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SCHEMES AND ALGEBRAIC GROUPS IN CHARACTERISTIC ONE

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Contents

Introduction

The idea of the field with one element was first suggested by Tits in relation with algebraic groups and finite geometries.
Define the *n-th q-number* n_q by $n_q = 1 + q + q^2 + ... + q^{n-1}$, clearly $n_1 = n$. Also define the q-factorial $!_q$ as $n!_q = 1_q \cdot 2_q ... \cdot n_q$, again when $q = 1$ this is just the traditional factorial. Finally define the q-binomial coefficients as

$$\left(\begin{array}{c} n \\ m \end{array} \right)_q = \frac{n!_q}{m!_q (n-m)!_q}. \tag{0.0.1}$$

Consider the Grassmannian variety $G_{n,m}(\mathbb{F}_q)$ with $q = p^k$, it is an easy exercise to compute its cardinality, resulting in

$$|G_{n,m}(\mathbb{F}_q)| = \left(\begin{array}{c} n \\ m \end{array} \right)_q \tag{0.0.2}$$

one immediately notices the simplicity of this formula, and one might wonder about its meaning when $q = 1$. Is there some sort of "variety" which corresponds to the case $q = 1$?
In the special case of projective space ($m = 1$) Tits noticed that this corresponds to a degenerate case of classical axiomatic projective geometry. Namely, if one substitutes the axiom that says that every line contains more than two points, by an axiom asserting that every line contains exactly two points, one gets a coherent degenerate projective geometry in which n-dimensional projective space contains exactly $n+1$ points. This should correspond to projective space over a 'field with one element' $\mathbb{F}_1$.
Moreover, in his work with algebraic groups, Tits suggests the following. Given a Chevalley group scheme $\mathcal{G}$, one considers its Weyl group W, then W should correspond to the group of $\mathbb{F}_1$-rational points of $\mathcal{G}$. With the growth of interest in the study of $\mathbb{F}_1$ geometry, this became a desired property of an adequate definition of variety over $\mathbb{F}_1$. Namely, $\mathbb{F}_1$ should be something lying below $\mathbb{Z}$, and for every Chevalley group scheme $\mathcal{G}$ there should be a group scheme $\underline{\mathcal{G}}$ over $\mathbb{F}_1$ such that, after extension of scalars to $\mathbb{Z}$ gives $\mathcal{G}$ and such that $\underline{\mathcal{G}}(\mathbb{F}_1) = W$.
A totally different story, and probably the reason why the interest in $\mathbb{F}_1$ grew so much in the last years, is that of the Riemann Hypothesis. As big as it sounds, I would guess that in the bottom of every $\mathbb{F}_1$ geometer lies the hope that, some day, a proof of the Riemann Hypothesis will come out from this theory. At least if it doesn't, new insights in its study might come out of this.
The story starts with André Weil's proof of the Riemann Hypothesis for curves over finite fields. Let X be a scheme of finite type over $\mathbb{Z}$, for every closed point x one can consider its residue field F_x. Then one can define the zeta function of X as

$$\zeta_X(s) = \prod_{x \in X,\ x \text{ closed}} \frac{1}{1 - |F_x|^{-s}}. \tag{0.0.3}$$

In the case that $X = \mathrm{Spec}\mathbb{Z}$ one recovers the Riemann zeta function. A generalized Riemann Hypothesis would be the assertion that all the zeroes of this zeta function have real part 1/2. In practice this is not so easy, one has to be more careful and take into account the dimension of the variety, but this gives an idea of the kind of things one is looking to prove.
Upon looking at the fibers over each closed point in $\mathrm{Spec}\mathbb{Z}$ it is easy to see that

$$\zeta_X(s) = \prod_p \zeta_{X_p}(s) \tag{0.0.4}$$

where ζ_{X_p} is the so called *local zeta function* of the fiber. This local zeta function has the exact same definition as in 0.0.3 but is defined for schemes of finite type over the finite field $\mathbb{F}_p$.
In view of equation 0.0.3 and the RH, one wonders what happens to the zeros of the local zeta functions, do they have real part equal to 1/2? In the case of complete smooth curves over $\mathbb{F}_q$ this is exactly the case, that is Weil's theorem or the acclaimed Riemann Hypothesis for curves over finite fields.
Based on his work in the case of dimension one Weil went even further and formulated a set of conjectures which came to be known as the Weil conjectures which were a generalization of the case of dimension one to higher dimensions. After many struggles by many different mathematicians they were finally proved for all dimensions by Dwork, Grothendieck and Deligne.
Recently there has been an interest in $\mathbb{F}_1$ geometry because Manin, based on work of Kurokawa and Deninger has proposed adapting the proof of Weil for curves over finite fields to "curves" over $\mathbb{F}_1$ and using that to prove the original Riemann Hypothesis!
Why is this plausible? First notice that the Krull dimension of $\mathbb{Z}$ is one so it can be thought of as some sort of curve. Second, as $\mathbb{Z}$ contains points of all the different characteristics the only possible characteristic for a field over which the hypothetical curve would be defined would be a "field of characteristic one".
This thesis consists of three chapters.
In the first chapter we give all the theory which is needed for chapters two and three. In the first section we discuss sites and sheaves over them (topoi). Then in the second section we provide the interpretation of schemes as functors on the category of rings, which will be later used to define $\mathbb{F}_1$ schemes. In the third section we give a background on Monoidal categories and some bicategory theory.
Chapter two is concerned with foundations for $\mathbb{F}_1$ geometry. We study a variation to Deitmar's geometry of monoids (which was given in [7]) leading to the definition we adopt of $\mathbb{F}_1$-schemes which was first given by Connes and Consani in [7]. In the next section we show a relationship between Toën and Vaquié's geometries with our $\mathbb{F}_1$-schemes. This part was mainly inspired by some personal communications with Andrew Salch.
Chapter three is more concrete in flavor, it is all about algebraic groups. In the first section we give a definition of GL_n in any cosmos in the framework of Toën and Vaquié. In the final section we give the proof due to Connes and Consani in [6] but in a slightly different language of the result asserting that any Chevalley group scheme can be descended to $\mathbb{F}_{1^2}$.

There have been given many other definitions to the notion of geometry over $\mathbb{F}_1$, notably Borger's Λ-ring geometries (which is also more general), Deitmar's monoidal spaces, Soulé's $\mathbb{F}_1$ varieties, and Lopez Peña and Lorscheid's torified varieties. The interested reader may consult the excellent survey paper [8].

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A Farid

Chapter 1

Preliminaries

1.1 Grothendieck Topologies

One of the reasons that the Weil conjectures where difficult to prove, at least in the way that Weil himself suggested, was that there was a need for a cohomology theory with certain 'good' properties. The Zariski topology for algebraic varieties provides a topological space that one can use to define cohomology, for example Serre's coherent sheaf cohomology. But for proving the Weil conjectures, it is too coarse. Using the Zariski topology to define coherent sheaf cohomology doesn't give a characteristic zero theory, which was one of the properties that, according to Weil, such a theory should have. This lack of a 'good' topological space to work with led Grothendieck to define sites.
Sites where invented by Grothendieck as a mean to mimic a topological space in cases where a sufficiently good topological space is not present. More explicitly, he invented them to be able to define good cohomology theories such as étale cohomology or crystalline cohomology, this cohomology theories where shown to have the 'good' properties so much wanted. This invention finally led, in Deligne's hands, to the full proof of the Weil conjectures.
Sites also help to give a nice characterization of schemes as functors. This is what we will mimic latter to define $\mathbb{F}_1$-schemes.
Let $\mathfrak{C}$ be any category, for a given object X we denote by h_X the functor that sends each Y in $\mathfrak{C}$ to the set $\mathrm{Hom}(Y, X)$ and each morphism f to the function induced by composition with f, $h_\bullet$ is itself a functor from $\mathfrak{C}$ to $\mathrm{Hom}(\mathfrak{C}^{op}, \mathfrak{S})$.
We recall Yoneda's Lemma:

Proposition 1.1 (Yoneda's Lemma) *For any functor $F : \mathfrak{C}^{op} \to \mathfrak{S}$ we have a bijective natural transformation between $h_F \circ h_\bullet$ and F.*

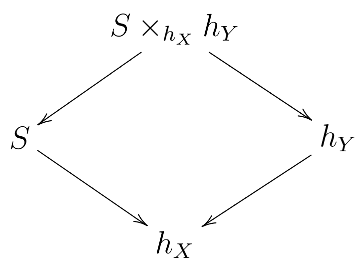
If we apply this to h_Y we get a bijection between $\mathrm{Hom}(h_X, h_Y)$ and $\mathrm{Hom}(X, Y)$, this means that $h_\bullet$ is fully faithful which is usually called the "weak version" of Yoneda's Lemma.
If a functor F is in the essential image of $h_\bullet$ then it is called *representable* and if $F \simeq h_X$ we say that X represents F
Next, we give the concept of a sieve.

Definition 1.1 *Let $\mathfrak{C}$ be a category and X an object of it, a* ***sieve*** *S over X is a subfunctor of h_X.*

Given a sieve S over an object X of $\mathfrak{C}$ and a morphism $\tau : Y \to X$ we get, by the Yoneda embedding, a natural transformation $h_\tau : h_Y \to h_X$. Using this, toghether with the inclusion $S \hookrightarrow h_X$, we can form the fibered product functor $S \times_{h_X} h_Y$.

Proposition 1.2 *The functor $S \times_{h_X} h_Y$ constructed above is a sieve over Y. It is called the **pullback** of S along τ and is denoted by $\tau^* S$.*

Proof. We just have to prove that it is a subfunctor of h_Y, or more precisely, that the projection to the second factor is injective. Take any object Z of $\mathfrak{C}$, by evaluating the diagram



on Z we get a corresponding diagram of sets and functions. Now it is clear that the upper right function is injective so it is a sieve. ■

Our next definition is central.

