of data.

Maps	Linear Systems
$X \dashrightarrow \mathbf{P}^n_k$	$\mathscr{L}$; $s_0, \ldots, s_n \in H^0(X, \mathscr{L})$.
$X \to \mathbf{P}^n_k$	$\mathscr{L}$; $s_0, \ldots, s_n \in H^0(X, \mathscr{L})$ generate $\mathscr{L}$.
$X \hookrightarrow \mathbf{P}^n_k$	$\mathscr{L}$ very ample; $s_0, \ldots, s_n \in H^0(X, \mathscr{L})$ generate $\mathscr{L}$.

A divisor D on X is said to be very ample in case $\mathscr{O}_X(D)$ is very ample. Hence, very ample divisors are exactly those which allow embeddings in the projective space (provided that they are globally generated).

1.5 Bertini in any characteristic

In this section we present the proof of Bertini's Theorem (Bertini 2), over an arbitrary algebraically closed field k. We need to keep in mind the following two results.

Proposition 1.5.1. The second projection $\mathbf{P}^n_k \times \mathbf{P}^m_k \to \mathbf{P}^m_k$ is a closed map.

Proof. The projective space $\mathbf{P}^n_k$ is proper over a point (Spec k) and properness is closed under base extension. It follows that the second projection is proper, and hence closed. □

Proposition 1.5.2. Let X and Y be projective varieties of dimension n and m respectively. Let $f : X \to Y$ be a surjective morphism. Then $m \leqslant n$, and

(♡) For any $y \in Y$, every irreducible component of $f^{-1}(y)$ has dimension $\geqslant n - m$.

(◊) There is a nonempty open set $U \subset Y$ such that $\dim f^{-1}(y) = n - m$ for all $y \in U$.

Proof. See [Shafarevich, I, Theorem 7 on p. 76]. □

THEOREM 1.7 (Bertini 2). Let $X \subset \mathbf{P}^n_k$ be a smooth projective variety over k. Then the set of hyperplanes $H \subset \mathbf{P}^n_k$ such that $X \cap H$ is a smooth scheme is a Zariski dense open subset of $\mathbf{P}^{n*}_k$.

Proof. Let d be the dimension of X. We look for a dense open subset $U \subset \mathbf{P}^{n*}_k$ such that the corresponding hyperplanes H have the *good* property of letting $X \cap H$ be a smooth scheme. We may assume that $d < n - 1$. Indeed, if X is a smooth hypersurface, the required good locus is easily seen to be the open subset $\mathbf{P}^{n*}_k - X^*$ (because x is smooth in $X \cap H$ if and only if $H \neq T_{X,x}$). Moreover, we may assume that X is not contained in any hyperplane. We define the set of "bad hyperplanes" at a closed[7] point $p \in X$ to be

$$B_p = \{ H \in \mathbf{P}^{n*}_k \mid p \text{ is singular in } X \cap H \} \subset \mathbf{P}^{n*}_k.$$

Now, if we let p range the closed points of X, we can define a set

$$B = \{ (p, H) \mid p \in X \text{ is a closed point and } H \in B_p \} \subset X \times \mathbf{P}^{n*}_k.$$

Our strategy is the following:

(1) Showing that B has the structure of a projective variety, and

(2) showing that the first projection $\pi_1 : B \to X$ is surjective, and that in this case we have $\dim B \leqslant n - 1$.

[7] Nonsingularity can be checked on closed points.

Suppose these statements are both proved. Let us explain why we can conclude. As $\dim B < n$ we get that the second projection $\pi_2 : B \to \mathbf{P}_k^{n*}$ sends B onto a *proper* closed subset of the dual projective space. We can then define $U = \mathbf{P}_k^{n*} - \pi_2(B)$ and see that it is the required dense (in particular, nonempty) open subset in the statement.

To prove (1), we note that we have a family (see Definition 1.6.1) $\mathscr{B} \subset X \times \mathbf{P}_k^{n*}$ of subvarieties of $\mathbf{P}_k^{n*}$ with base X. Concretely,

$$\mathscr{B} = \bigcup_{p \in X} \{ p \} \times B_p.$$

The members of this family are the fibers $q^{-1}(p) = B_p$ where $q : \mathscr{B} \to X$ is the pullback projection. Hence $B = \mathscr{B} \cap \mathcal{H}$, where $\mathcal{H}$ is the universal hyperplane (see (1.15)).

To prove (2), it is enough to show that, for every closed point $p \in X$, the fiber $\pi_1^{-1}(p)$ is a projective space of dimension $n-d-1$. If this is done then π_1 is automatically surjective because $n-d-1$ is a strictly positive integer, as we assumed $d < n-1$. In this case, we can apply Proposition 1.5.2 (♡) to see that for every closed point $p \in X$ we have $n - d - 1 = \dim \pi_1^{-1}(p) \geqslant \dim B - d$, hence $\dim B \leqslant n - 1$.[8] This would conclude the second step, so our last claim is that $\pi_1^{-1}(p) \cong \mathbf{P}^{n-d-1}$ for every closed point $p \in X$.

To prove our claim, let us fix a closed point $p \in X$, and homogeneous coordinates $x_0, \dots, x_n$ on $\mathbf{P}_k^n$ and $a_0, \dots, a_n$ on $\mathbf{P}_k^{n*}$; name $W = H^0(\mathbf{P}_k^n, \mathscr{O}(1))$, the k-vector space of homogeneous linear polynomials. Now, there exists a hyperplane that does not pass through p; without loss of generality we may assume it is $H_0 = V_+(x_0)$. Define a morphism of k-vector spaces

$$\begin{array}{ccc} W & \xrightarrow{\phi_p} & \mathscr{O}_{X,p}/\mathfrak{m}_p^2 \\ \displaystyle\sum_{i=0}^{n} a_i x_i & \longmapsto & \displaystyle\sum_{i=0}^{n} a_i \frac{x_i}{x_0} \end{array}$$

This is well defined because $x_0(p) \neq 0$. Note that ϕ_p is surjective: indeed, since p is a closed point and k is algebraically closed, the maximal ideal $\mathfrak{m}_p \subset \mathscr{O}_{X,p}$ is generated by linear polynomials in the coordinates $x_1/x_0, \dots, x_n/x_0$. This says that the image of ϕ_p

[8] In fact, $\dim B = n - 1$. This is not important. However, it follows by Proposition 1.5.2 (◊): there is a nonempty open subset of X such that all its points y satisfy $n - d - 1 = \dim \pi_1^{-1}(y) = \dim B - \dim X = \dim B - d$, which implies $\dim B = n - 1$.

contains a basis of $\mathscr{O}_{X,p}/\mathfrak{m}_p^2$, so ϕ_p is surjective.
As X is smooth, $\dim_k \mathfrak{m}_p/\mathfrak{m}_p^2 = \dim X = d$. Note that

$$\frac{\mathscr{O}_{X,p}/\mathfrak{m}_p^2}{\mathfrak{m}_p/\mathfrak{m}_p^2} \cong k \text{ implies that } \dim_k \mathscr{O}_{X,p}/\mathfrak{m}_p^2 = d+1.$$

Combined with the fact that $\dim_k W = n+1$ and with surjectivity of ϕ_p, it implies

$$\begin{aligned} \dim_k \ker \phi_p &= \dim_k W - \dim_k \mathscr{O}_{X,p}/\mathfrak{m}_p^2 \\ &= (n+1) - (d+1) = n - d. \end{aligned} \tag{1.10}$$

Notice that if $f \in W$ and $H_f = V_+(f)$ then, whenever $f \in \ker \phi_p$, we have that f vanishes in a neighborhood of p in X, and hence on the whole X (by irreducibility); thus $X \subset H_f$. Conversely, if $X \subset H_f$ for some $f \in W$ then $f|_X = 0$ and $f \in \ker \phi_p$. Summarizing,

$$f \in \ker \phi_p \quad \Longleftrightarrow \quad X \subset H_f. \tag{1.11}$$

However, this will never happen, by our assumption on X. Next, observe that for $f \in W$ our closed point p is in $X \cap H_f$ if and only if $f(p) = 0$, if and only if f is not invertible in a neighborhood of p, if and only if $\phi_p(f) \in \mathfrak{m}_p/\mathfrak{m}_p^2$. Assume this is the situation and write then $\phi_p(f) = y_p + \mathfrak{m}_p^2$ for $y_p = y \in \mathfrak{m}_p$. We can define the local ring A at $p \in X \cap H_f$: if $R = \mathscr{O}_{X,p}/\mathfrak{m}_p^2$ and $\mathfrak{a} = y\mathscr{O}_{X,p}$,

$$A = \mathscr{O}_{X \cap H_f, p} = R/\phi_p(f)R \cong \mathscr{O}_{X,p}/\mathfrak{a}.$$

By definition A is a local ring. When is it regular? The answer will tell us when p is a regular point of $X \cap H_f$. The maximal ideal of A is $\mathfrak{m}_A = \mathfrak{m}_p/\mathfrak{a}$ and

$$\mathfrak{m}_A^2 = (\mathfrak{m}_p/\mathfrak{a})^2 = (\mathfrak{m}_p^2 + \mathfrak{a})/\mathfrak{a} \qquad (\mathfrak{m}_p^2 + \mathfrak{a})/\mathfrak{m}_p^2 \cong \mathfrak{a}/(\mathfrak{m}_p^2 \cap \mathfrak{a}).$$

For the regularity of A, we are interested in comparing the dimension $\dim A$ to $\dim_k \mathfrak{m}_A/\mathfrak{m}_A^2$. First, we obtain

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \cong \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2 + \mathfrak{a}} \cong \frac{\mathfrak{m}_p/\mathfrak{m}_p^2}{(\mathfrak{m}_p^2 + \mathfrak{a})/\mathfrak{m}_p^2} \cong \frac{\mathfrak{m}_p/\mathfrak{m}_p^2}{\mathfrak{a}/(\mathfrak{m}_p^2 \cap \mathfrak{a})}$$

and by computing dimensions we get

$$\dim_k \mathfrak{m}_A/\mathfrak{m}_A^2 = d - \dim_k \mathfrak{a}/(\mathfrak{m}_p^2 \cap \mathfrak{a}).$$

By regularity of X at p and by the fact that $A \cong \mathscr{O}_{X,p}/\mathfrak{a}$ (where $\mathfrak{a}$ is principal) we have $\dim A = d - 1$. So A is regular if and only if $\dim A = \dim_k \mathfrak{m}_A/\mathfrak{m}_A^2$, if and only if $\dim_k \mathfrak{a}/(\mathfrak{m}_p^2 \cap \mathfrak{a}) = 1$. This happens exactly when $\mathfrak{y}$ is not contained in $\mathfrak{m}_p^2$:

$$p \text{ is regular in } X \cap H_f \iff \mathfrak{y}_p \in \mathfrak{m}_p - \mathfrak{m}_p^2 \tag{1.12}$$

Now we can use (1.11) and (1.12) to characterize the bad hyperplanes at p.

$$\begin{aligned} B_p &= \{ H_f \mid p \text{ is singular in } X \cap H_f, \text{ with } f \in W \} \\ &= \{ H_f \mid \mathfrak{y}_p \in \mathfrak{m}_p^2, \text{ with } f \in W \}. \end{aligned}$$

But again, $\mathfrak{y}_p \in \mathfrak{m}_p^2$ if and only if $\phi_p(f) = 0$, i.e. $f \in \ker \phi_p$. In addition, it is clear that $H_f = H_{\lambda f}$ for every $\lambda \in k^\times$, so we can identify $B_p = \mathbf{P}(\ker \phi_p)$, proving that $\dim B_p = n - d - 1$ thanks to (1.10). It remains to observe that $\pi_1^{-1}(p) \cong B_p$ to conclude that $\pi_1^{-1}(p)$ is a projective space of dimension $n - d - 1$, as claimed. This completes the proof. □

1.6 Bertini in characteristic zero

In this section we present a proof of Bertini's Theorem for a pencil over $\mathbb{C}$, and a proof over a field of characteristic zero which is valid for any linear system. The first one is quite easy; the second one relies on some fundamental theorems, like the generic smoothness theorem.

THEOREM 1.8 (Bertini 1). Let X be a smooth complex variety and let ℓ be a pencil on X. Then the general element of ℓ is smooth away from the base locus B. That is, the set

$$\{ \lambda \in \ell \mid D_\lambda \text{ is smooth away from } B \} \tag{1.13}$$

is a Zariski dense open subset of ℓ.

Proof. To any $\lambda \in \ell \cong \mathbf{P}^1$ there corresponds an effective divisor D_λ on X; let us assume that $\dim X = d$ and let us work with our pencil locally inside a polydisc $\Delta \subset X$ (the analytic analog of an affine open subset in the Zariski topology). If f, g are linearly independent sections generating ℓ inside Δ, then every divisor D_λ has the following local representation, in the analytic topology:

$$D_\lambda : \; f(z_1, \ldots, z_d) + \lambda g(z_1, \ldots, z_d) = 0 \tag{1.14}$$

where z_i are the local coordinates on Δ. Here we are using that a smooth complex algebraic variety has the structure of a complex manifold, so we can work locally in the analytic topology. Let us see what happens if for some $\lambda \in \mathbf{P}^1$ (different from $0, \infty$), p_λ is a singular point for D_λ *outside* the base locus of the system. For all $i = 1, \ldots, d$ we then have

$$f(p_\lambda) + \lambda g(p_\lambda) = 0$$
$$\frac{\partial f}{\partial z_i}(p_\lambda) + \lambda \frac{\partial g}{\partial z_i}(p_\lambda) = 0.$$

Note that if $f(p_\lambda) = 0 = g(p_\lambda)$ then $p_\lambda \in \bigcap_{\nu \in \mathbf{P}^1} D_\nu = B$; but $p_\lambda \notin B$ so we get that f and g do not both vanish at p_λ, hence neither one does, whence for all $i = 1, \ldots, d$ we must have

$$\lambda = -\frac{f(p_\lambda)}{g(p_\lambda)} \quad \text{and} \quad \frac{\partial f}{\partial z_i}(p_\lambda) - \frac{f(p_\lambda)}{g(p_\lambda)} \frac{\partial g}{\partial z_i}(p_\lambda) = 0.$$

Now, let us calculate the derivatives of the ratio f/g, and evaluate them at p_λ:

$$\frac{\partial}{\partial z_i}\left(\frac{f}{g}\right)(p_\lambda) = \frac{\frac{\partial f}{\partial z_i}(p_\lambda) g(p_\lambda) - f(p_\lambda) \frac{\partial g}{\partial z_i}(p_\lambda)}{g(p_\lambda)^2} = 0.$$

Next, call V the locus of singular points (in Δ) of the divisors in ℓ, and call S the subvariety of $\Delta \times \mathbf{P}^1$ cut out by the equations

$$f + \lambda g = 0$$
$$\frac{\partial f}{\partial z_i} + \lambda \frac{\partial g}{\partial z_i} = 0$$

where λ ranges $\mathbf{P}^1$ and $i = 1, \ldots, d$. Then V is the image of S inside Δ (under the pullback projection), that is V is locally cut out by the above equations. But now, by a calculation we made, the ratio f/g is locally constant on $V - B$. This means that here it takes only finitely many values. Translation: there are only finitely many $\lambda \in \mathbf{P}^1$ such that D_λ is singular away from $B_{\mathfrak{D}}$ (in fact, $B \cap \Delta$). Thus D_λ is smooth away from the base locus for almost all $\lambda \in \mathbf{P}^1$, proving that there is a dense open subset $U \subset \mathbf{P}^1$ such that the corresponding divisors are smooth away from $B \cap \Delta$. Of course, we worked locally on Δ, but there are finitely many such polydiscs covering our variety X, so the result follows. □

Remark 1.7. Note how by tacit agreement we used the map $\psi : X - B \to \mathbf{P}^1$. If q is a point outside the base locus, we define $\psi(q)$ to be the *unique* λ such that q lies on D_λ. This is unique just by the *linearity* condition of (1.14). In fact the key point in our proof is that $\psi|_{V-B}$ has finite image.

REMARK 1.8. Perhaps, it is possible to generalize this proof to a linear system of arbitrary dimension. Bertini himself asserts (see [5]) that it is so. But the details are not so easy to handle, and we do not dispose of a precise proof for the moment.

The next subsection is a digression, devoted to formalize what we already said in the Introduction about varieties "parameterizing" collections of varieties: a motivation for this digression is that a central object in the study of Bertini's Theorem is the *set* of hyperplane sections Σ_X of a variety $X \subset \mathbf{P}^n_k$; such a set is in fact a variety and, even more, it constitutes an example of a universal family over $\mathbf{P}^{n*}_k$. We already encountered this family during the proof of Bertini in any characteristic, and we *already* used all of its geometric structure. More precisely, we still have to justify the fact that the bad hyperplanes form a subvariety $B \subset X \times \mathbf{P}^{n*}_k$. We just asserted that B is the intersection (inside $\mathbf{P}^n_k \times \mathbf{P}^{n*}_k$) between a family $\mathscr{B} \to X$ and the universal hyperplane $\mathcal{H} \subset \mathbf{P}^n_k \times \mathbf{P}^{n*}_k$. Hopefully, this will become clearer soon.

1.6.1 Universal Families

In the following two subsections k is a field of any characteristic, at least until the new proof of Bertini's Theorem in characteristic zero. It might not be algebraically closed. Some elementary examples of universal families are provided; the most important one, that of parameter space, is omitted, but the reader can find a nice discussion on it in [12].

We now think of $\mathbf{P}^n_k$ as our big ambient variety. Let B be any variety, and $(V_b)_{b \in B}$ a collection of algebraic varieties $V_b \subset \mathbf{P}^n_k$. First, we want to answer this question: what does it mean that such a collection *varies algebraically and continuously* with parameters? Let us give an intuition on that: it is a possible point of view to regard a morphism of schemes $\pi : X \to Y$ as a collection of schemes (X_y), where X_y is the fiber $X \times_Y \operatorname{Spec} k(y)$; such a collection of fibers is thought of as a family of schemes varying *algebraically* with parameters (the points $y \in Y$) because each fiber is a *pullback* of two arrows depending only on y. Thus the answer to our question is (partially) inside the following definition.

Definition 1.6.1. Let B be a variety. A FAMILY of projective varieties in $\mathbf{P}^n_k$ with base B is a closed subvariety $\mathscr{V} \subset B \times \mathbf{P}^n_k$ together with the induced projection $\pi : \mathscr{V} \to B$. The fibers $V_b = \pi^{-1}(b)$ are called the members of the family.

But what about continuity? The fibers of π might be very different one from each other (some might be empty, there can be dimension jumps...). The collection of fibers $\mathscr{V} = (X_y)_{y \in Y}$ is a family of subschemes of X with base Y. It turns out that the right notion to express the continuity of $\mathscr{V}$ is *flatness*. We see its importance in the definition of a UNIVERSAL FAMILY: a family $\mathscr{V} \to B$ is universal if for every flat family $\mathscr{V}' \to B'$ whose members belong to $\mathscr{V}$ there exists a unique regular map $B' \to B$ such that $\mathscr{V}' = B' \times_B \mathscr{V}$. So in particular any universal family is flat.