Definition 1.2 *A Grothendieck topology on $\mathfrak{C}$ is given by a colection of sieves called **covering sieves** satisfying the following axioms:*

- *For each covering sieve S over X and each morphism $Y \to X$ the pull back sieve is a covering sieve.*
- *For each X in $\mathfrak{C}$ the functor h_X is a covering sieve.*
- *If S is a sieve and T is a covering sieve over X such that for each morphism $Y \to X \in T(Y)$ the pull back of S is a covering sieve then S is a covering sieve.*

Definition 1.3 *Let $\mathfrak{C}$ be a category, then a Grothendieck pretopology on $\mathfrak{C}$ is a collection of sets of the form $\{U_i \xrightarrow{\sigma_i} U \in \mathrm{Hom}(U_i, U)\}_{i \in I}$ which are called **open coverings** such that:*

- *If $V \to U$ is an isomorphism then $\{V \to U\}$ is an open covering.*
- *If $\{U_i \xrightarrow{\sigma_i} U\}_{i \in I}$ is an open covering and $V \to U$ is any morphism then the fibered product $U_i \times_U V$ exists for all $i \in I$ and the induced morphisms $\{U_i \times_U V \to V\}_{i \in I}$ form an open covering.*
- *If $\{U_i \xrightarrow{\sigma_i} U\}_{i \in I}$ is an open covering and for each $i \in I$ we have an open covering $\{V_{ij} \xrightarrow{\rho_{ij}} U_i\}_{j \in J_i}$ then $\{V_{ij} \xrightarrow{\sigma_i \circ \rho_{ij}} U\}_{i \in I j \in J_i}$ is an open covering.*

The following is the fundamental and motivating example

Example 1.1 *If T is a topological space we can consider the category $\mathfrak{T}$ whose objects are the open subsets of T and the morphisms are just the inclusions. If we take for open coverings the usual ones (that is jointly surjective maps), this is a Grothendieck pretopology on $\mathfrak{T}$.*

This is clear because every isomorphism is just an identity, the fibered products are given by intersections and the third condition is clearly true.
The reason for the name 'pretopology' is that every pretopology gives rise to a topology in a natural way:

Proposition 1.3 *Given a pretopology $\mathcal{P}$. Let $\mathcal{T}$ denote the set of sieves S such that, there exists an open covering $\{U_i \xrightarrow{\sigma_i} U\}_{i\in I}$ belonging to $\mathcal{P}$ such that $\sigma_i \in S(U_i)$ for all $i \in I$. Then $\mathcal{T}$ is a Grothendieck topology.*

This $\mathcal{T}$ is called the *topology generated by* $\mathcal{P}$.
Conversely, given a Grothendieck topology $\mathcal{T}$, we call a set C of the form $\{U_i \xrightarrow{\sigma_i} U\}$ an open covering of $\mathcal{T}$ if the functor S, which sends X to the set of arrows $f : X \to U$ such that there exists an arrow α satisfying $f = \sigma_i \circ \alpha$ for some $\sigma_i \in C$, is a covering sieve.
A category with a Grothendieck topology on it is called a *site*.
Now we want to define sheaves over sites, the idea is to generalize the notion of sheaves over topological spaces to sheaves over sites. Let T be a topological space. It is a standard interpretation that of a presheaf over T as a contravariant functor from the associated $\mathfrak{T}$ from last example to $\mathfrak{S}$. So, analogously, we define a presheaf F on a site $\mathfrak{C}$ as a functor $F : \mathfrak{C}^{op} \to \mathfrak{S}$.
Given a presheaf F on $\mathfrak{C}$ then for any morphism $V \xrightarrow{\sigma} U$ and any element a of $F(U)$ we will denote the element $F\sigma(a)$ of $F(V)$ by $a|_V^\sigma$ and when there is no confusion about σ we will just write $a|_V$.
Now given any set of morphisms $\{U_i \xrightarrow{\sigma_i} U\}$ and a presheaf F on a site $\mathfrak{C}$ we get a set of functions $F\sigma_i$ from $F(U)$ to the various $F(U_i)$ so by universal property of products we obtain a function $\Sigma : F(U) \to \prod F(U_i)$. Also for a fixed k we have that, if π_1 denotes the projection to the first factor from $U_k \times_U U_j$ to U_k, we obtain, for all j, a function $F\pi_1$ from $F(U_k)$ to $F(U_k \times_U U_j)$; if we compose this with the projection from $\prod F(U_i)$ to $F(U_k)$ and let k run through all possible indices we get functions from $\prod F(U_i)$ to each of the $F(U_i \times_U U_j)$ so again by universal property we obtain a function, which we will call Π_1, from $\prod F(U_i)$ to $\prod F(U_i \times_U U_j)$. Analogously using the projection π_2 to the second factor of $U_i \times_U U_k$ we obtain a function called Π_2 from $\prod F(U_i)$ to $\prod F(U_i \times_U U_j)$. Summarizing we have constructed the diagram

$$F(U) \xrightarrow{\Sigma} \prod F(U_i) \underset{\Pi_2}{\overset{\Pi_1}{\rightrightarrows}} \prod F(U_i \times_U U_j) \tag{1.1.1}$$

for every presheaf F. Now we can define what is a sheaf

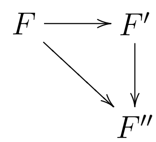
Definition 1.4 *A sheaf on a site $\mathfrak{C}$ is a presheaf $S : \mathfrak{C}^{op} \to \mathfrak{S}$ such that, for every open covering, the corresponding diagram*

$$S(U) \xrightarrow{\Sigma} \prod S(U_i) \underset{\Pi_2}{\overset{\Pi_1}{\rightrightarrows}} \prod S(U_i \times_U U_j) \tag{1.1.2}$$

is an equalizer.

When $\mathfrak{C}$ is $\mathfrak{T}$ from the example this definition goes down to the usual definition of sheaf on a topological space.
Given a presheaf F on a site $\mathfrak{C}$ we would like to make an analogous construction to the traditional construction of the sheafification of a presheaf.

Proposition 1.4 *For every presheaf F there is a unique (up to isomorphism) sheaf F' with a morphism $F \to F'$ such that for every other sheaf F'' with morphism $F \to F''$ there is a unique morphism $F' \to F''$ making the following diagram commute*



This F' is called the sheafification of F. Another way to see this is that the functor Sh from presheaves to sheaves which sends a presheaf to its sheafification is a left adjoint to the forgetful functor. For a reference on this and related subjects check [3] Chapter II.

1.2 Schemes as functors

Throughout this section we will call usual schemes *geometric schemes.* The category of geometric schemes will be denoted by $\underline{\mathfrak{Sch}}$.
Schemes can be viewed as sheaves on an adequate site. Consider the category $\mathfrak{R}$ of commutative rings with unity, we will make its opposite category into a site.
Given a ring A and an element f of it, we will denote by A_f the localization of A with respect to f.

Definition 1.5 *The Zariski topology on $\mathfrak{R}^{op}$ is the Grothendieck topology generated by the coverings of the form $\{A_{f_i} \to A\}_{i\in I}$ which are induced by sets of ring homomorphisms $\{A_{f_i} \leftarrow A\}_{i\in I}$ where the set $\{f_i\}_{i\in I}$ generates A.*

The category $\mathfrak{R}^{op}$ together with the Zariski topology forms a site which we will denote by $\mathfrak{Aff}$.
For a ring R when we want to talk of it as an element of $\mathfrak{Aff}$ we will write it as Spec$R$.
We will need the concept of an open covering of functors.

Definition 1.6 *Let $\mathfrak{C}$ be a site and $m_i : F_i \to F$ be morphisms of presheaves, we say they form an* open covering *if, for every representable functor h_X with morphism $h_X \to F$ the fibered product $h_X \times_F F_i$ is representable for all i by some Y_i in $\mathfrak{C}$ and the morphisms $Y_i \to X$ which induce the projections to the first factor form an open covering in $\mathfrak{C}$*

Using this, we can now give the main definition of this section:

Definition 1.7 *A scheme is a sheaf on $\mathfrak{Aff}$ which can be covered by representable functors. The category of schemes is denoted by $\mathfrak{Sch}$.*

Proposition 1.5 *There is an equivalence of categories between $\mathfrak{Sch}$ and $\underline{\mathfrak{Sch}}$.*

Proof. Given a geometric scheme $\underline{S}$ we construct a functor $S : (\mathfrak{R}^{op})^{op} \to \mathfrak{S}$ given by $S(A) = \mathrm{Hom}(Spec(A), \underline{S})$. We have to show this is a sheaf, so let's take a covering $\{A_{f_i} \to A\}_{i \in I}$ and elements g_i of the $S(A_i)$ such that for all i, j the restrictions $g_i|_{A_{f_i} \otimes_A A_{f_j}}$ and $g_j|_{A_{f_i} \otimes_A A_{f_j}}$ coincide. By well known facts the A_{f_i} correspond to open subschemes D_{f_i} of $SpecA$ and those generate it's topology, so this comes down to gluing scheme morphisms. This proves that S is a sheaf. Now $\underline{S}$ can be covered by affine schemes and one check that the representable functors associated to their respective rings cover S.
Conversely one takes a scheme S. If we assume that it is representable by A then it is clear that S is isomorphic to the functor associated to $Spec(A)$ this gives a functor G from affine schemes to geometrical schemes. If it's not representable, then we consider the category of diagrams $S' \to S$ where S' is affine, then using the restriction of G to this category one can show that S is ismomorphic to the scheme associated to the direct limit of G. ∎

Corollary 1.1 *For every ring R, $h_{\mathrm{Spec}R}$ is a sheaf.*

The definition we will adopt in the next chapter for $\mathbb{F}_1$-schemes is based on this result, so we define $\mathbb{F}_1$ schemes as functors on certain categories (together with some natural transformation).

1.3 Monoidal categories

Toën and Vaquié's construction [9] is based on *monoidal categories*. In fact, what they start with is a closed, symmetric, complete and cocomplete monoidal category. We will call such a category a *cosmos*.

Definition 1.8 *A* ***monoidal category*** *is a category $\mathfrak{M}$ together with a functor*

$$\otimes : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$$

called tensor product*, an object $\mathbb{I}$ called* identity *and natural isomorphisms*

$$\alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

$$\rho : A \otimes \mathbb{I} \to A$$

and

$$\lambda : \mathbb{I} \otimes A \to A.$$

Satisfying compatibility conditions, namely the diagrams

$$\begin{array}{ccccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha} & A \otimes (B \otimes (C \otimes D)) \\ \downarrow{\scriptstyle \alpha \otimes D} & & & & \uparrow{\scriptstyle A \otimes \alpha} \\ (A \otimes (B \otimes C)) \otimes D & & \xrightarrow[\alpha]{\qquad\qquad} & & A \otimes ((B \otimes C) \otimes D) \end{array} \tag{1.3.1}$$

and

$$\begin{array}{ccc} A \otimes (\mathbb{I} \otimes B) & \xrightarrow{\alpha} & (A \otimes \mathbb{I}) \otimes B \\ & {\scriptstyle A \otimes \lambda} \searrow & \downarrow{\scriptstyle \rho \otimes B} \\ & & A \otimes B \end{array} \tag{1.3.2}$$

commute.

Definition 1.9 *A monoidal category* $\mathfrak{M}$ *is called* closed *if, for every object* A *in* $\mathfrak{M}$*, the functor* $A \otimes \bullet : \mathfrak{M} \to \mathfrak{M}$ *has a right adjoint.*

If such a right adjoint exists, it is called the *internal Hom* and we will denote it by $A^\bullet : \mathfrak{M} \to \mathfrak{M}$.