1. THE UNIVERSAL HYPERPLANE. Consider the projective variety $\mathbf{P}_k^{n*}$ of hyperplanes in $\mathbf{P}_k^n$, and the subset

$$\mathcal{H} = \{ (H, p) \mid p \in H \text{ and } H \in \mathbf{P}_k^{n*} \} \subset \mathbf{P}_k^{n*} \times \mathbf{P}_k^n. \qquad (1.15)$$

If z denotes the homogeneous coordinates on $\mathbf{P}_k^n$ and w are those on $\mathbf{P}_k^{n*}$, then it is clear that $\mathcal{H}$ is given by the single polynomial

$$f(z, w) = \sum_{i=0}^{n} z_i w_i.$$

Thus it defines a hypersurface in $\mathbf{P}_k^{n*} \times \mathbf{P}_k^n$. If one puts $B = \mathbf{P}_k^{n*}$ and considers the first projection $\pi_1 : \mathcal{H} \to B$, then each hyperplane H has fiber $\pi^{-1}(H) \cong H \subset \mathbf{P}_k^n$, so $\mathcal{H}$ may be viewed as a family of hyperplanes in $\mathbf{P}_k^n$ with base $\mathbf{P}_k^{n*}$. Symmetrically, if one takes $\mathbf{P}_k^n$ as base, and considers the projection $\pi_2 : \mathcal{H} \to \mathbf{P}_k^n$, then the fiber at a point $p \in \mathbf{P}_k^n$ is $\pi_2^{-1}(p) = \{ (H, p) \mid H \ni p \}$, which is a hyperplane in $\mathbf{P}_k^{n*}$; so in this case $\mathcal{H}$ may be viewed as a family of hyperplanes in $\mathbf{P}_k^{n*}$ with base $\mathbf{P}_k^n$. It is called the *universal* hyperplane because it has the following property:

> For every flat family $\mathscr{V}' \subset B' \times \mathbf{P}_k^n$ of hyperplanes there exists a unique regular map $B' \to \mathbf{P}_k^{n*}$ such that $\mathscr{V}' = B' \times_{\mathbf{P}_k^{n*}} \mathcal{H}$. That is, there is a pullback diagram
>
> $$\begin{array}{ccc} \mathscr{V}' & \longrightarrow & \mathcal{H} \\ \downarrow & & \downarrow \\ B' & \longrightarrow & \mathbf{P}_k^{n*} \end{array}$$
>
> where the bottom arrow is the unique morphism sending $b \in B'$ to the member V'_b of $\mathscr{V}'$.

2. THE UNIVERSAL HYPERPLANE SECTION. Let $X \subset \mathbf{P}_k^n$ be a projective variety. View $\mathcal{H}$ as a family of hyperplanes with base $\mathbf{P}_k^n$, i.e. together with the second projection $\pi_2 : \mathcal{H} \to \mathbf{P}_k^n$. Define

$$\Sigma_X = \{ (H, p) \mid p \in X \cap H \text{ and } H \in \mathbf{P}_k^{n*} \} \subset \mathbf{P}_k^{n*} \times X.$$

Then $\Sigma_X = \pi_2^{-1}(X)$ is a subvariety of $\mathbf{P}_k^{n*} \times X$ and hence it is a family (of subvarieties of X) over $\mathbf{P}_k^{n*}$. It is called the *universal hyperplane section*. Note that

$$\Sigma_X = \mathcal{H} \cap (\mathbf{P}_k^{n*} \times X),$$

the intersection being inside $\mathbf{P}_k^{n*} \times \mathbf{P}_k^n$. And, most important, the fiber at each point $H \in \mathbf{P}_k^{n*}$ is isomorphic to $X \cap H$.

With the same spirit, one can define the universal conic $\mathfrak{C} \subset \mathbf{P}_k^5 \times \mathbf{P}_k^2$ and the universal line.

1.6.2 Constructible Properties

The next aim is to briefly state some basic facts about constructible subsets of Zariski spaces. A Zariski space is a noetherian topological space such that every closed nonempty irreducible subset has a unique generic point. Afterwards, we introduce constructible properties of morphisms of schemes; the reference for this subject is [EGA, tome IV_3]. So in this section all schemes considered are noetherian. To simplify, one may also assume they are algebraic varieties.

Definition 1.6.2. Let X be a Zariski space. A subset $E \subset X$ is said to be CONSTRUCTIBLE if it is a finite disjoint union of locally closed subsets of X. It is called locally constructible if X has a covering $X = \bigcup_i X_i$ such that $E \cap X_i$ is constructible in X_i for every i. If X is any scheme, this is the same as asserting that $E \cap V$ is constructible for every open affine subset $V \subset X$.

REMARK 1.9. If X is a quasi-compact and quasi-separated scheme (e.g. an algebraic variety), then locally constructible implies constructible.

Proposition 1.6.1. The collection of constructible subsets of X coincides with the collection $\mathscr{C}$ characterized by the following properties:

(1) any open subset of X is in $\mathscr{C}$;

(2) the class $\mathscr{C}$ is closed under complements (thus it contains the closed subsets);

(3) the class $\mathscr{C}$ is closed under finite unions (and thus under finite intersections).

Proof. The collection $\mathscr{C}$ contains every constructible set, by how it is defined. Conversely, open and closed subsets are constructible, and so are also finite unions of constructible sets, thus constructible subsets satisfy (1) and (3). So it remains to show that in $\mathscr{C}$ we find all complements E^c where E is constructible. In fact, as $(\coprod_{1\leqslant i\leqslant r}(U_i \cap F_i))^c = \bigcap_{1\leqslant i\leqslant r}(U_i \cap F_i)^c$ and a finite intersection of locally closed is locally closed, it is enough to prove that E^c is constructible for E locally closed. Let us write $E = U \cap F$, where U is open and F is closed. Then $E^c = F^c \cup U^c = F^c \coprod (U^c - F^c)$, which is a union of an open set with a closed subset, hence both constructible. □

Definition 1.6.3. Let X be a topological space. A subset $E \subset X$ is said to be NOWHERE DENSE (in french: *rare*) if $X - \overline{E}$ is dense in X.

Note that if E is not closed then its complement $X - E$ contains (properly) the open subset $X - \overline{E}$.

Proposition 1.6.2. Let X be a Zariski space. A subset $E \subset X$ is constructible if and only if for every closed irreducible subset $Y \subset X$, the intersection $E \cap Y$ either contains a nonempty open subset of Y, or is nowhere dense in Y.

Proof. See [13], Proposition 1.3, p. 2. □

REMARK 1.10. Assume we have a subset E of an *irreducible* Zariski space X with generic point η. In this situation, every nonempty open subset of X is dense. And E is dense if and only if $\eta \in E$, if and only if E contains an open subset. Assume now that E is constructible: as either E contains an open subset or (*aut*, in fact) $X - \overline{E}$ is dense (= nonempty), one concludes that:

> If E is constructible in $X = \{\eta\}^-$, then either E contains a nonempty open subset, or $X - E$ does. Equivalently, either E or $X - E$ is a neighborhood of η.

⋆ ⋆ ⋆

Let us switch to schemes.

Here is the aim of the following discussion: suppose we are given a morphism of schemes $f : X \to S$ of finite presentation, and a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$. First, some notation: if $s \in S$ is any point, we denote by X_s the fiber $X \times_S \operatorname{Spec} k(s)$ and we set $\mathscr{F}_s = \mathscr{F} \otimes_{\mathscr{O}_S} k(s)$.[9] It is an $\mathscr{O}_{X_s}$-module. Suppose Q is a property

[9] This is nothing but another notation for $j^*\mathscr{F} = j^{-1}\mathscr{F} \otimes_{j^{-1}\mathscr{O}_X} \mathscr{O}_{X_s}$, if $j : X_s \to X$ is the canonical projection. For instance, if $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$ and $\mathscr{F} = M^\sim$, then $X_s = \operatorname{Spec}(B \otimes_A k(s))$ and $\mathscr{F}_s = j^*(M^\sim) = (M \otimes_A k(s))^\sim$.

of schemes. We then look for conditions under which the locus

$$E = \{ s \in S \mid X_s \text{ has } Q \}$$

is locally constructible (and hence constructible, when S is an algebraic variety). If one assumes that f is flat, then very often E turns out to be *open* inside S. The formal discussion starts from the following definition.

Definition 1.6.4. Let $Q(X, \mathscr{F}, k)$ be a relation. It is said to be CONSTRUCTIBLE if the following two conditions hold:

(C1) Assume X is a k-scheme, $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module and k'/k a field extension. Then we have

$$Q(X, \mathscr{F}, k) \text{ is true } \iff Q(X_{k'}, \mathscr{F} \otimes_k k', k') \text{ is true.}$$

(C2) Let S be an integral noetherian scheme, with generic point ξ. Let $u : X \to S$ be a morphism of finite type and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Set, as before, $X_s = u^{-1}(s) = X \times_S \operatorname{Spec} k(s)$ and $\mathscr{F}_s = \mathscr{F} \otimes_{\mathscr{O}_S} k(s)$ for every point $s \in S$. Put

$$E = \{ s \in S \mid Q(X_s, \mathscr{F}_s, k(s)) \text{ is true} \}.$$

Then one between E and $S - E$ contains a nonempty open subset (and hence is a neighborhood of ξ).

Proposition 1.6.3. Let Q be constructible, S any scheme and X of finite presentation over S. Let $\mathscr{F}$ be a quasi-coherent finitely presented $\mathscr{O}_X$-module. Then

$$E = \{ s \in S \mid Q(X_s, \mathscr{F}_s, k(s)) \text{ is true} \} \tag{1.16}$$

is locally constructible. If S is irreducible with generic point ξ, then (as we know by Remark 1.10) one between E and $S - E$ is a neighborhood of ξ.

Proof. See [10], Proposition 9.2.3, p. 58. □

REMARK 1.11. We have a kind of converse of Proposition 1.6.3: if Q is a property satisfying the conclusion of the Proposition (that is, the set in (1.16) is locally constructible), then Q satisfies the condition (C2) in Definition 1.6.4. In fact, take S integral and noetherian as in the definition, and use the fact that in an irreducible noetherian space, a constructible set is either nowhere dense or it contains a nonempty open subset, according to Remark 1.10. Thus if it does not contain an open subset, its complement does.

We have the following results.

Proposition 1.6.4. Let $X \to S$ be finitely presented and Q one of the following properties:

- geometrically regular;
- geometrically normal;
- geometrically reduced.

Then the set $\{ s \in S \mid X_s \text{ has } Q \}$ is locally constructible in S.

Proof. See [10], Corollary 9.9.5, p. 94. □

As we announced, it turns out that the flatness property often lets constructible sets become open. In fact, if $f : X \to S$ is a morphism of finite presentation, then the set of points $x \in X$ at which f is flat is *open* in X. This observation, together with a strong result (See [10], Theorem 12.1.6, p. 178), allow to prove the following.

Theorem 1.9. Let $f : X \to S$ be locally of finite presentation. Then the set of points $x \in X$ where f has one of the following properties: being

1. smooth;
2. normal;
3. reduced;
4. Cohen-Macaulay.

is open in X.

Proof. See [10], Corollary 12.1.7, p. 179. □

⋆ ⋆ ⋆

It is time to approach, as promised, the new proof of Bertini's Theorem in characteristic zero. Let X be an integral and normal[10] algebraic variety over a field k of characteristic zero. At the very beginning of this section, one took a linear subspace $V \subset H^0(X, \mathscr{L})$, where $\mathscr{L}$ is a line bundle on X, and proved that k-linear equivalence in V (that is, the operation of identifying v to λv for all $\lambda \in k^\times$) corresponds to linear equivalence of effective

[10] Normality is required to get the bijection $\bar{v} \mapsto D(v)$, see below.

Cartier divisors on X. In particular, one constructed the divisor of zeros $(s) = (s)_0$ associated to any global section $s \in V$ and by the above consideration (in fact, Proposition 1.3.1) one identified $(V - \{0\})/k^\times = \mathbf{P}V^*$ to $|D|$, where $D = (v_0)$ is the divisor of some nonzero $v_0 \in V$. This allowed to assert that points in $\mathbf{P}V^*$ parameterize divisors in $|D|$. Now, as a notation, one writes $D(v)$ instead of the old $(v) + (v_0)$. As always, $D(v)$ can be viewed as a hypersurface in X determined by a linear combination of the shape $\lambda_0 v_0 + \cdots + \lambda_r v_r$, if the v_i's constitute a k-basis of V. Now, writing $\bar{v}$ for the class of v modulo $k^\times$, then the bijective correspondence becomes

$$\begin{aligned} \mathbf{P}V^* &\longrightarrow |D(v_0)| \\ \bar{v} &\longmapsto D(v). \end{aligned}$$

Each divisor $E = D(v)$ in the system has a structure of closed subscheme of X, given by the sheaf of ideals $\mathcal{O}_X(-E) \subset \mathcal{O}_X$ (See Remark 1.4). More precisely, recall that $\mathcal{O}_X(-E)(U) = f_U.\mathcal{O}_X(U)$ if E is defined by local data $\{U, f_U\}$.

One considered $\mathbf{P}V^*$ as a projective space $\mathbf{P}^{r*}$, and this was the variety of parameters. We now make something a little more subtle. However, the underlying idea is always the same: putting all the divisors $D(v)$ together in a new algebraic variety Z, which will look like a projective bundle at all points outside the base locus of $\mathbf{P}V^*$. In what follows, we denote by $\mathbf{P}(\mathscr{E})$ the X-scheme called the projectivization of the coherent $\mathcal{O}_X$-module $\mathscr{E}$. It is defined as **Proj** Sym $\mathscr{E}$, the global Proj of the (graded) symmetric algebra of $\mathscr{E}$. Let us define

$$Z = \{ (x, \bar{v}) \in X \times_k \mathbf{P}V^* \mid x \in D(v) \} \subset X \times_k \mathbf{P}V^* = \mathbf{P}(V^* \otimes_k \mathcal{O}_X)$$

where the last equality holds because **Proj** behaves well under base change (Proposition A.0.5). Of course, we called V^* the constant sheaf on Spec k with values in V^* (here we view V and V^* as vector bundles over Spec k).

If one shows that Z is a subvariety of $X \times_k \mathbf{P}V^*$, then one obtains that $q : Z \to \mathbf{P}V^*$ is a *family* of subvarieties of X with base $\mathbf{P}V^*$, where q is the restriction of the canonical projection $X \times_k \mathbf{P}V^* \to \mathbf{P}V^*$, and where the members of the family are exactly the closed subschemes $D(v)$. In fact, proving $q^{-1}(\bar{v}) = D(v)$ will require some computation; it is Lemma 1.2.

Proposition 1.6.5. The subset Z is a subvariety of $\mathbf{P}(V^* \otimes_k \mathcal{O}_X)$.

Proof. Let $\phi : V \otimes_k \mathcal{O}_X \to \mathscr{L}$ be the morphism of (coherent) sheaves that on every open subset $U \subset X$ sends $v \otimes f$ to $f.v|_U$, for every $v \in V$ and $f \in \mathscr{O}_X(U)$: just to fix notations, this means that $\phi(U)(v \otimes f) = f.v|_U$. We can consider the dual

$$\phi^* : \mathscr{L}^* \longrightarrow V^* \otimes_k \mathscr{O}_X$$

and observe that $Z = \mathbf{P}(\operatorname{coker}\phi^*)$. If this is proved and we name $\mathscr{E} = \operatorname{coker}\phi^*$, then by exactness of

$$\mathscr{L}^* \xrightarrow{\phi^*} V^* \otimes_k \mathscr{O}_X \longrightarrow \mathscr{E} \longrightarrow 0$$

we get the desired inclusion $\mathbf{P}(\mathscr{E}) = Z \hookrightarrow \mathbf{P}(V^* \otimes_k \mathscr{O}_X)$.

We can prove $Z = \mathbf{P}(\mathscr{E})$ affine-locally, so let us fix an open affine subset $U \subset X$. We also fix a basis $v_0, \ldots, v_r$ of V and write $v_i|_U = e.f_i$, for unique $f_i \in \mathscr{O}_X(U)$ (that is, we are translating $\mathscr{L}|_U \cong e.\mathscr{O}_X|_U$). Our claim is then $Z|_U = \mathbf{P}(\mathscr{E})|_U$, where

$$Z|_U = \{ (x, \overline{v}) \in X \times_k \mathbf{P}V^* \mid x \in U \cap D(v) \} \subset U \times_k \mathbf{P}V^*$$

and $\mathbf{P}(\mathscr{E})|_U = \mathbf{P}(\mathscr{E}|_U) = \operatorname{Proj}\operatorname{Sym}\mathscr{E}(U)$.

As $\mathscr{L}(U) = e.\mathscr{O}_X(U)$, we have $\mathscr{L}^*(U) = \mathscr{L}(U)^* = e^*.\mathscr{O}_X(U)$, and regarding the exact sequence

$$0 \longrightarrow \mathscr{L}^*(U) \longrightarrow V^* \otimes_k \mathscr{O}_X(U) \longrightarrow \mathscr{E}(U) \longrightarrow 0 \qquad (1.17)$$

it is important to know where the generator e^* goes under (the second arrow) $\phi^*(U)$. In general, if $\theta \in \mathscr{L}^*(U)$, then θ is an $\mathscr{O}_U$-morphism $\mathscr{L}|_U \to \mathscr{O}_X|_U$ and $\phi^*(U)(\theta)$ is the linear map in $V^* \otimes_k \mathscr{O}_X(U) = (V \otimes_k \mathscr{O}_X(U))^*$ sending $v \otimes 1$ to $\theta(U)(\phi(U)(v \otimes 1)) = \theta(U)(v|_U)$. In particular, e^* goes to the linear map $\phi^*(U)(e^*)$ sending $v \otimes 1$ to $e^{-1}.v|_U = e^{-1}.\sum_i \lambda_i v_i|_U = \sum_i \lambda_i f_i$ (if $v = \sum_i \lambda_i v_i$). In fact, simply by using the definition of the dual basis, one sees that

$$\phi^*(U)(e^*) = \sum_i v_i^* \otimes f_i.$$

Thus, looking again at exactness of (1.17), and considering the ideal $(\phi^*(U)(e^*))$ generated by e^* inside $\operatorname{Sym}(V^* \otimes_k \mathscr{O}_X(U)) \supset V^* \otimes_k \mathscr{O}_X(U)$, one has

$$\operatorname{Sym}\mathscr{E}(U) = \frac{\operatorname{Sym}(V^* \otimes_k \mathscr{O}_X(U))}{(\phi^*(U)(e^*))} = \frac{(\operatorname{Sym} V^*) \otimes_k \mathscr{O}_X(U)}{(\sum_i v_i^* \otimes f_i)}.$$

Now, $\mathbf{P}(\mathscr{E})|_U = \operatorname{Proj}\operatorname{Sym}\mathscr{E}(U) = \operatorname{Proj}\mathscr{O}_X(U)[t_0, \ldots, t_r]/(\sum_i f_i t_i)$ where we have made correspond v_i^* to the indeterminate t_i. To conclude, let us choose a k-rational point $x \in U(k)$, and let us compare $Z_x = \{ (x, \overline{v}) \mid D(v) \cap U \ni x \}$ and

$$\mathbf{P}(\mathscr{E}|_U)_x = \mathbf{P}(\mathscr{E}|_U) \times_U \operatorname{Spec} k(x) = \operatorname{Proj}\frac{k(x)[t_0, \ldots, t_r]}{(\sum_i f_i(x) t_i)}.$$

Finally, choosing $v = \sum_i \lambda_i v_i$ with $x \in D(v)$ is equivalent to selecting a point in $\mathbf{P}(\mathscr{E}|_U)_x$, via the identification $\lambda_i = f_i(x)$. □

So $q : Z \to \mathbf{P}V^*$ is a family of subvarieties of X. The following lemma states it is universal, in the sense that what lies above a point $\overline{v} \in \mathbf{P}V^*$ is exactly the subscheme $D(v)$.