Definition 1.10 *A* symmetric monoidal category *is a monoidal category* $\mathfrak{M}$ *together with a natural isomorphism*

$$\sigma : A \otimes B \to B \otimes A$$

such that the diagrams

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\sigma} & B \otimes A \\ & \searrow^{\mathrm{Id}} & \downarrow \sigma \\ & & A \otimes B \end{array} \tag{1.3.3}$$

$$\begin{array}{ccccc} & A \otimes (B \otimes C) & \xrightarrow{A \otimes \sigma} & A \otimes (C \otimes B) & \\ \swarrow^{\alpha} & & & & \searrow^{\alpha} \\ (A \otimes B) \otimes C & & & & (A \otimes C) \otimes B \\ \searrow^{\sigma} & & & & \swarrow^{\sigma \otimes B} \\ & C \otimes (A \otimes B) & \xrightarrow{\alpha} & (C \otimes A) \otimes B & \end{array} \tag{1.3.4}$$

and

$$\begin{array}{ccc} A \otimes \mathbb{I} & \xrightarrow{\sigma} & \mathbb{I} \otimes A \\ & \searrow^{\rho} & \downarrow \lambda \\ & & A \end{array} \tag{1.3.5}$$

commute.

We will now study the morphisms between monoidal categories. At first sight, one is tempted to think of functors between monoidal categories as functors preserving the tensor product and the identity. It turns out that this kind of functor is too restrictive and does not appear very much in "nature". The following definition is something at first sight a little awkward but behaves better as we will see later.

Definition 1.11 *A* ***lax monoidal functor*** *is a functor* $F : \mathfrak{C}_1 \to \mathfrak{C}_2$ *between the monoidal categories* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$*, together with a natural transformation*

$$m_{A,B} : F(A) \otimes_2 F(B) \to F(A \otimes_1 B)$$

and a morphism

$$m : \mathbb{I}_2 \to F(\mathbb{I}_1).$$

If these are isomorphisms then F *is called a* ***strong monoidal functor****. If the arrows are reversed in the definition then it is called a* ***colax monoidal functor****.*

This is related to a standard issue in higher category theory. There a distinction is made between strict functors and lax functors. It's the same sort of distinction as the one between bicategories and 2-categories. This is even more evident when one sees a monoidal category as a bicategory with only one object (in the same way as monoids can be seen as categories with just one object).
Clearly if F is strong then it is both lax and colax. From now on we will denote both the natural transformation and the morphism of the definition with the same letter.
Monoidal categories are a natural environment to define *monoids*.

Definition 1.12 *Let $\mathfrak{C}$ be a monoidal category. A monoid in $\mathfrak{C}$ is an object M together with a morphism $\mu : M \otimes M \to M$ called multiplication and a morphism $e : \mathbb{I} \to M$ called identitiy such that the following diagrams commute*

$$\begin{array}{ccc} M \otimes M \otimes M & \xrightarrow{M\otimes\mu} & M \otimes M \\ {\scriptstyle \mu\otimes M}\downarrow & & \downarrow{\scriptstyle \mu} \\ M \otimes M & \xrightarrow[\mu]{} & M \end{array} \tag{1.3.6}$$

Chapter 2

$\mathbb{F}_1$-schemes

2.1 Geometry of monoids

Consider the category $\mathfrak{M}_0$ of monoids with zero, whose objects are commutative monoids containing an absorbent element ($0 \cdot a = 0$ for all a). Morphisms in $\mathfrak{M}_0$ are monoid homomorphisms which preserve the absorbent element.
For the sake of shortness, throughout this work, the word monoid will refer exclusively to an object of the category $\mathfrak{M}_0$.
We can give now a (possible) definition of the field with one element.

Example 2.1 *The field with one element is the monoid $\mathbb{F}_1 = \{0, 1\}$ with usual multiplication from the ring $\mathbb{Z}/2\mathbb{Z}$.*

Let M be an object of $\mathfrak{M}_0$, an ideal I of M is a subset of it such that $0 \in I$ and $IM = I$. A prime ideal is an ideal such that its complement is a non empty multiplicatively closed set. If S is a multiplicatively closed set we can form the localization $S^{-1}M$ whose elements are $\frac{a}{s}$ with $a \in M$ and $s \in S$, modulo the equivalence relation $\frac{a}{s} = \frac{b}{t}$ iff there is a $u \in S$ such that $uta = usb$. When S is the complement of a prime ideal $\mathfrak{p}$ we will denote it by $M_{\mathfrak{p}}$.
A field is a ring without nontrivial ideals, in analogy one could define the analogous to a field to be a monoid without nontrivial ideals. It is easy to see that this gives monoids of the form $A \cup \{0\}$ where A is an abelian group.

Example 2.2 *The monoid $\mathbb{F}_{1^n} = \mathbb{Z}/n\mathbb{Z} \cup \{0\}$ is called the* algebraic extension of degree n *of $\mathbb{F}_1$.*

Example 2.3 *Let M be a monoid, then we can form another monoid $M[X_1, ..., X_n]$ which consists of all monomials of the form*

$$aX_1^{e_1} X_2^{e_2} ... X_n^{e_n}$$

with $a \in M$ and nonnegative exponents with the obvious multiplication.

Every monoid M has a unique maximal ideal (or otherwise said, every monoid is local). This is easy to see as it consists of all non invertible elements of M. We denote this by $\mathfrak{m}_M$.

Definition 2.1 *A morphism $F : M_1 \to M_2$ is said to be a* local morphism *if $F(\mathfrak{m}_{M_1}) \subset \mathfrak{m}_{M_2}$.*

Given a monoid M we define $Spec(M)$ to be the set of its prime ideals. For each ideal I we consider the set $V(I)$ of prime ideals which contain it, and we define a topology on $Spec(M)$ which has those as its closed sets. Finally we take the sheaf of monoids which assigns to every open set U the monoid $\mathcal{O}(U)$ which is formed by the functions $f : U \to \prod_{\mathfrak{p}\in U} M_{\mathfrak{p}}$ such that $F(\mathfrak{p}) \in M_{\mathfrak{p}}$ and for each $\mathfrak{p} \in U$ there exists an open neighborhood V of $\mathfrak{p}$, an h not belonging to any $\mathfrak{q} \in V$ and a g such that $f(\mathfrak{q}) = g/h$ for every $\mathfrak{q} \in V$.

Definition 2.2 *A monoidal space is a topological space L together with a sheaf of monoids $\mathcal{O}_L$ on it. A morphism between the monoidal spaces L and K consists of a continuous function $f : L \to K$ and a sheaf morphism $\mathcal{O}_K \to f_*(\mathcal{O}_L)$ which induces a local morphism (in the sense of the previous definition) on the stalks.*

It is immediate that, for every monoid M, $Spec(M)$ is a monoidal space.

Definition 2.3 *An* affine $\mathfrak{M}_0$-scheme *is a monoidal space isomorphic to $SpecM$ for some M.*

Now we can define $\mathfrak{M}_0$-schemes

Definition 2.4 *An* $\mathfrak{M}_0$-scheme *is a monoidal space which can be covered by affine $\mathfrak{M}_0$-schemes. A morphism of $\mathfrak{M}_0$-schemes is a morphism of monoidal spaces.*

Analogously to the last section in the previous chapter an $\mathfrak{M}_0$-Scheme X gives a functor $\mathfrak{M}_0 \to \mathfrak{S}$ given by

$$X(M) = \mathrm{Hom}(\mathrm{Spec}M, X)$$

and is called the functor of points.

Clearly, $\mathbb{F}_1$ is an initial object of $\mathfrak{M}_0$, so, in analogy with the case of the category $\mathfrak{R}$, one can think of the objects of $\mathfrak{M}_0$ as "$\mathbb{F}_1$-algebras". Even after saying this, later we will give a slightly different definition of $\mathbb{F}_1$-algebra.

Following this line of thought, one is pushed to think of $\mathfrak{M}_0$-schemes as "$\mathbb{F}_1$-schemes". The trouble with this definition is that the relation with usual schemes is not directly obvious, and there is no reasonable or natural way to say, for example, when a $\mathbb{Z}$-scheme is defined over $\mathbb{F}_1$. This is very similar to Deitmar's seminal definition but he uses the category $\mathfrak{M}$ of monoids instead of monoids with zero. In his work he overcomes this issue by using the analogous of the functor

$$\beta : \mathfrak{M}_0 \to \mathfrak{R}$$

which is a left adjoint to the obvious forgetful functor $\beta* : \mathfrak{R} \to \mathfrak{M}_0$. It is defined by

$$\beta(M) = \mathbb{Z}[M]$$

where $\mathbb{Z}[M]$ is the quotient of the ring of polynomials in variables from M by the ideal generated by all polynomials of the form $ab - c$ where $ab = c$ in M, $1_M - 1$ and the polynomial 0_M (of degree one, not to be confused with the zero polynomial).

One way of keeping track of the link between $\mathbb{F}_1$-schemes and regular ones was proposed by Soulé, and subsequently refined by Connes and Consani. In this work we will adopt their definition as the definition of $\mathbb{F}_1$-schemes.

Definition 2.5 *An* $\mathbb{F}_1$-scheme *is a triple* $\mathcal{X} = (\underline{X}, X, e_X)$ *where* $\underline{X}$ *is the functor of points of an* $\mathfrak{M}_0$*-scheme,* X *is the functor of points of a scheme and* e_X *is a natural transformation between* $\underline{X} \circ \beta*$ *and* X *such that for every field* K*,* $e_X(K)$ *is bijective.*

2.2 Gluing categories

Consider a pair of categories $\mathfrak{C}_1$ and $\mathfrak{C}_2$ and a pair of adjoint functors F_1, F_2 between them. In such a way that we have a functorial bijection $ad : \mathrm{Hom}(F_1(X_1), X_2) \to \mathrm{Hom}(X_1, F_2(X_2))$ for all X_1, X_2 in $\mathfrak{C}_1$ and $\mathfrak{C}_2$ respectively.
The functoriality of this bijection means, more explicitly, that for any X_1, Y_1 in $\mathfrak{C}_1$ and X_2, Y_2 in $\mathfrak{C}_2$, we have the following equalities:

$$ad(f \circ g \circ F_1(h)) = F_2(f) \circ ad(g) \circ h \tag{2.2.1}$$

$$ad^{-1}(F_2(f) \circ g \circ h) = f \circ ad^{-1}(g) \circ F_1(h) \tag{2.2.2}$$

Definition 2.6 *The* ***gluing*** *of* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$*, denoted by* $\mathfrak{C}_1\mathfrak{C}_2$ *is the category whose objects are the disjoint union of the objects of* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$*, and whose morphisms consist of:*