Lemma 1.2. The members of the family $q : Z \to \mathbf{P}V^*$ are the subschemes $D(v)$. That is, for every nonzero section $v \in V$ we have $Z_{\overline{v}} = D(v)$ as closed subschemes of X.

Proof. The claim can be verified locally on any open affine subset $U = \operatorname{Spec} A \subset X$. Suppose that $v_i|_U = e.f_i$ with $f_i \in A$, for $0 \leqslant i \leqslant r$. Then $v = \sum_i \lambda_i v_i = e \sum_i \lambda_i f_i$, where $\lambda_i \in k$ are coefficients.

$$\begin{aligned} Z_{\overline{v}} \cap (U \times_k \mathbf{P}V^*) &= Z_{\overline{v}} \cap (U \times_k \operatorname{Spec} k(\overline{v})) \\ &= \operatorname{Proj} A[t_0, \dots, t_r]/(\lambda_i t_j - \lambda_j t_i, \textstyle\sum f_i t_i) \\ &= \operatorname{Spec} A/f_U = \operatorname{Spec} \mathscr{O}_{D(v)}(U). \end{aligned}$$

□

Remark 1.12. Perhaps now it is a good moment to revisit Remark 1.6. At that time, it could seem obscure why divisors can be interpreted as preimages of hyperplanes in $\mathbf{P}^r$ under $\psi : X \dashrightarrow \mathbf{P}V^*$, but now, after Lemma 1.2, it should be clear that points of $\mathbf{P}^{r*}$ are exactly the $\overline{v} \in \mathbf{P}V^*$ we are to consider.

Lemma 1.3. If the linear system $\mathbf{P}V^*$ is base-point-free, then the canonical morphism $p : Z \to X$ is a projective bundle. In particular, it is smooth.[11]

Proof. The absence of base points says that ϕ is surjective, so $\operatorname{coker} \phi^* \cong (\ker \phi)^*$. Hence $Z = \mathbf{P}(\operatorname{coker} \phi^*) = \mathbf{P}((\ker \phi)^*)$, where $(\ker \phi)^*$ must be locally free of rank r (by looking at the exact sequence $0 \to \ker \phi \to V \otimes_k \mathscr{O}_X \to \mathscr{L} \to 0$). So Z is a projective bundle over X, via the morphism p. □

Theorem 1.10. Let k be a field of characteristic zero and X an integral algebraic variety over k. Let $\mathbf{P}V^*$ be a linear system on X with base locus B. Suppose that X satisfies one of the following properties Q: being

1. smooth ($\rightsquigarrow$ Bertini);

2. normal;

[11] This is proved in [EGA IV_4, p. 62].

3. reduced.

Then there exists a dense open subset $\Omega \subset \mathbf{P}V^*$ such that for every rational point $\overline{v} \in \Omega(k)$ the corresponding divisor $D(v)$ (i.e. the fiber of $q : Z \to \mathbf{P}V^*$ at $\overline{v}$) satisfies Q away from B.

Proof. The goal is to find Ω such that the fibers of $q : q^{-1}(\Omega) \to \Omega$ satisfy Q. First, by replacing X with the open subset $X - B$, one may assume the linear system is base-point-free. Indeed, working with $X - B$ we deal with the fibers $D(v)|_{X-B}$ instead of $D(v)$, and because $X - B$ is open in X one has that $\mathbf{P}(V^*|_{X-B})$ is open in $\mathbf{P}V^*$. Thus, a nonempty open subset $\Omega \subset \mathbf{P}(V^*|_{X-B})$ doing the job would be open in $\mathbf{P}V^*$ as well, and saying that $D(v)|_{X-B}$ satisfies Q is exactly our final claim. So by the absence of base points we know that the projective bundle Z is smooth over its base scheme X, namely, the projection $p : Z \to X$ is smooth. This implies that all the fibers of p are smooth, in particular Z itself is smooth. Hence Z satisfies Q.

- If Q is the smoothness property, this is the situation: there is morphism of varieties $q : Z \to \mathbf{P}V^*$ over a field of characteristic zero, and Z is smooth. Hence $q^{-1}(\Omega) \to \Omega$ is smooth for some nonempty open subset $\Omega \subset \mathbf{P}V^*$. This is exactly the generic smoothness theorem.

- If Q is any of the above properties, then by Proposition 1.6.4 (which we apply because q is of finite type) the set

  $$E = \{\, \overline{v} \in \mathbf{P}V^* \mid Z_{\overline{v}} \text{ is (geometrically) } Q \,\}$$

  is constructible. But now, as $\operatorname{char} k = 0$ and Z satisfies Q, the generic fiber Z_η of q is (geometrically) Q. So (see for example our discussion on generic $\rightsquigarrow$ general) E contains a nonempty open subset Ω because it contains the generic point η.

This concludes the proof. □

Corollary 1. If, in addition to the assumptions in the Theorem, $\dim V \geqslant 2$, then there are infinitely many members in $\mathbf{P}V^*$ satisfying Q.

Proof. Indeed, then $\mathbf{P}V^*$ is "at least" a $\mathbf{P}^1$ over an infinite field, so any nonempty open subset contains infinitely many rational points. □

Final Remark. Now k is any field and X is an algebraic variety over k. Suppose we are given a k-morphism $f : X \to \mathbf{P}^r$ (induced by some linear system). In proving Theorem 1.10, and in the preceding discussion, the central object was the family

$$q : Z \to \mathbf{P}V^* = \mathbf{P}^{r*}.$$

We did show that its generic fiber satisfies Q, and then used the constructibility of Q to see that for some nonempty open subset $\Omega \subset \mathbf{P}V^*$ the fibers of $q^{-1}(\Omega) \to \Omega$ satisfy Q. We can in fact generalize this procedure. A hyperplane is a particular linear subvariety of the projective space: it has codimension one. But for all $1 \leqslant d < r$ we can consider

$$\mathrm{Gr}(d, r) = \{\, \text{Linear projective } \mathcal{L} \subset \mathbf{P}^r \text{ of dimension } d \,\}.$$ [12]

Now we can construct the analog object of our family Z, namely

$$\mathscr{Z} = \{\, (x, \mathcal{L}) \in X \times_k \mathrm{Gr}(d, r) \mid f(x) \in \mathcal{L} \,\}.$$

Here the condition $f(x) \in \mathcal{L}$ replaces $x \in D(\nu)$ in the old Z. This construction is more general because we take linear subvarieties (intersections of finitely many hyperplanes) instead of hyperplanes. One shows that the projection morphism

$$\hat{q} : \mathscr{Z} \longrightarrow \mathrm{Gr}(d, r)$$

satisfies $\mathscr{Z}_{\mathcal{L}} = \hat{q}^{-1}(\mathcal{L}) \cong f^{-1}(\mathcal{L}) \subset X$ for every $\mathcal{L} \in \mathrm{Gr}(d, r)$, the analog of our old $Z_{\overline{\nu}} = D(\nu)$. One might ask: is there an analog of Theorem 1.10, for linear subvarieties of $\mathbf{P}^r$? The answer is yes, so here is the result.

Theorem 1.11. Let k be a field, X a k-scheme of finite type, let $1 \leqslant d < r$ and $f : X \to \mathbf{P}^r$ a k-morphism.

(1) If X is irreducible, then $\hat{q}$ is dominant if and only if $\dim \overline{f(X)} \geqslant d$.

(2) Suppose $\operatorname{char} k = 0$ or f not ramified. If X is smooth (resp. geometrically reduced) then so is also the generic fiber of $\hat{q}$. Moreover, if k is infinite, then $f^{-1}(\mathcal{L})$ is smooth (resp. geometrically reduced) for almost all $\mathcal{L} \in \mathrm{Gr}(d, r)$.

(3) If $\dim \overline{f(X)} \geqslant d + 1$ and X is geometrically irreducible then the generic fiber of $\hat{q}$ is geometrically irreducible.

[12] For instance, $\mathrm{Gr}(1, r) = \{\, [\ell] \mid \ell \subset \mathbf{P}^r \text{ line} \,\} \cong \mathbf{P}H^0(\mathbf{P}^r, \mathscr{O}(1))^*$, and $\mathrm{Gr}(r-1, r) = \{\, [H] \mid H \subset \mathbf{P}^r \text{ hyperplane} \,\} \cong \mathbf{P}H^0(\mathbf{P}^r, \mathscr{O}(1))$.

Proof. See [13], Théorème 6.10, p. 89. □

Note that the above theorem already answers a question that one could ask: is there a hope to let the strong Bertini's Theorem become true in positive characteristic? The answer provided in (2) is yes, by requiring that f be unramified. We will discuss again about this topic later.

CHAPTER 2

Applications and Pathologies

In this chapter we make some remarks and examples of application of Bertini's Theorem: most of all, we will deal with the version holding in any characteristic, and we will point out some basic consequences, like the existence of smooth hypersurfaces of any degree contained in a smooth variety. For example, any smooth surface will contain (many) smooth curves. A few explicit computations will be provided, and we will also focus on bad situations, namely when Bertini cannot hold, because of some missing hypothesis. We will be able to explain the failure of the strong Bertini in characteristic zero, with an explicit example in characteristic 2.

2.1 First positive results

Before beginning with examples, let us describe what we mean by the *dual variety* of an algebraic variety X. Assume $X \subset \mathbf{P}^n_k = \operatorname{Proj} k[x_0, \dots, x_n]$ is a smooth algebraic variety over a field k. We define the dual variety, as a set, to be

$$X^* = \{ T_{X,x} \mid x \in X \} .$$

If X is not smooth, then X^* is defined to be the Zariski closure of $\delta(X_{sm})$ where $\delta : X_{sm} \to \mathbf{P}^{n*}_k$ is the morphism $x \mapsto T_{X,x}$ and X_{sm} is the set of smooth points of X. In any case, X^* is a closed subvariety of $\mathbf{P}^{n*}_k$, hence itself a projective variety; it has the reduced subscheme structure but it might neither be irreducible nor of pure dimension.

For example, let X be a curve in $\mathbf{P}^2_k$. Then $X^* \subset \mathbf{P}^{2*}_k$ is the set of tangent lines through the (smooth) points of X. It need not be smooth even if X is. As an example, consider a smooth, i.e. non degenerate conic $C = V_+(f)$ in $\mathbf{P}^2_k$. Then $C \cong C^*$. Indeed, by looking at the gradient map $p \mapsto \nabla_p f$ one sees that C^* is also a conic (unless $\operatorname{char} k = 2$). But this is a very special case: in general, the dual curve of a degree d curve has degree $d(d-1)$, so it is not isomorphic to the original curve.

Example 2.1.1. (*More on the Space of Conics*) Let $k = \overline{k}$ be a field of characteristic $\neq 2$. Recall that on $\mathbf{P}^2$ we have the complete linear system of conics $\mathfrak{D} = \mathbf{P}H^0(\mathbf{P}^2, \mathscr{O}(2)) = \mathbf{P}^5$. Let us verify directly that Bertini's Theorem holds. Let us choose, for instance, the (smooth = non degenerate) conic X given by

$$f(x_0, x_1, x_2) = x_0^2 + x_1^2 + x_2^2$$

and let us see that, for a sufficiently general $H \in \mathbf{P}^{2*}$, the section $H \cap X$ remains smooth. So let us take a hyperplane $H_a : a_0x_0 + a_1x_1 + a_2x_2 = 0$, corresponding to the point $a = (a_0, a_1, a_2) \in \mathbf{P}^2$. We may assume $a_1 \neq 0$. We are interested in checking that the set $\aleph$ of hyperplanes such that their intersection with X is smooth is a dense open subset of $\mathbf{P}^{2*}$. Let us call $Y_a = H_a \cap X$. It is smooth at $p = (\alpha, \beta, \gamma) \in Y_a$ if and only if the Jacobian matrix

$$J_p = \begin{pmatrix} \frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ a_0 & a_1 & a_2 \end{pmatrix} (p) = \begin{pmatrix} 2\alpha & 2\beta & 2\gamma \\ a_0 & a_1 & a_2 \end{pmatrix}$$

has maximal rank,[1] i.e. $\operatorname{rank} J_p = 2$. Assuming $\beta \neq 0$, this happens if and only if

$$\begin{cases} 2\alpha a_1 - 2\beta a_0 \neq 0 \\ 2\alpha a_2 - 2\gamma a_0 \neq 0 \\ 2\beta a_2 - 2\gamma a_1 \neq 0 \end{cases} \iff \begin{cases} \alpha/\beta \neq a_0/a_1 \\ \alpha/\gamma \neq a_0/a_2 \\ \gamma/\beta \neq a_2/a_1 \end{cases}$$

where we have assumed that $\gamma \neq 0$ (indeed $\alpha = \gamma = 0$ would lead to a contradiction). So $p = (\alpha/\beta, 1, \gamma/\beta)$ is a smooth point of Y_a if and only if $p \neq (a_0, a_1, a_2) = a$. We deduce that Y_a is smooth precisely when $a \notin Y_a$, which is equivalent to $a \notin X$. So we find

$$\aleph = \{\, H_a \mid Y_a \text{ is smooth} \,\} \cong \mathbf{P}^{2*} - X^*$$

which of course is dense.

[1] Vocabulary: the good property of having smooth intersection with X is sometimes referred to as "H intersects X transversally".

EXAMPLE 2.1.2. More generally, let $X \subset \mathbf{P}^n_k$ be a smooth hypersurface, so that $\dim X = n - 1$, which is the dimension of any hyperplane $H \in \mathbf{P}^{n*}_k$. Consider, as above,

$$\aleph = \{ \text{Hyperplanes } H \text{ such that } H \cap X \text{ is smooth} \}.$$

Recall (perhaps, by looking at the proof of Bertini's Theorem) that $x \in H \cap X$ is a smooth point if and only if $T_{X,x} \not\subset H$, if and only if $T_{X,x} \neq H$, which is equivalent to $H \notin X^*$. The first equivalence comes from the fact that, as X is a smooth hypersurface, $T_{X,x}$ is a hyperplane, that is, a hypersurface given by a linear form, hence of dimension $n - 1 = \dim X$, so being contained means being equal. We conclude that

$$\aleph = \mathbf{P}^{n*}_k - X^*$$

and $\aleph$ is not empty because $\dim X^* = \dim X < n$.

EXAMPLE 2.1.3. We prove that in $\mathbf{P}^n_k$ there exists "many" smooth hypersurfaces of any fixed degree. Let us consider the smooth variety $X = \mathbf{P}^n_k$, over any algebraically closed field k. Consider the Veronese d-uple embedding

$$\nu_d : \mathbf{P}^n_k \hookrightarrow \mathbf{P}^m_k$$

where $m = \binom{n+d}{d} - 1$. Assume that $x_0, \dots, x_n$ are the coordinates on $\mathbf{P}^n_k$ and $T_0, \dots, T_m$ are those on $\mathbf{P}^m_k$. The morphism ν_d is given, locally, by a morphism $\theta : k[T_0, \dots, T_m] \to k[x_0, \dots, x_n]$ sending T_i to some monomial of degree d in the x_j's. Thus if $g(T_0, \dots, T_m)$ is any homogeneous polynomial of degree r and V is the hypersurface it defines in $\mathbf{P}^m_k$, then $V \cap \mathbf{P}^n_k$ is the hypersurface given by the polynomial $g(\theta(T_0), \dots, \theta(T_m))$, hence of degree dr. Hence, if H is any hyperplane in $\mathbf{P}^m_k$, the variety $H \cap \mathbf{P}^n_k$ is a smooth hypersurface of degree d inside $\mathbf{P}^n_k$. There exists one for every positive integer d. What Bertini says is that the set of these smooth hypersurfaces is a dense open subset of the complete linear system $|\mathscr{O}(d)|$ on $\mathbf{P}^n_k$.

EXAMPLE 2.1.4. As a last example, we prove that Bertini's Theorem continues to hold if X has finitely many singular points, say $X^s = \{ p_1, \dots, p_t \}$. The set of transversal hyperplanes $U = \{ H \mid H \not\supset T_{X,x} \text{ for all } x \in X \} \subset \mathbf{P}^{n*}_k$ is a nonempty open subset, because it is obtained as the intersection of finitely many (just by assumption) open subsets of the dual projective space. Consider

$$U' = \{ H \mid H \cap X^s \neq \varnothing \} = \bigcup_{i=1}^{t} \{ H \mid p_i \in H \}.$$

Every $\{H \mid p_i \in H\}$ is a hyperplane in $\mathbf{P}_k^{n*}$, so U' is a finite union of hyperplanes, and hence a closed subset of $\mathbf{P}_k^{n*}$. Thus $U - U'$ is still open (thus dense), and every H_0 inside it satisfies $H_0 \not\supset T_{X,x}$ for any point $x \in X$ and $H_0 \cap X^s = \varnothing$, thus $H_0 \cap X$ is smooth, and Bertini's statement then follows. If, instead, $\dim X^s \geqslant 1$, then by the *Dimension Theorem* we get $H \cap X^s \neq \varnothing$ for every hyperplane H, thus $H \cap X$ is never a smooth scheme.

To conclude this section, we give the following result. It will be useful later.

THEOREM 2.1. Let X be an integral, normal projective variety of dimension at least 2. Let $Y \subset X$ be a closed subset of codimension one, which is the support of an effective ample divisor. Then Y is connected.

Proof. See [14], III, Corollary 7.9, p. 244. □

A consequence of Theorem 2.1 is that when $\dim X \geqslant 2$ in Bertini's Theorem (any characteristic), the hyperplane sections $Y = H \cap X$ are in fact *connected*, and since they are smooth by Bertini, they are irreducible. In fact, a hyperplane section is by definition an effective (it corresponds to the pullback of $\mathcal{O}(1)$) ample divisor on X, once one has fixed an embedding $X \hookrightarrow \mathbf{P}_k^n$.

2.2 Something goes wrong: Frobenius

Let X be an algebraic variety. When $\operatorname{char} k = p > 0$, strange things can happen. In fact, there are two possibilities:

- The field k is infinite. Then one always finds a hyperplane, defined over k, cutting X transversally. If k, in addition, is algebraically closed, then the (weak) Bertini Theorem applies and we know that there are many good hyperplanes, intersecting a smooth X transversally. But strong-Bertini might fail. In fact, even for $k = \overline{k}$, weak-Bertini might fail as well, if one does not embed X in $\mathbf{P}_k^n$ in the right way (that is, via a closed immersion).

- If k is finite, we are in a trouble. All the finitely many hyperplanes might fail to cut X transversally, even if X is smooth. So even weak-Bertini might be false.

Zariski gave the following example: consider, for any (perfect) field k of positive characteristic, the pencil of curves inside $\mathbf{P}^2$

generated by x^p and y^p. Then every curve is of the form $C_\lambda : x^p + \lambda y^p = 0$, for λ ranging $\mathbf{P}^1$. But then $0 = x^p + \lambda y^p = (x + \lambda^{1/p} y)^p$ says that every point in C_λ is a p-fold point, in particular it is singular. This is an example of a linear system on $\mathbf{P}^2$ with (lots of) singularities outside the base locus (which is one point). So we proved the possible failure of strong-Bertini, and the above example applies both to finite and infinite (perfect) fields.

We now illustrate the possible failure of weak-Bertini over any $k = \overline{k}$ of characteristic $p > 0$. Consider any smooth $X = V_+(f_1, \ldots, f_r) \subset \mathbf{P}^n$ where $f_i(x_0, \ldots, x_n)$ are homogeneous. We show that for a particular embedding of X in $\mathbf{P}^n$ (via Frobenius) there is no smooth hyperplane section. More precisely, we consider the composition of a fixed closed immersion with the Frobenius morphism (see Appendix for the construction of the *relative Frobenius* $F = F_{X/k}$). More precisely, call f the composition

$$X \xrightarrow{F} X^{(p)} = X \hookrightarrow \mathbf{P}^n.$$

The point is that f is not a closed immersion (F is not). We show that there is no hyperplane $V_+(h) = H \in \mathbf{P}^{n*}$ such that $H \cap X = f^{-1}(H)$ is a smooth variety. Let H be any hyperplane, and let us verify our claim locally, say on $U = D_+(x_0) \cong \operatorname{Spec} k[u_1, \ldots, u_n]$, where u_j denotes the affine coordinate x_j/x_0. The corestriction $f^{|U}$, still denoted by f, induces

$$\begin{array}{ccc} k[u_1,\ldots,u_n] & \xrightarrow{f^\sharp} & k[u_1,\ldots,u_n]/(\tilde{f}_1,\ldots,\tilde{f}_r) \\ \| & & \| \\ \mathscr{O}_{\mathbf{P}^n_k}(U) & & \mathscr{O}_X(f^{-1}(U)) \end{array}$$

where $\tilde{f}_i(u_1, \ldots, u_n) = f_i(1, u_1, \ldots, u_n)$. Thus

$$\begin{aligned} f^{-1}(U \cap H) &= f^{-1}(V(\tilde{h})) = V(f^\sharp(\tilde{h})) = V(\tilde{h}^p) \\ &= \operatorname{Spec} k[u_1, \ldots, u_n]/(\tilde{f}_1, \ldots, \tilde{f}_r, \tilde{h}^p), \end{aligned}$$

which is not even reduced.

2.3 Smooth Complete Intersection

Throughout this section, one refers to weak-Bertini (in any characteristic). So k is any algebraically closed field. The proofs of the main assertions in Commutative Algebra can all be found in [3]. The main topic will be complete intersection in projective space: one will be able to apply Bertini's Theorem to show that for every $r < n$ there exists a codimension r subvariety $Y \subset \mathbf{P}^n_k$ which is a smooth irreducible complete intersection (of hypersurfaces of prescribed degree).

Definition 2.3.1. A closed subscheme $Y \subset \mathbf{P}^n_k$ is said to be a (global) COMPLETE INTERSECTION if its homogeneous ideal $I(Y) \subset k[x_0, \ldots, x_n]$ can be generated by $r = \operatorname{codim}(Y, \mathbf{P}^n_k)$ elements.

REMARK 2.1. We take as a definition $I(Y) = \Gamma_*(\mathscr{I}_Y)$, for every Y projective over $\operatorname{Spec} A$. In fact, $\Gamma_*(\mathscr{I}_Y)$ is the largest ideal defining Y inside $\mathbf{P}^n_A$.

The aim of this section is resumable in three points.

1. We will show this result:

    THEOREM 2.2. A closed subscheme $Y \subset \mathbf{P}^n_k$ of codimension r is a complete intersection if and only if there are r hypersurfaces $H_1, \ldots, H_r$ in $\mathbf{P}^n_k$ such that $Y = H_1 \cap \cdots \cap H_r$, scheme-theoretically. On the level of sheaves, this means that $\mathscr{I}_Y = \mathscr{I}_{H_1} + \cdots + \mathscr{I}_{H_r}$.

2. If $Y \subset \mathbf{P}^n_k$ is a complete intersection of dimension > 0 and Y is normal, then it is projectively normal; moreover, for all $\ell \geqslant 0$, the natural map $H^0(\mathbf{P}^n_k, \mathscr{O}(\ell)) \to H^0(Y, \mathscr{O}_Y(\ell))$ is surjective. When $\ell = 0$ this says that Y is connected.

3. For any $r < n$ choose $d_1, \ldots, d_r \geqslant 1$ to be arbitrary integers. Then there exist smooth hypersurfaces $H_i \subset \mathbf{P}^n_k$, of degree d_i, such that the scheme $Y := H_1 \cap \cdots \cap H_r$ is irreducible, nonsingular and of codimension r in $\mathbf{P}^n_k$.

One can already show one direction of Theorem 2.2, the 'only if' one. Indeed, if Y is a complete intersection then $I(Y)$ can be generated by r elements, say $I(Y) = (f_1, \ldots, f_r)$ where r is the codimension of Y in $\mathbf{P}^n_k$. Hence $Y = H_1 \cap \cdots \cap H_r$, scheme-theoretically, where $H_i = V_+(f_i)$ for $i = 1, \ldots, r$. Another way to see the 'only if' part is by considering sheaves: we have $\Gamma_*(\mathscr{I}_Y) = I(Y) = (f_1, \ldots, f_r)$ and $I(Y)^\sim \cong \mathscr{I}_Y$, so

$$\mathscr{I}_Y = (f_1, \ldots, f_r)^\sim = \left(\sum (f_i)\right)^\sim = \sum (f_i)^\sim = \sum \mathscr{I}_{H_i}$$

as the ideal sheaf of $H_i = V_+(f_i)$ is $(f_i)^\sim$. The next efforts are devoted to show the converse.

§ TOOL I: **Some Primary Decomposition.** The following holds for every commutative unitary ring A, if one replaces 'ideal' by 'decomposable ideal'. But we assume that A is a noetherian ring, thus every ideal $\mathfrak{a} \subset A$ admits a primary decomposition, i.e. one can write

$$\mathfrak{a} = \bigcap_{i=1}^{m} \mathfrak{q}_i \tag{2.1}$$

where $\mathfrak{q}_i$ are primary ideals. Saying that $\mathfrak{q}$ is primary means that $\mathfrak{q} \neq A$ and, whenever $xy \in \mathfrak{q}$ and $x \notin \mathfrak{q}$, one has $y^r \in \mathfrak{q}$ for some $r > 0$. (Equivalently, $A/\mathfrak{q} \neq 0$ and every zero divisor of $A/\mathfrak{q}$ is nilpotent.) Note that the radical of a primary ideal is necessarily prime, but the power of a prime need not be primary; we say that $\mathfrak{q}$ is $\mathfrak{p}$-primary if $\mathfrak{p} = \sqrt{\mathfrak{q}}$.

Remark 2.2. Because a finite intersection of $\mathfrak{p}$-primary ideals is $\mathfrak{p}$-primary, one can always find a primary decomposition in which all the radicals $\sqrt{\mathfrak{q}_i}$ are distinct; if, in addition, for every $i = 1, \dots, m$ one has $\mathfrak{q}_i \not\supset \bigcap_{j \neq i} \mathfrak{q}_j$, then the decomposition is said to be *minimal* (we always have one such, of course).

Theorem 2.3. Given a minimal decomposition like in (2.1), with $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, one has the following identification:

$$\{ \mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_m \} = \{ \text{Primes associated to } \mathfrak{a} \} = \mathrm{Ass}_A(A/\mathfrak{a}).$$

Proof. See [3], Theorem 4.5, p. 52. □

Definition 2.3.2. A prime ideal $\mathfrak{p} \subset A$ is said to be an ASSOCIATED PRIME for $\mathfrak{a}$ in case $\mathfrak{p} = \mathrm{Ann}(x + \mathfrak{a})$ for some $x \in A$. The minimal elements in $\mathrm{Ass}_A(A/\mathfrak{a})$ are called ISOLATED/MINIMAL PRIMES, while all others are called EMBEDDED PRIMES.

Example 2.3.1. Consider the ideal $\mathfrak{a} = (x^2, xy)$ in $A = k[x, y]$. Then we obtain $\mathfrak{a}$ as $\mathfrak{p}_1 \cap \mathfrak{p}_2^2$, where $\mathfrak{p}_1 = (x) = \mathfrak{q}_1$ and $\mathfrak{p}_2 = (x, y)$. The ideal $\mathfrak{q}_2 = \mathfrak{p}_2^2$ is $\mathfrak{p}_2$-primary because it is a power of the maximal ideal $\mathfrak{p}_2$. Hence $\mathfrak{q}_1 \cap \mathfrak{q}_2$ is a (minimal) primary decomposition for $\mathfrak{a}$ and $\mathrm{Ass}_A(A/\mathfrak{a}) = \{ \mathfrak{p}_1, \mathfrak{p}_2 \}$, where $\mathfrak{p}_1$ is isolated and $\mathfrak{p}_2$ is embedded (in this case it is easy to see, as $\mathfrak{p}_1 \subset \mathfrak{p}_2$). The geometric meaning of the terminology *isolated/embedded prime* is the following: consider the closed subvariety X of $\mathbf{A}^2$ defined by the ideal $\mathfrak{a}$. Then, in this example, $X = V(x)$ is the y-axis, which is irreducible and thus corresponds to the unique irreducible component (that is, minimal prime in the usual sense) given by (x). Instead, the embedded ideal (x, y) corresponds to the origin, clearly lying in X. So we have to keep in mind that an isolated/minimal prime corresponds, ideally, to an irreducible component of the variety defined by $\mathfrak{a}$, and an embedded prime corresponds to a variety embedded in some irreducible component. Algebraically, an isolated prime $\mathfrak{p}$ is characterized by the properties

$$\mathfrak{p} \supset \mathfrak{a}, \text{ and for every } \mathfrak{b} \supset \mathfrak{a} \text{ one has } \mathfrak{p} \not\supset \mathfrak{b}.$$

As $\mathrm{Ass}_A(A/\mathfrak{a}) = \{\text{Isolated primes}\} \coprod \{\text{Embedded primes}\}$, an embedded prime is characterized by

$$\mathfrak{p} \supset \mathfrak{a}, \text{ and for some } \mathfrak{b} \supset \mathfrak{a} \text{ one has } \mathfrak{p} \supset \mathfrak{b}.$$

Finally, because any prime ideal $\mathfrak{p} \supset \mathfrak{a}$ also contains an isolated prime associated to $\mathfrak{a}$, we have that

$$\{\text{Isolated primes associated to } \mathfrak{a}\} = \{\text{Minimal primes above } \mathfrak{a}\}$$

and we are very happy with this, because we recover the usual correspondence between minimal primes above $\mathfrak{a}$ and irreducible components of the variety $\mathrm{Spec}\, A/\mathfrak{a}$.

§ TOOL II: **Cohen-Macaulay rings and unmixed ideals.** Let A be a ring and M an A-module. We say that a finite sequence $\{x_1, \dots, x_r\} \subset A$ is *regular* for M if x_1 is not a zero divisor in M (that is, there exists $m \neq 0$ such that $x_1.m = 0$) and for all $i = 2, \dots, r$ the element x_i is not a zero divisor in $M/\mathfrak{a}_i M$, where $\mathfrak{a}_i = (x_1, \dots, x_{i-1})$. The latter condition means that there exists a nonzero $m + \mathfrak{a}_i M$, namely $m \notin \mathfrak{a}_i M$, such that $x_i.m \in \mathfrak{a}_i M$.

Definition 2.3.3. Let $(A, \mathfrak{m})$ be a local ring and M an A-module. One writes $\mathrm{depth}_A M$ for the depth of M, the maximum length of a regular sequence $\{x_1, \dots, x_r\} \subset \mathfrak{m}$ for M. If A is noetherian (and local) one defines its depth in the same way, and we call A a COHEN-MACAULAY local ring if $\mathrm{depth}\, A = \dim A$. A noetherian (not necessarily local) ring A is said to be Cohen-Macaulay (and we write CM) in case every localization $A_\mathfrak{p}$ at a prime (equivalently, maximal) ideal of A is a CM local ring. This suggests that, on the scheme side, the CM property will be stalk-local, and in fact it is so.

Definition 2.3.4. Let A be a noetherian ring and $\mathfrak{a}$ an ideal with associated primes $\mathrm{Ass}_A(A/\mathfrak{a}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. Then $\mathfrak{a}$ is said to be UNMIXED if it shares its height with all of its associated primes, i.e. $\mathrm{ht}\, \mathfrak{a} = \mathrm{ht}\, \mathfrak{p}_i$ for all $i = 1, \dots, m$.

Now let us consider the following property.

> (†) Every ideal $\mathfrak{a} \subset A$ of height r and generated by r elements is unmixed.

One has the following results.

THEOREM 2.4. A noetherian ring A is CM if and only if (†) holds in A.

THEOREM 2.5. A noetherian ring A is CM if and only if the polynomial ring $A[x]$ is CM. In particular, if A is CM (e.g. a field) then the ring $A[x_1, \dots, x_n]$ is also CM.

Proof. (of Theorem 2.2) Recall that we have some $Y \subset \mathbf{P}^n_k$ of codimension r. We still have to show that if $Y = H_1 \cap \cdots \cap H_r$ scheme-theoretically, where $H_i = V_+(f_i)$ are hypersurfaces, then Y is a complete intersection. We have to show that, if $\mathfrak{a} = (f_1, \dots, f_r)$ and $\mathfrak{b} = I(Y)$, then $\mathfrak{a} = \mathfrak{b}$. By Theorem 2.5, the ring $S = k[x_0, \dots, x_n]$ is CM, so (†) holds in S by Theorem 2.4. This implies that the ideal $\mathfrak{a} \subset S$, which is of height r because $\operatorname{codim}(Y, \mathbf{P}^n_k) = r$, is unmixed. Hence every $\mathfrak{p} \in \operatorname{Ass}_S(S/\mathfrak{a})$ has height r. Now, we are in a situation where $\mathfrak{a}$ and $\mathfrak{b}$ define the same projective k-scheme and in fact $\widetilde{\mathfrak{b}} = \mathscr{I}_Y = \mathscr{I}_{H_1} + \cdots + \mathscr{I}_{H_r} = \widetilde{\mathfrak{a}}$ (this translates the scheme-theoretic equality), so that the f_i's all belong to $\mathfrak{b}$ and hence $\mathfrak{a} \subset \mathfrak{b}$. For every $j = 0, 1, \dots, n$ we have $\mathfrak{a}_{(x_j)} = \mathfrak{b}_{(x_j)}$ and this implies that $x_j^{N_j}\mathfrak{b} \subset \mathfrak{a}$ for some positive integer N_j. Indeed, for every $d > 0$ and for every homogeneous element $\alpha \in \mathfrak{b}_d$ there exists some $\alpha' \in \mathfrak{a}_d$ such that $\alpha/x_j^d = \alpha'/x_j^d$. This implies that $x_j^d(\alpha - \alpha') = 0$, that is, $x_j^d\alpha = x_j^d\alpha' \in \mathfrak{a}$. This holds for every d and for every α, thus there is some $N_j \gg 0$ such that $x_j^{N_j}\mathfrak{b} \subset \mathfrak{a}$. By setting $N = \max_j N_j$ one gets that

$$S_+^N\mathfrak{b} \subset \mathfrak{a} \subset \mathfrak{b}.$$

Now we denote by a bar the reduction mod $\mathfrak{a}$. Let us notice that $\overline{\mathfrak{b}} = \mathfrak{b}/\mathfrak{a}$ is an ideal of $S/\mathfrak{a}$ satisfying $\overline{S}_+^N\overline{\mathfrak{b}} = 0$. Let us suppose, by contradiction, that $\overline{\mathfrak{b}}$ is nonzero. Then there exists some nonzero element $\overline{\alpha} \in \overline{\mathfrak{b}}$ and by the above we have $\overline{S}_+^N\overline{\alpha} = 0$, which says

$$S_+^N \subset \{\, x \in S \mid x\alpha \in \mathfrak{a} \,\} = \operatorname{Ann}(\alpha + \mathfrak{a}) = \mathfrak{p} \in \operatorname{Ass}_S(S/\mathfrak{a}).$$

As S_+ is prime, we have in fact that S_+ is contained in the minimal prime $\mathfrak{p}$. But as it is maximal we must have $S_+ = \mathfrak{p}$, which is a contradiction because $n + 1 = \operatorname{ht} S_+ > r = \operatorname{ht}\mathfrak{a}$. Thus we conclude that $\mathfrak{a} = \mathfrak{b}$. □

So Theorem 2.2 is proved and the first part of our purpose is complete. Let us pass to part 2, about projectively normal varieties.

Definition 2.3.5. Let A be a ring. A projective A-scheme $Y \subset \mathbf{P}^n_A$ is said to be PROJECTIVELY NORMAL if its homogeneous coordinate ring $S(Y) = A[x_0, \dots, x_n]/I(Y)$ is normal (integrally closed), where $I(Y) = \Gamma_*(\mathscr{I}_Y)$.

Let us show that a normal complete intersection $Y \subset \mathbf{P}^n_k$ of strictly positive dimension must be projectively normal. We will use the following result, applied to the affine cone $C(Y) \subset \mathbf{A}^{n+1}_k$.

THEOREM 2.6. Let X be a smooth variety over an algebraically closed field k and let $Y \subset X$ be a local complete intersection subvariety. Then Y (is CM and) is normal if and only if it is regular in codimension one.

Proof. See [14], II, Proposition 8.23, p. 186. □

Let us call r the codimension of Y in $\mathbf{P}^n_k$ and, using our hypothesis, let us write $Y = H_1 \cap \cdots \cap H_r$, remembering that $I(Y) = (f_1, \dots, f_r)$. By considering the projection $\theta : \mathbf{A}^{n+1}_k - \{0\} \to \mathbf{P}^n_k$ then the affine cone $C = C(Y)$ over Y is defined to be

$$C = \theta^{-1}(Y) \cup \{0\} = \operatorname{Spec} S(Y) \subset \mathbf{A}^{n+1}_k.$$

Here $S(Y)$ is $k[x_0, \dots, x_n]/I(Y)$, the homogeneous coordinate ring of Y. It coincides with the *affine* coordinate ring of C. Thus our goal is to prove that C is normal. If one shows that it is a local complete intersection in $\mathbf{A}^{n+1}_k$ then one is left to prove that it is regular in codimension one. In fact, C is even a global complete intersection, because its ideal $I(Y) = (f_1, \dots, f_r)$ is generated by r elements, where $\operatorname{codim}(Y, \mathbf{P}^n_k) = r = \operatorname{codim}(C, \mathbf{A}^{n+1}_k)$ because $\dim C = \dim Y + 1$. It remains to prove that C is regular in codimension one. Of course Y has this property because it is normal by assumption, but we will not use this, we will just use that Y is normal. Let us start by noticing that if we corestrict θ to, say, $D_+(x_0)$ we obtain

$$\theta^{|D_+(x_0)} : D(x_0) \longrightarrow D_+(x_0).$$

This corresponds to the ring homomorphism

$$k\Big[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\Big] \longrightarrow k\Big[x_0, \dots, x_n, \frac{1}{x_0}\Big] = k\Big[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\Big]\Big[x_0, \frac{1}{x_0}\Big].$$

Now if we write y_i for the class $x_i \bmod I(Y)$ then $C - \{0\} = \theta^{-1}(Y) = \bigcup_i D(y_i)$ and the corestriction of θ to $D_+(y_i) \subset Y$ is $D(y_i) \to D_+(y_i)$, and this holds for all $i = 0, \dots, n$. By the above, we remark that $C - \{0\}$ is, locally on each $D(y_i)$, the product of the affine schemes $D_+(y_i) \times_k \mathbf{G}_m$, where we identify $\mathbf{G}_m$ to $\operatorname{Spec} k[t, t^{-1}]$. Hence

$$\begin{aligned}\mathscr{O}(D(y_i)) &= \mathscr{O}(D_+(y_i) \times_k \mathbf{G}_m) \cong \mathscr{O}(D_+(y_i)) \otimes_k \mathscr{O}(\mathbf{G}_m) \\ &= \mathscr{O}(D_+(y_i)) \otimes_k k[t, t^{-1}] = \mathscr{O}(D_+(y_i))[t, t^{-1}]\end{aligned}$$

and the latter is a normal ring as, in general, if A is normal then $A[t]$ is normal, and the same holds for any localization of A. Here of course we used that each $\mathscr{O}(D_+(y_i))$ is normal. So we conclude that $C - \{0\}$ is normal. At the origin, we have that

$$\dim \mathscr{O}_{C,0} = \dim C = \dim Y + 1 > 1$$

so we get that C is regular in codimension one, as claimed.