- $\mathrm{Hom}(X, Y)_{\mathfrak{C}_1\mathfrak{C}_2} = \mathrm{Hom}(X, Y)_{\mathfrak{C}_i}$ *if* X *and* Y *both lie in* $\mathfrak{C}_i$ *(*$i = 1$ *or* 2*),*
- $\mathrm{Hom}(X, Y)_{\mathfrak{C}_1\mathfrak{C}_2} = \mathrm{Hom}(F_1(X), Y)_{\mathfrak{C}_2}$ *if* $X \in \mathfrak{C}_1$ *and* $Y \in \mathfrak{C}_2$*,*
- $\mathrm{Hom}(X, Y)_{\mathfrak{C}_1\mathfrak{C}_2} = \emptyset$ *if* $X \in \mathfrak{C}_2$ *and* $Y \in \mathfrak{C}_1$*.*

Furthermore, suppose $g : X \to Y$ *and* $f : Y \to Z$*, composition is defined as follows:*

- $f \circ g = f \circ_i g$ *if all of* X*,* Y *and* Z *lie in* $\mathfrak{C}_i$ *(*$i = 1$ *or* 2*).*
- $f \circ g = f \circ_2 g$ *if* X *belongs to* $\mathfrak{C}_1$ *and* Y *and* Z *to* $\mathfrak{C}_2$*.*
- $f \circ g = f \circ_2 F_1(g)$ *if* X *and* Y *belong to* $\mathfrak{C}_1$ *and* Z *to* $\mathfrak{C}_2$

Proposition 2.1 *Given categories* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *with adjoint functors* F_1 *and* F_2 *as before, and given an arbitrary category* $\mathfrak{C}$*, there is an isomorphism of categories between the category of functors* $\mathrm{Hom}(\mathfrak{C}_1\mathfrak{C}_2, \mathfrak{C})$ *and the category of triples* (A_1, A_2, α) *where* A_1 *and* A_2 *are functors from* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *to* $\mathfrak{C}$ *respectively and* α *is a natural transformation between* $A_1 \circ F_2$ *and* A_2*.*

Proof. First let's take a functor $A : \mathfrak{C}_1\mathfrak{C}_2 \to \mathfrak{C}$, then restricting it to C_1 and C_2 we obtain functors A_1 and A_2 as the ones we want. Now, let's take an object X of $\mathfrak{C}_2$. We define $\alpha_X : A_1F_2(X) \to A_2(X)$ as $Aad^{-1}(\mathrm{Id}_{F_2(X)})$ in this definition we are viewing $ad^{-1}(\mathrm{Id}_{F_2(X)})$ both as an element of $\mathrm{Hom}(F_1F_2(X), X)_{\mathfrak{C}_2}$ and as an element of $\mathrm{Hom}(F_2(X), X)_{\mathfrak{C}_1\mathfrak{C}_2}$.
Let's now show that α is a natural transformation. Take a morphism $f : X \to Y$ in $\mathfrak{C}_2$, then

$$A_2(f) \circ \alpha_X = A(f) \circ (Aad^{-1}(\mathrm{Id}_{F_2(X)})) = A(f \circ ad^{-1}(\mathrm{Id}_{F_2(X)})) \tag{2.2.3}$$

this, by 2.2.2 with $X_2 = X$, $Y_2 = Y$ and $X_1 = F_2(X)$, equals

$$\begin{aligned} Aad^{-1}(F_2(f)\circ \mathrm{Id}_{F_2(X)}) &= Aad^{-1}F_2(f) \\ &= Aad^{-1}(\mathrm{Id}_{F_2(Y)}\circ F_2(f)) \\ &= A(ad^{-1}(\mathrm{Id}_{F_2(f)})\circ F_1F_2(f)) \end{aligned}$$

again by 2.2.2. By definition of composition in $\mathfrak{C}_1\mathfrak{C}_2$ this equals the following:

$$\begin{aligned} A(ad^{-1}(\mathrm{Id}_{F_2(Y)})\circ F_2(f)) &= Aad^{-1}(\mathrm{Id}_{F_2(Y)})\circ AF_2(f) \\ &= \alpha_Y\circ A_1F_2(f). \end{aligned}$$

Conversely, given a triple (A_1, A_2, α), we define a functor $A:\mathfrak{C}_1\mathfrak{C}_2\to\mathfrak{C}$ as follows: for every object X we define

$$A(X)=\begin{cases} A_1(X) & X\in\mathfrak{C}_1 \\ A_2(X) & X\in\mathfrak{C}_2 \end{cases} \tag{2.2.4}$$

and for every morphism $f:X\to Y$

$$A(f)=\begin{cases} A_1(f) & X,Y\in\mathfrak{C}_1 \\ A_2(f) & X,Y\in\mathfrak{C}_2 \\ \alpha_Y\circ A_1ad(f) & X\in\mathfrak{C}_1\quad Y\in\mathfrak{C}_2 \end{cases} \tag{2.2.5}$$

in the third condition we are using the fact that f is both a morphism in $\mathfrak{C}_1\mathfrak{C}_2$ and in $\mathfrak{C}_2$.
Let's show A is a functor. It is clear that A takes identity morphisms to identity morphisms. Now, take $X\xrightarrow{g}Y\xrightarrow{f}Z$ in $\mathfrak{C}_1\mathfrak{C}_2$. If all of X, Y and Z lie in one of $\mathfrak{C}_1$ or $\mathfrak{C}_2$ there's nothing to prove. In the case that X, Y are in $\mathfrak{C}_1$ and Z in $\mathfrak{C}_2$ we have

$$A(f\circ g)=\alpha_Z\circ A_1ad(f\circ g)$$

by 2.2.5. Now, by definition of composition in $\mathfrak{C}_1\mathfrak{C}_2$ this equals

$$\begin{aligned} \alpha_Z\circ A_1ad(f\circ F_1(g)) &= \alpha_Z\circ A_1(ad(f)\circ g) \\ &= \alpha_Z\circ A_1ad(f)\circ A_1(g) \\ &= A(f)\circ A(g) \end{aligned}$$

where we used once more the adjointness. In the case that X is in $\mathfrak{C}_1$ and Y and Z are in $\mathfrak{C}_2$ we have

$$A(f\circ g)=\alpha_Z\circ A_1ad(f\circ g)$$

viewing g as a morphism from $F_1(X)$ to Y in $\mathfrak{C}_2$ and using 2.2.1 this equals

$$\alpha_Z\circ A_1(F_2(f)\circ ad(g))=\alpha_Z\circ A_1F_2(f)\circ A_1ad(g)$$

using that α is a natural transformation this equals

$$A_2(f)\circ\alpha_Y\circ A_1ad(g)=A(f)\circ A(g).$$

■

Now suppose that $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are both monoidal categories with tensor products $\otimes_1$ and $\otimes_2$ respectively and unit objects $\mathbb{I}_1$ and $\mathbb{I}_2$ respectively. We want to give conditions over F_1 and F_2 so that $\mathfrak{C}_1\mathfrak{C}_2$ becomes a monoidal category in a natural way.
Recall that a bicategory $\mathcal{B}$ consists of the following data:

- A set Obj$\mathcal{B}$ whose elements are called *objects*.
- For each pair (A, B) of objects a (small) set $\mathrm{Hom}(A, B)$ whose elements are called *1-cells*. If $f \in \mathrm{Hom}(A, B)$ we say A is the source and B is the target of f.
- For each pair of 1-cells (f, g) having the same source and target a (smaller) set whose elements are called 2-cells.

If two 1-cells f and g satisfy that the target of f is the same as the source of g then one can form the horizontal composition $g \circ f$. Also with 2-cells one can form the vertical composition.These operations are associative and have identities, well, in te case of 2-cells this is exact but in the case of 1-cells this properties are satisfied "up to isomorphism".
It turns out that the "good" notion of morphism between monoidal categories is that of a lax monoidal functor. We will denote the bicategory of monoidal categories with 1-cells lax monoidal functors and 2-cells monoidal natural transformations by $\mathcal{MON}$.

Proposition 2.2 *Let $\mathfrak{C}_1$, $\mathfrak{C}_2$, F_2, F_1 be two monoidal categories, F_1 left adjoint to F_2 as above and suppose that F_2 is lax monoidal with natural transformation $n_{A,B}$ and morphism n , then F_1 is colax monoidal.*

Proof. Just consider the natural transformation given by

$$m_{A,B} = ad^{-1}(n_{F_1(A),F_1(B)} \circ ad(\mathrm{Id}_{F_1(A)}) \otimes_1 ad(\mathrm{Id}_{F_1(B)})) \tag{2.2.6}$$

from $F_1(A \otimes_1 B)$ to $F_1(A) \otimes_2 F_1(B)$ and the morphism

$$m = ad^{-1}(n) \tag{2.2.7}$$

from $F_1(\mathbb{I}_1)$ to $\mathbb{I}_2$. ■
Moreover, suppose that F_1 is also lax monoidal, that is, all the adjunction lies inside $\mathcal{MON}$. Then we have the following

Proposition 2.3 *Suppose $\mathfrak{C}_1$, $\mathfrak{C}_2$, F_1, F_2 are as above with F_1 lax monoidal, then $\mathfrak{C}_1\mathfrak{C}_2$ becomes a monoidal category.*

Proof. We have to define a bifunctor

$$\otimes : \mathfrak{C}_1\mathfrak{C}_2 \times \mathfrak{C}_1\mathfrak{C}_2 \to \mathfrak{C}_1\mathfrak{C}_2$$

so take $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$ then

$$f \otimes g = \begin{cases} f \otimes_i g & A, B, C, D \in \mathfrak{C}_i & i = 1, 2 \\ F_1(f) \otimes_2 g & A, B \in \mathfrak{C}_1 & C, D \in \mathfrak{C}_2 \\ f \otimes_2 F_1(g) & C, D \in \mathfrak{C}_1 & A, B \in \mathfrak{C}_2 \\ F_1(f) \otimes_2 g \circ_2 m_{A,C} & A, B, C \in \mathfrak{C}_1 & D \in \mathfrak{C}_2 \\ f \otimes_2 F_1(g) \circ_2 m_{A,C} & A, C, D \in \mathfrak{C}_1 & B \in \mathfrak{C}_2 \\ f \otimes_2 g & A \in \mathfrak{C}_1 & B, C, D \in \mathfrak{C}_2 \\ f \otimes_2 g & C \in \mathfrak{C}_1 & A, B, D \in \mathfrak{C}_2 \\ f \otimes_2 g \circ_2 m_{A,C} & A, C \in \mathfrak{C}_1 & B, D \in \mathfrak{C}_2 \end{cases} . \tag{2.2.8}$$

Lets show this is indeed a functor so take $A_0 \xrightarrow{g_1} A_1 \xrightarrow{f_1} A_2$ and $B_0 \xrightarrow{g_2} B_1 \xrightarrow{f_2} B_2$. If all of A_0, A_1, A_2, B_0, B_1, B_2 lie in one of $\mathfrak{C}_1$ or $\mathfrak{C}_2$ it follows immediately from the functoriality of $\otimes_1$ or $\otimes_2$ respectively. Suppose all lie in $\mathfrak{C}_1$ except A_2 then we have

$$\begin{aligned}
f_1 \otimes f_2 \circ g_1 \otimes g_2 &= (f_1 \otimes_2 F_1(f_2) \circ_2 m_{A_1,B_1}) \circ (g_1 \otimes_1 g_2) \\
&= f_1 \otimes_2 F_1(f_2) \circ_2 m_{A_1,B_1} \circ_2 F_1(g_1 \otimes_1 g_2) \\
&= f_1 \otimes_2 F_1(f_2) \circ_2 F_1(g_1) \otimes_2 F_1(g_2) \circ_2 m_{A_0,B_0} \\
&= (f_1 \circ_2 F_1(g_1)) \otimes_2 (F_1(f_2) \circ_2 F_1(g_2)) \circ_2 m_{A_0,B_0} \\
&= (f_1 \circ g_1) \otimes_2 F_1(f_2 \circ_1 g_2) \circ_2 m_{A_0,B_0} \\
&= (f_1 \circ g_1) \otimes (f_2 \circ g_2)
\end{aligned}$$

If A_1 and A_2 are in $\mathfrak{C}_2$ and the rest in $\mathfrak{C}_1$ then

$$\begin{aligned}
f_1 \otimes f_2 \circ g_1 \otimes g_2 &= f_1 \otimes_2 F_1(f_2) \circ (g_1 \otimes_2 F_1(g_2) \circ_2 m_{A_0,B_0}) \\
&= f_1 \otimes_2 F_1(f_2) \circ_2 g_1 \otimes_2 F_1(g_2) \circ_2 m_{A_0,B_0} \\
&= (f_1 \circ_2 g_1) \otimes_2 (F_1(f_2) \circ_2 F_1(g_2)) \circ_2 m_{A_0,B_0} \\
&= (f_1 \circ_2 g_1) \otimes_2 F_1(f_2 \circ_1 g_2) \circ_2 m_{A_0,B_0} \\
&= (f_1 \circ g_1) \otimes (f_2 \circ g_2)
\end{aligned}$$

If A_0, A_1, A_2 in $\mathfrak{C}_2$ and the others in $\mathfrak{C}_1$ then

$$\begin{aligned}
f_1 \otimes f_2 \circ g_1 \otimes g_2 &= f_1 \otimes_2 F_1(f_2) \circ_2 g_1 \otimes_2 F_1(g_2) \\
&= (f_1 \circ_2 g_1) \otimes_2 (F_1(f_2) \circ F_1(g_2)) \\
&= (f_1 \circ_2 g_1) \otimes_2 F_1(f_2 \circ_1 g_2) \\
&= (f_1 \circ g_1) \otimes (f_2 \circ g_2)
\end{aligned}$$

If B_0 and B_1 are in $\mathfrak{C}_1$ and the rest are in $\mathfrak{C}_2$ then

$$\begin{aligned}
f_1 \otimes f_2 \circ g_1 \otimes g_2 &= f_1 \otimes_2 f_2 \circ_2 g_1 \otimes_2 F_1(g_2) \\
&= (f_1 \circ_2 g_1) \otimes_2 (f_2 \circ_2 F_1(g_2)) \\
&= (f_1 \circ g_1) \otimes_2 (f_2 \circ g_2) \\
&= (f_1 \circ g_1) \otimes (f_2 \circ g_2)
\end{aligned}$$

If B_0 is in $\mathfrak{C}_1$ the rest in $\mathfrak{C}_2$ then it is trivial. If A_2 and B_2 are in $\mathfrak{C}_2$ and the rest in $\mathfrak{C}_1$ then

$$\begin{aligned}
f_1 \otimes f_2 \circ g_1 \otimes g_2 &= (f_1 \otimes_2 f_2 \circ_2 m_{A_1,B_1}) \circ g_1 \otimes_1 g_2 \\
&= f_1 \otimes_2 f_2 \circ_2 m_{A_1,B_1} \circ_2 F_1(g_1 \otimes_1 g_2) \\
&= f_1 \otimes_2 f_2 \circ_2 F_1(g_1) \otimes_2 F_1(g_2) \circ_2 m_{A_0,B_0} \\
&= (f_1 \circ_2 F_1(g_1)) \otimes_2 (f_2 \circ_2 F_1(g_2)) \circ_2 m_{A_0,B_0} \\
&= (f_1 \circ_2 F_1(g_1)) \otimes (f_2 \circ_2 F_1(g_2)) \\
&= (f_1 \circ g_1) \otimes (f_1 \circ g_2)
\end{aligned}$$

If A_1, A_2, and B_2 are in $\mathfrak{C}_2$ and the rest in $\mathfrak{C}_1$ then

$$\begin{aligned} f_1 \otimes f_2 \circ g_1 \otimes g_2 &= f_1 \otimes_2 f_2 \circ (g_1 \otimes_2 F_1(g_2) \circ_2 m_{A_0,B_0}) \\ &= f_1 \otimes_2 f_2 \circ_2 g_1 \otimes_2 F_1(g_2) \circ_2 m_{A_0,B_0} \\ &= (f_1 \circ_2 g_1) \otimes_2 (f_2 \circ_2 F_1(g_2)) \circ_2 m_{A_0,B_0} \\ &= (f_1 \circ g_1) \otimes_2 (f_2 \circ g_2) \circ_2 m_{A_0,B_0} \\ &= (f_1 \circ g_1) \otimes (f_2 \circ g_2) \end{aligned}$$

If A_0 and B_0 are in $\mathfrak{C}_1$ and the rest in $\mathfrak{C}_2$ then

$$\begin{aligned} f_1 \otimes f_2 \circ g_1 \otimes g_2 &= f_1 \otimes_2 f_2 \circ (g_1 \otimes_2 g_2 \circ_2 m_{A_0,B_0}) \\ &= f_1 \otimes_2 f_2 \circ_2 g_1 \otimes_2 g_2 \circ_2 m_{A_0,B_0} \\ &= (f_1 \circ_2 g_1) \otimes_2 (f_2 \circ_2 g_2) \circ_2 m_{A_0,B_0} \\ &= (f_1 \circ g_1) \otimes (f_2 \circ g_2) \end{aligned}$$

The identity object is

$$\mathbb{I} = \mathbb{I}_1$$

and we define the natural isomorphisms $\kappa : A \otimes \mathbb{I} \to A$ by

$$\kappa_A = \begin{cases} \kappa^1_A & A \in \mathfrak{C}_1 \\ \kappa^2_A \circ \mathrm{Id}_A \otimes m & A \in \mathfrak{C}_2 \end{cases}, \tag{2.2.9}$$

$\lambda : \mathbb{I} \otimes A \to A$ defined by

$$\lambda_A = \begin{cases} \lambda^1_A & A \in \mathfrak{C}_1 \\ \lambda^2_A \circ m \otimes \mathrm{Id}_A & A \in \mathfrak{C}_2 \end{cases} \tag{2.2.10}$$

and $\alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ by

$$\alpha_{A,B,C} = \begin{cases} \alpha^i_{A,B,C} & A,B,C \in \mathfrak{C}_i & i = 1,2 \\ \alpha^2_{F_1(A),F_1(B),C} \circ m_{A,B} \otimes \mathrm{Id}_C & A,B \in \mathfrak{C}_1 & C \in \mathfrak{C}_2 \\ \alpha^2_{F_1(A),B,F_1(C)} & A,C \in \mathfrak{C}_1 & B \in \mathfrak{C}_2 \\ \mathrm{Id}_A \otimes m^{-1}_{B,C} \circ \alpha^2_{A,F_1(B),F_1(C)} & B,C \in \mathfrak{C}_1 & A \in \mathfrak{C}_2 \\ \alpha^2_{A,B,F_1(C)} & C \in \mathfrak{C}_1 & A,B \in \mathfrak{C}_2 \\ \alpha^2_{A,F_1(B),C} & B \in \mathfrak{C}_1 & A,C \in \mathfrak{C}_2 \\ \alpha^2_{F_1(A),B,C} & A \in \mathfrak{C}_1 & B,C \in \mathfrak{C}_2 \end{cases}. \tag{2.2.11}$$

We have to check that these are indeed natural isomorphisms, the isomorphism part follows directly from the definitions. We won't do all of them because it's a tedious exercise. Nevertheless let's do it for α.
Take $A_1 \xrightarrow{f} A_2$, $B_1 \xrightarrow{g} B_2$ and $C_1 \xrightarrow{h} C_2$. Suppose $A_1, A_2, B_1 \in \mathfrak{C}_1$ and $B_1, C_1, C_2 \in \mathfrak{C}_2$ then

$$\begin{aligned} \alpha_{A_2,B_2,C_2} \circ (f \otimes g) \otimes h &= \alpha^2_{F_1(A_2),B_2,C_2} \circ (F_1(f) \otimes_2 g \circ_2 m_{A_1,B_1}) \otimes h \\ &= \alpha^2_{F_1(A_2),B_2,C_2} \circ_2 (F_1(f) \otimes_2 g \circ_2 m_{A_1,B_1}) \otimes_2 h \\ &= \alpha^2_{F_1(A_2),B_2,C_2} \circ_2 (F_1(f) \otimes_2 g \circ_2 m_{A_1,B_1}) \otimes_2 (h \circ_2 \mathrm{Id}_{C_1}) \\ &= \alpha^2_{F_1(A_2),B_2,C_2} \circ_2 (F_1(f) \otimes_2 g) \otimes_2 h \circ_2 m_{A_1,B_1} \otimes_2 \mathrm{Id}_{C_1} \\ &= F_1(f) \otimes_2 (g \otimes_2 h) \circ_2 \alpha^2_{F_1(A_1),F_1(B_1),C_1} \circ_2 m_{A_1,B_1} \otimes_2 \mathrm{Id}_{C_1} \\ &= f \otimes (g \otimes h) \circ \alpha_{A_1,B_1,C_1} \end{aligned}$$

The rest of the cases are similar. The last thing we have to check are the coherence conditions but these, once more, are an easy (although lengthy) exercise. ■

2.3 The site $\mathfrak{M}_0\mathfrak{R}$

We will use some pieces of bicategory theory. If $\mathcal{B}$ is a bicategory we will denote horizontal composition by $\circ$ and vertical composition by $*$

Definition 2.7 *Let $\mathcal{B}$ be a bicategory. An* adjoint pair $(f_1 : X_1 \to X_2, f_2 : X_2 \to X_1)$ *in $\mathcal{B}$ is a pair of 1-cells such that there exist 2-cells $\eta_1 : X_1 \to f_2 \circ f_1$ and $\eta_2 : f_1 \circ f_2 \to X_2$ such that*

$$(\eta_2 \circ f_1) * (f_1 \circ \eta_1) = f_1$$

and

$$(f_2 \circ \eta_2) * (\eta_1 \circ f_2) = f_2$$

Let (f_1, f_2) be an adjoint pair as in the definition. In the case of categories this gives a usual pair of adjoint functors. We want to generalize the construction of the gluing of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ to this more general context.