To conclude part 2, we need the following result (which holds, more generally, if one replaces the field k with any ring A):

THEOREM 2.7. A projective k-scheme $Y \subset \mathbf{P}^n_k$ is projectively normal if and only if it is normal and, in addition, the natural map $H^0(\mathbf{P}^n_k, \mathscr{O}(\ell)) \to H^0(Y, \mathscr{O}_Y(\ell))$ is surjective, for every $\ell \geqslant 0$.

Proof. See [Hartshorne, II, Ex. 5.14(d)] □

So in particular we have that for a closed projectively normal subvariety $Y \subset \mathbf{P}^n_k$ all the morphisms $H^0(\mathbf{P}^n_k, \mathscr{O}(\ell)) \to H^0(Y, \mathscr{O}_Y(\ell))$ are surjective. When $\ell = 0$ we have that $H^0(\mathbf{P}^n_k, \mathscr{O}) = k$ surjects onto $H^0(Y, \mathscr{O}_Y) = \mathscr{O}_Y(Y)$, hence $\dim \mathscr{O}_Y(Y) \leqslant 1$, which says exactly that Y is connected. Indeed, if it were not connected, one could decompose $\mathscr{O}_Y(Y)$ as a direct sum of copies of k. Part 2 is now complete.

To prove the third point of our initial goal, we will iterate, in some sense, the reasoning we did in Example 2.1.3. So now we are assuming that $r < n$ and we have r positive integers $d_1, \dots, d_r$. We want to construct a smooth irreducible complete intersection $Y \subset \mathbf{P}^n_k$ given by the intersection of r hypersurfaces of the given degrees. To start with, we define the integer $m_i = \binom{n+d_i}{d_i} - 1$. In Example 2.1.3 we saw that if we consider the d_1-uple embedding $\nu_1 : \mathbf{P}^n_k \hookrightarrow \mathbf{P}^{m_1}_k$ then (by Bertini) there is (in fact, there are many) a hyperplane in $\mathbf{P}^{m_1}_k$ that pulls back to a smooth hypersurface $H_1 \subset \mathbf{P}^n_k$ of degree d_1. Now, $\nu_2(H_1) \subset \mathbf{P}^{m_2}_k$ is a smooth variety, and again by Bertini there is a hyperplane $H \subset \mathbf{P}^{m_2}_k$ such that $H \cap \nu_2(H_1)$ is smooth of dimension $\dim H_1 - 1$. The preimage of this intersection under ν_2 is some $H_1 \cap H_2$, where H_2 is a smooth hypersurface of degree d_2. Note that Bertini also ensures that the dimension drops by exactly one at each step. One can go on like this r times, and at the end $Y := H_1 \cap \cdots \cap H_r$ is smooth of codimension r in $\mathbf{P}^n_k$. So it is a normal complete intersection of positive dimension (because $r < n$) and by part 2 we get that Y is connected; as it is smooth, it is also irreducible, as claimed.

> Summary: Thanks to Bertini, we have just proved that for every $r < n$ there exists a codimension r subvariety $Y \subset \mathbf{P}^n_k$ which is a smooth irreducible complete intersection, given by intersecting r hypersurfaces of prescribed degree.

2.4 Moving singularities: failure in char 2

Throughout this section, one refers to the (stronger) version of Bertini in characteristic zero (a general member of a linear system is smooth away from the base locus), and one shows that it may fail in positive characteristic.

In the following example one deals with a linear system of cubics of dimension 2 (a net), all of whose members are singular, with singular points (inside and) outside the base locus. Let $k \supset \mathbb{F}_2$ be an algebraically closed field of characteristic 2. Let us name T the scheme $\mathbf{P}^2_{\mathbb{F}_2} = \{ P_1, \ldots, P_7 \}$ (see Figure 2.1 below). One defines $\mathfrak{D}$ to be the linear system (over k) of all the cubics passing through the P_i's.

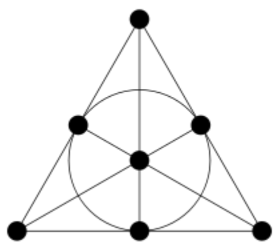


Figure 2.1: The projective plane over $\mathbb{F}_2$.

> Final goal. First, it will be shown that $\mathfrak{D}$ is a net with base locus equal to T, and the associated morphism $\mathbf{P}^2_k - T \to \mathbf{P}^2_k$ is inseparable of degree 2. After that, we will see that *all* the curves $C \in \mathfrak{D}$ are singular: either C consists of three lines passing through (exactly) one of the P_i's, or C is an irreducible cuspidal cubic curve

with cusp $S \notin T$. Furthermore, associating to $C \in \mathfrak{D}$ its (unique) singular point gives a 1-1 correspondence between $\mathfrak{D}$ and $\mathbf{P}^2_k$.

We recall the following:

Definition 2.4.1. A morphism of, say, integral schemes $f : X \to Y$ is said to be purely inseparable if it is injective and if for every $x \in X$, the extension of residue fields $k(f(x)) \hookrightarrow k(x)$ is purely inseparable. It is called INSEPARABLE if $K(Y) \hookrightarrow K(X)$ is inseparable (i.e. we drop the injectivity[2] and we just have to check at the generic point).

Let us construct explicitly $\mathfrak{D}$. Before that, let us list the points in T:

$$P_1 = (1,0,0) \quad P_2 = (0,1,0) \quad P_3 = (0,0,1)$$
$$P_4 = (1,1,0) \quad P_5 = (1,0,1) \quad P_6 = (0,1,1) \quad P_7 = (1,1,1).$$

If x, y, z are homogeneous coordinates on $\mathbf{P}^2_k$, the generic cubic $\mathcal{C} \subset \mathbf{P}^2_k$ is given by the vanishing of the polynomial

$$a_1x^3 + a_2x^2y + a_3x^2z + a_4y^3 + a_5xyz +$$
$$a_6xz^2 + a_7xy^2 + a_8z^3 + a_9y^2z + a_{10}yz^2.$$

Imposing that $\mathcal{C}$ passes through P_1, P_2, P_3 forces $a_1 = a_4 = a_8 = 0$. Passing through P_4 gives $a_2 + a_7 = 0$, which means $a_2 = a_7$. Similarly, for P_5 we get $a_3 = a_6$, while P_6 gives $a_9 = a_{10}$ and passing through P_7 forces $a_5 = 0$. Thus, after renaming the survived coefficients, we get that the general member of $\mathfrak{D}$ is

$$f(x,y,z) = a(x^2y + xy^2) + b(x^2z + xz^2) + c(y^2z + yz^2) \qquad (2.2)$$

so that $\mathfrak{D}$ is a two-dimensional linear system, generated by the three cubics appearing in the expression of f. One easily sees that

$$B_{\mathfrak{D}} = C_1 \cap C_2 \cap C_3 = T,$$

where C_i are the generators of $\mathfrak{D}$. Therefore there is a morphism

$$\begin{aligned} \psi : \mathbf{P}^2_k - T &\longrightarrow \mathbf{P}^2_k \\ (x,y,z) &\longmapsto (x^2y + xy^2, x^2z + xz^2, y^2z + yz^2) \end{aligned}$$

[2] The reason is that we want that a composition $Z \to X \xrightarrow{F} X^{(p)}$, with F the relative Frobenius, be an inseparable morphism, but it has no hope to be injective whenever the first arrow is not injective.

and one has to show it is inseparable of degree 2: it is enough to do this locally, for example in the open subset $U = D_+(z) \cong \mathbf{A}^2_k$, where $z = 1$. The coordinates on U are $s = x/z$ and $t = y/z$. The restriction of ψ becomes

$$\begin{array}{rcl} ((\mathbf{P}^2_k - T) \cap U) - V_+(y^2 + y) & \longrightarrow & \mathbf{A}^2_k \\ (x, y, 1) & \longmapsto & \left(\frac{x^2y + xy^2}{y^2 + y}, \frac{x^2 + x}{y^2 + y}\right) \end{array}$$

More precisely, we have

$$\frac{x^2y + xy^2}{y^2 + y} = \frac{x^2 + yx}{y + 1},$$

thus the induced morphism on function fields is defined as follows:

$$\begin{array}{rcl} j : k(s,t) & \hookrightarrow & k(x,y) \\ s & \longmapsto & \frac{x+y}{y+1}x \\ t & \longmapsto & \frac{x+1}{y+1}\frac{x}{y} \end{array}$$

One has to show that $k(s,t) \cong k(j(s), j(t)) =: K \subset k(x,y) =: L$ is an inseparable extension of degree 2. First, let us notice that

$$yj(t) + j(s) = \frac{x + 1 + x + y}{y + 1}x = x.$$

By replacing this value of x in, say, $j(s)$, we get

$$\begin{aligned} 0 = j(s) + j(s) &= j(s) + \frac{x + y}{y + 1}x \\ &= \frac{j(s)(y + 1) + y^2j(t)^2 + yj(ts) + y^2j(t) + yj(ts) + j(s)^2 + yj(s)}{y + 1} \end{aligned}$$

whence the relation

$$y^2j(t)(j(t) + 1) + j(s)(j(s) + 1) = 0 \ \rightsquigarrow \ y^2 + c = 0$$

with $c = j(s)(j(s)+1)/j(t)(j(t)+1) \in K$. So the minimal polynomial of y over K is $u^2 + c = (u + \sqrt{c})^2$, which is inseparable of degree 2.

Let us pass to the second part of the *final goal*, regarding singularities in $\mathfrak{D}$. First, let us show that every curve in $\mathfrak{D}$ is singular, by building the system

$$\begin{cases} \dfrac{\partial f}{\partial x} = ay^2 + bz^2 = 0 \\[2ex] \dfrac{\partial f}{\partial y} = ax^2 + cz^2 = 0 \\[2ex] \dfrac{\partial f}{\partial z} = bx^2 + cy^2 = 0 \end{cases} \quad \rightsquigarrow \quad \begin{cases} \sqrt{a}\, y = \sqrt{b}\, z \\[2ex] \sqrt{a}\, x = \sqrt{c}\, z \\[2ex] \sqrt{b}\, x = \sqrt{c}\, y \end{cases}$$

which gives the singular point $S = (\sqrt{c}, \sqrt{b}, \sqrt{a})$. One notes that every P_i is singular for *exactly* one cubic in $\mathfrak{D}$, obtained by letting a, b, c lie in $\mathbb{F}_2$. For example, P_1 is singular on the cubic curve $y^2z + yz^2$ which, in addition, is a union of three lines. More precisely:

Points in $\mathbb{F}_2$	Coefficients	Equation
P_1	$a = b = 0,\ c = 1$	$yz(y+z)$
P_2	$a = c = 0,\ b = 1$	$xz(x+z)$
P_3	$b = c = 0,\ a = 1$	$xy(x+y)$
P_4	$a = 0,\ b = c = 1$	$z(x+y)(x+y+z)$
P_5	$b = 0,\ a = c = 1$	$y(x+z)(x+y+z)$
P_6	$c = 0,\ a = b = 1$	$x(y+z)(x+y+z)$
P_7	$a = b = c = 1$	$(x+y)(x+z)(y+z)$

During the computations, one easily notices that the unique way of getting a union of three lines is in fact by choosing a, b, c in $\mathbb{F}_2$. So the 7 above equations classify all cubics with a singular point in T, and any other cubic will be singular in $S = (\sqrt{c}, \sqrt{b}, \sqrt{a})$ but will never split into three lines. Finally, two different triples of coefficients will necessarily give rise to different singular points, so one concludes that the association

$$\begin{aligned} \mathfrak{D} &\longrightarrow \mathbf{P}^2_k \\ C &\longmapsto S_C \end{aligned}$$

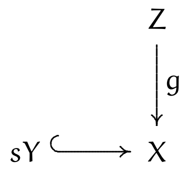
is a bijection, and the singularities in $\mathfrak{D}$ move all over.

2.5 Kleiman-Bertini Theorem

The aim of this section is to explain a generalization of Bertini's Theorem due to Kleiman (see [16] for the result in its original form, or: [14], III, Theorem 10.8; or [12], Theorem 17.22 on page 219).

From now on, the base field k is assumed to be algebraically closed, and all schemes considered are integral k-varieties. A homogeneous space is a couple (X, G) where X is a variety and G is a group variety acting on X, in such a way that the group $G(k)$ acts transitively on $X(k)$. Let us start by giving the friendly statement presented in [12], even if it will be generalized later:

Theorem 2.8 (Kleiman-Bertini, special case). Let (X, G) be a homogeneous space. Let $Y \subset X$ be a smooth subvariety and $g : Z \to X$ a scheme over X. For $s \in G$, define $V_s = sY \times_X Z \subset Z$, the pullback of the diagram

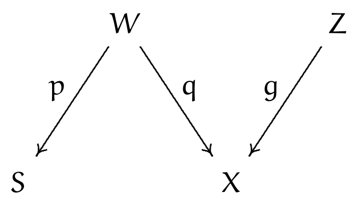


where sY is isomorphic to Y but is included in X in a different way, namely the embedding $sY \to X$ sends $y \mapsto sy$. Then, for a general s, the singularities of V_s are all contained in the singular locus of Z, that is $(V_s)_{\text{sing}} = V_s \cap Z_{\text{sing}}$.

As a consequence, consider the case where g is an inclusion. So what we have is two subvarieties Y, Z of X and one of them is regular. But *if both* are regular, then $sY \cap Z$ is regular for general $s \in G$. In words: the general translate of a smooth subvariety of a homogeneous space meets transversally any other smooth subvariety.

Kleiman starts its paper by proving the following:

Lemma 2.1. Given a diagram of integral schemes



the following facts are true:

(1) if q is flat then there is a dense open subset $U \subset S$ such that for every $s \in U$, either $p^{-1}(s) \times_X Z$ is empty, or it is equidimensional of dimension $\dim p^{-1}(s) + \dim Z - \dim X$. (Here, of course, $p^{-1}(s) = W \times_S \operatorname{Spec} k(s) \subset W$.)

(2) If q is smooth and Z is regular, then $p^{-1}(\xi) \times_X Z$ is regular, where ξ is the generic point of S. If $\operatorname{char} k = 0$ then $p^{-1}(s) \times_X Z$ is regular for s ranging a dense open subset of S.

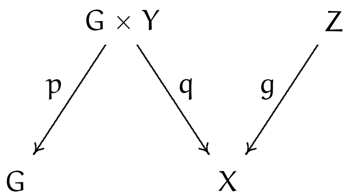
Essentially, everything that is inside Kleiman's paper follows from the above lemma. First of all, the following theorem (generalizing slightly Theorem 2.8). Recall that if $s \in G$ then for every X-scheme $f : Y \to X$ the translate sY is (isomorphic to) Y, viewed as an X-scheme via the map $y \mapsto sf(y)$.

Theorem 2.9 (Kleiman-Bertini). Let (X, G) be a homogeneous space. Consider $Y \xrightarrow{f} X \xleftarrow{g} Z$ two maps of integral k-varieties (k any algebraically closed field).

(1) There exists a dense open subset $U \subset G$ such that for every $s \in U$, either $sY \times_X Z$ is empty or it is equidimensional of dimension $\dim Y + \dim Z - \dim X$.

(2) If $\operatorname{char} k = 0$ and both Y, Z are regular then there is a dense open subset $U \subset G$ such that for every $s \in U$ the variety $sY \times_X Z$ is regular.

Proof. The strategy is to recover the hypothesis of the Lemma: Kleiman shows that in the diagram



the morphism q sending $(s, y) \mapsto sf(y)$ is flat (so (1) follows) and in fact - thanks to regularity of Y - it is smooth (so (2) follows, as $p^{-1}(s) \cong \{ s \} \times Y = sY$). The smoothness of q is proved by showing that, in addition to flatness, its fibers are geometrically regular and equidimensional. □

As a particular (and important) situation, consider when both $Y, Z \subset X$ are subvarieties of a homogeneous space (X, G). Then the pullback along X becomes the scheme-theoretic intersection, and we get that:

(1) for general $s \in G$, the scheme $sY \cap Z$ is purely of dimension $\dim Y + \dim Z - \dim X$.

(2) If $\operatorname{char} k = 0$ and both Y, Z are regular then for general $s \in G$, the scheme $sY \cap Z$ is smooth.

As another consequence, there is our old friend.

Corollary 2 (Bertini)**.** Let k be algebraically closed of characteristic zero. Suppose $\mathbf{P}^r$ is a linear system on an integral scheme Z. Then a general element in the system is smooth away from the base locus and the singularities of Z.

Proof. Here the homogeneous space X is $(\mathbf{P}^r, \mathrm{PGL}(r,k))$. By replacing Z by $Z - (B \cup Z_{\text{sing}})$ (the base locus B and the singular locus Z_{sing} being both closed in Z), we may assume that Z is smooth and the system is base-point-free. Then we have a morphism $\psi : Z \to \mathbf{P}^r$ and by construction the elements of the linear system are the preimages of the hyperplanes $H \subset \mathbf{P}^r$ under ψ, that is: $D_H = \psi^{-1}(H) = H \times_{\mathbf{P}^r} Z$ is the divisor corresponding to H. The two arrows appearing in Kleiman-Bertini Theorem are then $H \hookrightarrow \mathbf{P}^r \leftarrow Z$ where H is a fixed hyperplane. Because the action of $\mathrm{PGL}(r,k)$ is transitive on hyperplanes, we can conclude that for general H the element D_H is smooth. □

Remark 2.3. Every result that has been deduced in characteristic zero up to now (look at all the items labelled (2)) is still true if one replaces 'regular' (or smooth) by: reduced, normal, Cohen-Macaulay.

One might ask: what is the hurdle that one has to clear to let Bertini's Theorem (Corollary 2) become true over a field of positive characteristic?

Answer: | Ramification. In fact, inseparability. |

Just to be clear:

Definition 2.5.1. A morphism $f : S \to T$ of k-varieties is said to be UNRAMIFIED if for all $s \in S$, and letting $t = f(s)$, one has that $\mathfrak{m}_s = \mathfrak{m}_t \mathscr{O}_s$ (via $\mathscr{O}_t \to \mathscr{O}_s$) and the extension $k(t) \subset k(s)$ is finite and separable.

Theorem 2.10 (Bertini for $k = \overline{k}$ and $\operatorname{char} k = p$). Let Z be a regular integral scheme and $\mathbf{P}^r$ a base-point-free linear system on Z. Then a general element of the system is regular if Z *separates infinitely near points*, that is, if the associated morphism $\psi : Z \to \mathbf{P}^r$ is unramified.