Definition 2.8 *Let $(f_1 : X_1 \to X_2, f_2 : X_2 \to X_1)$ be an adjoint pair in a bicategory $\mathcal{B}$. Then we say that an object X is a* gluing *of X_1 and X_2 along (f_1, f_2) if there are essentially unique 1-cells $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$ with 2-cells $\alpha_1 : i_1 \to i_2 \circ f_1$ and $\alpha_2 : i_1 \circ f_2 \to i_2$ such that*

$$(i_2 \circ \eta_2) * (\alpha_1 \circ f_2) = \alpha_2$$

and such that for any other object Y with 1-cells j_1, j_2 and 2-cells β_1, β_2 satisfying the same properties, there exists an essentially unique 1-cell f (called the gluing of j_1 and j_2 along (β_1, β_2)) such that

$$f \circ i_1 \simeq j_1$$

and

$$f \circ i_2 \simeq j_2.$$

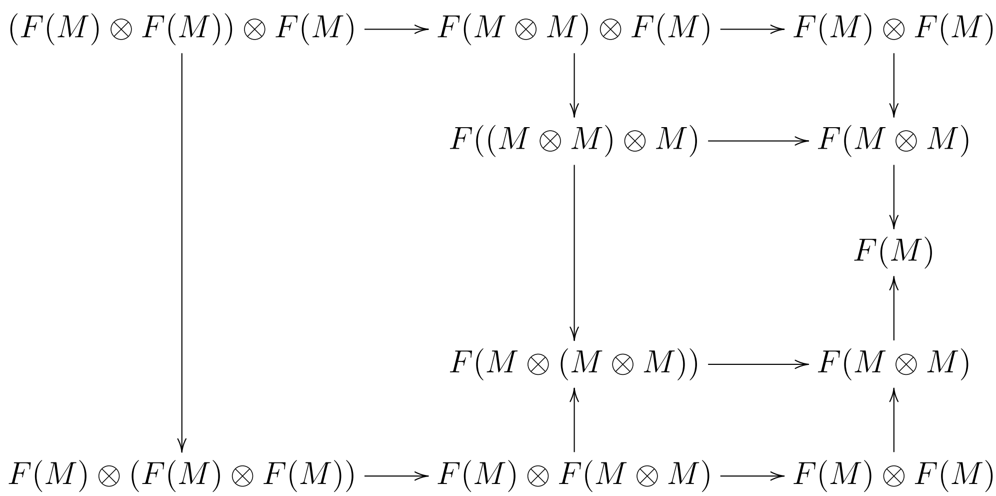
In the case of categories or monoidal categories we easily can prove the following

Proposition 2.4 *In the bicategories $\mathcal{CAT}$ and $\mathcal{MON}$ the gluing of categories* is *a gluing.*

Consider the bicategory $\mathcal{MON}$, the objects are monoidal categories, the 1-cells are lax monoidal functors and the 2-cells are monoidal natural transformations. A monoidal natural transformation is a natural transformation which correctly interacts with the lax monoidal structure of the functors.
We can see that an adjoint pair in this bicategory corresponds to the ones we used in the previous proposition. Also it is possible to see the construction of the gluing of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ as a lax colimit both in the bicategory $\mathcal{MON}$ and $\mathcal{CAT}$.
In the following we will put Connes and Consani's $\mathbb{F}_1$-schemes in the framework of Toën and Vaquié. The basic idea is that a triple $(\underline{X}, X, e_X)$ can be seen as a functor in the category of monoids of a certain cosmos.

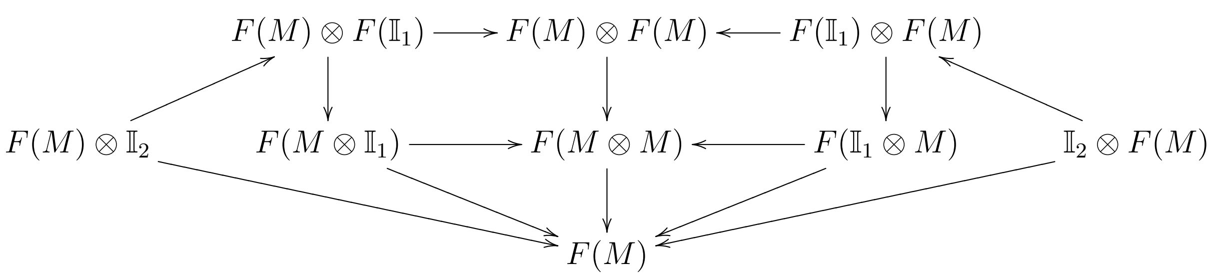
Proposition 2.5 *There is a bifunctor from $\mathcal{MON}$ to $\mathcal{CAT}$ which takes a symmetric monoidal category and gives the category of monoid objects inside it.*

Proof. The functor will be denoted by Ξ.
First we have to say how Ξ treats the 1-cells and 2-cells. So take a lax monoidal functor $F : \mathfrak{M}_1 \to \mathfrak{M}_2$ with natural transformation $\mu_{A,B} : F(A)\otimes F(B) \to F(A\otimes B)$ and $\mu : \mathbb{I}_2 \to F(\mathbb{I}_1)$, we have to check it induces a functor between $\Xi(\mathfrak{M}_1)$ and $\Xi(\mathfrak{M}_2)$.
Let M be a monoid in $\mathfrak{M}_1$ with product $p : M \otimes M \to M$ and identity $e : \mathbb{I}_1 \to M$, we have natural product and identity defined in $F(M)$ by $p' = F(p) \circ \mu_{M,M}$ and $e' = F(e) \circ \mu$. Associativity follows from the commutativity of the following diagram



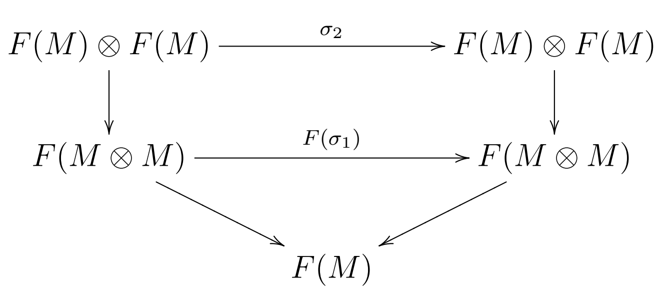
(2.3.1)

Identity follows from the commutativity of the following diagram



(2.3.2)

And symmetry comes from



(2.3.3)

Also, take a monoidal natural transformation $\eta : F_1 \to F_2 : \mathfrak{M}_1 \to \mathfrak{M}_2$ it induces a natural transformation from $\Xi(F_1)$ to $\Xi(F_2)$, the only thing we have to check is that for a monoid M

in $\Xi(\mathfrak{M}_1)$ η_M is a morphism of monoids, this comes from the commutativity of the following two diagrams

$$\begin{CD}
F_1(M)\otimes F_1(M) @>>> F_2(M)\otimes F_2(M)\\
@VVV @VVV\\
F_1(M\otimes M) @>>> F_2(M\otimes M)\\
@VVV @VVV\\
F_1(M) @>>> F_2(M)
\end{CD} \tag{2.3.4}$$

$$\begin{array}{ccc}
 & \mathbb{I}_2 & \\
\swarrow & & \searrow\\
F_1(\mathbb{I}_1) & \longrightarrow & F_2(\mathbb{I}_1)\\
\downarrow & & \downarrow\\
F_1(M) & \longrightarrow & F_2(M)
\end{array} \tag{2.3.5}$$

Finally it is clear that Ξ preserves vertical composition and horizontal composition strictly. So we have proved Ξ is a strict bifunctor. ■

Now we will apply the previous construction to a particular case. So consider the category $\mathfrak{S}_0$ of pointed sets. The morphisms are just functions which preserve the distinguished point.

Take S_0 and S_1 in $\mathfrak{S}_0$, with distinguished points 0_0 and 0_1 respectively. We define

$$S_0 \otimes S_1 = S_0 \times S_1/\sim \tag{2.3.6}$$

where $\sim$ is an equivalence relation where $(0_1, a) \sim (b, 0_2)$ for all a and b and the rest of the points are just equivalent to themselves.

Proposition 2.6 *The category $\mathfrak{S}_0$ with that tensor product and identity the single point forms a cosmos.*

The proof is trivial.

Consider also the category $\mathfrak{Ab}$ of abelian groups, it is known that it is also a monoidal category with the usual tensor product.

Also, we have a pair of adjoint functors between them: $For : \mathfrak{Ab} \to \mathfrak{S}_0$ the forgetful functor that sends each abelian group to its underlying pointed set (the identity is the point). This has a left adjoint $\mathbb{Z}[\_]$ which sends every pointed set S to the group $\mathbb{Z}[S]$ the free abelian group generated by the non-distinguished points of S.

Proposition 2.7 *The pair $(\mathbb{Z}[\_], For)$ is an adjoint pair in $\mathcal{MON}$.*

So we can apply all the above constructions to this pair and form the monoidal category $\mathfrak{S}_0\mathfrak{Ab}$. This is again a cosmos.

Even more, the category of monoids of this category is equivalent to the gluing of the categories $\mathfrak{M}_0$ and $\mathfrak{R}$ along the pair $(\mathbb{Z}[\_], For)$, since Ξ preserves adjunctions. And this category has $\mathbb{F}_1$ as initial object.

Now we can apply Toën and Vaquié's construction and obtain $\mathfrak{S}_0\mathfrak{Ab}$-schemes.
From Proposition 2.1 we immediately notice that an $\mathbb{F}_1$-scheme gives an $\mathfrak{S}_0\mathfrak{Ab}$-scheme, but not conversely since there's also the issue of the bijective natural transformation on fields.
So we can give an alternative, equivalent, definition of $\mathbb{F}_1$-schemes, but also one can generalize to $\mathbb{F}_1$-algebras.

Definition 2.9 *Let R be an object of $\mathfrak{M}_0\mathfrak{R}$. An A-algebra is an element A of $\mathfrak{M}_0\mathfrak{R}$ together with a morphism $R \to A$.*

Definition 2.10 *Let R be an object of $\mathfrak{M}_0\mathfrak{R}$. An R-scheme is a sheaf in the category of R-algebras such that , in every field, the natural transformation associated is bijective.*

We saw that not all $\mathfrak{S}_0\mathfrak{Ab}$ schemes are $\mathbb{F}_1$-schemes. In particular not every affine $\mathfrak{S}_0\mathfrak{Ab}$-scheme is an $\mathbb{F}_1$-scheme, but we do have the following.

Proposition 2.8 *For every M in $\mathfrak{M}_0$ $\text{Spec}M$ defines an $\mathbb{F}_1$-scheme.*

Proof. Clearly, by Proposition 2.1 $\text{Spec}M$ defines a triple $(\underline{X}, X_{\mathbb{Z}}, e)$ as we want. We just have to prove that for every field F, e induces a bijection between $\underline{X} \circ For(F)$ and $X_{\mathbb{Z}}(F)$ and in this case this is $\text{Hom}(M, For(F)) \to \text{Hom}(M, F)$ by definition of morphisms in $\mathfrak{M}_0\mathfrak{R}$ this is true. ■

Corollary 2.1 $\text{Spec}\mathbb{F}_1$ *and* $\text{Spec}\mathbb{F}_{1^n}$ *are $\mathbb{F}_1$-schemes.*

Let n be a positive integer and let $\mathbb{A}^n$ be the functor $\mathbb{A}^n : \mathfrak{M}_0\mathfrak{R} \to \mathfrak{S}$ given by $\mathbb{A}^n(X) = X^n$. It is called the affine space of dimension n.

Proposition 2.9 *The functor $\mathbb{A}^n$ is an $\mathbb{F}_1$-scheme.*

Proof. Just consider the, monoid $\mathbb{F}_1[X_1, X_2, ..., X_n]$, it doesn't take time to convince oneself that $\mathbb{A}^n$ is representable by it. So by Proposition 2.8 we are done. ■
Finally we define the extension of scalars.

Definition 2.11 *Let M_0 be an element of $\mathfrak{M}_0\mathfrak{R}$ and M_1 be an M_0-algebra. Let S be an M_0-scheme, then we define the extension of scalars of S to M_1 as the restriction of the functor S to the subcategory of M_1-algebras. We donote it by*

$$S \times_{M_0} \text{Spec}M_1$$

.

Chapter 3

Algebraic groups

3.1 Group objects

Consider an arbitrary category $\mathfrak{C}$ and form the category $\mathrm{Hom}(\mathfrak{C}^{op}, \mathfrak{G})$. For any functor F in this category we can compose it with the forgetful functor from $\mathfrak{G}$ to $\mathfrak{S}$ and obtain an element of $\mathrm{Hom}(\mathfrak{C}^{op}, \mathfrak{S})$. If this functor is representable by an object X then we call it a group object in the category $\mathfrak{C}$.
If $\mathfrak{C}$ has finite products this is equivalent to having morphisms μ, e and i such that:

$$\mu : X \times X \to X$$

$$i : X \to X$$

$$e : *_{\mathfrak{C}} \to X$$

$$\begin{array}{ccc} X \times *_{\mathfrak{C}} & \xrightarrow{\mathrm{Id}_X \times e} & X \times X \\ & \searrow & \downarrow{\scriptstyle \mu} \\ & & X \end{array} \tag{3.1.1}$$

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\mu \times \mathrm{Id}_X} & X \times X \\ {\scriptstyle \mathrm{Id}_X \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ X \times X & \xrightarrow[\mu]{} & X \end{array} \tag{3.1.2}$$

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{Id}_X \times i} & X \times X \\ \downarrow & & \downarrow{\scriptstyle \mu} \\ \mathfrak{C} & \xrightarrow[e]{} & X \end{array} \tag{3.1.3}$$

In the case that $\mathfrak{C}$ is the category of schemes we get group schemes, in the case that it is the category of $\mathbb{F}_1$-schemes we get $\mathbb{F}_1$ group schemes.
Examples:

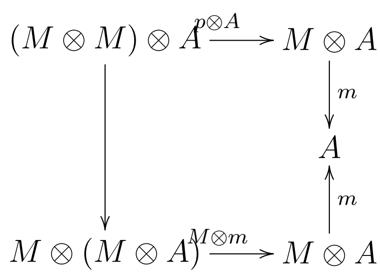
Example 3.1 *Consider the functor $\mathbb{G}_m$ which sends an object X of $\mathfrak{M}_0\mathfrak{R}$ to its multiplicative group $X^{\times}$. This is a group object in $\mathfrak{M}_0\mathfrak{R}$.*

This is a functor because for any morphism $f : X \to Y$ in $\mathfrak{M}_0\mathfrak{R}$ an invertible element of X has to go to an invertible element in Y so this is a group homomorphism. This is representable by $\mathbb{F}_1[X, X^{-1}]$ so by proposition 2.8 this is an $\mathbb{F}_1$-scheme. More generally, by restricting this functor to the smaller categorie of M-algebras one can define accordingly the multiplicative group over M.

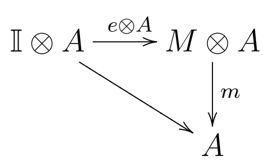
Definition 3.1 *An $\mathbb{F}_{1^n}$-group scheme is a group object in the category of $\mathbb{F}_{1^n}$-schemes.*

3.2 GL_n

Take and arbitrary symmetric monoidal closed complete and cocomplete category $\mathfrak{C}$ (i.e. a cosmos). For a monoid M in $\mathfrak{C}$ we define Mod_M as the category of modules over M, where a module is an object A of $\mathfrak{C}$ with a multiplication $M \otimes A \xrightarrow{m} A$ such that the following diagrams commute



(3.2.1)

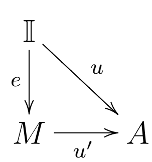


(3.2.2)

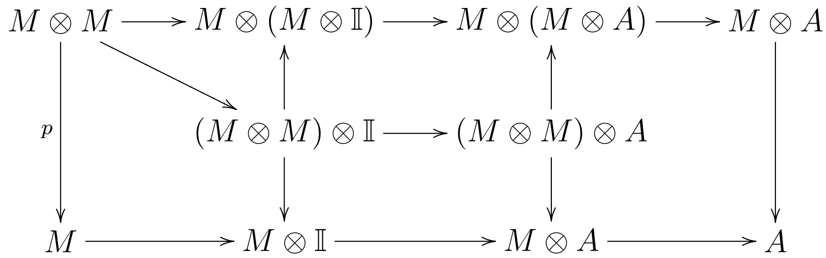
A morphism in Mod_M is a morphism in $\mathfrak{C}$ that preserves multiplication.

Example 3.2 *Every monoid M is naturally an M-module with multiplication given by the multiplication of the monoid.*

Lemma 3.1 *Given an M-module A and a morphism $u : \mathbb{I} \to A$ there is a unique morphism of M-modules $u' : M \to A$ extending u, or more explicitly, u' fits in the following commutative diagram*

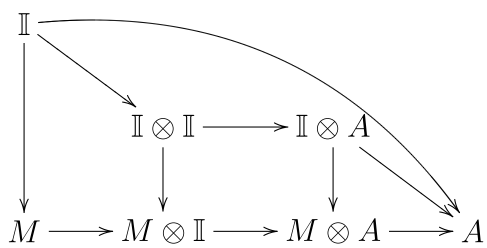


Proof. For existence we define $u' = m \circ M \otimes u \circ \rho^{-1}$. The fact that it is a morphism of monoids follows from the commutativity of



(3.2.3)

and it extends u because the diagram



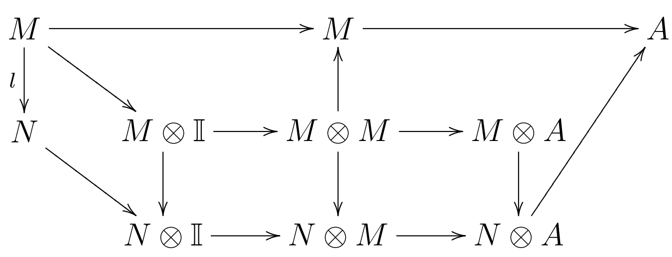
(3.2.4)

commutes. The unicity follows also from the previous diagram. ■

Let M and N be monoids of $\mathfrak{C}$ and l a monoid morphism from M to N then every N-module A is naturally an M-module with multiplication $m \circ l \otimes A$.

Lemma 3.2 *In the above situation, given an M-module morphism $F : M \to A$, it can be extended uniquely to an N-module homomorphism $F' : N \to A$.*

Proof. Existence and unicity follows from the previous lemma by composing F with $e : \mathbb{I} \to M$ because we get morphisms $l \circ e$ and $F \circ e$ satisfying the hypothesis of that lemma, we just have to check that it does extend F which follows from the commutative diagram



(3.2.5)

■

The aim of the section is to provide a definition of $GL_n(M)$ for every monoid M in every cosmos by defining a functor from the category of M-algebras to $\mathfrak{G}$.

Proposition 3.1 *The coproduct in Mod_M commutes with the forgetful functor from Mod_M to $\mathfrak{C}$.*

Proof. Let $\{A_i\}_{i\in I}$ be a family of M-modules. It is enough to check that $\coprod_{i\in I} A_i$ is an M-module. As the tensor product is a left adjoint it preserves coproducts so $M \otimes \coprod A_i$ is a coproduct of the $M \otimes A_i$, so the multiplication morphisms $m_i : M \otimes A_i \to A_i$ induce a multiplication $m : M \otimes \coprod A_i \to \coprod A_i$.
The axioms are satisfied immediately because they are satisfied by the A_i and using the universal property of coproducts. ■
Now we can formulate our definition

Definition 3.2 *Let $\mathfrak{C}$ be a cosmos, then the* General Linear Group GL_n *is the functor* $GL_n : \mathfrak{C} \to \mathfrak{G}$ *given by*

$$Gl_n(A) = \mathrm{Aut}(\coprod_1^n A) \tag{3.2.6}$$

where Aut *denotes the group of automorphisms (as A-modules).*

The main theorem of this section is

Theorem 3.1 *GL_n is a $\mathfrak{C}$-group functor.*

Proof. We have to show that the definition given is functorial, so take a morphism $f : M \to N$ in $\Xi(\mathfrak{C})$. Take also an automorphism α of $\coprod_1^n M$. Denote by $i_1, ..., i_n$ the immersions from M to $\coprod_1^n M$ and by $j_1, ..., j_n$ the immersions from N to $\coprod_1^n N$.
By composing f with the j's we get $j_1 \circ f, ..., j_n \circ f$, which are M-morphisms from M to $\coprod_1^n N$, so by the universal property of coproducts we get an M-morphism $F : \coprod_1^n M \to \coprod_1^n N$.
Now consider the M-morphisms $F \circ \alpha \circ i_1, ..., F \circ \alpha \circ i_n$, by Lemma 3.2 they can be extended to N-morphisms $b_1, ..., b_n$ from N to $\coprod_1^n N$ and by universality of coproducts this gives an endomorphism $\hat{f}(\alpha)$ of $\coprod_1^n N$. Moreover, by construction, it satisfies $\hat{f}(\alpha) \circ F = F \circ \alpha$. Using the same construction for the inverse of α we see that $\hat{\alpha}$ is an automorphism.
Even more, by the way we constructed it, one can see that $\hat{f}$ commutes with composition and preserves identities, so it is a group homomorphism. So if we set $GL_n(f) = \hat{f}$ we are done.
■

3.3 Chevalley groups as $\mathbb{F}_{1^2}$-schemes

The construction of Chevalley gives a group scheme $\mathcal{G}$ over $\mathbb{Z}$. This $\mathcal{G}$ can be seen as a functor from rings to groups, the objective of this section is to extend $\mathcal{G}$ to a functor from $\mathbb{F}_{1^2}$-$\mathfrak{A}$ to sets and show that it gives an $\mathbb{F}_{1^2}$-scheme following Connes and Consani.