Proof. See [16], Corollary 12, p. 296. □

Remark 2.4. In the previous section, when we dealt with the linear system of cubics passing through $\mathbf{P}^2_{\mathbb{F}_2}$, we proved that the morphism $\mathbf{P}^2 - T \to \mathbf{P}^2$ is inseparable by looking at the generic point of the projective plane. So in particular it is not unramified. This fact agrees with the above theorem.

2.6 Abelian varieties are quotients of jacobians

In this section the reference is [1]. The motivation for this short discussion is just that we have a chance to use Bertini's Theorem in order to prove something nontrivial, and moreover the proof here presented is an example of how Bertini can be useful in "reduction steps". The aim is showing:

Theorem 2.11. Any abelian variety A over an infinite field k is a quotient of a jacobian variety.

An abelian variety is a geometrically integral and projective algebraic group. Note that if $\dim A = 1$ then A is an elliptic curve, thus it is isomorphic to its own jacobian and the above statement is satisfied.

Let C be a nonsingular projective curve of genus $g > 0$ and let J be its jacobian. We first describe the universal property of an important arrow mapping to the jacobian. We will need it during the proof of Theorem 2.11. So, the situation is the following.

The map $F : C(\bar{k}) \times C(\bar{k}) \to J(\bar{k})$ sending $(Q, P) \mapsto [Q] - [P]$ happens to be defined over k *even if* C *has no rational points*. To prove this, one has to show that for every Galois extension of k and for every σ in the Galois group, one has $\sigma F = F$. This k-morphism is universal in the following sense: for any map $\theta : C \times C \to A$ to an abelian k-variety A, satisfying $\theta(\Delta) = 0$, there is a unique homomorphism $\psi : J \to A$ such that $\theta = \psi \circ F$. We will deal with a smooth curve $C \subset A$ so our θ will be the map $(Q, P) \mapsto Q - P$.

Lemma 2.2. Let X be a nonsingular projective k-variety of dimension at least 2, and let us fix a closed embedding $X \hookrightarrow \mathbf{P}^n_k$. Let $Z \subset X$ be a hyperplane section relative to this embedding and V a nonsingular variety with a finite morphism $\pi : V \to X$. Then $\pi^{-1}(Z)$ is geometrically connected.

Proof. One may assume $k = \bar{k}$, because closed immersions, finite morphisms, hyperplane sections all remain what they are after base extension. So, one has to show that $\pi^{-1}(Z)$ is connected. As

Z is ample on X (by definition), $\pi^{-1}(Z)$ is ample on V (the pullback of an ample line bundle by a finite morphism between complete varieties is still ample). By Theorem 2.1, which we apply because $\dim V \geqslant \dim X \geqslant 2$ and because $\pi^{-1}(Z) \subset V$ is again ample, we can conclude. □

Proof. (of Theorem 2.11) We may assume $\dim A = d \geqslant 2$. As A is projective, we may fix a closed embedding $A \hookrightarrow \mathbf{P}^n_k$. Let l denote the algebraic closure of k. Bertini says that there exists a dense open subset $U_1 \subset \mathbf{P}^{n*}_l$ such that every hyperplane $H \in U_1$ gives a smooth connected (again by Therorem 2.1) intersection $H \cap A_l$. Because k is infinite, $U_1(k)$ is nonempty (it is even infinite). So one can choose a good hyperplane H_1 with coordinates in k, and this will give a nonsingular (geometrically connected) variety $A \cap H_1 \subset \mathbf{P}^n_k$, of dimension $d-1$. To this new subvariety of $\mathbf{P}^n_k$ (and to its induced embedding) one applies the same procedure, in order to find a new open subset $U_2 \subset \mathbf{P}^{n*}_l$ and a new hyperplane H_2 defined over k, such that $A \cap H_1 \cap H_2 \subset \mathbf{P}^n_k$ is nonsingular, of dimension $d-2$. In total, if one does so $d-1$ times, one ends with a nonsingular curve $C \subset A$ obtained by intersecting A with a linear subspace $\ell = H_1 \cap \cdots \cap H_{d-1} \subset \mathbf{P}^n_k$. Now, remembering Lemma 2.2, one discovers that for each nonsingular variety V with a finite morphism $\pi : V \to A$, one can say for sure that $\pi^{-1}(C)$ is geometrically connected. One has to use the above Lemma with smaller and smaller hyperplane sections: at the first step one sets $Z_1 = A \cap H_1$, then one looks at $\pi^{-1}(Z_1) \to Z_1$ and sets $Z_2 = Z_1 \cap H_2$, and so on, until $Z_{d-1} = C$.

Now, by the universal property of the map $F : C \times C \to J$, there is a morphism $\psi : J \to A$ where J is the jacobian of C (note that C *might not* have a rational point). Let us call A_1 the abelian subvariety $\psi(J) \subset A$. Let us assume, by contradiction, that $A_1 \neq A$. It is a theorem that A_1 has a complement, that is, there exists an abelian subvariety $A_2 \subset A$ such that $A_1 + A_2 = A$, the intersection $A_1 \cap A_2$ is finite and the map $\pi_1 : A_1 \times A_2 \to A$ sending $(P_1, P_2) \mapsto P_1 + P_2$ is an isogeny (a surjective morphism with finite kernel), so in particular it is finite. The inverse image $\pi_1^{-1}(A_1) = \{ (P, Q) \in A_1 \times A_2 \mid P + Q \in A_1 \}$ decomposes (note that $Q \in A_1$) as a disjoint union of the following $m := \sharp(A_1 \cap A_2)$ irreducible components:

$$\pi_1^{-1}(A_1) = \coprod_{Q \in A_1 \cap A_2} (A_1 - Q) \times \{ Q \}.$$

Thus, if $m > 1$, then because $C \subset A_1$ one has that $\pi_1^{-1}(C)$ is also disconnected, and this is a contradiction. But if $m = 1$ (that is, π_1

is an isomorphism) one cannot conclude yet. However, one can choose an integer s coprime to $\operatorname{char} k$, and consider the composition

$$\begin{array}{ccc} A_1 \times A_2 & \xrightarrow{1 \times s} & A_1 \times A_2 \\ & {\scriptstyle \pi} \searrow & \downarrow {\scriptstyle \pi_1} \\ & & A \end{array} \qquad \begin{array}{ccc} (P,Q) & \longrightarrow & (P,sQ) \\ & \searrow & \downarrow \\ & & P+sQ \end{array}$$

In this case $\pi^{-1}(A_1) = \{ (P,Q) \in A_1 \times A_2 \mid P + sQ \in A_1 \}$ says that each such Q satisfies $sQ \in A_1$. Now, because s is coprime to $\operatorname{char} k$, the degree of $1 \times s$ is the power $s^{2 \dim A_2}$, and this is strictly bigger then 1. So, by the decomposition

$$\pi^{-1}(A_1) = \coprod_{Q \in A_2 \mid sQ \in A_1} (A_1 - sQ) \times \{ Q \}$$

one sees that there are $\deg(1 \times s) > 1$ components. Hence, as above, because $C \subset A_1$ one can conclude that neither $\pi^{-1}(C)$ is connected, contradiction. □

Remark 2.5. The main result of this section remains true over finite fields, and the strategy of proof is exactly the same, provided that one applies Bertini's Theorem for finite fields, replacing hyperplanes with hypersurfaces of higher degree (this is done in [18], and explained roughly in the Introduction.)

2.7 Arithmetic Bertini Theorem

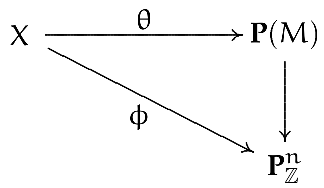
The main object of study in this section will be arithmetic varieties over the ring of integers of a number field. One will start with some notations and definitions, in order to make some simple but useful remarks.

Definition 2.7.1. An arithmetic variety of dimension m is an integral scheme X, projective and flat over $\mathbb{Z}$, such that the generic fiber $X_{\mathbb{Q}}$ is regular of dimension $m-1$. If K is a number field and $B = \operatorname{Spec} \mathcal{O}_K$, an arithmetic variety over $\mathcal{O}_K$ is a B-scheme X that is an arithmetic variety and whose generic fiber X_K is geometrically irreducible over K.

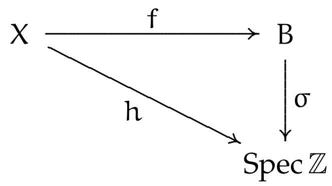
Proposition 2.7.1. Let Y be a Dedekind scheme and $f : X \to Y$ a morphism, with X a reduced scheme. Then f is flat if and only if every irreducible component of X dominates Y (i.e. its image is dense in Y).

Proof. See [17], Proposition 3.9, p. 137. □

REMARK 2.6. Let us prove that an arithmetic variety X over $\mathcal{O}_K$ is both projective and flat over $B = \operatorname{Spec} \mathcal{O}_K$. Because X is projective over $\mathbb{Z}$, there is a closed immersion $\phi : X \to \mathbf{P}^n_{\mathbb{Z}}$ for some n. The module $M = H^0(X, \phi^*\mathscr{O}(1))$ is finitely generated over $\mathbb{Z}$ (Theorem 1.5) so one has a surjection $\mathbb{Z}^{n+1} \twoheadrightarrow M$ sending $(a_0, \dots, a_n)$ to the sum $\sum_j \phi^*(s_j)a_j$ where $s_j \in H^0(\mathbf{P}^n_{\mathbb{Z}}, \mathscr{O}(1))$ are the global sections determining ϕ. Thus one has a commutative triangle



and since ϕ is a closed immersion, so must be θ. Note that M is locally free over $\mathcal{O}_K$ because it is projective.
For flatness one proceeds as follows. If $f : X \to B$ is the structural morphism, there is a commutative triangle



where σ is separated and h is proper (because it is projective), so f is proper. Hence $f(X)$ is either the whole B or it is one point. But if it is one point, then the same is true for $h(X)$, by commutativity of the triangle, and this is impossible as h is flat. So now apply Proposition 2.7.1 to the morphism f, to see that it must be flat. Indeed, $f(X) = B$ says that the unique irreducible component of X dominates B. Remark, finally, that an arithmetic $\mathcal{O}_K$-variety is necessarily *horizontal*, that is, it is flat and surjects onto the base.

⋆ ⋆ ⋆

The *Arithmetic Bertini Theorem* proved by Autissier in [2] is too advanced for our comprehension (it requires the theory of heights). So our final goal is to show the "algebraic part", which is however an important tool in the proof of Autissier main result. In the classical Bertini Theorem one calls *good* a hyperplane H giving a smooth intersection $X \cap H$. Recall that such a hyperplane section can be viewed as an element of the universal family $\Sigma_X \subset \mathbf{P}^{n*}_k \times X$, namely, as the set of those couples (H, p) where p

ranges $X \cap H$. Next, in this arithmetic version, one will call *good* those hyperplanes with two properties instead of just one (we describe them below): this will make our good locus relatively small.

– MAIN CHARACTERS. We fix a number field K and denote $B = \operatorname{Spec} \mathcal{O}_K$. This will be the base scheme, replacing the base field of classical Bertini Theorem. We also fix a locally free $\mathcal{O}_K$-module M of rank $n+1$ (this replaces the vector space k^{n+1}). By $\mathbf{P}$ we mean the projective bundle $\mathbf{P}(M) = \operatorname{Proj}(\operatorname{Sym} M)$ and by $\mathbf{P}^*$ its dual, both projective spaces of dimension n, not over a field but over $\mathcal{O}_K$ (and, as usual, points in $\mathbf{P}$ have to be thought of as hyperplanes in $\mathbf{P}^*$). Let $\beta : \mathbf{P} \to B$ be the structural morphism and $\mathcal{H} \subset \mathbf{P} \times_B \mathbf{P}^*$ the universal hyperplane. Our best friend will be an arithmetic $\mathcal{O}_K$-variety $X \subset \mathbf{P}^*$ of dimension $d \geqslant 3$, with structural morphism $f : X \to B$. By X_b we will always mean the fiber $f^{-1}(b)$. Note that these fibers are all of dimension $d-1$ (a flat morphism of irreducible algebraic varieties has fibers of constant dimension. See [Liu, Remark 3.15, p. 139] for the details); moreover, X_b is nonsingular over $k(b)$ for all but finitely many $b \in B$: indeed these bad points form a closed subset (of a scheme of dimension one), essentially by the openness condition in Theorem 1.9, and because saying that f is smooth at $x \in X$ amounts to asserting that f is flat at x and x is a smooth point in the fiber $X_{f(x)}$, over the field $k(f(x))$. Finally, the bad $b \in B$ are collected in a finite locus J, that is:

$$J = \{\, b \in B \mid X_b \text{ is singular over } k(b) \,\} \subset B.$$

Here is a short preview of the future discussion: as usual, we are interested in answering questions like:

- is there a smooth hyperplane section $X \cap H$?
- how many hyperplane sections are smooth?

– PREVIEW. For the first question: it turns out (Proposition 2.7.2) that what one can do *without extending* the base B is to find a closed horizontal subset $Z \subset \mathbf{P}$ containing the bad locus (where we still have to specify what will be good/bad for us); in other words, we will be able to lock the bad hyperplanes inside a quite small (= closed) subset Z. This is already very nice, but *it would count for nothing* if there were no B-valued points inside the good locus (in that case, no good hyperplane would be defined over B): this of course can happen, because we do not work over a field, so an open subset of a projective space might have no rational points. Instead, what one can do by (suitably) extending the base is to find a (arithmetic) hyperplane section $X' \subset X_{\mathcal{O}_L}$ over some $B' = \operatorname{Spec} \mathcal{O}_L$ such that the only singular fibers $X'_{b'}$ of $f' : X' \to B'$ correspond to those b' above points in J (thus there are finitely many of them).

For the second question: the answer is that *one* good hyperplane section is guaranteed if one fixes a suitable extension. If one wants a (infinite) family of good hyperplane sections one has to perform subsequent extensions $K \subset L_1 \subset L_2 \subset \dots$

Let us start with the details. First, let us "visualize" the hyperplanes inside $\mathbf{P}^*$. These are exactly the fibers $\mathcal{H}_y = \mathcal{H} \times_{\mathbf{P}} \operatorname{Spec} k(y) \subset \mathbf{P}^*$ of $\mathcal{H} \to \mathbf{P}$ at all points $y \in \mathbf{P}$.

Another important object is the universal hyperplane section $\mathcal{H}' = \mathcal{H} \times_{\mathbf{P}^*} X = \mathcal{H} \cap (\mathbf{P} \times_B X) \subset \mathcal{H}$ (a universal family over $\mathbf{P}$ whose members are the sections $X \cap \mathcal{H}_y$). If one calls π the composition $\mathcal{H}' \hookrightarrow \mathcal{H} \to \mathbf{P}$ then π is the morphism defining the family $\mathcal{H}'$. More precisely, if $y \in \mathbf{P}$ is any point, then $\mathcal{H}'_y = \pi^{-1}(y)$ coincides with the hyperplane section $X \cap \mathcal{H}_y$.

In the sequel, one will work inside the open subset

$$U_0 = \{ y \in \mathbf{P} \mid \pi \text{ is flat on } \mathcal{H}'_y \} \subset \mathbf{P}.$$

– Interlude. Before going on, it is perhaps useful to set up a comparison with the usual Bertini Theorem, just to check the analogies.

	Bertini over k	*Bertini over* $\mathcal{O}_K$
Main character:	$X \subset \mathbf{P}^n_k$ projective	$X \subset \mathbf{P}^*$ arithmetic
Hyperplanes:	$\mathbf{P}^{n*}_k$ $(H \subset \mathbf{P}^n_k)$	$\mathbf{P}$ $(\mathcal{H}_y \subset \mathbf{P}^*)$
Univ. Hyp. Section:	$\mathcal{H} \cap (\mathbf{P}^{n*}_k \times_k X)$	$\mathcal{H} \cap (\mathbf{P} \times_B X) = \mathcal{H}'$
Hyperplane Sections:	$X \cap H \subset \mathbf{P}^n_k$	$X \cap \mathcal{H}_y = \mathcal{H}'_y \subset \mathbf{P}^*$

As it was announced, there is more than a single condition to define a good $y \in \mathbf{P}$. Let us see this in detail, by looking at the following arrows:

$$\begin{array}{ccccccc} U_0 & \xrightarrow{\beta} & B & \qquad & y & \overset{\beta}{\longmapsto} & b \\ {\scriptstyle\pi}\uparrow & & \uparrow{\scriptstyle f} & & | & & | \\ \mathcal{H}' & & X & & \mathcal{H}'_y & & X_b \end{array}$$

One calls *good* those $y \in U_0$ such that: either

- one has $\beta(y) \in B - J$, and $\mathcal{H}'_y$ is smooth and geometrically irreducible over $k(y)$, or

•• when $y \in \beta^{-1}(b)$ for $b \in J$, then in this case $n(X_b) = n(\mathcal{H}'_y)$.

By resuming conditions • and ••, one recovers a good locus

$$W' = W \cup \bigcup_{b \in J} W_b \subset U_0 \subset \mathbf{P}$$

where $W_b = \{\, y \in U_0 \cap \beta^{-1}(b) \mid n(X_b) = n(\mathcal{H}'_y) \,\}$.

⋆ ⋆ ⋆

– *Notation:* If B is any scheme and $f : X \to Y$ is a B-morphism, one will denote by f_b the morphism $f \times 1 : X_b \to Y_b$.
A quasi-coherent sheaf $\mathscr{F}$ on X is said to be f-flat at $x \in X$ in case $\mathscr{F}_x$ is flat over $\mathscr{O}_{Y,f(x)}$ (via $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$). Let us recall the following result, whose proof can be found in [EGA IV_3, p. 138].

Theorem 2.12 (Fibral Flatness Criterion). Let B, X, Y be locally noetherian schemes, $\mathscr{F}$ a quasi-coherent sheaf on X, $f : X \to Y$ a B-morphism, and $x \in X$. Assume that

$$\begin{array}{ccccccc} Y & \xrightarrow{h} & B & \qquad & y & \longmapsto & b \\ {\scriptstyle f}\uparrow & \nearrow_{g} & & & \uparrow & \nearrow & \\ X & & & & x & & \end{array}$$

is commutative. Then, if $\mathscr{F}$ is coherent and $\mathscr{F}_x \neq 0$, the following are equivalent:

(1) The sheaf $\mathscr{F}$ is g-flat at x and $\mathscr{F}_b$ is f_b-flat at x.

(2) The morphism h is flat at y and $\mathscr{F}$ is f-flat at x.

Coming back to our situation, there is a commutative square

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\ \beta\ } & B \\ {\scriptstyle\pi}\uparrow & & \uparrow{\scriptstyle f} \\ \mathcal{H}' & \xrightarrow{\ \sigma\ } & X \end{array}$$

and if one denotes $\mathbf{P}_b = \beta^{-1}(b) = \{\eta_b\}^-$ and $\tau = f \circ \sigma$, one can use the criterion to show that for every $b \in B$ the generic point η_b of $\mathbf{P}_b$ lies in U_0: that is, π is flat at every $x \in \pi^{-1}(\eta_b)$. It is enough to consider

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\ \beta\ } & B \\ {\scriptstyle\pi}\uparrow & \nearrow{\scriptstyle\tau} & \\ \mathcal{H}' & & \end{array} \qquad\qquad \begin{array}{ccc} \eta_b & \longmapsto & b \\ \uparrow & \nearrow & \\ x & & \end{array}$$

for $b \in B$ fixed and $x \in \mathcal{H}'_b$ such that $\pi_b(x) = \eta_b$. So, taking the coherent sheaf $\mathscr{F} = \mathscr{O}_{\mathcal{H}'}$, the statement says that the following are equivalent:

(1) The morphism $\mathscr{O}_{B,b} \to \mathscr{O}_{\mathcal{H}'_b,x}$ is flat, and $\pi_b : \mathcal{H}'_b \to \mathbf{P}_b$ is flat at x.

(2) The morphism $\mathscr{O}_{B,b} \to \mathscr{O}_{\mathbf{P}_b,\eta_b}$ is flat and π is flat at x.

Let us verify that (1) is true: $\mathcal{H}' \to B$ is flat because it is a composition of flat morphisms (π is flat because f is and β is flat because σ is; and σ is flat because $\mathcal{H}'$ is a universal family). Finally, $\mathscr{O}_{\mathbf{P}_b,\eta_b} \to \mathscr{O}_x$ is flat as $\mathscr{O}_{\mathbf{P}_b,\eta_b}$ is a field. Hence (2) is true, in particular π is flat at all points $x \in \mathcal{H}'_b$ such that $\pi_b(x) = \eta_b$, which means exactly that $\eta_b \in U_0$. And this holds for all $b \in B$.

Proposition 2.7.2 (Autissier)**.** The locus $\mathbf{P} - W'$ is contained in a closed horizontal subset $Z \subset \mathbf{P}$, purely of codimension one. Hence the good locus W' contains the nonempty open subset $\mathbf{P} - Z$.

Proof. One has to show that $\eta_b \in W'$ for all $b \in B$. If this is done, one can conclude because W' is constructible (union of constructible subsets of $\mathbf{P}$) and a constructible set which is a neighborhood of all the generic points contains an open subset.
Start with $b \in B - J$. Let us note that $f : X \to B$ is birational projective and B is normal so, according to Zariski Main Theorem (cfr. [Hartshorne, p. 280]) the fibers X_b of f are geometrically connected. Those coming from $b \in B - J$ are also smooth over $k(b)$,

so they are geometrically irreducible as $k(b)$-varieties. Hence, by classical Bertini's Theorem, the generic hyperplane section $\mathcal{H}'_{\eta_b}$ is smooth and geometrically irreducible over $k(\eta_b)$ (recall that $\dim X_b \geqslant 2$) and this says that $\eta_b \in W$.
If $b \in J$ then $n(\mathcal{H}'_{\eta_b}) = n(X_b)$, so $\eta_b \in W_b$. □

REMARK 2.7. Note that one can discard the second condition defining W', in the following sense: instead of considering the loci W_b one might take the whole fiber $\mathbf{P}_b$ for those $y \in U_0$ such that $\beta(y) = b \in J$. This is a situation more close to the usual Bertini Theorem but the good locus become quite larger, so it is morally not so hard to find good hyperplanes. Instead, by considering W' as above, one gets more information.

THEOREM 2.13 (Autissier). There exists a finite extension L/K and a closed subscheme $X' \subset X_{\mathcal{O}_L}$ such that, by letting $B' = \operatorname{Spec} \mathcal{O}_L$ and $g : B' \to B$, the morphism induced by the inclusion $\mathcal{O}_K \subset \mathcal{O}_L$, one has:

- the scheme X' is an arithmetic variety over $\mathcal{O}_L$, of dimension $d - 1$;
- for every $b' \in B'$ such that $g(b') \in B - J$, the fiber $X'_{b'}$ (of the structural morphism $f' : X' \to B'$) is smooth and geometrically irreducible over $k(b')$;
- for every $b' \in B'$ such that $g(b') \in J$, $n(X'_{b'}) = n(X_{g(b')})$.

Morally, Autissier's theorem says that in the hyperplane section X' the only bad points come from (= live above) points of X which were already bad at the beginning.

APPENDIX A

Appendix

This chapter is devoted to fill the (main) gaps here and there left throughout the previous work. Giving complete proofs or definitions at the right moment would have diverted attention from the discussion.

Proof of Theorem 1.4

Theorem — *Let* X *be a variety and* r *a positive integer. Vector bundles of rank* r *over* X *correspond, up to isomorphism, to locally free* $\mathscr{O}_X$*-modules of rank* r.

Proof. Let us use the identification $\mathbb{C}^r = \mathbf{A}^r$. Start with a complex vector bundle $\pi : E \to X$ of rank r. Consider the sheaf $\Gamma(-, E)$ of sections of E, defined as follows: for every open subset $U \subset X$

$$\Gamma(U, E) = \{ U \xrightarrow{s} E \mid \pi \circ s = 1_U \}.$$

This amounts to saying that every point $x \in U$ is sent to a point of the $\mathbb{C}$-vector space E_x. Note that $\Gamma(U, E)$ is an $\mathscr{O}_X(U)$-module: first, it is an abelian group since we can add two sections s, t over U by declaring that $(s+t)(x)$ be the element $s(x)+t(x)$, sum of two vectors in E_x. We can also multiply by a regular function $f : U \to \mathbf{A}^1$, simply by evaluating: $(fs)(x) = f(x)s(x) \in E_x$. It is easy to check the compatibility of restrictions. Now we show that $\Gamma(-, E)$ is locally free. Recall that, together with (E, π), we are given an open covering (U_i) of X on which E is trivial, i.e. we have a family of trivializations $E|_{U_i} \xrightarrow{\sim} U_i \times \mathbf{A}^r$. If $\mathbf{A}^r = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_r]$, we can consider the local coordinate sections

$$\begin{aligned} x_i : U_i &\longrightarrow U_i \times \mathbf{A}^r \\ p &\longmapsto (p, e_i) \end{aligned}$$

where e_i is the vector with all zeros except for a 1 in the i-th component. This is a section: when composed with the projection, we get the identity on U_i. Moreover, any section $s \in \Gamma(U_i, E)$ can be written as $f_1x_1 + \cdots + f_rx_r$ for some regular functions f_i, so we can define a morphism of $\mathscr{O}_X(U_i)$-modules

$$\begin{aligned} \Gamma(U_i, E) &\longrightarrow \mathscr{O}_X(U_i)^r \\ s &\longmapsto (f_1, \ldots, f_r). \end{aligned}$$

This is the required isomorphism. Note that we have been able to check the locally free property on the open covering on which E becomes trivial, just because these special subsets U_i speak to us and say: "We trivialize E, so if you look at sections of the projection $U_i \times \mathbf{A}^r \to U_i$ then you are done". The way they say this is through the commutative diagrams

$$\begin{array}{ccc} E|_{U_i} & \xrightarrow{\sim} & U_i \times \mathbf{A}^r \\ & \searrow^{\pi} & \downarrow \mathrm{pr} \\ & & U_i \end{array}$$

So, our argument simply says: such a section of pr is identified by a n-tuple of regular functions.

Now start with a locally free $\mathscr{O}_X$-module $\mathscr{F}$ of rank r, and an open covering $\mathfrak{U} = (U_i)$ of X, so that $\mathscr{F}|_{U_i}$ is a free $\mathscr{O}_X|_{U_i}$-module of rank r for every i. Denote τ_i the isomorphism $\mathscr{F}|_{U_i} \cong \mathscr{O}_{U_i}^r$, and notice that $\mathfrak{U}$ can be chosen to be finite, since a variety is quasi-compact. Construct the disjoint union $F = \coprod(U_i \times \mathbf{A}^r)$, on which we will set an equivalence relation once we will have a suitable family of transition functions. For every i, j we have a couple of isomorphisms

$$\mathscr{F}|_{U_i} \xrightarrow{\tau_i} \mathscr{O}_{U_i}^r \qquad \mathscr{F}|_{U_j} \xrightarrow{\tau_j} \mathscr{O}_{U_j}^r$$

and if we restrict each of them to U_{ij} we get two new isomorphisms

$$\mathscr{F}|_{U_{ij}} \underset{f_j}{\overset{f_i}{\rightrightarrows}} \mathscr{O}_{U_{ij}}^r.$$

Hence, $g_{ij} = f_i f_j^{-1} \in \operatorname{Aut}\mathscr{F}|_{U_{ij}} \cong \operatorname{Aut}\mathscr{O}^r_{U_{ij}}$. That is, for every i, j we can identify each g_{ij} to an $r \times r$ matrix A_{ij} with entries in $\mathscr{O}_X(U_{ij})$. Call again g_{ij} the morphism of varieties $U_{ij} \to GL(r, \mathbb{C})$ sending a point $x \in U_{ij}$ to the matrix $A_{ij}(x)$. These g_{ij} are transition functions, so we can glue the $U_i \times \mathbb{C}^r$ together along their intersections, by identifying $(x, v) \in U_i \times \mathbb{C}^r$ to any $(y, w) \in U_j \times \mathbb{C}^r$ whenever $x = y$ and $w = g_{ij}(x)v$. The quotient of F that we get, together with the fibration $[(x, v)] \mapsto x$, is the required vector bundle on X. □

Čech Cohomology

Let X be a topological space and $\mathfrak{U} = (U_\alpha)_{\alpha \in A}$ an open covering. Put a well-ordering $\leqslant$ on A and define Σ_i, for $i \geqslant 0$, to be the set of simplexes $\underline{U} = [U_{\alpha_0}, \dots, U_{\alpha_i}]$ i.e. the set of those sets $\{ U_{\alpha_0}, \dots, U_{\alpha_i} \}$ such that $U_{\alpha_0 \dots \alpha_i} \neq \varnothing$. One can also interpret a simplex as a set of indices $\{ \alpha_0, \dots, \alpha_i \}$ such that $\alpha_0 < \dots < \alpha_i$. For any (pre)sheaf $\mathscr{F}$ on X, define the (additive, say) group of Čech i-cochains to be

$$C^i(\mathfrak{U}, \mathscr{F}) = \prod_{\alpha_0 < \dots < \alpha_i} \mathscr{F}(U_{\alpha_0 \dots \alpha_i}) = \prod_{\underline{U} \in \Sigma_i} \mathscr{F}(U_{\alpha_0 \dots \alpha_i})$$

In fact i-cochains are functions $s : \Sigma_i \to \coprod_{\alpha_0 < \dots < \alpha_i} \mathscr{F}(U_{\alpha_0 \dots \alpha_i})$. For any $i \geqslant 0$ there is a group homomorphism $\partial^i : C^i(\mathfrak{U}, \mathscr{F}) \to C^{i+1}(\mathfrak{U}, \mathscr{F})$ defined by sending $s = (s(\alpha_0, \dots, \alpha_i))_{\alpha_0 < \dots < \alpha_i}$ to the function $\partial^i s \in C^{i+1}(\mathfrak{U}, \mathscr{F})$ taking the $(i+1)$-simplex $\{ \alpha_0, \dots, \alpha_{i+1} \}$ to the element

$$\sum_{k=0}^{i+1} (-1)^k s(\alpha_0, \dots, \hat{\alpha}_k, \dots, \alpha_{i+1})|_{U_{\alpha_0 \dots \alpha_{i+1}}}$$

One can show that $\partial^{i+1} \circ \partial^i = 0$ so that $C^\bullet(\mathfrak{U}, \mathscr{F})$ is a complex. Let us make some examples in low dimension: if $i = 0$ then $C^0(\mathfrak{U}, \mathscr{F}) = \prod_\alpha \mathscr{F}(U_\alpha)$, so that a 0-cochain $s = (s(\alpha))$ is a function $s : A \to \coprod_\alpha \mathscr{F}(U_\alpha)$. Applying ∂^0 gives a 1-cochain $\partial^0 s$ sending the one-simplex $\{ \alpha, \beta \}$ to the element $s(\alpha)|_{U_{\alpha\beta}} - s(\beta)|_{U_{\alpha\beta}} \in \mathscr{F}(U_{\alpha\beta})$.

One verifies that the transition functions of a vector bundle are a Čech one-cochain in the complex $C^\bullet(\mathfrak{U}, \mathscr{O}_X^\times)$. First, $c = (g_{\alpha\beta}) : \Sigma_1 \to \coprod_{\alpha < \beta} \mathscr{O}_X^\times(U_{\alpha\beta})$ is an element of $C^1(\mathfrak{U}, \mathscr{O}_X^\times) = \prod_{\alpha < \beta} \mathscr{O}_X^\times(U_{\alpha\beta})$. It goes under ∂^1 to a function taking the two-simplex $\{ \alpha, \beta, \gamma \}$ to the element

$$c(\beta, \gamma) c(\alpha, \gamma)^{-1} c(\alpha, \beta)|_{U_{\alpha\beta\gamma}} = g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta}|_{U_{\alpha\beta\gamma}}$$

But the cocycle condition

$$g_{\alpha\gamma}(x) = g_{\alpha\beta}(x)g_{\beta\gamma}(x) \qquad \forall\, x \in U_{\alpha\beta\gamma}$$

says exactly that the c is a Čech one-cocycle.

* * *

Some relative constructions

Definition A.0.2. A morphism of schemes $f : X \to Y$ is said to be *affine* if there exists an open affine cover (Y_i) of Y such that $f^{-1}(Y_i)$ is affine for every i.

If f is finite type then it is affine (in fact, finite type is equivalent to proper and affine). An affine morphism is quasi-compact and separated.

§ I. The Global **Spec**.

We start with the following

> Initial data: A noetherian scheme X and a quasi-coherent sheaf of $\mathscr{O}_X$-algebras $\mathscr{A}$. That is: if $U \subset X$ is open, then $\mathscr{A}(U)$ is an $\mathscr{O}_X(U)$-algebra, and if U is affine then $\mathscr{A}|_U \cong \mathscr{A}(U)^\sim$.

In this situation, one proves that there exists a unique scheme over X, denoted $\mathbf{Spec}\,\mathscr{A}$, together with its structural morphism

$$\beta : \mathbf{Spec}\,\mathscr{A} \longrightarrow X,$$

satisfying the following properties: for every open *affine* subset $U \subset X$, one has $\beta^{-1}(U) \cong \operatorname{Spec}\mathscr{A}(U)$, and for every inclusion $V \subset U$ of open *affine* subsets of X, the inclusion $\beta^{-1}(V) \hookrightarrow \beta^{-1}(U)$ corresponds to the restriction $\mathscr{A}(U) \to \mathscr{A}(V)$.

– *Translation*: the object $\mathbf{Spec}\,\mathscr{A} \in \mathfrak{Sch}(X)$ is constructed ad hoc to satisfy this: if $U = \operatorname{Spec} B \subset X$ is an open subscheme, then $\mathscr{A}(\operatorname{Spec} B)$ is some B-algebra A_U, by our assumption on $\mathscr{A}$. Well, $\mathbf{Spec}\,\mathscr{A}$ is constructed so that above $\operatorname{Spec} B$ there lies exactly $\operatorname{Spec} A_U$.

$$\begin{array}{ccc}
\operatorname{Spec} A_U & \hookrightarrow & \mathbf{Spec}\,\mathscr{A} \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\
U = \operatorname{Spec} B & \hookrightarrow & X
\end{array}$$

Note the analogy with vector bundles: the philosophy is again to construct a space by glueing ”prescribed” fibers. The following result is a characterization of affine morphisms.

Proposition A.0.3. If $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_X$-algebra and β : $\mathbf{Spec}\,\mathscr{A} \to X$ is the structural morphism, then β is affine and $\mathscr{A} \cong \beta_*\mathscr{O}_{\mathbf{Spec}\,\mathscr{A}}$. Conversely, if $f : Y \to X$ is affine, then $\mathscr{A} := f_*\mathscr{O}_Y$ is a quasi-coherent $\mathscr{O}_X$-algebra, and $Y \cong \mathbf{Spec}\,\mathscr{A}$ as X-schemes.

By the above Proposition, $(\mathbf{Spec}\,\mathscr{A}, \beta)$ satisfies the following universal property:

> UP: Given a morphism $\pi : Y \to X$, the X-morphisms $Y \to \mathbf{Spec}\,\mathscr{A}$ are in Y-functorial bijection with morphisms α_π making the following diagram commute:

$$\begin{array}{ccc} \mathscr{O}_X & \longrightarrow & \pi_*\mathscr{O}_Y \\ \big\downarrow & \nearrow_{\alpha_\pi} & \\ \mathscr{A} & & \end{array}$$

– *Translation*: If one considers the functor $F : \mathfrak{Sch}(X) \longrightarrow \mathrm{Sh}(X)$ sending the X-scheme $\pi : Y \to X$ to $\pi_*\mathscr{O}_Y$, then one is just asserting that the object $\beta : \mathbf{Spec}\,\mathscr{A} \to X$ represents F, that is, $F \cong \hom_X(-, \mathbf{Spec}\,\mathscr{A})$. This natural isomorphism comes from the universal property that $(\mathbf{Spec}\,\mathscr{A}, \beta)$ does satisfy.

Remark A.1. If $X = \operatorname{Spec} B$ is affine and $\mathscr{A} = \widetilde{A}$ for some B-algebra A, then the canonical morphism $\operatorname{Spec} A \to X$ satisfies the universal property above.

Proposition A.0.4. Start with a couple $(X, \mathscr{A})$ with the usual assumptions. Assume there exists an object $(\mathbf{Spec}\,\mathscr{A}, \beta)$ satisfying UP. Then, for every open subset $U \subset X$, the morphism

$$\mathbf{Spec}\,\mathscr{A} \times_X U = (\mathbf{Spec}\,\mathscr{A})|_U \xrightarrow{\beta|_U} U$$

satisfies the UP with respect to $(U, \mathscr{A}|_U)$.

The latter Proposition, together with Remark A.1, shows that $(\mathbf{Spec}\,\mathscr{A}, \beta)$ exists for all open subschemes U of any affine scheme $X = \operatorname{Spec} B$. In fact, such an object exists in general.

Remark A.2. In terms of the UP described above, one sees that the natural isomorphism $\mathscr{A} \cong \beta_*\mathscr{O}_{\mathbf{Spec}\,\mathscr{A}}$ (let us call it ϕ) of Proposition A.0.3 comes from

$$\begin{array}{ccc} X & \xleftarrow{\ \beta\ } & \mathbf{Spec}\,\mathscr{A} \\ {\scriptstyle\beta}\uparrow & \swarrow{\scriptstyle \mathrm{id}} & \\ \mathbf{Spec}\,\mathscr{A} & & \end{array}$$

That is, the identity map on $\mathbf{Spec}\,\mathscr{A}$ over X corresponds to $\beta_*\mathscr{O}_{\mathbf{Spec}\,\mathscr{A}}$ under $\hom_X(-, \mathbf{Spec}\,\mathscr{A})$. It follows that in a situation like in the diagrams

$$\begin{array}{ccc} X & \xleftarrow{\ \pi\ } & Y \\ {\scriptstyle\beta}\uparrow & \swarrow{\scriptstyle\theta} & \downarrow{\scriptstyle\theta} \\ \mathbf{Spec}\,\mathscr{A} & \xleftarrow{\ \mathrm{id}\ } & \mathbf{Spec}\,\mathscr{A} \end{array} \qquad \begin{array}{ccc} \mathscr{O}_X & \longrightarrow & \pi_*\mathscr{O}_Y \\ \downarrow & \nearrow{\scriptstyle\alpha} & \uparrow \\ \mathscr{A} & \xrightarrow{\ \phi\ } & \beta_*\mathscr{O}_{\mathbf{Spec}\,\mathscr{A}} \end{array}$$

one recovers $\alpha = \alpha_\pi$ as the composition

$$\mathscr{A} \xrightarrow{\phi} \beta_*\mathscr{O}_{\mathbf{Spec}\,\mathscr{A}} \longrightarrow \beta_*\theta_*\mathscr{O}_Y = \pi_*\mathscr{O}_Y.$$

Good behavior under base change. Let $f : Z \to X$ be any morphism, and $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X$-algebra. There is a natural Z-isomorphism

$$Z \times_X \mathbf{Spec}\,\mathscr{A} \cong \mathbf{Spec}\, f^*\mathscr{A}.$$

This means that the canonical projection $Z \times_X \mathbf{Spec}\,\mathscr{A} \to Z$ is exactly the structural morphism $\mathbf{Spec}\, f^*\mathscr{A} \to Z$.