Definition 3.3 *A root system $\Omega = (L, \Phi, \Phi^\vee)$ is a triple consisting of:*

- *A lattice L, that is, a free module over $\mathbb{Z}$ of finite dimension.*
- *A finite subset Φ of L whose elements are called roots.*
- *A set $\Phi^\vee$ of group homomorphisms $\phi^\vee : L \to \mathbb{Z}$, one assigned to each root ϕ and called its co-root.*

Satisfying:

- Φ *generates* $L \otimes \mathbb{Q}$ *as a vector space.*
- $\phi^\vee(\phi) = 2$ *for every* ϕ *in* Φ.
- *If* ϕ *and* $k\phi$ *are in* Φ *with* k *in* $\mathbb{Q}$ *then* $k = \pm 1$.
- *For each* $\phi_1, \phi_2 \in \Phi$, $\phi_2 - \phi_1^\vee(\phi_2)\phi_1$ *belongs to* Φ

Root systems come form the theory of Lie groups, to every compact Lie group (more precisely to every Lie algebra) one can associate a root system. What Chevalley did was to build the analogous algebraic groups over finite fields. Chevalley associates to every root system a group scheme $\mathcal{G}$ over $\mathbb{Z}$ which, for diferent root systems and finite fields $\mathcal{G}(\mathbb{F}_q)$ gives one of the so-called finite groups of Lie type.
What Tits suggested was that one can also define such groups for the field with one element, and that that would give the Weyl group of the root system.
To each root ϕ one associates a reflection $\rho_\phi : L \to L$ defined by $\rho_\phi(x) = x - \phi^\vee(x)\phi$.
The group generated by all the ρ_ϕ, $\phi \in \Phi$, is called the *Weyl group* of the root system and is denoted by W. An element ρ of W is called a reflection if it is conjugate to one of the generating reflections for some $\phi \in \Phi$. The set of reflections in W will be denoted by R.

Proposition 3.2 *In a root system* Ω *the lattice* L *can be bijected with* $\mathbb{Z}$ *in such a way that it divides* Φ *in two, positive and negative. The set of positive roots is denoted by* Φ^+ *and it satisfies the following:*

- *If* ϕ_1 *and* ϕ_2 *are in* Φ^+ *and* $\phi_1 + \phi_2$ *is in* Φ *then* $\phi_1 + \phi_2$ *is in* Φ^+
- *Exactly one of* ϕ *and* $-\phi$ *belongs to* Φ^+ *for every* ϕ *in* Φ

Now we define Φ^0 to be the set $\{\phi_i | i \in I\}$ of indecomposable roots, that is, the set of roots in Φ^+ which can't be written as a sum of other positive roots with positive coefficients. Also we define m_{ij}, for $i, j \in I$, as the minimum integer such that $(\rho_{\phi_i} \circ \rho_{\phi_j})^{m_{ij}} = 1$.

Proposition 3.3 *With that definition* $m_{ij} \geq 1$, *it is equal to* 1 *if* $i = j$ *and it is greater than* 1 *if* i *is different from* j

Now we define the *extended Coxeter group* $V(\Omega)$ which is defined by the following generators and relations: the generators are q_i with $i \in I$ and g_ρ with $\rho \in R$, and the relations are

$$q_i q_j q_i \ldots = q_j q_i q_j \ldots \tag{3.3.1}$$

where there are m_{ij} factors on each side,

$$q_i^2 = g_{\rho_{\phi_i}}, \tag{3.3.2}$$

$$q_i g_\rho q_i^{-1} = g_{\rho_{\phi_i}(\rho)} \tag{3.3.3}$$

and

$$g_\rho g_{\rho'} = g_{\rho'} g_\rho. \tag{3.3.4}$$

We also define $U(\Omega) \subset V(\Omega)$ the subgroup generated by all the g_ρ with $\rho \in R$.
Let A be an $\mathbb{F}_{1^2}$-algebra (that is a monoid or ring A together with an element ϵ such that $\epsilon^2 = 1$). Then we define $T_\Omega(A) = \mathrm{Hom}(L, A^*)$. Also for each $\rho \in R$ we define h_ρ an element of $T_\Omega(A)$ defined by $h_\rho(x) = \epsilon^{\phi^\vee(x)}$ for $\rho = \rho_\phi$ and extended to all of R.

Definition 3.4 *The normalizer functor is the functor* $\mathcal{N}_\Omega$ *given by* $\mathcal{N}_\Omega(A) = T_\Omega(A) \rtimes V(\Omega)/H$ *where* H *is the graph of the homomorphism* $U(\Omega) \to T_{\Omega(A)}$ *defined in the generators of* $U(\Omega)$ *by* $g_\rho \mapsto h_\rho^{-1}$

We also have a projection $p : \mathcal{N}_\Omega(A) \to W(\Omega)$ which is induced by $\mathrm{Id} \times f$ where f is the group homomorphism $V(\Omega) \to W(\Omega)$ given by $f(q_i) = \rho_{\phi_i}$ and $f(g_\rho) = 1$.

Definition 3.5 *For* w *in* W *we define* $\Phi_w = \{\phi \in \Phi^+ | w(\phi) < 0, w \in W\}$

Let $\mathcal{G}$ be the Chevalley group scheme associated with the root system Ω. We recall the following standard construction. We have a maximal torus $\mathcal{T}$ of $\mathcal{G}$ and its normalizer $\mathcal{N}$. To every root r corresponds a subgroup $\mathcal{X}_r$ of $\mathcal{G}$ and an isomorphism $x_r : \mathbb{A} \to \mathcal{X}_r$.
Recall that the maximal unipotent group of $\mathcal{G}, \mathcal{U}$ is the subgroup generated by $\mathcal{X}_r$ with $r \in \Phi^+$ and $t \in A$. Define $\mathcal{U}_w$ as the subgroup generated by $\mathcal{X}_r$ with $r \in \Phi_w$.
The following is a theorem of Chevalley. Recall that the Weyl group of $\mathcal{G}$ is defined as $W = \mathcal{N}(K)/\mathcal{T}(K)$ for any field K (it doesn't depend on the choice of K).

Theorem 3.2 *Let* K *be a field, and let* $a \in \mathcal{G}(K)$ *then there exists a unique* $w \in W$ *and a unique triple* (x, n, x') *such that* $x \in \mathcal{U}(K)$, $n \in \mathcal{N}(K)$, $x' \in \mathcal{U}_w(K)$ *with* $p(n) = w$ *satisfying* $a = xnx'$.

Now we can define the functor $\underline{\mathcal{G}}$.

Definition 3.6 *Let* Ω *be a root system, then we define the* Chevalley functor $\underline{\mathcal{G}} : \mathbb{F}_{1^2}\text{-}\mathfrak{A} \to \mathfrak{S}$ *as*

$$\underline{\mathcal{G}}_\Omega = \underline{\mathbb{A}}^{\Phi^+} \times \coprod_{w \in W} (p^{-1}(w) \times \underline{\mathbb{A}}^{\Phi_w}) \tag{3.3.5}$$

where by abuse of notation we use $\underline{\mathbb{A}}$ *to denote the extension of scalars*

$$\underline{\mathbb{A}} \times_{\mathbb{F}_1} \mathrm{Spec}\mathbb{F}_{1^2}$$

We finally get to the main theorem of this section, which is a rephrasing of Theorem 5.1 of [6].

Theorem 3.3 $\underline{\mathcal{G}}_\Omega$ *is an* $\mathbb{F}_{1^2}$*-scheme and it satisfies*

$$\underline{\mathcal{G}}_\Omega \times_{\mathbb{F}_{1^2}} \mathrm{Spec}\mathbb{Z} \simeq \mathcal{G}. \tag{3.3.6}$$

Proof.
The fact that the restrictions to $\mathfrak{M}_0$ and $\mathfrak{R}$ are respectively an $\mathfrak{M}_0$-scheme and a scheme follows immediately from the definition by Proposition 2.9.
Given a field K we have to show that the natural transformation $e_{\mathcal{G}}$ induced by $\underline{\mathcal{G}}_{\Omega}$ is bijective on K-points, but this also follows readily by the bijectivity of the corresponding natural transformation of $\underline{\mathbb{A}}$.
Now, for the last part, let $\mathcal{G}' = \underline{\mathcal{G}} \times_{\mathbb{F}_{1^2}} \mathbb{Z}$. We know by Theorem 3.2 that for every field K every element of $\mathcal{G}$ can be written uniquely as a product xnx' where $x \in \mathcal{U}(K)$, $x' \in \mathcal{U}_w(K)$ and $n \in \mathcal{N}(K)$ with $p(n) = w$ for some w in the Weyl group. So we can define a morphism $\phi : \mathcal{G}' \to \mathcal{G}$ and clearly it is bijective on K-points. Now $\mathcal{G}$ and $\underline{\mathcal{G}} \times_{\mathbb{F}_1} \mathbb{Z}$ are schemes over $\mathbb{Z}$ so we can consider each of the fibers over $\mathbb{F}_p$ and over $\mathbb{Q}$.It is clear that both $\mathcal{G}$ and $\mathcal{G}'$ are smooth. So the fibers are also smooth. As ϕ is an isomorphism on the geometric points for every fiber, we see that ϕ is an isomorphism on the fibers, so it is a closed immersion.
By the fiberwise criterion for flatness ([1], exp. IV) ϕ is flat. A closed immersion which is flat is also open, so ϕ is an isomorphism onto one of the connected components of $\mathcal{G}$,but $\mathcal{G}$ is connected so ϕ is an isomorphism.

■

One notices that this doesn't say anything about $\underline{\mathcal{G}}$ being a group scheme and in fact one cannot prove so much, nonetheless this is the closest result to Tits proposal.

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