Example A.0.1 (*Total spaces and vector bundles*). Let $\mathscr{F}$ be a locally free sheaf on X, of finite rank r. To $\mathscr{F}$ we can associate the (graded) sheaf of $\mathscr{O}_X$-algebras

$$S(\mathscr{F}) = \mathrm{Sym}\ \mathscr{F}^*$$

which (is also locally free and) assigns to each open subset $U \subset X$ the $\mathscr{O}_X(U)$-algebra $\mathrm{Sym}\ \mathscr{F}(U)^*$. Then $\mathbf{Spec}\, S(\mathscr{F})$ is a vector bundle on X: for every $p \in X$ there exists an open neighborhood U of p such that $\beta^{-1}(U) = (\mathbf{Spec}\, S(\mathscr{F}))|_U \cong \mathbf{A}^r_U$. This vector bundle is called the geometric vector bundle associated to $\mathscr{F}$, and we denote it $\mathbf{V}(\mathscr{F})$. Every vector bundle on X arises in this way (see Theorem 1.4). Note that for every open subset U of X one has

$$\Gamma(U, \mathscr{O}_{\mathbf{Spec}\, S(\mathscr{F})}) = \Gamma(\mathbf{A}^r_U, \mathscr{O}_{\mathbf{A}^r_U}) = \mathscr{O}_X(U)[x_1, \ldots, x_r] \cong \operatorname{Sym} \mathscr{F}(U)^*.$$

If, more generally, $\mathscr{F}$ is any quasi-coherent $\mathscr{O}_X$-module, then so is $\operatorname{Sym} \mathscr{F}^*$ and thus the latter may be computed affine-locally (Proposition 1.1.2). In this case we call $\mathbf{Spec}\operatorname{Sym} \mathscr{F}^*$ the TOTAL SPACE of $\mathscr{F}$.

EXAMPLE A.0.2. In the above example, take $\mathscr{F} = \mathscr{O}_X^n$ (direct sum of n copies of $\mathscr{O}_X$). We define the trivial geometric vector bundle to be the X-scheme

$$\mathbf{V}(\mathscr{O}_X^n) = \mathbf{A}^n_X = \mathbf{Spec}\operatorname{Sym}(\mathscr{O}_X^n)^*.$$

It generalizes the affine n-space over a ring $\mathbf{A}^n_A = \operatorname{Spec} A[x_1, \ldots, x_n]$.[1] Suppose we have a morphism $f : X \to Y$ for some scheme Y. We can consider the $\mathscr{O}_Y$-algebra $S(\mathscr{O}_Y^n) = \operatorname{Sym}(\mathscr{O}_Y^n)^*$. Then, by what we said about the good behavior under base change, we have a natural isomorphism $\mathbf{A}^n_X \cong X \times_Y \mathbf{A}^n_Y$:

$$\begin{aligned} X \times_Y \mathbf{A}^n_Y &= X \times_Y \mathbf{Spec}\, S(\mathscr{O}_Y^n) \cong \mathbf{Spec}\, f^* S(\mathscr{O}_Y^n) \\ &= \mathbf{Spec}\, f^*(\operatorname{Sym}(\mathscr{O}_Y^n)^*) \cong \mathbf{Spec}\operatorname{Sym}(\mathscr{O}_X^n)^* = \mathbf{A}^n_X. \end{aligned}$$

We used that f^* commutes with Sym and that $f^*\mathscr{O}_Y = \mathscr{O}_X$.

§ II. THE GLOBAL **Proj**.

Exactly in the same way as **Spec** specializes to Spec when X is affine (see Remark A.1), we now perform a construction that will specialize to Proj of a graded ring when X is affine. More precisely, we will generalize the following fact: when we take the Proj of a graded ring $S = \bigoplus_{d \geqslant 0} S_d$, we get naturally a morphism $Y = \operatorname{Proj} S \to \operatorname{Spec} S_0$. We now replace $\operatorname{Spec} S_0$ by an arbitrary noetherian scheme X. To globalise our Proj we need also to replace the S_0-algeba S by a sheaf of graded algebras $\mathscr{S}$ on X. At the end, we would like a structural morphism $\mathbf{Proj}\, \mathscr{S} \to X$.

> INITIAL DATA: A noetherian scheme X and a quasi-coherent sheaf of graded $\mathscr{O}_X$-algebras $\mathscr{S} = \bigoplus_{d \geqslant 0} \mathscr{S}_d$, satisfying $\mathscr{S}_0 = \mathscr{O}_X$. Then, for any affine open subset $U = \operatorname{Spec} A \subset X$, we have $\mathscr{S}(U) = A \oplus \bigoplus_{d \geqslant 1} \mathscr{S}_d(U)$.

[1] When $X = \operatorname{Spec} A$, of course $\mathbf{A}^n_X = \mathbf{A}^n_A$, indeed $\mathbf{A}^n_X = \mathbf{Spec}\operatorname{Sym} \mathscr{O}_X^{n*} = \mathbf{Spec}\operatorname{Sym} \widetilde{A}^n = \mathbf{Spec}(\operatorname{Sym} A^n)^\sim = \operatorname{Spec} A[x_1, \ldots, x_n]$, the last equality holding by Remark A.1.

To construct $\mathbf{Proj}\,\mathscr{S} \to X$, let us fix an affine open subset $U = \operatorname{Spec} A$ of X. Let us look at the scheme $\operatorname{Proj}\mathscr{S}(U)$ and its structural morphism $\sigma_U : \operatorname{Proj}\mathscr{S}(U) \to U$. Now we use the quasi-coherence of $\mathscr{S}$: for every distinguished inclusion $U_f \subset U$, where $U_f = \operatorname{Spec} A_f$ for some $f \in A$, we have an isomorphism

$$\mathscr{S}(U) \otimes_A A_f = \mathscr{S}(U)_f \cong \mathscr{S}(U_f).$$

Hence, remembering that Proj commutes with tensor product,

$$\operatorname{Proj}\mathscr{S}(U_f) = \operatorname{Proj}(\mathscr{S}(U) \otimes_A A_f) = \operatorname{Proj}\mathscr{S}(U) \times_U U_f = \sigma_U^{-1}(U_f).$$

In addition, for every inclusion $U \subset V$ of open subsets of X, the restriction $\mathscr{S}(V) \to \mathscr{S}(U)$ coincides, in zero degree, with the restriction $\mathscr{O}_X(V) \to \mathscr{O}_X(U)$; this induces a commutative diagram

$$\begin{array}{ccc} \operatorname{Proj}\mathscr{S}(U) & \longrightarrow & \operatorname{Proj}\mathscr{S}(V) \\ \Big\downarrow \sigma_U & & \Big\downarrow \sigma_V \\ U & \hookrightarrow & V \end{array}$$

Finally, one can check that for every two open affine subsets U and V there is a natural isomorphism $\sigma_U^{-1}(U \cap V) \cong \sigma_V^{-1}(U \cap V)$. All these information allow to glue the schemes $\operatorname{Proj}\mathscr{S}(U) \to U$ together to get a global object

$$\sigma : \mathbf{Proj}\,\mathscr{S} \to X.$$

One can show that, by the construction, $\mathbf{Proj}\,\mathscr{S}$ comes equipped with an invertible sheaf $\mathscr{O}(1)$, which comes from glueing the various $\mathscr{O}(1)$ on $\operatorname{Proj}\mathscr{S}(U)$.

Remark A.3. One often requires that $\mathscr{S}_1$ be coherent and generate $\mathscr{S}$ as an $\mathscr{S}_0$-algebra. In that case, any homogeneous part $\mathscr{S}_d$ is also coherent. To see this, fix an open affine subset $U \subset X$. Then, for any $d \geqslant 1$, the typical element of $\mathscr{S}_d(U)$ is a product of d elements of $\mathscr{S}_1(U)$, so we see that $\mathscr{S}_d|_U$ is finitely generated over $\mathscr{S}_0 = \mathscr{O}_X$. It follows (by Remark 1.1, for example, and using that X is noetherian) that $\mathscr{S}_d$ is coherent. One can show that if $\mathscr{S}_1$ is coherent, the structural morphism $\mathbf{Proj}\,\mathscr{S} \to X$ is proper.

Proposition A.0.5. Let $g : Z \to X$ be a morphism of noetherian schemes and let $\mathscr{S}$ be a quasi-coherent sheaf of graded $\mathscr{O}_X$-algebras as in our assumptions. Then $\mathbf{Proj}\,\mathscr{S} \times_X Z \cong \mathbf{Proj}\, g^*\mathscr{S}$ as Z-schemes.

We have a characterization of projective morphisms similar to the one we gave for affine morphisms. It is just the following definition.

Definition A.0.3. A morphism of schemes $f : X \to Y$ is said to be *projective* (and we say that X is projective over Y) if there is a Y-isomorphism $X \cong \mathbf{Proj}\,\mathscr{S}$ for some quasi-coherent sheaf of graded $\mathscr{O}_Y$-algebras, which is generated in degree 1 over $\mathscr{S}_0$.

EXAMPLE A.0.3. For any scheme Y, the projective n-space over Y, defined to be $\mathbf{P}^n_Y = \mathbf{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} Y$, can be obtained as the global Proj of the sheaf $S(\mathscr{O}_Y^{n+1}) = \operatorname{Sym}(\mathscr{O}_Y^{n+1})$:

$$\mathbf{P}^n_Y = \mathbf{Proj}\, S(\mathscr{O}_Y^{n+1}).$$

EXAMPLE A.0.4 (*Projectivization and Projective Bundles*[2]). Let X be a noetherian scheme and $\mathscr{E}$ a coherent $\mathscr{O}_X$-module. Define

$$\mathbf{P}(\mathscr{E}) = \mathbf{Proj}\, S(\mathscr{E}^*) = \mathbf{Proj}\ \operatorname{Sym}\mathscr{E}.$$

It is an X-scheme called the PROJECTIVIZATION of $\mathscr{E}$.

EXAMPLE A.0.5. Any n-dimensional vector space V over a field k may be viewed as a vector bundle $V \to \operatorname{Spec} k = X$. Writing V and V^* for the corresponding constant sheaves, we have their projectivizations, which are related but completely different spaces (as V and V^* themselves, in fact). The first one, $\mathbf{P}(V)$, is of course a projective space of dimension n, and its k-rational points correspond to one-dimensional quotients (hyperplanes) of V; dually, the k-rational points of $\mathbf{P}(V^*)$ correspond to one-dimensional subspaces of V. Summary:

$$\mathbf{P}^{n*} = \mathbf{P}(V) = \mathbf{Proj}\, S(V^*) \text{ and } \mathbf{P}^n = \mathbf{P}(V^*) = \mathbf{Proj}\, S(V).$$

In general, if $\mathscr{E}$ is a locally free $\mathscr{O}_X$-module of rank $n+1$, then $\mathbf{P}(\mathscr{E})$ is a PROJECTIVE BUNDLE: above any affine open subset $U = \operatorname{Spec} A \subset X$ there lies (under the structural morphism) exactly the U-scheme $\mathbf{P}^n_U = \mathbf{P}^n_A$.

REMARK A.4. Any closed subscheme $Z \subset \mathbf{P}^n_X$ can be written as $Z = \mathbf{Proj}\,\mathscr{S}$, for some quasi-coherent sheaf $\mathscr{S}$ of graded $\mathscr{O}_X$-algebras. More generally, if $\mathscr{E}$ is coherent, any closed subscheme $Z \subset \mathbf{P}(\mathscr{E})$ is the **Proj** of something; conversely, any quasi-coherent sheaf $\mathscr{S}$ of graded $\mathscr{O}_X$-algebras, generated by a coherent $\mathscr{S}_1$, is an epimorphic image of $\operatorname{Sym}\mathscr{S}_1$, and such a surjection $\operatorname{Sym}\mathscr{S}_1 \twoheadrightarrow \mathscr{S}$ induces a closed immersion $X = \mathbf{Proj}\,\mathscr{S} \hookrightarrow \mathbf{P}(\mathscr{S}_1)$.

[2] See Example A.0.1.

Finally, a comparison between **Spec** and **Proj**: a projective bundle is a particular case of the projectivization construction, exactly like a vector bundle is a particular case of a total space (the only observation here is 'locally free of finite rank' ⇒ 'coherent').

THE FROBENIUS MORPHISM

Let k be any field of characteristic $p > 0$ and denote by $\phi : k \to k$ the Frobenius endomorphism sending $x \mapsto x^p$. For an algebraic variety X over k, consider the map $f : X \to X$ which is the identity on points and, at the level of structure sheaves, is defined as follows:

$$\begin{aligned} f^\sharp(U) : \mathscr{O}_X(U) &\longrightarrow \mathscr{O}_X(U) \\ a &\longmapsto a^p. \end{aligned}$$

Note that $F_X = (f, f^\sharp)$ is a morphism of ringed spaces but not a morphism of k-varieties, unless $k = \mathbb{F}_p$, because $f^\sharp$ might not preserve the structure of k-algebra. F_X is called the ABSOLUTE FROBENIUS of X; note that it commutes with any morphism $f : X \to Y$ of k-schemes, in the sense that $f \circ F_X = F_Y \circ f$. We now change the structure of k-algebra on k to make F_X into a k-morphism. To do so, call k' the field k whose structure of k-algebra is defined by a new multiplication by scalars, namely, for every $\lambda \in k$ and $x \in k'$, we set $\lambda \cdot x = \phi(\lambda)x = \lambda^p x$. For a finitely generated k-algebra A, define a k-algebra structure on

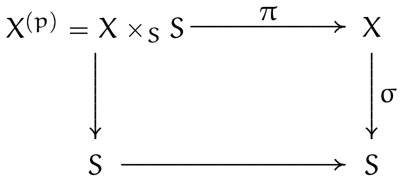
$$A^{(p)} = A \otimes_k k'$$

by letting $\lambda(a \otimes \mu) = a \otimes (\lambda\mu)$, and note that the map

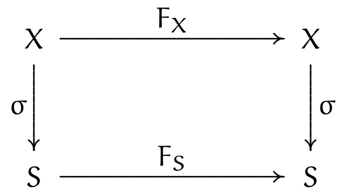
$$\begin{aligned} \rho : A^{(p)} &\longrightarrow A \\ a \otimes \mu &\longmapsto \mu a^p \end{aligned}$$

is a k-morphism, indeed, $\rho(\lambda(a \otimes \mu)) = \rho(a \otimes (\lambda\mu)) = \lambda\rho(a \otimes \mu)$. For $X = \operatorname{Spm} A$ and $X^{(p)} = \operatorname{Spm} A^{(p)}$, define $F_{X/k} : X \to X^{(p)}$ to be the k-morphism corresponding to ρ. It is called the RELATIVE FROBENIUS of X. For example, if $A = k[x_1, \dots, x_n]$, then $\rho : A^{(p)} = A \to A$ sends $x_i \otimes 1 \mapsto x_i^p$ and hence $\sum_\nu a_\nu x^\nu \mapsto \sum_\nu a_\nu x^{p\nu}$.

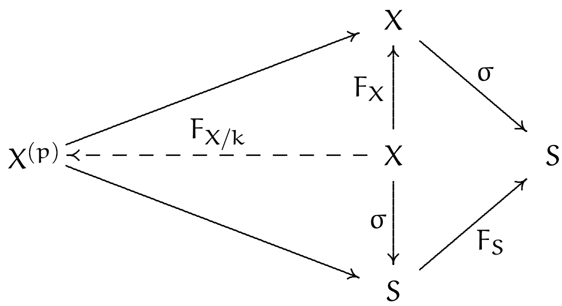
We now define the relative Frobenius of an arbitrary algebraic variety X. In what follows S denotes the base variety $\operatorname{Spm} k = \operatorname{Spm} k'$ and $\sigma : X \to S$ the structural morphism. Define $X^{(p)} = X_{k'}$. Defining the new structure of k-algebra on k' is the same (in the category of schemes over S) as viewing $k \to k'$ as a base change, relative to which we have a pullback diagram



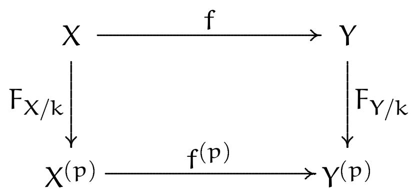
Moreover, coming back to the absolute Frobenius of X, we know that the following square commutes:



Putting all this together, we use the universal property of pullback in the diagram



to deduce the existence of a unique canonical morphism $F_{X/k} : X \to X^{(p)}$, making all commutative. We call it the relative Frobenius of X. If (X_i) is an open affine covering of X, then $F_{X/k}$ results from the glueing of the affine relative morphisms $F_{X_i/k} : X_i \to X_i^{(p)}$. Note that, because pullback is functorial, so is $X \mapsto X^{(p)}$. In fact, for any k-morphism $f : X \to Y$ there is a commutative diagram



Fix a field extension K/k. Then the functor $-^{(p)}$ commutes with this base extension, i.e. $(X^{(p)})_K = (X_K)^{(p)}$. Moreover $F_{X/k}$ induces a map on rational points $X(K) \to X^{(p)}(K)$. If X is $\mathbf{A}_k^n$ or $\mathbf{P}_k^n$ then this map is simply $(t_1,\dots,t_n) \mapsto (t_1^p,\dots,t_n^p)$, or $(t_0,\dots,t_n) \mapsto (t_0^p,\dots,t_n^p)$, defined on K-rational points. In particular, by reducing to the standard affine covering on $X = \mathbf{P}_k^n$, we see that $F_{X/k}$ is defined on $k[x_0,\dots,x_n]$ by $\sum_\nu a_\nu x^\nu \mapsto \sum_\nu a_\nu x^{p\nu}$.

Lemma A.1. Let $X = V(I) \subset \mathbf{A}_n^k$ be an algebraic variety over $k = \mathbb{F}_p$. Then $X^{(p)} = \operatorname{Spm} k[x_1,\dots,x_n]/I^{(p)}$ where $I^{(p)}$ is the ideal generated by polynomials $\sum_\nu a_\nu^p x^\nu$, where $\sum_\nu a_\nu x^\nu \in I$. The relative Frobenius $F_{X/k} : X \to X^{(p)}$ is induced by $x_i \mapsto x_i^p$.

Proof. See [17], Lemma 2.25, p. 95. □

Corollary 3. Let X be an algebraic variety over $k = \mathbb{F}_p$.

(a) We have $X = X^{(p)}$ and $F_X = F_{X/k}$ (absolute and relative Frobenius coincide).

(b) Let $\bar{k}$ be the algebraic closure of k. Let $\bar{F}_X : X(\bar{k}) \to X(\bar{k})$ be the map induced by composition with F_X. If $X = V(I)$ is a closed subvariety of $\mathbf{A}_k^n = \operatorname{Spm} k[x_1,\dots,x_n]$, then we have $\bar{F}_X(t_1,\dots,t_n) = (t_1^p,\dots,t_n^p)$.

Proof. See [17], Corollary 2.26, p. 95. □

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